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Introduction

Initial-boundary value problems for higher order linear hyperbolic equations with two independent variables were studied in [1] and [2], where necessary and sufficient conditions of solvability and unique solvability of well-posed cases, as well as in ill-posed cases, were established.

In our research, we study the case with three independent variables. For that, the methods developed in [1] and [2] were modified.

In the set $\Omega_T = [0,\omega_1]\times[0,\omega_2]\times[0,T]$, consider the equation

$$u^{(m,n,l)} = p_1(y,t)u^{(m,0,l)} + p_2(x,t)u^{(0,n,l)} + p_0(x,y,t)u^{(0,0,l)} + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sum_{k=0}^{l-1} p_{ijk}(x,y,t)u^{(i,j,k)} + q(x,y,t) \qquad (1)$$

subject to the boundary and initial conditions

$$A_i[u(\cdot,y,t)] = 0 \ (i=1,\cdots,m), \qquad (2)$$

$$B_j[u(x,\cdot,t)] = 0 \ (j=1,\cdots,n), \qquad (3)$$

$$u^{(0,0,k)}(x,y,0) = \varphi_k(x,y) \quad (k=1,\cdots,l). \qquad (4)$$

Here m, n and l are positive integers, $\omega_1 > 0$, $\omega_2 > 0$, $T > 0$, $D = [0,\omega_1]\times[0,\omega_2]$, p_0, p_1, p_2, p_{ijk} and $q \in C(\Omega_T)$, $\varphi_k \in C^{m,n}(D)$, and $A_i: C^{m-1}([0,\omega_1]) \to \mathbb{R}$ and $B_j: C^{n-1}([0,\omega_2]) \to \mathbb{R}$ are bounded linear functionals.

Particular cases of the boundary conditions (2),(3) are the periodic boundary conditions

$$u^{(i-1,0,0)}(0,y,t) = u^{(i-1,0,0)}(\omega_1,y,t) \quad (i=1,\cdots,m), \qquad (2_P)$$

$$u^{(0,j-1,0)}(x,0,t) = u^{(0,j-1,0)}(x,\omega_2,t) \quad (j=1,\cdots,n), \qquad (3_P)$$

and the Dirichlet boundary conditions

$$u^{(i-1,0,0)}(0,y,t) = u^{(i-1,0,0)}(\omega_1,y,t) = 0 \quad (i=1,\cdots,m_0), \qquad (2_D)$$

$$u^{(0,j-1,0)}(x,0,t) = u^{(0,j-1,0)}(x,\omega_2,t) = 0 \quad (j=1,\cdots,n_0). \qquad (3_D)$$

In our research, the primary focus was on well-posed cases. In Theorem 1 we established the relation between the well-posedness of problem (1),(2),(3),(4) and unique solvability of problems (I),(II) and (III).

Examples 1, 2 and 3 demonstrate what may happen if one of the problems (I),(II) and (III) has a nontrivial solution. Also, Example 3 shows that problem (1),(2),(3),(4) may be uniquely solvable even in ill-posed case, i.e. unique solvability of problem (1),(2),(3),(4) does not guarantee its well-posedness.

The Main Results

Theorem 1

Let the following boundary value problems

$$v^{(m)} = p_2(x,t)v, \quad A_i[v] = 0 \quad (i=1,\cdots,m); \qquad (I)$$

$$w^{(n)} = p_1(y,t)w, \quad B_j[w] = 0 \quad (j=1,\cdots,n); \qquad (II)$$

and

$$z^{(m,n)} = p_1(y,t)z^{(m,0)} + p_2(x,t)z^{(0,n)} + p_0(x,y,t)z \qquad (III)$$

$$A_i[z(\cdot,y)] = 0 \ (i=1,\cdots,m), \qquad B_j[z(x,\cdot)] = 0 \ (j=1,\cdots,n)$$

have only trivial solutions ***for every*** $t \in [0,T]$. *Then problem* (1),(2),(3),(4) *is well-posed, i.e., it has a unique solution* u *admitting the estimate*

$$\| u \|_{C^{m,n,l}(\Omega_t)} \le M\left(\sum_{k=1}^{l} \| \varphi_k \|_{C^{m,n}(D)} + \| q \|_{C(\Omega_t)}\right) \qquad t \in [0,T],$$

where M *is a positive constant independent of* φ_k *and* q.

Remark 1

The conditions of Theorem 1 are sharp and cannot be weakened. In fact, if any of the problems (I), (II) and (III) have a nontrivial solution for at least one $t \in [0,T]$, then problem (1),(2)(3),(4) may not be solvable. More precisely, violation of the conditions of Theorem 1 may result in:

1) Existence of a local, but not a global solution;
2) Existence of unique classical and infinite dimensional set of generalized solutions;
3) Existence of infinite dimensional set of classical solutions;
4) Nonexistence of solutions.

Corollary 1

Let $m = 2m_0$, $n = 2n_0$,

$$(-1)^{n_0} p_1(y,t) < 0 \quad \text{for} \quad y \in [0,\omega_2], \quad t \in [0,T],$$

$$(-1)^{m_0} p_2(x,t) < 0 \quad \text{for} \quad x \in [0,\omega_1], \quad t \in [0,T],$$

$$(-1)^{m_0+n_0} p_0(x,y,t) < 0 \quad \text{for} \quad (x,y,t) \in \Omega_T.$$

Then problem $(1),(2_P),(3_P),(4)$ is well-posed.

Corollary 2

Let $m = 2m_0$, $n = 2n_0$,

$$(-1)^{n_0} p_1(y,t) \le -p_1^*(t) \le 0 \ \text{for}\ y \in [0,\omega_2], \quad t \in [0,T],$$

$$(-1)^{m_0} p_2(x,t) \le -p_2^*(t) \le 0 \ \text{for}\ x \in [0,\omega_1], \quad t \in [0,T],$$

$$(-1)^{m_0+n_0} p_0(x,y,t)\left(\frac{\omega_1\,\omega_2}{\pi^2}\right)^{mn} < 1 + p_1^*(t)\left(\frac{\omega_2}{\pi}\right)^{n} + p_2^*(t)\left(\frac{\omega_1}{\pi}\right)^{m} \ \text{for}\ (x,y,t) \in \Omega_T. \qquad (5)$$

Then problem $(1),(2_D),(3_D),(4)$ is well-posed.

Corollary 3

Let $m = 2m_0$, $n = 2n_0+1$, $p_1(y,t)$ *be either positive or negative*

$$(-1)^{m_0} p_2(x,t) < 0 \quad \text{for} \quad x \in [0,\omega_1], \quad t \in [0,T],$$

$$(-1)^{m_0} p_0(x,y,t)\, p_1(x,t) < 0 \quad \text{for} \quad (x,y,t) \in \Omega.$$

Then problem $(1),(2_P),(3_P),(4)$ is well-posed.

Corollary 4

Let $m = 2m_0+1$, $n = 2n_0+1$, $p_1(y,t) \equiv p_1(t)$, $p_2(x,t) \equiv p_2(t)$, $p_0(x,y,t) \equiv p_0(t)$, *and*

$$p_0(t)\ p_1(t)\, p_2(t) < 0.$$

Then problem $(1),(2_P),(3_P),(4)$ is well-posed.

Example 1

The problem

$$u^{(2,2,1)} = u^{(2,0,1)} + u^{(0,2,1)} + (t-1)u^{(0,0,1)} + 1$$

$$u^{(i-1,0,0)}(0,y,t) = u^{(i-1,0,0)}(\omega_1,y,t) \quad (i=1,2),$$

$$u^{(0,j-1,0)}(x,0,t) = u^{(0,j-1,0)}(x,\omega_2,t) \quad (j=1,2),$$

$$u(x,y,0) = 0$$

has a unique solution $u(x,y,t) = -\ln(1-t)$, which blows-up at $t=1$. This happens because problem (III) has a nontrivial solution when $t=1$.

Example 2

The problem

$$u^{(2,2,1)} = u^{(2,0,1)} + u^{(0,2,1)} + (t-1)u^{(0,0,1)} + \sin x \sin y$$

$$u(0,y,t) = u(\pi,y,t) = 0, \ u(x,0,t) = u(x,\pi,t) = 0,$$

$$u(x,y,0) = 0$$

has a unique solution $u(x,y,t) = -\sin x \sin y \ln(4-t)$, which blows-up at $t=4$. This happens because problem (III) has a nontrivial solution when $t=4$. Also, this example demonstrates that in condition (5) the strict inequality cannot be replaced by a nonstrict one.

Example 3

Let $p(t) \ge 0$ and $I_p = \{t \in [0,T] \mid p(t) = 0\}$. Then the problem

$$u^{(1,1,1)} = u^{(1,0,1)} + u^{(0,1,1)} - p(t)u^{(0,0,1)} + p(t)q(t)$$

$$u(0,y,t) = u(\omega_1,y,t), \ u(x,0,t) = u(x,\omega_2,t), \ u(x,y,0) = 0$$

has a solution $u(x,y,t) = \int_0^t h(\tau)\, d\tau$, where

$$h(t) = \begin{cases} q(t) & \text{if } p(t) > 0 \\ q(t) + \Theta(t) & \text{if } p(t) = 0 \end{cases}$$

and $\Theta(t)$ is an arbitrary continuous function such that $p(t)\Theta(t) \equiv 0$. If I_p is nowhere dense in [0,T], then the solution is unique (and $\Theta(t) \equiv 0$). Otherwise the problem has an infinite dimensional set of solutions.

References

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2. T. Kiguradze and T. Kusano, On ill-posed initial-boundary value problems for higher order linear hyperbolic equations with two independent variables. *Differ. Equ.* **39** (2003), N 10, 1454-1470.