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Ravi P. Agarwal

Donal O'Regan

Baoqiang Yan

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POSITIVE SOLUTIONS FOR SINGULAR THREE-POINT BOUNDARY-VALUE PROBLEMS

RAVI P. AGARWAL, DONAL O'REGAN, BAOQIANG YAN

Abstract. Using the theory of fixed point index, this paper discusses the existence of at least one positive solution and the existence of multiple positive solutions for the singular three-point boundary value problem:

$$y''(t) + a(t)f(t, y(t), y'(t)) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta),$$

where $0 < \alpha < 1$, $0 < \eta < 1$, and f may be singular at $y = 0$ and $y' = 0$.

1. Introduction

In this paper, we consider the singular three-point boundary-value problem (BVP):

$$y''(t) + a(t)f(t, y(t), y'(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta), \tag{1.2}$$

where $0 < \alpha < 1$, $0 < \eta < 1$, f may be singular at $y = 0$ and $y' = 0$, and $a \in C((0, 1), (0, \infty))$.

When $f(t, x, z)$ has no singularity at $x = 0$ and $z = 0$, there are many results on the existence of solutions to (1.1)-(1.2) with different boundary conditions such as $x(0) = 0$, $x(1) = \delta x(\eta)$, or $x(0) = x_0$, $x(\eta) - x(1) = x_1$ (see [4, 5, 7, 8, 9]). Also when $f(t, x, z) = f(t, x)$ has no singularity at $x = 0$, using Krasnoselkii's fixed point theorem, Liu citel1 discussed the existence of positive solutions to (1.1)-(1.2). In [3, 14], the authors obtained the existence of at least one positive solutions to (1.1)-(1.2) when $f(t, x, z)$ is singular at $x = 0$ and $z = 0$.

The features in this article, that differ from those in [3, 14], are as follows. Firstly, the nonlinearity $f(t, x, z)$ may be sublinear in x at $x = +\infty$ and the degree of singularity in x and z may be arbitrary; i.e., $f(t, x, z)$ contains $\frac{1}{x^\alpha}$, x^β and $\frac{1}{(-z)^{-\gamma}}$ for any $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Secondly, (1.1)-(1.2) may have at least two positive solutions. Thirdly, (1.1)-(1.2) may have no positive solutions.

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There are main five sections in our paper. In sections 2, we discuss a special Banach space and define a new cone in this space, and some lemmas are proved for convenience. In section 3, we discuss the nonexistence of positive solutions to (1.1)-(1.2). In section 4, the existence of at least one positive solution to (1.1)-(1.2) is presented when $f(t,x,z)$ is singular at $x=0$ and $z=0$. In section 5, we consider the existence of at least two positive solutions to (1.1)-(1.2) when $f(t,x,z)$ is singular at $x=0$ and $z=0$ and f is suplinear at $x=+\infty$. Some of the ideas in this paper were motivated from [1, 2, 12, 13].

2. Preliminaries

Let

$$C^1[0,1]=\{y:[0,1]\to R:\ y(t) \text{ and } y'(t) \text{are continuous on } [0,1]\}$$

with norm $\|y\|=\max\{\max_{t\in[0,1]}|y(t)|,\max_{t\in[0,1]}|y'(t)|\}$ and

$$P=\{y\in C^1[0,1]:y(t)\geq 0,\forall t\in[0,1]\}.$$

Obviously $C^1[0,1]$ is a Banach space and P is a cone in $C^1[0,1]$. The following lemmas are needed later.

Lemma 2.1 (citeg3)**.** *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\bar{\Omega}\cap P\to P$ be continuous and compact. Suppose*

$$\lambda Ax\neq x,\quad \forall x\in\partial\Omega\cap P,\ \lambda\in(0,1], \tag{2.1}$$

then $i(A,\Omega\cap P,P)=1$.

Lemma 2.2 ([6])**.** *Let Ω be a bounded open set in real Banach space E, P be a cone of E, $\theta\in\Omega$ and $A:\bar{\Omega}\cap P\to P$ be continuous and compact. Suppose*

$$Ax\neq x,\quad \forall\ x\in\partial\Omega\cap P, \tag{2.2}$$

then $i(A,\Omega\cap P,P)=0$.

Lemma 2.3 ([11])**.** *Let $0<\alpha<1$, a, $h\in C((0,1),(0,\infty))$, $a,h\in L^1[0,1]$ and*

$$\begin{aligned}y(t)=&\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)h(\tau)\,d\tau\,ds-\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)h(\tau)\,d\tau\,ds\\&-\int_0^t\int_0^s a(\tau)h(\tau)\,d\tau\,ds.\end{aligned}$$

Then

$$\min_{t\in[0,1]}y(t)\geq\frac{\alpha(1-\eta)}{1-\alpha\eta}\max_{t\in[0,1]}|y(t)|. \tag{2.3}$$

Lemma 2.4. *Assume that $f\in C((0,1)\times(0,+\infty)\times(-\infty,0),(0,+\infty))$, that $a\in C((0,1),(0,+\infty))$, and that for any constant $H>0$ there exists a function $\Psi_H(t)$ continuous on $(0,1)$, and positive on $(0,1)$ and a constant $0\leq\gamma<1$ such that*

$$f(t,x,z)\geq\Psi_H(t)(-z)^\gamma,\quad \forall t\geq 0,\ 0<x\leq H,\ z<0, \tag{2.4}$$

where $\int_0^1 a(s)\Psi_H(s)ds<+\infty$. Then there is a $c_0>0$ such that for any positive solution $x\in C[0,1]$ with $x'(t)<0$ for all $t\in(0,1)$ to (1.1)-(1.2) we have

$$x(t)\geq c_0,\quad t\in[0,1]. \tag{2.5}$$

Moreover, if $x_0 \in C[0,1]$ *is a positive solution to*

$$y''(t) + a(t)f(t, \max\{c_0, y(t)\}, -|y'(t)| - \frac{1}{n}) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta),$$

where $\frac{\alpha}{1-\alpha}(1-\eta)\frac{1}{n} < c_0$, x_0 *is a positive solution to*

$$y''(t) + a(t)f(t, y(t), -|y'(t)| - \frac{1}{n}) = 0, \quad 0 < t < 1,$$
$$y'(0) = 0, \quad y(1) = \alpha y(\eta).$$

Proof. Assume that x is a positive solution to (1.1)-(1.2) with $x'(t) < 0$ for $t \in (0,1)$. Then Lemma 2.3 implies $\min_{t\in[0,1]} x(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| > 0$.

Let $H = 1$. Then there exists a function $\Psi_1(t)$ continuous on $[0,1]$, and positive on $(0,1)$ a constant $0 \leq \gamma < 1$ such that

$$f(t,x,z) \geq \Psi_1(t)(-z)^{\gamma}, \quad \forall t \geq 0,\ 0 < x \leq 1,\ z < 0.$$

There are two cases to be considered: (1) $x(t) \geq 1$ for all $t \in [0,1]$. (2) $x(1) < 1$. Let $t_* = \inf\{t | x(t) < 1 \text{ for all } s \in [t,1]\}$. If $t_* > 0$, we have $x(t_*) = 1$ and $x(0) \geq 1$. Then, Lemma 2.3 yields

$$\min_{t\in[0,1]} |x(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}. \tag{2.6}$$

If $t_* = 0$ and $x(t_*) = 1$, Lemma 2.3 implies

$$\min_{t\in[0,1]} |x(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x(t)| \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \tag{2.7}$$

also. If $t_* = 0$ and $x(t_*) < 1$, from (2.4), we have

$$-x''(t) = a(t)f(t, x(t), x'(t)) \geq a(t)\Psi_1(t)(-x'(t))^{\gamma}, \quad t \in (0,1).$$

Also note

$$-\frac{x''(t)}{(-x'(t))^{\gamma}} \geq a(t)\Psi_1(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}(-x'(t))^{1-\gamma} \geq \int_0^t a(s)\Psi_1(s)ds, t \in (0,1),$$

which implies

$$-x'(t) \geq [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Integration from η to 1 yields

$$x(\eta) - x(1) \geq \int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}} dt.$$

Since $x(1) = \alpha x(\eta)$, we have

$$x(1) \geq \frac{\alpha}{1-\alpha}\int_\eta^1 [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}} dt. \tag{2.8}$$

Let $c_0 = \frac{1}{2}\min\{1, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \frac{\alpha}{1-\alpha}\int_\eta^1 [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}} dt\}$. Combining (2.6), (2.7) and (2.8), we have

$$\min_{t\in[0,1]} x(t) \ge c_0.$$

Suppose that x_0 satisfies

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, -|x_0'(t)| - \frac{1}{n}) = 0, \quad t \in (0,1),$$
$$x_0'(0) = 0, \quad x_0(\eta) = \alpha x_0(1).$$

Then $x_0''(t) < 0$ and so $x_0(t) < 0$ for $t \in (0,1)$. Then x_0 satisfies

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = 0, \quad t \in (0,1),$$
$$x_0'(0) = 0, \quad x_0(\eta) = \alpha x_0(1).$$

There are are two cases to be considered:
(1) $x_0(t) \ge 1$ for all $t \in [0,1]$. In this case, since $c_0 \le 1$, we have

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = x_0''(t) + a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0,$$

for $0 < t < 1$.
(2) $x_0(1) < 1$. Let $t_* = \inf\{t | x_0(t) < 1 \text{ for all } s \in [t,1]\}$. If $t_* > 0$, we have $x_0(t_*) = 1$ and $x_0(0) \ge 1$. Then

$$\min_{t\in[0,1]} x_0(t) \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x_0(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta}.$$

If $t_* = 0$ and $x_0(t_*) = 1$, we have

$$\min_{t\in[0,1]} x_0(t) \ge \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |x_0(t)| \ge \frac{\alpha(1-\eta)}{1-\alpha\eta}$$

also. If $t_* = 0$ and $x_0(t_*) < 1$, from (2.4), we have

$$-x_0''(t) = a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) \ge a(t)\Psi_1(t)(-x_0'(t) + \frac{1}{n})^\gamma, \quad t \in (0,1).$$

Also note

$$-\frac{x_0''(t)}{(-x_0'(t) + \frac{1}{n})^\gamma} \ge a(t)\Psi_1(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}[(-x_0'(t) + \frac{1}{n})^{1-\gamma} - (\frac{1}{n})^{1-\gamma}] \ge \int_0^t a(s)\Psi_1(s)ds, t \in (0,1),$$

which implies

$$-x_0'(t) + \frac{1}{n} \ge [(1-\gamma)\int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}, t \in (0,1).$$

Integration from η to 1 yields

$$x_0(\eta) - x_0(1) \ge \int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}]dt - (1-\eta)\frac{1}{n}.$$

Since $x_0(1) = \alpha x_0(\eta)$, we have

$$x_0(1) \ge \frac{\alpha}{1-\alpha}\int_\eta^1 [(1-\gamma)\int_\eta^1 \int_0^t a(s)\Psi_1(s)ds]^{\frac{1}{1-\gamma}}]dt - \frac{\alpha}{1-\alpha}(1-\eta)\frac{1}{n} \ge c_0.$$

Consequently, the definition of c_0 implies that $x_0(t) \geq c_0$ for all $t \in [0,1]$. Therefore,

$$x_0''(t) + a(t)f(t, \max\{c_0, x_0(t)\}, x_0'(t) - \frac{1}{n}) = x_0''(t) + a(t)f(t, x_0(t), x_0'(t) - \frac{1}{n}) = 0,$$

for $0 < t < 1$. The proof is complete. □

To discuss the existence of multiple positive solutions, we construct a new space. Let $q(t) = 1 - t$, $t \in [0,1]$ and

$$C_q^1[0,1] = \{y : [0,1] \to R : y(t) \text{ and } q(t)y'(t) \text{ are continuous on } [0,1]\}$$

with norm $\|y\|_q = \max\{\max_{t\in[0,1]} |y(t)|, \max_{t\in[0,1]} q(t)|y'(t)|\}$ and

$$P_q = \{y \in C_q^1[0,1] : \min_{t\in[0,1]} y(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} \max_{t\in[0,1]} |y(t)| \text{ and } y(0) \geq \max_{t\in[0,1]} q(t)|y'(t)|\}.$$

Lemma 2.5. *The set $C_q^1[0,1]$ is a Banach space and P_q is cone in $C_q^1[0,1]$*

Proof. It is easy to see that $\|\cdot\|_q$ is a norm of the space C_q^1. Now we show that C_q^1 is a Banach space. Assume that $\{x_n\}_{n=1}^{\infty} \subseteq C_q^1$ is a Cauchy sequence; i.e., for each $\varepsilon > 0$, there is a $N > 0$ such that

$$\|x_n - x_m\|_q < \varepsilon, \quad \forall n > N,\ m > N\,. \tag{2.9}$$

Then

$$\max_{t\in[0,1]} |x_n(t) - x_m(t)| < \varepsilon, \quad \forall n > N,\ m > N.$$

Thus, there is a $x_0 \in C[0,1]$ such that

$$\lim_{n\to+\infty} \max_{t\in[0,1]} |x_n(t) - x_0(t)| = 0. \tag{2.10}$$

For $1 > \delta > 0$, since $(1-\delta)\max_{t\in[0,1-\delta]} |x_n'(t) - x_m'(t)| \leq \max_{t\in[0,1-\delta]} q(t)|x_n'(t) - x_m'(t)|$, we have

$$\max_{t\in[0,1-\delta]} |x_n'(t) - x_m'(t)| \leq \frac{1}{1-\delta} \max_{t\in[0,1-\delta]} q(t)|x_n'(t) - x_m'(t)| < \frac{1}{1-\delta}\varepsilon,$$

which implies that for any $\delta > 0$, $x_n'(t)$ is uniformly convergent on $[0, 1-\delta]$. Hence, $x_0(t)$ is continuously differentiable on $[0,1)$. And since $q(t)x_{N+1}'(t)$ is uniformly continuous on $[0,1]$, there exists a $\delta' > 0$ such that

$$|q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2)| < \varepsilon \quad \text{for } |t_1 - t_2| < \delta, t_1, t_2 \in [0,1).$$

Then

$$\begin{aligned}
&|q(t_1)x_0'(t_1) - q(t_2)x_0'(t_2)| \\
&= |q(t_1)x_0'(t_1) - q(t_1)x_{N+1}'(t_1) \\
&\quad + q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2) + q(t_2)x_{N+1}'(t_2) - q(t_2)x_0'(t_1)| \\
&\leq |q(t_1)x_0'(t_1) - q(t_1)x_{N+1}'(t_1)| \\
&\quad + |q(t_1)x_{N+1}'(t_1) - q(t_2)x_{N+1}'(t_2)| + |q(t_2)x_{N+1}'(t_2) - q(t_2)x_0'(t_1)| \\
&< 3\varepsilon, \quad \text{for } |t_1 - t_2| < \delta',\ t_1, t_2 \in [0,1),
\end{aligned}$$

which implies that $\lim_{t\to 1^-} q(t)x_0'(t)$ exists. Let $q(1)x_0(1) = \lim_{t\to 1^-} q(t)x_0'(t)$. Now from (2.9), we have for any $t \in [0,1]$,

$$q(t)|x_n'(t) - x_m'(t)| < \varepsilon, \quad \forall n > N, m > N.$$

Letting $m \to +\infty$, for all $t \in [0,1]$, we have

$$q(t)|x_n'(t) - x_0'(t)| \le \varepsilon, \quad \forall n > N. \tag{2.11}$$

Combining (2.10) and (2.11) shows $C_q^1[0,1]$ is a Banach space.

Clearly P_q is a cone of $C_q^1[0,1]$. The proof is complete. □

Lemma 2.6. *For each $y \in P_q$, $\|y\|_q = \max_{t\in[0,1]} |y(t)|$.*

Proof. For $y \in P$, obviously $\|y\|_q \ge \max_{t\in[0,1]} |y(t)|$. On the other hand, since $y \in P_q$,

$$\max_{t\in[0,1]} |y(t)| \ge y(0) \ge \max_{t\in[0,1]} q(t)|y'(t)|.$$

Then

$$\begin{aligned} \|y\|_q &= \max\{\max_{t\in[0,1]} |y(t)|, \max_{t\in[0,1]} q(t)|y'(t)|\} \\ &\le \max\{\max_{t\in[0,1]} |y(t)|, y(0)\} = \max_{t\in[0,1]} |y(t)|. \end{aligned}$$

Consequently, $\|y\|_q = \max_{t\in[0,1]} |y(t)|$. The proof is complete. □

Now we list the following conditions to be used in this article.

(H) $f \in C((0,1)\times(0,\infty)\times(-\infty,0),(0,\infty))$ and there are three functions $g, h \in C((0,+\infty),(0,+\infty))$, $\Phi \in C((0,1),[0,+\infty))$, with $\Phi(t) > 0$ for all $t \in (0,1)$, and

$$f(t,x,z) \le \Phi(t)h(x)g(|z|) \quad \forall\,(t,x,z) \in (0,1)\times(0,+\infty)\times(-\infty,0). \tag{2.12}$$

(H') For any constant $H > 0$ there exists a function $\Psi_H(t)$ continuous on $(0,1)$ and positive on $(0,1)$, and a constant $0 \le \gamma < 1$ such that

$$f(t,x,z) \ge \Psi_H(t)(-z)^\gamma, \quad \forall t \in (0,1),\ 0 < x \le H,\ z < 0, \tag{2.13}$$

where $\int_0^1 a(s)\Psi_H(s)ds < +\infty$.

For each $n \in N = \{1,2,\dots\}$, for $y \in P$ (or $y \in P_q$), define operators

$$\begin{aligned} (A_n y)(t) = {}& \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds \\ &- \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds \\ &- \int_0^t\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds, \end{aligned} \tag{2.14}$$

for $t \in [0,1]$ and $c_0 > 0$.

Suppose that (H) and (H') hold. A standard argument (see [6, 7]) applied to (2.14) yields that $A_n : P \to P$ is continuous and completely continuous for each $n \in N$.

Lemma 2.7. *Suppose* (H) *and* (H') *holds and $\int_0^1 a(t)\Phi(t)\sup_{\frac{1}{c}\le u\le\frac{1}{c}+\frac{1}{1-t}c} g(u)dt < +\infty$ for all $c > 1$. Then $A_n : P_q \to P_q$ is a continuous and completely continuous for each $n \in N$.*

Proof. For $y \in P_q$, it is easy to see that $|y'(t)| \le \frac{1}{1-t}\|y\|_q$ for all $t \in [0,1)$. Also (H) and Lemma 2.3 yield

$$\frac{\alpha(1-\eta)}{1-\alpha\eta}\max_{t\in[0,1]} |(A_n y)(t)| \le (A_n y)(t) < +\infty, \quad \forall t \in [0,1]$$

and $(A_ny)'(t) > -\infty$ for all $t \in [0,1)$. Moreover, since

$$\begin{aligned}
(A_ny)(0) &= \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds \\
&\quad - \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})\,d\tau\,ds \\
&\geq \int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|)\,d\tau\,ds \\
&= \int_0^1 (1-s)a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds
\end{aligned}$$

and

$$\begin{aligned}
q(t)|(A_nx)'(t)| &= (1-t)\int_0^t a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds \\
&\leq \int_0^1 (1-s)a(s)f(s,\max\{c_0,y(s)\},-|y'(s)|-\frac{1}{n})ds,
\end{aligned}$$

we have

$$(A_nx)(0) \geq \max_{t\in[0,1]} q(t)|(A_nx)'(t)|.$$

Consequently, $A_nP_q \subseteq P_q$ for each $n \in N = \{1,2,\dots\}$. Moreover, since

$$\lim_{t\to 1^-}|(A_ny)'(t)| = \int_0^1 a(s)f(s,\max\{c_0,y(s)\},|y'(s)|-\frac{1}{n})ds,$$

we can assume that $A_ny \in C^1[0,1]$.

Next we show that $A_n : P_q \to P_q$ is continuous and completely continuous. Suppose that $\{y_m\} \subseteq P_q$, $y_0 \in P_q$ with $\lim_{m\to+\infty}\|y_m - y_0\|_q = 0$. Then, there is an $M > c_0$ such that

$$\|y_m\|_q \leq M, \|y_0\|_q \leq M, \quad m \in N.$$

Then $|y_m'(t)| \leq M/(1-t)$ for $m \in \{1,2,\dots\}$ and so

$$f(t,\max\{c_0,y_m(t)\},-|y_m'(t)|-\frac{1}{n}) \leq \Phi(t)\max_{c_0\leq u\leq M}h(u)\sup_{\frac{1}{n}\leq u\leq\frac{1}{n}+\frac{1}{1-t}M}g(u),$$

for $t \in (0,1)$. Moreover, since

$$\lim_{m\to+\infty} f(t,\max\{c_0,y_m(t)\},-|y_m'(t)|-\frac{1}{n}) = f(t,\max\{c_0,y_0(t)\},-|y_0'(t)|-\frac{1}{n}),$$

for $t \in (0,1)$, the Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned}
&\max_{t\in[0,1]}|(A_ny_m)(t)-(A_ny_0)(t)| \\
&\leq \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)\Big|f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)|-\frac{1}{n}) \\
&\quad - f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})\Big|\,d\tau\,ds \\
&\quad + \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)\Big||f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)|-\frac{1}{n}) \\
&\quad - f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})Big|\,d\tau\,ds
\end{aligned}$$

$$+\int\int_0^1\int_0^s a(\tau)\Big|f(\tau,\max\{c_0,y_m(\tau)\},-|y_m'(\tau)| - \frac{1}{n}) - f(\tau,\max\{c_0,y_0(\tau)\},-|y_0'(\tau)|-\frac{1}{n})\Big|\,d\tau\,ds \to 0, \quad \text{as } m\to+\infty.$$

Since A_ny_m, $A_ny_0\in P_q$, Lemma 2.6 yields

$$\lim_{m\to+\infty}\|A_ny_m-A_ny_0\|_q=\max_{t\in[0,1]}|(A_ny_m)(t)-(A_ny_0)(t)|=0,$$

which implies that $A_n:P_q\to P_q$ is continuous.

Suppose $D\subseteq P_q$ is bounded. Then, there is an $M>c_0$ such that $\|y\|_q\le M$ for all $y\in D$. Then $|y'(t)|\le M/(1-t)$ for all $y\in D$, and so

$$f(t,\max\{c_0,y(t)\},-|y'(t)|-\frac{1}{n})\le\Phi(t)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-t}M}g(u),\quad t\in(0,1).$$

Thus

$$\begin{aligned}
&\max_{t\in[0,1]}|(A_ny)(t)|\\
&\le\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\quad+\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\quad+\int_0^1\int_0^s a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds\\
&\le\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)\Phi(s)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds\\
&\quad+\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s\Phi(s)a(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds\\
&\quad+\int_0^1\int_0^s a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\max_{t\in[0,1]}|(A_ny)'(t)|\\
&\le\max_{t\in[0,1]}\int_0^t a(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|d\tau\\
&\le\max_{t\in[0,1]}\int_0^t a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)d\tau.
\end{aligned}$$

Also A_nD is bounded in the norm $\|x\|_0=\max\{\max_{t\in[0,1]}|x(t)|,\max_{t\in[0,1]}|x'(t)|\}$. For t_1, $t_2\in[0,1]$, $y\in D$, we have

$$\begin{aligned}
&|(A_ny)(t_1)-(A_ny)(t_2)|\\
&=|\int_{t_1}^{t_2}\int_0^s a(\tau)\Phi(\tau)|f(\tau,\max\{c_0,y(\tau)\},-|y'(\tau)|-\frac{1}{n})|\,d\tau\,ds|\\
&\le|\int_{t_1}^{t_2}\int_0^s a(\tau)\Phi(\tau)\max_{c_0\le u\le M}h(u)\sup_{\frac{1}{n}\le u\le\frac{1}{n}+\frac{1}{1-\tau}M}g(u)\,d\tau\,ds|
\end{aligned}$$

and

$$\begin{aligned}|(A_n y)'(t_1) - (A_n y)'(t_2)| &= |\int_{t_1}^{t_2} a(\tau)|f(\tau, \max\{c_0, y(\tau)\}, -|y'(\tau)| - \frac{1}{n})|d\tau| \\ &\le |\int_{t_1}^{t_2} a(\tau)\Phi(\tau) \max_{c_0 \le u \le M} h(u) \sup_{\frac{1}{n} \le u \le \frac{1}{n} + \frac{1}{1-\tau}M} g(u) d\tau|,\end{aligned}$$

which implies that $\{(A_n y)(t)|y \in D\}$ and $\{(A_n y)'(t)|y \in D\}$ are equicontinuous on $[0, 1]$.

The Arzela-Ascoli Theorem guarantees that $A_n D$ and $(A_n D)'$ are relatively compact in $C[0, 1]$. Since

$$\begin{aligned}\|A_n y\|_q &= \max\{\max_{t\in[0,1]} |(A_n y)(t)|, \max_{t\in[0,1]} (1 - t)|(A_n y)'(t)|\} \\ &\le \max\{\max_{t\in[0,1]} |(A_n y)(t)|, \max_{t\in[0,1]} |(A_n y)'(t)|\},\end{aligned}$$

the set $A_n D$ is relatively compact in $C^1_q[0, 1]$. Consequently, $A_n : P_q \to P_q$ is continuous and completely continuous for each $n \in \{1, 2, \dots\}$. The proof is complete. □

3. Nonexistence of positive solutions to (1.1)-(1.2)

In this section, we notice that the presence of z in $f(t, x, z)$ can lead to the nonexistence of positive solutions to (1.1)-(1.2).

Theorem 3.1. *Suppose* (H) *holds and* $\int_0^z \frac{1}{g(r)} dr = +\infty$ *for all* $z \in (0, +\infty)$ *and* $\int_0^1 a(s)\Phi(s)ds < +\infty$. *Then* (1.1)-(1.2) *has no positive solution.*

Proof. Suppose $x_0(t)$ is a positive solution to (1.1)-(1.2). Then

$$\begin{gathered}x_0''(t) + a(t)f(t, x_0(t), x_0'(t)) = 0, \quad t \in (0, 1) \\ x_0'(0) = 0, \quad x_0(1) = \alpha x_0(\eta),\end{gathered}$$

which means that there is a $t_0 \in (0, 1)$ with $x_0'(t_0) < 0$, $x_0(t_0) > 0$ (otherwise $x'(t) \ge 0$ for all $t \in (0, 1)$ which would contradict $x(1) = \alpha x(\eta) < x(\eta)$). Let $t_* = \inf\{t < t_0 | x_0'(s) < 0 \text{ for all } s \in [t, t_0]\}$. Obviously, $t_* \ge 0$ and $x_0'(t_*) = 0$ and $x_0'(t) < 0$ for all $t \in (t_*, t_0]$. Condition (H) implies

$$-x_0''(t) \le a(t)f(t, x_0(t), x_0'(t)) \le a(t)\Phi(t)h(x_0(t))g(|x_0'(t)|), \quad \forall t \in (t_*, t_0),$$

and so

$$\frac{-x_0''(t)}{g(-x_0'(t))} \le a(t)\Phi(t)h(x_0(t)) \le a(t)\Phi(t)h(x_0(t)), \quad \forall t \in (t_*, t_0).$$

Integration from t to t_0 yields

$$\int_{-x_0'(t)}^{-x_0'(t_0)} \frac{1}{g(r)} dr = \int_t^{t_0} \frac{1}{g(-x_0'(s))} d(-x_0'(s)) \le \max_{u\in[x_0(t_0), x_0(t)]} h(u) \int_0^1 a(s)\Phi(s)ds.$$

Letting $t \to t_*$, we have

$$+\infty = \int_0^{-x_0'(t_0)} \frac{1}{g(r)} dr \le \max_{u\in[x_0(t_0), x_0(t_*)]} h(u) \int_0^1 a(s)\Phi(s)ds < +\infty,$$

a contradiction. Consequently, (1.1)-(1.2) has no positive solution. □

Example 3.2. Consider the boundary-value problem

$$x'' + (1-t)^a(|x'|)^a[x^b + (x+1)^{-d} + 1] = 0, \quad t \in (0,1),$$
$$x(0) = 0, \quad x(1) = \frac{1}{2}x(\frac{1}{2}),$$

where $a \geq 1$, $b > 1$, $d > 0$. This problem has no positive solution. It is easy to see that $f(t,x,z) = (1-t)^a(|z|)^a[x^b + (x+1)^{-d} + 1]$ for all $(t,x,z) \in [0,1] \times [0,+\infty) \times (-\infty,+\infty)$. Obviously, $g(r) = r^a$ and $\int_0^z \frac{1}{g(r)}dr = +\infty$ for all $z \in (0,+\infty)$. Then Theorem 3.1 guarantees that (3.2)-(3.2) has no positive solution.

4. Existence of at least one positive solution to (1.1)-(1.2)

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$ and Φ . Throughout this section we will assume that the following conditions hold:

- (H1) $a(t) \in C(0,1)$, $a(t) > 0$ for all $t \in (0,1)$;
- (H2) Conditions (H) and (H') hold and $I(z) = \int_0^z \frac{1}{g(r)}dr < +\infty$ for all $z \in [0,+\infty)$ with
$$\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds < \int_0^\infty \frac{dr}{g(r)}$$
for all $c \in [c_0,+\infty)$ and suppose
$$\sup_{c_0 \leq c < +\infty} \frac{c}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1,$$
where c_0 is defined in Lemma 2.4.

Theorem 4.1. *Suppose that* (H1)–(H2) *hold. Then* (1.1)-(1.2) *has at least one positive solution* $y_0 \in C[0,1] \cap C^2(0,1)$ *with* $y_0(t) > 0$ *on* $[0,1]$ *and* $y_0'(t) < 0$ *on* $(0,1)$.

Proof. Choose $R_1 > 0$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1. \tag{4.1}$$

From the continuity of I^{-1} and I, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s))\Phi(s)ds) + I(\varepsilon))} > 1. \tag{4.2}$$

Let $n_0 \in \{1,2,\dots\}$ with $\frac{1}{n_0} < \min\{\varepsilon, \frac{1}{2}\frac{1-\alpha}{\alpha(1-\eta)}c_0\}\}$ and let $N_0 = \{n_0, n_0+1, \dots\}$. Now (H1)–(H2) guarantee that for each $n \in N_0$, $A_n : P \to P$ is a continuous and completely continuous operator.

Let $\Omega_1 = \{y \in C^1[0,1] : \|y\| < R_1\}$. We now show that

$$y \neq \mu A_n y, \quad \forall y \in P \cap \partial\Omega_1, \mu \in (0,1], n \in N_0. \tag{4.3}$$

Suppose there exists a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $y_0 = \mu_0 A_n y_0$. It is easy to see that $y_0'(t) \leq 0$ and

$$y_0'(t) = -\mu_0 \int_0^t a(s)f(s, \max\{c_0, y_0(s)\}, y_0'(s) - \frac{1}{n})ds, t \in (0,1). \tag{4.4}$$

Also

$$y_0''(t)+\mu_0 a(t)f(t,\max\{c_0,y_0(t)\},y_0'(t)-\frac{1}{n})=0,\quad 0<t<1, \tag{4.5}$$

$$y_0'(0)=0, y_0(1)=\alpha y_0(\eta). \tag{4.6}$$

Therefore,

$$\begin{aligned}-y_0''(t)&=\mu_0 a(t)f(t,\max\{c_0,y_0(t)\},y_0'(t)-\frac{1}{n})\\&\le a(t)\Phi(t)h(\max\{c_0,y_0(t)\})g(-y_0'(t)+\frac{1}{n}),\forall t\in(0,1),\end{aligned}$$

which yields

$$\frac{-y_0''(t)}{g(-y_0'(t)+\frac{1}{n})}\le a(t)\Phi(t)h(\max\{c_0,y_0(t)\}),\quad \forall t\in(0,1).$$

Integration from 0 to t yields

$$\begin{aligned}I(-y_0'(t)+\frac{1}{n})-I(\frac{1}{n})&\le\int_0^t a(s)\Phi(s)h(\max\{c_0,y_0(s)\})ds\\&\le\sup_{c_0\le r\le R_1}h(r)\int_0^t a(s)\Phi(s)ds,\end{aligned}$$

and so

$$I(-y_0'(t)+\frac{1}{n})\le\sup_{c_0\le r\le R_1}h(r)\int_0^t a(s)\Phi(s)ds+I(\varepsilon).$$

Thus

$$-y_0'(t)\le I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big),\quad t\in(0,1). \tag{4.7}$$

Therefore,

$$y_0(t)-y_0(1)\le(1-t)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big). \tag{4.8}$$

Let $t=\eta$ in (4.8). Then

$$y_0(\eta)-y_0(1)\le(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big).$$

Since $y_0(1)=\alpha y_0(\eta)$, one has

$$(\frac{1}{\alpha}-1)y_0(1)\le(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big),$$

which yields

$$y_0(1)\le\frac{\alpha}{1-\alpha}(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big).$$

Then (4.8) implies

$$\begin{aligned}y_0(0)&\le y_0(1)+I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big)\\&=\frac{1-\alpha\eta}{1-\alpha}I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big).\end{aligned} \tag{4.9}$$

Now (4.7) and (4.9) guarantee that

$$\begin{aligned} R_1 &= \max\{\max_{t\in[0,1]}|y_0(t)|, \max_{t\in[0,1]}|y_0'(t)|\} \\ &\le \frac{1-\alpha\eta}{1-\alpha} I^{-1}\Big(\sup_{c_0\le r\le R_1} h(r)\int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big) \end{aligned}$$

which implies

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha}I^{-1}(\sup_{c_0\le r\le R_1} h(r)\int_0^1 a(s)\Phi(s)ds + I(\varepsilon))} \le 1.$$

This contradicts (4.2). Thus (4.3) is true.

From Lemma 2.1, for each $n \in N_0$, we have $i(A_n, \Omega_1 \cap P, P) = 1$. As a result, for each $n \in N_0$, there exists a $y_n \in \Omega_1 \cap P$, such that $y_n = A_n y_n$; i.e.,

$$\begin{aligned} y_n(t) &= \frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\,d\tau\,ds \\ &\quad - \frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\,d\tau\,ds \\ &\quad - \int_0^t\int_0^s a(\tau)f(\tau, \max\{c_0, y_n(\tau)\}, -|y_n'(\tau)| - \frac{1}{n})\,d\tau\,ds. \end{aligned}$$

It is easy to see that $y_n'(t) < 0$, and

$$y_n'(t) = -\int_0^t a(s)f(s, \max\{c_0, y_n(s)\}, y_n'(s) - \frac{1}{n})ds, \quad n \in N_0,\ t \in (0,1).$$

Now we consider $\{y_n(t)\}_{n\in N_0}$ and $\{y_n'(t)\}_{n\in N_0}$. Since $\|y_n\| \le R_1$, one has

$$\text{the functions belonging to } \{y_n\} \text{ are uniformly bounded on } [0,1], \tag{4.10}$$

$$\text{the functions belonging to } \{y_n'\} \text{ are uniformly bounded on } [0,1]. \tag{4.11}$$

Thus

$$\text{the functions belonging to } \{y_n\} \text{ are equicontinuous on } [0,1]. \tag{4.12}$$

A similar argument to that used to show (4.5) yields

$$\begin{gathered} y_n''(t) + a(t)f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\ y_n'(0) = 0, \quad y_n(1) = \alpha y_n(\eta). \end{gathered} \tag{4.13}$$

By Lemma 2.4, we have

$$y_n(t) \ge c_0, \quad \forall n \in N_0. \tag{4.14}$$

Now we claim that for any $t_1, t_2 \in [0,1]$,

$$|I(y_n'(t_2) - \frac{1}{n}) - I(y_n'(t_1) - \frac{1}{n})| \le \sup_{c_0\le r\le R_1} h(r))|\int_{t_1}^{t_2} a(t)\Phi(t))dt|. \tag{4.15}$$

Notice that

$$\begin{aligned} -y_n''(t) &= a(t)f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n}) \\ &\le a(t)|f(t, \max\{c_0, y_n(t)\}, y_n'(t) - \frac{1}{n})| \\ &\le a(t)\Phi(t)h(\max\{c_0, y_n(t)\})g(y_n'(t) - \frac{1}{n}), \quad \forall t \in (0,1), \end{aligned}$$

and

$$\begin{aligned} y_n''(t) &= -a(t)f(t,\max\{c_0,y_n(t)\},y_n'(t)-\frac{1}{n}) \\ &\le a(t)|f(t,\max\{c_0,y_n(t)\},y_n'(t)-\frac{1}{n})| \\ &\le a(t)\Phi(t)h(\max\{c_0,y_n(t)\})g(y_n'(t)-\frac{1}{n}), \forall t\in(0,1), \end{aligned}$$

which yields

$$\frac{-y_n''(t)}{g(y_n'(t)-\frac{1}{n})} \le a(t)\Phi(t)h(\max\{c_0,y_n(t)\}), \forall t\in(0,1), \tag{4.16}$$

$$\frac{y_n''(t)}{g(y_n'(t)-\frac{1}{n})} \le a(t)\Phi(t)h(\max\{c_0,y_n(t)\}), \forall t\in(0,1). \tag{4.17}$$

Note that the right hand sides are always positive in (4.16) and (4.17). For any t_1, $t_2\in[0,1]$ with $t_1<t_2$, we have

$$|\int_{t_1}^{t_2}\frac{1}{g(-y_n'(s)+\frac{1}{n})}d(-y_n'(s)+\frac{1}{n})| \le \sup_{c_0\le r\le R_1} h(r)|\int_{t_1}^{t_2}a(t)\Phi(t)dt|;$$

i.e., (4.15) is true.

Since I^{-1} is uniformly continuous on $[0,I(R_1)]$, for any $\bar\varepsilon>0$, there is a $\varepsilon'>0$ such that

$$|I^{-1}(s_1)-I^{-1}(s_2)|<\bar\varepsilon, \forall\ |s_1-s_2|<\varepsilon', \quad s_1,s_2\in[0,I(R_1)]. \tag{4.18}$$

Also (4.15) guarantees that, for $\varepsilon'>0$, there is a $\delta'>0$ such that

$$|I(y_n'(t_2)-\frac{1}{n})-I(y_n'(t_1)-\frac{1}{n})|<\varepsilon', \forall\ |t_1-t_2|<\delta', \quad t_1,t_2\in[0,1]. \tag{4.19}$$

Now (4.18) and (4.19) yield

$$\begin{aligned} |y_n'(t_2)-y_n'(t_1)| &= |-y_n'(t_2)+\frac{1}{n}+y_n'(t_1)-\frac{1}{n}| \\ &= |I^{-1}(I(-y_n'(t_2)+\frac{1}{n}))-I^{-1}(I(-y_n'(t_1)+\frac{1}{n}))| \\ &< \bar\varepsilon, \quad \forall\ |t_1-t_2|<\delta',\ t_1,t_2\in[0,1], \end{aligned}$$

which implies

$$\text{the functions belonging to } \{y_n'\} \text{ are equicontinuous on } [0,1]. \tag{4.20}$$

Consequently (4.10), (4.11), (4.12) and (4.20), the Arzela-Ascoli Theorem guarantees that $\{y_n\}$ and $\{y_n'\}$ are relatively compact in $C[0,1]$; i.e., there is a function $y_0\in C^1[0,1]$, and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to+\infty}\max_{t\in[0,1]}|y_{n_j}(t)-y_0(t)|=0, \quad \lim_{j\to+\infty}\max_{t\in[0,1]}|y_{n_j}'(t)-y_0'(t)|=0.$$

Since $y_{n_j}'(0)=0$, $y_{n_j}(1)=\alpha y_{n_j}(\eta)$, $y_{n_j}'(t)<0$, $y_{n_j}(t)>0$, $t\in(0,1)$, $j\in\{1,2,\dots\}$, then one has

$$y_0'(0)=0, y_0(1)=\alpha y_0(\eta), y_0'(t)\le 0, y_0(t)\ge 0, t\in(0,1). \tag{4.21}$$

Now since $\sup_{n\ge1}\|y_n\|\le R_1$, (H') guarantees that there exists a $\Psi_{R_1}(t)$ continuous and $\Psi_{R_1}(t)>0$ on $(0,1)$ such that

$$f(t,x,z)\ge\Psi_{R_1}(t)(-z)^\gamma, \quad t\in(0,1),\ x\in(0,R_1],\ z<0.$$

Then

$$-y''_{n_j}(t) = a(t)f(t, \max\{c_0, y_{n_j}(t)\}, y'_{n_j}(t) - \frac{1}{n_j})) \ge a(t)\Psi_{R_1}(t)(-y'_{n_j}(t) + \frac{1}{n_j})^{\gamma},$$

for $t \in (0,1)$. Also note that

$$-\frac{y''(t)}{(-y'_{n_j}(t) + \frac{1}{n_j})^{\gamma}} \ge a(t)\Psi_{R_1}(t), \quad t \in (0,1).$$

Integrating from 0 to t, we have

$$\frac{1}{1-\gamma}(-y'_{n_j}(t) + \frac{1}{n_j})^{1-\gamma} - \frac{1}{1-\gamma}(\frac{1}{n_j})^{1-\gamma} \ge \int_0^t a(s)\Psi_1(s)ds, \quad t \in (0,1),$$

which implies

$$-y'_{n_j}(t) + \frac{1}{n_j} \ge [(1-\gamma)(\int_0^t a(s)\Psi_1(s)ds + \frac{1}{1-\gamma}(\frac{1}{n_j})^{1-\gamma})]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Letting $j \to +\infty$, we have

$$-y'_0(t) \ge [(1-\gamma)(\int_0^t a(s)\Psi_1(s)ds)]^{\frac{1}{1-\gamma}}, \quad t \in (0,1).$$

Consequently, $y'_0(t) < 0$ for all $t \in (0,1)$, which together with $y_0(1) > 0$ guarantees that $y_0(t) > 0$ for all $t \in [0,1]$. Therefore,

$$\min\{\min_{s\in[\frac{1}{2},t]} y_0(s), \min_{s\in[\frac{1}{2},t]} |y'_0(s)|\} > 0, \quad \text{for all } t \in [\frac{1}{2}, 1),$$

$$\min\{\min_{s\in[t,\frac{1}{2}]} y_0(s), \min_{s\in[t,\frac{1}{2}]} |y'_0(s)|\} > 0, \quad \text{for all } t \in (0, \frac{1}{2}].$$

Since

$$y'_{n_j}(t) - y'_{n_j}(\frac{1}{2}) = -\int_{\frac{1}{2}}^t a(s)f(s, \max\{c_0, y_{n_j}(s)\}, y'_{n_j}(s) - \frac{1}{n_j})ds, \quad t \in (0,1),$$

letting $j \to +\infty$, one has

$$y'_0(t) - y'_0(\frac{1}{2}) = -\int_{\frac{1}{2}}^t a(s)f(s, \{c_0, y_0(s)\}, y'_0(s))ds, t \in (0,1).$$

Now by direct differentiation, we have

$$y''_0(t) + a(t)f(t, \{c_0, y_0(t)\}, y'_0(t)) = 0, 0 < t < 1.$$

Now (4.14) guarantees that $y_0(t) \ge c_0$ for all $t \in [0,1]$ and so

$$y''_0(t) + a(t)f(t, y_0(t), y'_0(t)) = 0, \quad 0 < t < 1.$$

From (4.21), we have $y_0 \in C[0,1] \cap C^2(0,1)$ and y_0 is a positive solution to (1.1)-(1.2). □

Example 4.2. Consider the three-point boundary value problems

$$y'' + \alpha[(-y')^{\frac{1}{2}} + (-y')^{-a}][y^b + (\frac{1}{\alpha})^{\frac{1}{2}d}y^{-d}] = 0, t \in (0,1),$$

$$y'(0) = 0, y(1) = \frac{1}{2}y(\frac{1}{2}),$$

where $\alpha > 0$, $a > 0$, $1 > \gamma \geq 0$, $b \geq 0$ and $d > 0$. Then, there is a $\alpha_0 > 0$ such that (4.2)-(4.2) has one positive solution $y_0 \in C[0,1] \cap C^2(0,1)$ with $y_0(t) > 0$ on $[0,1]$ and $y_0'(t) < 0$ on $(0,1)$ for all $0 < \alpha < \alpha_0$.

Let $a(t) \equiv \mu$, $\Phi(t) \equiv 1$ for all $t \in [0,1]$, $h(x) = x^b + (\frac{1}{\alpha})^{\frac{1}{2}d} x^{-d}$ for $x \in (0,+\infty)$ and $g(z) = z^{\frac{1}{2}} + z^{-a}$ for $z \in (0,+\infty)$. From the proof of Lemma 2.4, we have $c_0 = \frac{7}{192}\alpha^{\frac{1}{2}}$ with $\alpha \leq 1$, and then $\alpha[y^b + (\frac{1}{\alpha})^{\frac{1}{2}d} y^{-d}] \leq \alpha y^b + (\frac{192}{7})^d$ for all $y \in [c_0,+\infty)$. Let $I(z) = \int_0^z \frac{1}{r^{\frac{1}{2}} + r^{-a}} dr$. Thus there exists an α_0 such that

$$\frac{I(1/3)}{\alpha \sup_{c_0 \leq r \leq 1} h(r)} > 1, \quad \forall \alpha \in (0,\alpha_0]$$

and then

$$\sup_{c_0 \leq c < +\infty} \frac{c}{3I^{-1}(\sup_{c_0 \leq r \leq c} h(r)\alpha)} > 1.$$

Hence, the conditions (H1) and (H2) hold. Thus Theorem 4.1 guarantees that (4.2) and (4.2) has at least one positive solution.

5. Multiple positive solutions to (1.1)-(1.2)

In this section our nonlinearity f may be singular at $y' = 0$ and $y = 0$. Throughout this section we will assume that the following conditions hold:

- (P1) $a(t) \in C(0,1)$, $a(t) > 0$ for all $t \in (0,1)$;
- (P2) Conditions (H) and (H') hold and $I(z) = \int_0^z \frac{1}{g(r)} dr < +\infty$ for all $z \in [0,+\infty)$ with $\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds < \int_0^\infty \frac{dr}{g(r)}$ for all $c \in [c_0,+\infty)$ and suppose
$$\sup_{c_0 \leq c < +\infty} \frac{c}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1,$$
where c_0 is defined by Lemma 2.4;
- (P3) $\lim_{u \to +\infty} f(t,u,z)/u = +\infty$ uniformly for $(t,z) \in [\frac{1}{4},\frac{3}{4}] \times (0,+\infty)$.

Theorem 5.1. *Suppose that* (P1)–(P3) *hold. Then* (1.1)-(1.2) *has at least two positive solutions* $y_{1,0}$, $y_{1,0} \in C[0,1] \cap C^2(0,1)$ *with* $y_{1,0}(t) > 0$, $y_{2,0}(t) > 0$ *on* $[0,1]$ *and* $y_{1,0}'(t) < 0$, $y_{2,0}'(t) < 0$ *on* $(0,1)$.

Proof. Choose $R_1 > 0$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}\big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds\big)} > 1. \tag{5.1}$$

From the continuity of I^{-1} and I, we can choose $\varepsilon > 0$ and $\varepsilon < R_1$ with

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha} I^{-1}(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon))} > 1. \tag{5.2}$$

Let $n_0 \in \{1,2,\dots\}$ so that $\frac{1}{n_0} < \min\{\varepsilon, \frac{1}{2}\frac{1-\alpha}{\alpha(1-\eta)} c_0\}$ and let $N_0 = \{n_0, n_0+1, \dots\}$. Lemma 2.7 guarantees that for each $n \in N_0$, $A_n : P_q \to P_q$ is a continuous and completely continuous operator. From (P_3), there is a $R' > R_1$ such that

$$f(t,x,y) \geq N^* x, \quad \forall x \geq R',$$

where $N^* > (\int_{1/4}^{3/4}(1-s)a(s)ds\frac{\alpha(1-\eta)}{1-\alpha\eta})^{-1}$. Let

$$R_2 > \max\{R', \frac{1-\alpha\eta}{\alpha(1-\eta)}R'\}.$$

Now let

$$\Omega_1 = \{y \in C_q^1[0,1] : \|y\|_q < R_1\}, \quad \Omega_2 = \{y \in C_q^1[0,1] : \|y\|_q < R_2\}.$$

We now show that

$$y \neq \mu A_n y, \quad \forall y \in P \cap \partial\Omega_1,\ \mu \in (0,1],\ n \in N_0, \tag{5.3}$$

and

$$A_n x \not\leq x, \quad \forall\ x \in \partial\Omega_2 \cap P, n \in N_0. \tag{5.4}$$

Suppose there exists a $y_0 \in P \cap \partial\Omega_1$ and a $\mu_0 \in (0,1]$ such that $y_0 = \mu_0 A_n y_0$. It is easy to see that $y_0'(t) \leq 0$ and

$$y_0'(t) = -\mu_0 \int_0^t a(s)f(s, \max\{c_0, y_0(s)\}, y_0'(s) - \frac{1}{n})ds, t \in (0,1). \tag{5.5}$$

Also

$$y_0''(t) + \mu_0 a(t)f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) = 0, \quad 0 < t < 1,$$
$$y_0'(0) = 0, y_0(1) = \alpha y_0(\eta).$$

Therefore,

$$\begin{aligned}
-y_0''(t) &= \mu_0 a(t)f(t, \max\{c_0, y_0(t)\}, y_0'(t) - \frac{1}{n}) \\
&\leq a(t)\Phi(t)h(\max\{c_0, y_0(t)\})g(-y_0'(t) + \frac{1}{n}), \quad \forall t \in (0,1).
\end{aligned}$$

which yields

$$\frac{-y_0''(t)}{g(-y_0'(t) + \frac{1}{n})} \leq a(t)\Phi(t)h(\max\{c_0, y_0(t)\}), \quad \forall t \in (0,1).$$

Integration from 0 to t yields

$$\begin{aligned}
I(-y_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) &\leq \int_0^t a(s)\Phi(s)h(\max\{c_0, y_0(s)\})ds \\
&\leq \sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds,
\end{aligned}$$

and so

$$I(-y_0'(t) + \frac{1}{n}) \leq \sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon).$$

Thus

$$-y_0'(t) \leq I^{-1}\Big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1). \tag{5.6}$$

Integration from t to 1 yields

$$y_0(t) - y_0(1) \leq (1-t)I^{-1}\Big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1). \tag{5.7}$$

Let $t=\eta$ in (5.7). Then

$$y_0(\eta)-y_0(1)\le(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big).$$

Since $y_0(1)=\alpha y_0(\eta)$, one has

$$(\frac{1}{\alpha}-1)y_0(1)\le(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big),$$

which yields

$$y_0(1)\le\frac{\alpha}{1-\alpha}(1-\eta)I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big).$$

Then (5.7) implies

$$\begin{aligned} y_0(0)&\le y_0(1)+I^{-1}(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon))\\ &=\frac{1-\alpha\eta}{1-\alpha}I^{-1}(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)).\end{aligned}\tag{5.8}$$

Now (5.6) and (5.8) guarantees

$$\begin{aligned} R_1&=\max\{\max_{t\in[0,1]}|y_0(t)|,\max_{t\in[0,1]}(1-t)|y_0'(t)|\}\\ &\le\frac{1-\alpha\eta}{1-\alpha}I^{-1}\Big(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon)\Big)\end{aligned}$$

which implies

$$\frac{R_1}{\frac{1-\alpha\eta}{1-\alpha}I^{-1}(\sup_{c_0\le r\le R_1}h(r)\int_0^1 a(s)\Phi(s)ds+I(\varepsilon))}\le 1.$$

This contradicts (5.2). Thus (5.3) is true.

From Lemma 2.1, for each $n\in N_0$, we have

$$i(A_n,\Omega_1\cap P,P)=1,\quad n\in N_0.\tag{5.9}$$

Suppose there is a $x_0\in\partial\Omega_2\cap P$ such that $A_nx_0\le x_0$. Then $\|x_0\|_q=R_2$. Also Lemma 2.3 implies

$$\min_{t\in[0,1]}x_0(t)\ge\frac{\alpha(1-\eta)}{1-\alpha\eta}\max_{t\in[0,1]}|x_0(t)|=\frac{\alpha(1-\eta)}{1-\alpha\eta}\|x_0\|_q=\frac{\alpha(1-\eta)}{1-\alpha\eta}R_2>R'.$$

Then, we have

$$\begin{aligned} x_0(0)&\ge(Ax_0)(0)\\ &=\frac{1}{1-\alpha}\int_0^1\int_0^s a(\tau)f(\tau,\max\{c_0,x_0(\tau)\},-|x_0'(\tau)|-\frac{1}{n})\,d\tau\,ds\\ &\quad-\frac{\alpha}{1-\alpha}\int_0^\eta\int_0^s a(\tau)f(\tau,\max\{c_0,x_0(\tau)\},-|x_0'(\tau)|-\frac{1}{n})\,d\tau\,ds\\ &\ge\int_{1/4}^{3/4}(1-s)a(s)f(\tau,\max\{c_0,x_0(s)\},-|x_0'(s)|-\frac{1}{n})ds\\ &\ge\int_{1/4}^{3/4}(1-s)a(s)N*\max\{c_0,x_0(s)\}ds\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{1/4}^{3/4} (1-s)a(s)dsN * \frac{\alpha(1-\eta)}{1-\alpha\eta} R_2 \\
&> \|x_0\|_q,
\end{aligned}
$$

which is a contradiction. Thus, (5.4) is true. Then Lemma 2.2 implies

$$
i(A_n, \Omega_2 \cap P, P) = 0, \quad n \in N_0. \tag{5.10}
$$

From (5.9) and (5.10), we have

$$
i(A_n, (\Omega_2 - \overline{\Omega}_1) \cap P, P) = -1, \quad n \in N_0. \tag{5.11}
$$

By (5.9), (5.11), there is a $x_{1,n} \in \Omega_1 \cap P$ and another $x_{2,n} \in \Omega_2 \cap P$ such that

$$
A_n x_{1,n} = x_{1,n}, \quad A_n x_{2,n} = x_{2,n}, \quad n \in N_0.
$$

Now we consider $\{x_{1,n}\}_{n\in N_0}$ and $\{x_{2,n}\}_{n\in N_0}$. By Lemma 2.4, we have $x_{1,n}(t) \geq c_0$ and $x_{2,n} \geq c_0$.

We consider $\{x_{1,n}\}_{n\in N_0}$. Obviously $\max_{t\in[0,1]} |x_{1,n}(t)| \leq R_1$ for all $n \in N_0$ and $\max_{t\in[0,1]}(1-t)|x'_{1,n}(t)| \leq R_1$ for all $n \in N_0$. Also $|x'_{1,n}(t)| \leq \frac{1}{1-t} R_1$ for all $t \in [0,1)$ and $n \in N_0$. Hence, the functions belonging to $\{x_{1,n}\}$ are uniformly bounded on $[0,1]$.

Since $x_{1,n}(t)$ satisfies

$$
\begin{gathered}
x''_{1,n}(t) + a(t) f(t, \max\{c_0, x_{1,n}(t)\}, x'_{1,n}(t) - \frac{1}{n}) = 0, \quad 0 < t < 1, \\
x'_{1,n}(0) = 0, x_{1,n}(1) = \alpha x_{1,n}(\eta).
\end{gathered}
$$

A similar argument to that used to show (5.6) yields that

$$
-x'_{1,n}(t) \leq I^{-1}\Big(\sup_{c_0 \leq r \leq R_1} h(r) \int_0^1 a(s)\Phi(s)ds + I(\varepsilon)\Big), \quad t \in (0,1),
$$

which implies that the functions belonging to $\{x'_{1,n}\}$ are uniformly bounded on $[0,1]$ and so the functions belonging to $\{x_{1,n}\}$ are equicontinuous on $[0,1]$.

A similar argument to that to used to show (4.15) yields the functions belonging to $\{x'_{1,n}\}$ are equicontinuous on $[0,1]$.

Consequently, the Arzela-Ascoli Theorem guarantees that $\{x_{1,n}(t)\}$ and $\{x'_{1,n}(t)\}$ are relatively compact in $C[0,1]$; i.e., there is a function $x_{1,0} \in C^1[0,1]$, and a subsequence $\{x_{1,n_j}\}$ of $\{x_{1,n}\}$ such that

$$
\lim_{j\to+\infty} \max_{t\in[0,1]} |x_{1,n_j}(t) - x_{1,0}(t)| = 0, \quad \lim_{j\to+\infty} \max_{t\in[0,1]} |x'_{1,n_j}(t) - x'_{1,0}(t)| = 0.
$$

Similar reasoning as in the proof of Theorem 4.1 establishes that $x_{1,0}$ is a positive solution to (1.1) and (1.2).

Similarly, there is a convergent subsequence $\{x_{2,n_k}\}$ of $\{x_{2,n}\}$ such that

$$
\lim_{k\to+\infty} |x_{2,n_k}(t) - x_{2,0}(t)| = 0, \quad \lim_{k\to+\infty} |x'_{2,n_j}(t) - x'_{2,0}(t)| = 0
$$

and $x_{2,0}$ satisfies (1.1)-(1.2).

Since $\|x_{1,0}\|_q = \max_{t\in[0,1]} |x_{1,0}(t)| \leq R_1$ and $\|x_{2,0}\|_q = \max_{t\in[0,1]} |x_{2,0}(t)| \geq R_1$, a similar argument to that used to show (5.3) yields that $x_{1,0}$, $x_{2,0} \notin P \cap \partial\Omega_1$; i.e.,

$$
\|x_{1,0}\|_q = \max_{t\in[0,1]} |x_{1,0}(t)| < R_1, \quad \|x_{2,0}\|_q = \max_{t\in[0,1]} |x_{2,0}(t)| > R_1.
$$

Consequently, $x_{1,0}$ and $x_{2,0}$ are different positive solutions to (1.1)-(1.2). □

Example 5.2. Consider the three-point boundary value problems

$$y'' + \alpha(1-t)^a[1 + (-y')^e + (-y')^{-a}][1 + y^b + y^{-d}] = 0, \quad t \in (0,1),$$
$$y'(0) = 0, y(1) = \frac{1}{2}y(\frac{1}{2})$$

where $1 \geq e \geq 0$, $a > 0$, $b \geq 0$, $d > 0$ and $\alpha > 0$. Then there is a $\alpha_0 > 0$ such that (5.11)-(5.2) has at least two positive solutions $y_{1,0}$, $y_{2,0} \in C[0,1] \cap C^2(0,1)$ with $y_{1,0}(t) > 0$, $y_{2,0}(t) > 0$ on $[0,1]$ and $y'_{1,0}(t) < 0$, $y'_{2,0}(t) < 0$ on (0,1) for all $0 < \alpha \leq \alpha_0$.

Let $a(t) \equiv \mu$, $\Phi(t) = (1-t)^a$ for all $t \in [0,1]$, $h(x) = 1 + x^b + x^{-d}$ for $x \in (0,+\infty)$ and $g(z) = 1 + z^e + z^{-a}$ for $z \in (0,+\infty)$. From the proof of Lemma 2.4, we have $c_0 = \frac{1}{2}\min\{\frac{1}{3}, \frac{1}{a+1}(\frac{1}{2} - \frac{1}{a+2}(\frac{1}{2})^{a+2})\}$, and then $\alpha(1-t)^a[1 + y^b + y^{-d}] \leq \alpha(1-t)^a[1 + y^b + {c_0}^{-d}]$ for all $y \in [c_0, +\infty)$. Let $I(z) = \int_0^z \frac{1}{1+r^e+r^{-a}}dr$. Thus there exists an α_0 such that

$$\frac{I(\frac{1}{3})}{\alpha\frac{1}{a+1}\sup_{c_0 \leq r \leq 1} h(r)} > 1, \quad \forall \alpha \in (0, \alpha_0]$$

and then

$$\sup_{c_0 \leq c < +\infty} \frac{c}{3I^{-1}(\sup_{c_0 \leq r \leq c} h(r) \int_0^1 a(s)\Phi(s)ds)} > 1.$$

Hence, the conditions (P1), (P2) and (P3) hold. Thus Theorem 5.1 guarantees that (5.2)-(5.2) has at least two positive solutions.

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Ravi P. Agarwal
Department of Mathematical Science, Florida Institute of Technology, Melbourne, Florida 32901, USA
E-mail address: agarwal@fit.edu

Donal O'Regan
Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie

Baoqiang Yan
Department of Mathematics, Shandong Normal University, Ji-nan, 250014, China
E-mail address: yanbqcn@yahoo.com.cn