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THE KOLMOGOROV EQUATION WITH TIME-MEASURABLE COEFFICIENTS

JAY KOVATS

Abstract. Using both probabilistic and classical analytic techniques, we investigate the parabolic Kolmogorov equation

$$L_t v + \frac{\partial v}{\partial t} \equiv \frac{1}{2} a^{ij}(t) v_{x^i x^j} + b^i(t) v_{x^i} - c(t)v + f(t) + \frac{\partial v}{\partial t} = 0$$

in $H_T := (0,T) \times E_d$ and its solutions when the coefficients are bounded Borel measurable functions of t. We show that the probabilistic solution $v(t,x)$ defined in $\bar{H}_T$, is twice differentiable with respect to x, continuously in (t,x), once differentiable with respect to t, a.e. $t \in [0,T)$ and satisfies the Kolmogorov equation $L_t v + \frac{\partial v}{\partial t} = 0$ a.e. in $\bar{H}_T$. Our main tool will be the Aleksandrov-Busemann-Feller Theorem. We also examine the probabilistic solution to the fully nonlinear Bellman equation with time-measurable coefficients in the simple case $b \equiv 0$, $c \equiv 0$. We show that when the terminal data function is a paraboloid, the payoff function has a particularly simple form.

1. Introduction

It is well-known in the theory of diffusion processes [2, 3] that when $g \in C^2(E_d)$ and the coefficients $a(t,x)$, $b(t,x)$, $c(t,x)$ and free term $f(t,x)$ are sufficiently smooth in (t,x) and satisfy certain growth conditions, with $c(t,x) \geq 0$, then the function

$$\begin{gathered} v(t,x) = \mathbf{E}\Big[\int_t^T f(r,\xi_r(t,x)) e^{-\varphi_r(t,x)}\,dr + e^{-\varphi_T(t,x)} g(\xi_T(t,x))\Big], \\ \varphi_s(t,x) = \int_t^s c(r,\xi_r(t,x))\,dr \end{gathered} \tag{1.1}$$

belongs to $C^{1,2}(H_T)$ and satisfies the Kolmogorov equation $Lv(t,x) + \frac{\partial v}{\partial t}(t,x) = 0, \forall\, (t,x) \in \bar{H}_T$, where $Lv := \frac{1}{2} a^{ij}(t,x) v_{x^i x^j} + b^i(t,x) v_{x^i} - c(t,x)v + f(t,x)$, with $v(T,x) = g(x)$. In (1.1), for fixed $(t,x) \in \bar{H}_T$, $\omega \in \Omega$ and $s \geq t$, $\xi_s(t,x) = \xi_s(\omega,t,x)$ is the solution of the stochastic equation $\xi_s = x + \int_t^s \sigma(r,\xi_r)\,d\mathbf{w}_r + \int_t^s b(r,\xi_r)\,dr$, where $(\Omega,\mathcal{F},P)$ is a complete probability space on which $(\mathbf{w}_t,\mathcal{F}_t)$ is a d_1-dimensional Wiener process, defined for $t \geq 0$. Furthermore, $\sigma(t,x)$ and $b(t,x)$ are assumed continuous in (t,x) and have values in the set of $d \times d_1$ matrices, E_d respectively, with $a = \sigma\sigma^*$. The fact that the probabilistic solution v satisfies the

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Kolmogorov equation throughout $\bar{H}_T$ is proved using Itô's formula and relies heavily on the continuity in t of the coefficients to establish the existence and continuity in (t,x) of $\frac{\partial v}{\partial t}$ [3, Chapter 5]. In this paper, we show that if the coefficients are only bounded Borel measurable functions of t, the second derivatives $v_{x^i x^j}(t,x)$ exist and are continuous in (t,x) (Theorem 2.1) but in general, $\frac{\partial v}{\partial t}$ exists only in the generalized sense (Theorem 2.3) and the Kolmogorov equation will be satisfied only in the almost everywhere sense (Theorem 2.5). For example, consider the function $v(t,x) = |x|^2 + 2d(\frac{1}{2} - t)_+$. For $t \neq \frac{1}{2}$, $\frac{\partial v}{\partial t}(t,x)$ exists and equals $-2d\, I_{0 \le t < \frac{1}{2}}$ and hence for $t \neq \frac{1}{2}$, v is a solution of the degenerate equation $I_{0\le t<\frac{1}{2}} \Delta v + \frac{\partial v}{\partial t} = 0$ in $[0,1) \times E_d$. Note $\frac{\partial v}{\partial t}(t,x)$ is discontinuous in t.

When the coefficients and free term are independent of x, the right hand side of our stochastic equation is independent of $\xi_{\cdot}$ and the probabilistic solution (1.1) takes a decidedly more convenient form (see (3.4)). Since the other terms in (3.4) are independent of x and their derivatives with respect to t can be explicitly calculated (almost everywhere) it suffices to investigate the function $v(t,x) = \mathbf{E} g(\xi_T(t,x))$.

We do this in two ways. In section 1, we use probabilistic arguments to show that for $g \in C^2(E_d)$, the function $v(t,x) = \mathbf{E} g(\xi_T(t,x))$ is twice differentiable with respect to x, continuously in (t,x) and once differentiable with respect to t, a.e. $t \in [0,T)$. We then apply the Aleksandrov-Busemann-Feller theorem to a variant of v to show that v satisfies the Kolmogorov equation $\frac{1}{2} a^{ij}(t) v_{x^i x^j} + b^i(t) v_{x^i} + \frac{\partial v}{\partial t} = 0$ a.e. in H_T. From this it follows (by our previous remark) that the simplified version of (1.1), given by (3.4) satisfies the more general Kolmogorov equation a.e. in H_T. In section 2, we use the fact that $\xi_T(t,x)$ is a Gaussian vector to express v as a convolution (in x) of g with a kernel p which is the fundamental solution of the Kolmogorov equation (a.e. t). Our proof that this convolution satisfies the Kolmogorov equation amounts to showing that we can differentiate the kernel under the integral sign. Here we assume only that g is continuous and slowly increasing, that is $|g(x)| \le C_1 e^{C_2 |x|^2}$. Our derivative estimates are done under the assumption that the coefficient matrix $a(t)$ is non-degenerate. This assumption was not needed in section 1, (due to the assumption $g \in C^2(E_d)$) yet we do get a slightly more refined result here, namely $v(t,x) = \mathbf{E} g(\xi_T(t,x))$ satisfies the Kolmogorov equation for almost every $t \in [0,T)$ and any $x \in E_d$. Finally in section 4, we examine the payoff function for the fully nonlinear Bellman equation in the simple case $b \equiv 0$, $c \equiv 0$. It turns out that when g is a paraboloid, the probabilistic solution of the Bellman equation has a particularly simple form.

2. The Probabilistic Approach

Throughout this section, we assume the following.

Let $g \in C^2(E_d)$ and assume that for all $x, y \in E_d$, $|g(x)|$, $|g_{(y)}(x)|$, $|g_{(y)(y)}(x)| \le K(1 + |x|^m)$, where for any twice differentiable function $u(x)$ and $l \in E_d$, $u_{(l)}(x) = |l|^{-1} u_x(x) \cdot l$, $u_{(l)(l)}(x) = |l|^{-2} l^* u_{xx}(x) l$. For $t \in [0,T]$ and $x \in E_d$, we define, for $s \in [t,T]$, the diffusion process $\xi_s(t,x) = x + \int_t^s \sigma(r)\, d\mathbf{w}_r + \int_t^s b(r)\, dr$, where the Borel measurable coefficients $\sigma(t)$, $b(t)$ are defined on $[0,T]$, independent of $\omega \in \Omega$ and satisfy

$$\int_0^T \left[\|\sigma(t)\|^2 + |b(t)| \right] dt < \infty. \tag{2.1}$$

Under these assumptions, we prove our first theorem.

Theorem 2.1. *For $(t,x) \in \bar{H}_T$, the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ is twice differentiable with respect to x, continuously in (t,x) and for any $y, \bar{y} \in E_d$, $v_{y\bar{y}}(t,x) = \mathbf{E}g_{y\bar{y}}(\xi_T(t,x))$.*

Proof. We show that $v(t,x)$ is differentiable with respect to x. Writing $\xi_T(t,x) = x + \eta_T(t)$, where $\eta_T(t) := \int_t^T \sigma(r)\,d\mathbf{w}_r + \int_t^T b(r)\,dr$, note that for any $y \in E_d$ and any sequence $h_n \to 0$ as $n \to \infty$

$$\Delta^1_{h_n,y} v(t,x) := \frac{v(t,x+h_n y) - v(t,x)}{h_n} = \mathbf{E}\Delta^1_{h_n,y} g(x + \eta_T(t)) = \mathbf{E}\Delta^1_{h_n,y} g(\xi_T(t,x)).$$

Since g_y is continuous, the Mean Value Theorem yields

$$\Delta^1_{h_n,y} g(\xi_T(t,x)) = \int_0^1 g_y(\xi_T(t,x) + r h_n y)\,dr = g_y(\xi_T(t,x)) + \theta h_n y),$$

for some $\theta \in [0,1]$. Since $g \in C^1(E_d)$, $\Delta^1_{h_n,y} g(\xi_T(t,x)) \to g_y(\xi_T(t,x))$ as $n \to \infty$. Furthermore, as $n \to \infty$

$$\mathbf{E}\Delta^1_{h_n,y} g(\xi_T(t,x)) \to \mathbf{E}g_y(\xi_T(t,x)). \tag{2.2}$$

To see this observe that

$$\begin{aligned} |\Delta^1_{h_n,y} g(\xi_T(t,x))| &= |g_y(\xi_T(t,x) + \theta h_n y)| \\ &\le |y| K \left(1 + |\xi_T(t,x) + \theta h_n y|^m\right) \\ &\le 2^m K |y| \left(1 + |\xi_T(t,x)|^m + |\theta h_n y|^m\right) \\ &\le N|y| \Big(1 + |x|^m + \Big|\int_t^T \sigma(r) d\mathbf{w}_r\Big|^m + \Big|\int_t^T b(r)\,dr\Big|^m + |y|^m\Big), \end{aligned}$$

where $N = N(m,K)$. By (2.1), the Burkholder-Davis-Gundy inequalities and the fact that σ, b are independent of ω, the last expression above has finite expectation. Hence by [3, Lemma III.6.13 (f)], (2.2) holds. Since $\{h_n\}$ was an arbitrary sequence converging to 0 as $n \to \infty$, we conclude

$$\lim_{h \to 0} \mathbf{E}\Delta^1_{h,y} g(\xi_T(t,x)) = \mathbf{E}g_y(\xi_T(t,x)).$$

Thus $v(t,x)$ is differentiable with respect to x and for any $y \in E_d$, $v_y(t,x) = \lim_{h\to 0} \mathbf{E}\Delta^1_{h,y} g(\xi_T(t,x)) = \mathbf{E}g_y(\xi_T(t,x))$. We now show that $v(t,x)$ is twice differentiable with respect to x. By the above expression for $v_y(t,x)$, we have, for any $\bar{y} \in E_d$

$$\frac{v_y(t,x+h\bar{y}) - v_y(t,x)}{h} = \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)). \tag{2.3}$$

But since $g_{y\bar{y}}$ is continuous, for any sequence $h_n \to 0$ as $n \to \infty$, the Mean Value Theorem yields

$$\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x)) = \int_0^1 g_{y\bar{y}}(\xi_T(t,x) + r h_n y)\,dr = g_{y\bar{y}}(\xi_T(t,x)) + \theta h_n y),$$

for some $\theta \in [0,1]$. Since $g \in C^2(E_d)$, $\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x)) \to g_{y\bar{y}}(\xi_T(t,x))$ as $n \to \infty$. By the argument immediately following (2.2), except with $|y|^2 + |\bar{y}|^2$ in place of $|y|$ and using the growth condition on $|g_{(y)(y)}(x)|$, we see that $|\Delta^1_{h_n,\bar{y}}\, g_y(\xi_T(t,x))|$ is bounded above (independently of n) by a random variable which has finite expectation. Hence

$$\mathbf{E}\Delta^1_{h_n,\bar{y}} g_y(\xi_T(t,x)) \to \mathbf{E}g_{y\bar{y}}(\xi_T(t,x)) \quad \text{as } n \to \infty.$$

Since $\{h_n\}$ was an arbitrary sequence converging to 0 as $n \to \infty$,

$$\lim_{h\to 0} \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)) = \mathbf{E} g_{y\bar{y}}(\xi_T(t,x)).$$

Thus by (2.3), $v_{y\bar{y}}(t,x)$ exists and since $y, \bar{y} \in E_d$ were arbitrary, $v(t,x)$ is twice differentiable with respect to x and

$$v_{y\bar{y}}(t,x) = \lim_{h\to 0} \mathbf{E}\Delta^1_{h,\bar{y}}\, g_y(\xi_T(t,x)) = \mathbf{E} g_{y\bar{y}}(\xi_T(t,x)).$$

We now show the continuity of $v_{y\bar{y}}(t,x)$ in (t,x). To this end, fix (t,x) and let $t^n \to t^+, x^n \to x$. It suffices to show $v_{y\bar{y}}(t^n,x^n) \to v_{y\bar{y}}(t,x)$. We have

$$|v_{y\bar{y}}(t^n,x^n) - v_{y\bar{y}}(t,x)| \le \mathbf{E}|g_{y\bar{y}}(\xi_T(t^n,x^n)) - g_{y\bar{y}}(\xi_T(t,x))|. \tag{2.4}$$

Observe that $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$ and since $g_{y\bar{y}}$ is continuous, $g_{y\bar{y}}(\xi_T(t^n,x^n)) \xrightarrow{P} g_{y\bar{y}}(\xi_T(t,x))$. Since $|g_{y\bar{y}}(\xi_T(t^n,x^n))| \le \eta$ with $\mathbf{E}\eta < \infty$, the right hand side of (2.5) tends to zero as $n \to \infty$. The details are as follows. To see that $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$, observe that

$$\begin{aligned} &|\xi_T(t^n,x^n) - \xi_T(t,x)| \\ &\le |x^n - x| + \Big|\int_{t^n}^T \sigma(r) d\mathbf{w}_r - \int_t^T \sigma(r) d\mathbf{w}_r\Big| + \Big|\int_{t^n}^T b(r)\,dr - \int_t^T b(r)\,dr\Big|. \end{aligned} \tag{2.5}$$

The middle summand tends to zero in probability as $n \to \infty$ by [3, Theorem III.6.6] and the fact that

$$\int_0^T \|I_{t^n \le r}\sigma(r) - I_{t\le r}\sigma(r)\|^2\,dr = \int_0^T \|\sigma(r)\|^2 I_{t\le r<t^n}\,dr \to 0 \quad \text{as } n \to \infty$$

by (2.1) and the Dominated Convergence Theorem. The third summand on the right hand side of (2.5) tends to zero by the Dominated Convergence Theorem. Since $x^n \to x$, we have $\xi_T(t^n,x^n) \xrightarrow{P} \xi_T(t,x)$. Since $g_{y\bar{y}}(x)$ is continuous,

$$g_{y\bar{y}}(\xi_T(t^n,x^n)) \xrightarrow{P} g_{y\bar{y}}(\xi_T(t,x)),$$

by [3, Theorem III.6.13 (c)]. Finally,

$$\begin{aligned} &|g_{y\bar{y}}(\xi_T(t^n,x^n))| \\ &\le K(|y|^2 + |\bar{y}|^2)(1 + |\xi_T(t^n,x^n)|^m) \\ &\le 3^m K(|y|^2 + |\bar{y}|^2)\Big\{1 + |x^n|^m + \Big|\int_{t^n}^T \sigma(r)\,d\mathbf{w}_r\Big|^m + \Big|\int_{t^n}^T b(r)\,dr\Big|^m\Big\}. \end{aligned} \tag{2.6}$$

Since

$$\Big|\int_{t^n}^T \sigma(r)\,d\mathbf{w}_r\Big|^m \le 2^m \sup_s \Big|\int_0^{s\wedge T} \sigma(r)\,d\mathbf{w}_r\Big|^m$$

as $x^n \to x$ and $\big|\int_{t^n}^T b(r)\,dr\big|^m \le \big(\int_0^T |b(r)|\,dr\big)^m$, the right hand side of (2.6) is bounded uniformly in n by a random variable, which, by the Burkholder-Davis-Gundy inequalities and (2.1), has finite expectation. Hence, by [3, Theorem III.6.13 (f)],

$$\mathbf{E}|g_{y\bar{y}}(\xi_T(t^n,x^n)) - g_{y\bar{y}}(\xi_T(t,x))| \to 0 \quad \text{as } n \to \infty$$

and hence by (2.4), $v_{y\bar{y}}(t^n,x^n) \to v_{y\bar{y}}(t,x)$. □

The proof that $v(t,x)$ and $v_y(t,x)$ are continuous in $\bar{H}_T$ follow same the technique shown here, except we use the respective assumptions $|g(x)|$, $|g_{(y)}(x)| \le K(1+|x|^m)$. Observe that by (2.6) and the Burkholder-Davis-Gundy inequalities, we obtain the following estimate, which holds for $(t,x) \in \bar{H}_T$

$$\begin{aligned}&\|v_{xx}(t,x)\| \\ &\le N(d,m,K)\Big\{1+|x|^m + \Big(\int_t^T \|\sigma(r)\|^2\,dr\Big)^{m/2} + \Big(\int_t^T |b(r)|\,dr\Big)^m\Big\}.\end{aligned} \tag{2.7}$$

If in addition, σ, b satisfy $\sup_{t\le T}(\|\sigma(t)\| + |b(t)|) \le K$, inequality (2.7) yields, with $N_1 = N_1(d,m,K)$

$$\begin{aligned}\|v_{xx}(t,x)\| &\le 2N(1\vee K^m)(1+|x|^m)\{1+(T-t)^m\} \\ &\le 4N(1\vee K^m)(1+|x|^m)e^{(T-t)m} \\ &\le N_1(1+|x|)^m e^{N_1(T-t)}.\end{aligned} \tag{2.8}$$

The following lemma appears in [3, p. 195]. We will use this lemma and the fact that v, v_x, v_{xx} are continuous in (t,x) to show that when $\sigma(t), b(t)$ are bounded, $v(t,x)$ is differentiable with respect to t for almost every $t \in [0,T]$.

Lemma 2.2. *Let* $\xi_s(t,x) = x + \int_t^s \sigma(r)\,d\mathbf{w}_r + \int_t^s b(r)\,dr$*, where* $\sup_{t\le T}(\|\sigma(t)\| + |b(t)|) \le K$*. For* $\epsilon > 0$ *and* $(t,x) \in Q$*, let*

$$\tau_\epsilon(t,x) = \inf\{s \ge t : (s, \xi_s(t,x)) \notin Q_\epsilon(t,x)\},$$

where $Q_\epsilon(t,x) = (t-\epsilon^3, t+\epsilon^3) \times B_\epsilon(x)$*. Then for any compact set* $\Gamma \subset Q_+ := Q \cap \{t \ge 0\}$*,*

$$\epsilon^{-3}P\{\tau_\epsilon(t,x) - t < \epsilon^3\} \to 0, \quad \epsilon^{-3}\mathbf{E}[\tau_\epsilon(t,x) - t] \to 1,$$

uniformly in $(t,x) \in \Gamma$*, as* $\epsilon \to 0^+$*.*

Theorem 2.3. *Under the hypotheses of Theorem 2.1 suppose that* $\sup_{t\le T}(\|\sigma(t)\| + |b(t)|) \le K$*. Then for any* $x \in E_d$*, the function* $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ *is differentiable with respect to* t *for almost every* $t \in [0,T)$*.*

Proof. Fix any $(t,x) \in H_T$ and choose ϵ so small that $t+\epsilon^3 < T$. Since absolutely continuous functions of a single real variable are differentiable almost everywhere, it suffices to show that $v(t,x)$ is Lipschitz in t. By the strong Markov property we can write

$$v(t,x) = \mathbf{E}v(\tau_\epsilon(t,x), \xi_{\tau_\epsilon(t,x)}(t,x)), \tag{2.9}$$

which we henceforth abbreviate as $\mathbf{E}v(\tau_\epsilon, \xi_{\tau_\epsilon})$. By Itô's formula applied to the C^2 function (of x) $v(t+\epsilon^3, \cdot)$, we have

$$\begin{aligned}&v(t,x) - v(t+\epsilon^3, x) \\ &= \mathbf{E}[v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})] + \mathbf{E}[v(t+\epsilon^3, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_t)] \\ &= \mathbf{E}\,I_{\tau_\epsilon < t+\epsilon^3}[v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})] + \mathbf{E}\int_t^{\tau_\epsilon} L_r v(t+\epsilon^3, \xi_r)\,dr\end{aligned} \tag{2.10}$$

Certainly $|v(\tau_\epsilon, \xi_{\tau_\epsilon}) - v(t+\epsilon^3, \xi_{\tau_\epsilon})| \le 2\sup_{[t,t+\epsilon^3]\times\overline{B_\epsilon(x)}} |v|$. We recall that v, v_x, v_{xx} are continuous and hence bounded in any compact set. By definition, $L_r v(t +$

$\epsilon^3,\xi_r) = \frac{1}{2}\operatorname{tr}[a(r)\, v_{xx}(t+\epsilon^3,\xi_r)]+b(r)\cdot v_x(t+\epsilon^3,\xi_r)$. From the elementary inequality $|\operatorname{tr}[a\cdot m]| \le \|a\|\,\|m\|$ and the fact that $\|\sigma(t)\| + |b(t)| \le K$, we get, for $r \in [t,\tau_\epsilon]$,

$$\begin{aligned} \left|L_r v(t+\epsilon^3,\xi_r)\right| &\le \frac{K^2}{2}\|v_{xx}(t+\epsilon^3,\xi_r)\| + K|v_x(t+\epsilon^3,\xi_r)| \\ &\le N(K)\big(\sup_{B_\epsilon(x)}\|v_{xx}(t+\epsilon^3,\cdot)\| + \sup_{B_\epsilon(x)}|v_x(t+\epsilon^3,\cdot)|\big). \end{aligned} \tag{2.11}$$

So in any small closed cylinder $\widetilde{Q} \supset [t,t+\epsilon^3]\times\overline{B_\epsilon(x)}$, we have, by (2.10) and Lemma 2.2, for sufficiently small ϵ,

$$\begin{aligned} &|v(t,x) - v(t+\epsilon^3,x)| \\ &\le 2\sup_{\widetilde{Q}}|v|\cdot P\{\tau_\epsilon - t < \epsilon^3\} + N(K)(\sup_{\widetilde{Q}}\|v_{xx}\| + \sup_{\widetilde{Q}}|v_x|)\mathbf{E}[\tau_\epsilon - t] \\ &\le N_1(K)(\sup_{\widetilde{Q}}|v| + \sup_{\widetilde{Q}}|v_x| + \sup_{\widetilde{Q}}\|v_{xx}\|)\epsilon^3. \end{aligned}$$

Since t,ϵ were arbitrary (such that $t+\epsilon^3 < T$), we get, for any $s,t\in[0,T)$ and any fixed $x\in E_d$,

$$|v(t,x) - v(s,x)| \le N_2|t-s|,$$

where N_2 is independent of s,t,x. Hence the generalized derivative $\frac{\partial v}{\partial t}$ exists and $\left|\frac{\partial v}{\partial t}(t,x)\right| \le N_2$. □

We will now show that the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ satisfies the Kolmogorov equation almost everywhere in H_T, under the assumptions of Theorem 2.3. Our main tool will be the Aleksandrov-Busemann-Feller (ABF) theorem (see [4, Theorem 1.1]) for continuous functions which are convex in x and non-increasing in t.

Theorem 2.4 (Aleksandrov-Busemann-Feller)**.** *Let $u(t,x)$ be convex in x, non-increasing in t and continuous in $\bar{H}_T$. Let $P(s,x,t,y) = u(s,x) + u_s^{(0)}(s,x)t + u_x(s,x)\cdot y + \frac{1}{2}y^* u_{xx}^{(0)}(s,x)y$, where $u_s^{(0)}, u_{x^ix^j}^{(0)}$ denote generalized derivatives. Then for almost all $(s,x)\in E_{d+1}$, $u(s+t,x+y) = P(s,x,t,y) + o(|t| + |y|^2)$ as $(t,y)\to(0,0)$.*

Equivalently, for almost all $(t_0,x_0)\in E_{d+1}, u(t,x) = P_{(t_0,x_0)}(t,x) + o(|t-t_0| + |x-x_0|^2)$ as $(t,x)\to(t_0,x_0)$, where $P_{(t_0,x_0)}(t,x) = u(t_0,x_0) + u_t^{(0)}(t_0,x_0)(t-t_0) + u_x(t_0,x_0)\cdot(x-x_0) + \frac{1}{2}(x-x_0)^* u_{xx}^{(0)}(t_0,x_0)(x-x_0)$. We want to apply the ABF theorem to a variant of v. To this end, note that by (2.6), for any $l\in E_d$, we have

$$\left|v_{(l)(l)}(t,x)\right| \le \mathbf{E}\left|g_{(l)(l)}(\xi_T(t,x))\right| \le Ne^{N(T-t)}(1+|x|)^m,$$

where $N = N(m,K)$. Direct calculation shows that for any $l,x\in E_d$, $(m+2)2^{-\frac{m}{2}}(1+|x|)^m \le \left[(1+|x|^2)^{\frac{m}{2}+1}\right]_{(l)(l)}$. Hence

$$\left|v_{(l)(l)}(t,x)\right| \le \frac{Ne^{N(T-t)}2^{\frac{m}{2}}}{m+2}\left[(1+|x|^2)^{\frac{m}{2}+1}\right]_{(l)(l)} \le Ne^{N(T-s)}\left[(1+|x|^2)^{\frac{m}{2}+1}\right]_{(l)(l)}$$

which yields

$$0 \le \Big(v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}\Big)_{(l)(l)} \qquad \forall (t,x)\in H_T,\, l\in E_d.$$

That is, the function $v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}$ is convex in x. We may also consider this function to be decreasing in t by the following argument. By Lemma

2.2, the first summand on the right hand side of (2.10) is $o(\epsilon^3)$ as $\epsilon \to 0$. By the continuity of $v_{xx}(t,x), v_x(t,x)$, the last factor on the right hand side of (2.11) tends to $\|v_{xx}(t,x)\|+|v_x(t,x)|$ as $\epsilon \to 0$. Since the estimate $|v_x(t,x)| \le Ne^{N(T-t)}(1+|x|)^m$ also holds, dividing (2.10) by ϵ^3, letting $\epsilon \to 0$, using (2.8) and applying the second result in Lemma 2.2, we get for almost every $t \in [0,T)$ and any $x \in E_d$

$$\Big|\frac{\partial v}{\partial t}(t,x)\Big| \le Ne^{N(T-t)}(1+|x|)^m \le Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1}, \tag{2.12}$$

where $N = N(d,m,K)$. From (2.12) it follows, as before, that for some $N = N(d,m,K)$, $v(t,x) + Ne^{N(T-t)}(1+|x|^2)^{\frac{m}{2}+1} := v + v_0$ is decreasing in t.

Theorem 2.5. *Under the assumptions of Theorem 2.3, the function $v(t,x) = \mathbf{E}g(\xi_T(t,x))$ satisfies the Kolmogorov equation almost everywhere in H_T.*

Proof. Since the ABF theorem holds for the function $v + v_0$ and v_0 is smooth, the ABF theorem also holds for v. Since v has continuous second derivatives (by Theorem 2.1), $v^{(0)}_{xx} = v_{xx}$ almost everywhere. So fix any $(t,x) \in H_T$ for which the assertion of the ABF theorem holds for v, $v^{(0)}_{xx}(t,x) = v_{xx}(t,x)$ and t is in the Lebesgue set of the operator $L_s \equiv \frac{1}{2}a^{ij}(s)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(s)\frac{\partial}{\partial x^i}$. By the strong Markov property, $v(t,x) = \mathbf{E}v(\tau_\epsilon, \xi_{\tau_\epsilon})$, where $\tau_\epsilon(t,x)$ is as in Lemma 2.2. By the ABF theorem, $v(\tau_\epsilon, \xi_{\tau_\epsilon}) = P_{(t,x)}(\tau_\epsilon, \xi_{\tau_\epsilon}) + o(|\tau_\epsilon - t| + |\xi_{\tau_\epsilon} - x|^2)$ as $\epsilon \to 0$. Since $\xi_t(t,x) = x$ and $P_{(t,x)}(t,x) = v(t,x)$, applying Itô's formula to the paraboloid $P_{(t,x)}$ yields

$$0 = \mathbf{E}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr + \mathbf{E}[o(|\tau_\epsilon - t| + |\xi_{\tau_\epsilon} - x|^2)]. \tag{2.13}$$

Since $0 \le \tau_\epsilon - t \le \epsilon^3$, the estimates $\mathbf{E}|\xi_{\tau_\epsilon} - x|^p \le N(p,K)\epsilon^{\frac{3p}{2}}(1+\epsilon^{\frac{3p}{2}})$ and $|v(t,x)| \le N(T,m,K)(1+|x|)^m$ imply that the second summand on the right of (2.13) is $o(\epsilon^3)$. Let us write the first summand on the right of (2.13) as

$$\mathbf{E}I_{\tau_\epsilon < t+\epsilon^3}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr + \mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\int_t^{t+\epsilon^3} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr. \tag{2.14}$$

Since the coefficients of L_r are uniformly bounded and $r \in [t,\tau_\epsilon]$ implies $|\xi_r - x| < \epsilon$, the integrand in the first summand of (2.14) satisfies

$$\big|\big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\big| \le (K^2 + K\epsilon)\|v_{xx}(t,x)\| + K|v_x(t,x)| + |v^{(0)}_t(t,x)|.$$

Since $\epsilon \in (0,1)$,

$$\begin{aligned}&\Big|\mathbf{E}I_{\tau_\epsilon < t+\epsilon^3}\int_t^{\tau_\epsilon} \big(L_r P + \frac{\partial P}{\partial r}\big)(r,\xi_r)\,dr\Big| \\ &\le N(K)(\|v_{xx}(t,x)\| + |v_x(t,x)| + |v^{(0)}_t(t,x)|)\cdot P\{\tau_\epsilon < t+\epsilon^3\},\end{aligned}$$

and hence the first expectation in (2.14) is $o(\epsilon^3)$ by Lemma 2.2. Dividing the second expectation in (2.14) by ϵ^3 and evaluating it explicitly yields

$$\begin{aligned}&\mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} \Big(L_r v(t,x) + v^{(0)}_t(t,x)\Big)\,dr \\ &+ \mathbf{E}I_{\tau_\epsilon = t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} b(r)^* v_{xx}(t,x)(\xi_r - x)\,dr.\end{aligned} \tag{2.15}$$

Since t is a Lebesgue point for L_s, we have (almost surely)

$$I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3}\Big(L_r v(t,x)+v_t^{(0)}(t,x)\Big)\,dr \to L_t v(t,x)+v_t^{(0)}(t,x) \quad \text{as } \epsilon\to 0$$

and since

$$\begin{aligned}&\Big|I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3}\Big(L_r v(t,x)+v_t^{(0)}(t,x)\Big)\,dr\Big|\\ &\le K^2\|v_{xx}(t,x)\|+K|v_x(t,x)|+|v_t^{(0)}(t,x)|,\end{aligned}$$

the first expectation in (2.15) converges to $L_t v(t,x)+v_t^{(0)}(t,x)$ as $\epsilon\to 0$. The second expectation in (2.15) converges to 0 as $\epsilon\to 0$. Recalling that $0\le\tau_\epsilon-t\le\epsilon^3$ and that $r\in[t,\tau_\epsilon]$ implies $|\xi_r-x|<\epsilon$, we immediately get the bound

$$\Big|I_{\tau_\epsilon=t+\epsilon^3}\frac{1}{\epsilon^3}\int_t^{t+\epsilon^3} b(r)^*v_{xx}(t,x)(\xi_r-x)\,dr\Big|\le\frac{1}{\epsilon^3}\|v_{xx}(t,x)\|K\epsilon^4=\|v_{xx}(t,x)\|K\epsilon.$$

Hence dividing (2.13) by ϵ^3 and letting $\epsilon\to 0$, we get $L_t v(t,x)+v_t^{(0)}(t,x)=0$. $\square$

3. Fundamental solutions of the Kolmogorov equation - the analytic approach

Even the "analytic" proof that $v(t,x)=\mathbf{E}g(\xi_T(t,x))$ is a solution of the Kolmogorov equation relies on the well known probabilistic fact that since coefficients $\sigma(t)$, $b(t)$ are independent of ω, the vector $\xi_T(t,x)$ is a Gaussian vector with parameters $\big(x+\int_t^T b(r)\,dr,\int_t^T a(r)\,dr\big)$. Hence, the distribution $P\xi_T(t,x)^{-1}$ has density function

$$p(T,t,y)=\frac{e^{-\frac12\langle C^{-1}(t)(y-x-\int_t^T b(r)dr),y-x-\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}},$$

where $C(t)=C_T(t)=\int_t^T a(r)\,dr$. From this, it follows that a solution to the problem

$$\begin{cases}\frac12 a^{ij}(t)v_{x^ix^j}(t,x)+b^i(t)v_{x^i}(t,x)+\frac{\partial v}{\partial t}(t,x)=0 & \text{a. e. } t\in[0,T)\\ v(T,x)=g(x) & x\in E_d\end{cases}\tag{3.1}$$

is given by

$$\begin{aligned}v(t,x)&=\mathbf{E}g(\xi_T(t,x))\\&=\int_{E_d}g(y)\,P\,\xi_T^{-1}(t,x)\,(dy)\\&=\int_{E_d}g(y)\frac{e^{-\frac12\langle C^{-1}(t)(y-x-\int_t^T b(r)dr),y-x-\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}}\,dy,\end{aligned}\tag{3.2}$$

where $a(r)=\sigma(r)\sigma^*(r)$ is non-degenerate. We prove this in Theorem 3.1 below, for slowly increasing $g\in C^0(E_d)$. Viewed analytically, since the function

$$p(T,t,x)=\frac{e^{-\frac12\langle C^{-1}(t)(x+\int_t^T b(r)dr),x+\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac d2}\sqrt{\det C(t)}}\tag{3.3}$$

is a fundamental solution (in x) of the equation $L_t p(t,x) + \frac{\partial p}{\partial t}(t,x) = 0$ a.e. $t \in [0,T)$, all $x \neq -\int_t^T b(r)dr$, where $L_t \equiv \frac{1}{2}a^{ij}(t)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t)\frac{\partial}{\partial x^i}$, a solution to (3.1) will be given by the convolution

$$\begin{aligned} v(t,x) &= [g * p(T,t,\cdot)](x) \\ &= \int_{E_d} g(y)\, p(T,t,x-y)\, dy \\ &= \int_{E_d} g(y) \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(x-y+\int_t^T b(r)dr), x-y+\int_t^T b(r)dr\rangle}}{(2\pi)^{\frac{d}{2}}\sqrt{\det C(t)}}\, dy, \end{aligned}$$

providing, of course, we can differentiate under the integral sign. Regarding notation, by *fundamental solution*, we mean that for all $t \in [0,T)$, $p(T,t,x)$ is infinitely differentiable in x and $\int_{E_d} p(T,t,x)\, dx = 1$.

By Lebesgue's differentiation theorem, $p(T,t,x)$ in (2) is differentiable with respect to t, only in the *almost everywhere* sense. This is in contrast to the case where $a(t) = I_d, b(t) = b\,(const.)$ and the Kolmogorov equation is simply $\frac{1}{2}\Delta u(t,x) + b \cdot u_x(t,x) + \frac{\partial u}{\partial t}(t,x) = 0$ for all $(t,x) \in H_T$. In this case, $p(T,t,x) = (2\pi(T-t))^{-\frac{d}{2}} e^{\frac{-|x+b(T-t)|^2}{2(T-t)}}$ is infinitely differentiable in both t and x.

Theorem 3.1. *For $t \in [0,T]$ and $x \in E_d$ and $s \in [t,T]$, let $\xi_s(t,x) = x + \int_t^s \sigma(r)\, d\mathbf{w}_r + \int_t^s b(r)\, dr$, where $\sup_{t \le T}(\|\sigma(t)\| + |b(t)|) \le K$. Assume $\exists \delta > 0$ for which $\delta I_d \le a(t)$, for all $t \in [0,T]$, where $a(t) = \sigma(t)\sigma^*(t)$. Then for $p(T,t,x)$ as in (2) and g continuous and slowly increasing, the function*

$$v(t,x) = \mathbf{E}g(\xi_T(t,x)) = \int_{E_d} g(y)\, p(T,t,x-y)\, dy$$

satisfies the Kolmogorov equation $\frac{1}{2}a^{ij}(t)v_{x^i x^j}(t,x) + b^i(t)v_{x^i}(t,x) + \frac{\partial v}{\partial t}(t,x) = 0$ a.e. $t \in [0,T)$ and any $x \in E_d$.

Proof. Direct calculation shows that for almost every $t \in [0,T)$ and any $x \neq -\int_t^T b(r)dr \in E_d$, $p(T,t,x)$ is a solution of the Kolmogorov equation. Thus we need only show that we can differentiate under the integral sign. Omitting the constant factor of $(2\pi)^{-d/2}$, direct calculation shows that for almost every $t \in [0,T)$, with $z = y - x$ and $\eta_t := \int_t^T b(r)\, dr$,

$$\begin{aligned} &\frac{\partial p}{\partial t}(T,t,x-y) \\ &= \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(z-\eta_t), z-\eta_t\rangle}}{2\sqrt{\det C(t)}}\Big\{ \operatorname{tr}[a(t)C^{-1}(t)] - \langle C^{-1}(t)\, a(t)\, C^{-1}(t)(z-\eta_t), z-\eta_t\rangle \\ &\quad + 2\langle C^{-1}(t)(y-x), b(t)\rangle + 2\langle C^{-1}(t)\, b(t), \eta_t\rangle \Big\} \end{aligned}$$

and hence

$$\begin{aligned} &\Big|\frac{\partial p}{\partial t}(T,t,x-y)\Big| \\ &\le \frac{e^{-\frac{1}{2}\langle C^{-1}(t)(z-\eta_t), z-\eta_t\rangle}}{\sqrt{\det C(t)}}\Big\{ \|a(t)\|\|C^{-1}(t)\| \\ &\quad + \|C^{-1}(t)\, a(t)\, C^{-1}(t)\||z-\eta_t|^2 + \|C^{-1}(t)\||z||b(t)| + \|C^{-1}(t)\||b(t)||\eta_t| \Big\}. \end{aligned}$$

Since $\sup_{t\le T}(\|\sigma(t)\|+|b(t)|)\le K$ and $\|ab\|\le\|a\|\,\|b\|$, $\|a(t)\|=\|\sigma(t)\sigma^*(t)\|\le K^2$. From the estimate $\|C(t)\|\le\sqrt{T-t}\sqrt{\int_t^T\|a(r)\|^2\,dr}$, we have $\|C(t)\|\le K^2(T-t)$. Moreover, by the uniform non-degeneracy condition $\delta|\lambda|^2\le a^{ij}(t)\lambda^i\lambda^j$, which holds for all $t\in[0,T]$ and all $\lambda\in E_d$, we get $\|C^{-1}(t)\|\le\frac{\sqrt{d}}{\delta(T-t)}$. We also have $C^{ij}(t)\lambda^i\lambda^j\ge\delta|\lambda|^2(T-t)$, from which it immediately follows that $\det C(t)\ge[\delta(T-t)]^d$. Obviously, $|\eta_t|\le K(T-t)$. This gives

$$
\begin{aligned}
&\Big|\frac{\partial p}{\partial t}(T,t,x-y)\Big|\\
&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\\
&\times\Big\{\frac{K^2\sqrt{d}}{\delta(T-t)}+\frac{2dK^2}{\delta^2(T-t)^2}\left(|y-x|^2+K^2(T-t)^2\right)+\frac{K\sqrt{d}}{\delta(T-t)}|x-y|+\frac{K^2\sqrt{d}}{\delta}\Big\}.
\end{aligned}
$$

Similarly, the gradient and hessian of $p(T,t,x-y)$ satisfy

$$
p_x(T,t,x-y)=\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{\sqrt{\det C(t)}}\;C^{-1}(t)\cdot(z-\eta_t)
$$

$$
\begin{aligned}
&p_{xx}(T,t,x-y)\\
&=\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{\sqrt{\det C(t)}}\left\{C^{-1}(t)(z-\eta_t)[C^{-1}(t)(z-\eta_t)]^*-C^{-1}(t)\right\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|p_x(T,t,x-y)|&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\|C^{-1}(t)\|\cdot|z-\eta_t|\\
&\le\frac{\sqrt{d}\,e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2+1}}\Big\{|y-x|+K(T-t)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\|p_{xx}(T,t,x-y)\|\\
&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\left\{\|C^{-1}(t)\|^2\cdot|z-\eta_t|^2+\|C^{-1}(t)\|\right\}\\
&\le\frac{e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}}{(\delta(T-t))^{d/2}}\Big\{\frac{2d}{\delta^2(T-t)^2}\left(|y-x|^2+K^2(T-t)^2\right)+\frac{\sqrt{d}}{\delta(T-t)}\Big\}.
\end{aligned}
$$

To estimate the exponential term in each derivative, we use the inequality $\frac{|z-\eta_t|^2}{K^2(T-t)}\le\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle$ and Young's inequality (twice): $|z-\eta_t|^2\ge\big||z|-|\eta_t|\big|^2\ge\frac12|z|^2-|\eta_t|^2\ge\frac12|y-x|^2-K^2(T-t)^2\ge\frac14|y|^2-\frac12|x|^2-K^2(T-t)^2$ to conclude

$$
e^{-\frac12\langle C^{-1}(t)(z-\eta_t),z-\eta_t\rangle}\le e^{-\frac{|y|^2}{8K^2(T-t)}+\frac{|x|^2}{4K^2(T-t)}+\frac{T-t}{2}}.
$$

Denoting any of the derivatives p_t,p_x,p_{xx} by $p'(T,t,x-y)$, we see that

$$
|p'(T,t,x-y)|\le\frac{N\cdot e^{-\frac{|y|^2}{8K^2(T-t)}+\frac{|x|^2}{4K^2(T-t)}+\frac{T-t}{2}}}{(T-t)^{\frac d2+2}}q(T-t,|y-x|),
$$

where $N = N(\delta, d, K)$ and $q(a, b)$ is a paraboloid in a and b. Hence if $(t, x) \in [0, t_0] \times B_R$, where $0 \le t_0 < T$, then

$$|p'(T, t, x - y)| \le \frac{N \cdot e^{-\frac{|y|^2}{8K^2T} + \frac{R^2}{4K^2(T-t_0)} + T}}{(T - t_0)^{\frac{d}{2}+2}} q(T, |y| + R).$$

So if we require that $|g(x)| \le Ne^{\frac{|x|^2}{16K^2T}}$, we see that the integrals $\int_{E_d} g(y)p'(T, t, x - y)\,dy$ converge uniformly with respect to $(t, x) \in [0, t_0] \times B_R$. This implies $v(t, x)$ is twice differentiable with respect to x, once differentiable with respect to t (almost everywhere) and its derivatives can be evaluated by differentiating under the integral sign. Since $p(T, t, x - y)$ satisfies the Kolmogorov equation for almost every $t \in [0, T)$, so does $v(t, x)$. □

Remark. The above growth condition for g is obviously satisfied when g has polynomial growth, $|g(x)| \le K(1 + |x|^m)$. Furthermore, direct calculation shows that for any d-dimensional multi-index α, any derivative of $p(T, t, x-y)$ with respect to x satisfies

$$|D_x^\alpha p(T, t, x - y)| \le \frac{N \cdot e^{-\frac{|y-x|^2}{4K^2(T-t)} + \frac{T-t}{2}}}{(T - t)^{\frac{d}{2}+|\alpha|}} \cdot q_\alpha(T - t, |y - x|),$$

where $N = N(\delta, d, K, |\alpha|)$ and $q_\alpha(a, b)$ is a polynomial of degree less than or equal to $|\alpha|$ in a and b, from which it follows, as above, that $v(t, x)$ is infinitely differentiable with respect to x.

More generally, if $c(t) \ge 0$ is bounded and measurable in $[0, T]$ and we define $\phi_s(t) = \int_t^s c(r)dr$, the function $\widetilde{p}(T, t, x) := p(T, t, x)e^{-\phi_T(t)}$ is an infinitely differentiable solution (in x) of the equation $L_t u(t, x) - c(t)u(t, x) + \frac{\partial u}{\partial t}(t, x) = 0$ a.e. $t \in [0, T)$. Since $[g * \widetilde{p}(T, t, \cdot)](x) = e^{-\phi_T(t)}[g * p(T, t, \cdot)](x)$, a solution to the problem

$$\begin{cases} \frac{1}{2}a^{ij}(t)v_{x^ix^j}(t, x) + b^i(t)v_{x^i}(t, x) - c(t)v(t, x) + \frac{\partial v}{\partial t}(t, x) = 0 \\ \qquad \text{a.e. } t \in [0, T), \text{ all } x \in E_d \\ \qquad v(T, x) = g(x) \quad x \in E_d \end{cases}$$

is given by

$$v(t, x) = e^{-\phi_T(t)}\mathbf{E}g(\xi_T(t, x)),$$

while if $\int_0^T |f(r)|e^{-\phi_r(t)}\,dr < \infty$, direct calculation shows that the function

$$v(t, x) = e^{-\phi_T(t)}\mathbf{E}g(\xi_T(t, x)) + \int_t^T f(r)e^{-\phi_r(t)}\,dr \tag{3.4}$$

satisfies

$$\begin{cases} \frac{1}{2}a^{ij}(t)v_{x^ix^j}(t, x) + b^i(t)v_{x^i}(t, x) - c(t)v(t, x) + f(t) + \frac{\partial v}{\partial t}(t, x) = 0 \\ \qquad \text{a.e. } t \in [0, T) \text{ all } x \in E_d \\ \qquad v(T, x) = g(x) \quad x \in E_d. \end{cases}$$

4. Paraboloid solutions of the Simplest Time-Measurable Bellman equations

In this section, we prove a result about the payoff function for the Bellman equation in the simple case where the equation depends only on second derivatives and t and the coefficients are Borel measurable functions of t. Let A be a separable metric space, where for $(\alpha, t) \in A \times [0, T]$, $\sigma(\alpha, t)$ is a $d \times d_1$ matrix and $f^\alpha(t)$ is a function, both continuous in α and Borel measurable in t. Now let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which $(\mathbf{w}_t, \mathcal{F}_t)$ is a d_1-dimensional Wiener process. We consider the controlled diffusion process $\xi_s(\alpha, t, x)$, defined for $s \in [0, T]$ by $\xi_s(\alpha, t, x) = x + \int_0^s \sigma(\alpha_r, t + r)\, d\mathbf{w}_r$, where $t \in [0, T], x \in E_d$ are fixed and α_t is a strategy in class U, that is, progressively measurable with values in A.

Suppose $g \in C^2(E_d)$ and satisfies $|g(x)|$, $|g_{(y)}(x)|$, $|g_{(y)(y)}(x)| \le K(1 + |x|^m)$, $\forall\, x, y \in E_d$, where K, m are nonnegative constants. It is known [4] that if for any $\alpha \in A$, $f^\alpha, \sigma(\alpha, \cdot)$ are differentiable with respect to t with derivatives not exceeding K, then the payoff function

$$v(t, x) = \sup_{\alpha \in U} \mathbf{E}\Big[\int_0^{T-t} f^{\alpha_r}(r + t)\, dr + g(\xi^\alpha_{T-t}(t, x))\Big] \tag{4.1}$$

satisfies the Bellman equation

$$\sup_{\alpha \in A}\{a^{ij}(\alpha, t) v_{x^i x^j}(t, x) + f^\alpha(t)\} + \frac{\partial v}{\partial t}(t, x) = 0 \quad \text{a. e. } H_T, \quad v(T, x) = g(x).$$

In the special case where g is a paraboloid, the payoff function takes a very convenient form and clearly satisfies the Bellman equation under the weak assumption that $\sup_{\alpha \in A} f^\alpha$, $\sup_{\alpha \in A} \sigma(\alpha, \cdot) \in L_1([0, T]), L_2([0, T])$, respectively.

Theorem 4.1. *Let $p(x)$ be any paraboloid defined on E_d, i.e. $p(x) = \frac{1}{2}x^* m x + l \cdot x + l_0$, where $m \in E_{d^2}, l \in E_d, l_0 \in E_1$. Then the probabilistic solution of the Bellman equation*

$$\begin{cases} \sup_{\alpha \in A}\{a^{ij}(\alpha, t) v_{x^i x^j}(t, x) + f^\alpha(t)\} + \dfrac{\partial v}{\partial t}(t, x) = 0 & \text{a.e. } t \in [0, T) \\ v(T, x) = p(x) \quad x \in E_d. \end{cases}$$

is given by

$$v(t, x) = p(x) + \int_t^T \sup_{\alpha \in A}\{\operatorname{tr}[a(\alpha, r) m] + f^\alpha(r)\}\, dr. \tag{4.2}$$

Proof. From the theory of controlled diffusion processes [2], the probabilistic solution to this Bellman equation is the payoff function (4.1) with $g = p$ and $a(\alpha, t) = \frac{1}{2}\sigma(\alpha, t)\sigma(\alpha, t)^*$. It immediately follows from Itô's formula that $\forall \alpha \in U$, $t \in [0, T]$ and $x \in E_d$, we have

$$\mathbf{E} p(\xi^\alpha_{T-t}(t, x)) = p(x) + \mathbf{E}\int_0^{T-t} \operatorname{tr}[a(\alpha_r, t + r) m]\, dr. \tag{4.3}$$

We give a more direct proof of (4.3) using Wald's identity. Writing $\xi^\alpha_{T-t} = \xi^\alpha_{T-t}(t, x)$, we have $p(\xi^\alpha_{T-t}(t, x)) = \dfrac{\xi^{\alpha\,*}_{T-t} m \xi^\alpha_{T-t}}{2} + l \cdot \xi^\alpha_{T-t} + l_0$ and

$$\xi^{\alpha\,*}_{T-t} m \xi^\alpha_{T-t} = \langle m\xi^\alpha_{T-t}, \xi^\alpha_{T-t}\rangle = \langle mx, x\rangle + 2\langle mx, \eta^{\alpha,t}_{T-t}\rangle + \langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle,$$

where $\eta^{\alpha,t}_{T-t} := \int_0^{T-t} \sigma(\alpha_r, t+r)\, d\mathbf{w}_r$. Writing $m = ODO^*$, where $D = (\lambda^i \delta^{ij})$, we get

$$\langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle = \langle ODO^*\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle = \langle Dz^{\alpha,t}_{T-t}, z^{\alpha,t}_{T-t}\rangle = \sum_{i=1}^{d} \lambda^i \left(z^{\alpha,t,i}_{T-t}\right)^2$$

where

$$z^{\alpha,t}_{T-t} := O^* \cdot \eta^{\alpha,t}_{T-t} = \int_0^{T-t} O^* \cdot \sigma(\alpha_r, t+r)\, d\mathbf{w}_r := \int_0^{T-t} \widetilde{\sigma}(\alpha_r, t+r)\, d\mathbf{w}_r.$$

Orthogonality and the Wald identity yield

$$\mathbf{E}\big(z^{\alpha,t,i}_{T-t}\big)^2 = \mathbf{E}\sum_{k=1}^{d}\Big(\int_0^{T-t} \widetilde{\sigma}^{ik}(\alpha_r, t+r)\, dw^k_r\Big)^2 = \sum_{k=1}^{d}\mathbf{E}\int_0^{T-t}[\widetilde{\sigma}^{ik}(\alpha_r, t+r)]^2\, dr,$$

and hence

$$\begin{aligned}\mathbf{E}\langle m\eta^{\alpha,t}_{T-t}, \eta^{\alpha,t}_{T-t}\rangle &= \sum_{i=1}^{d}\lambda^i\sum_{k=1}^{d}\mathbf{E}\int_0^{T-t}[\widetilde{\sigma}^{ik}(\alpha_r, t+r)]^2\, dr\\ &= 2\mathbf{E}\int_0^{T-t}\operatorname{tr}[a(\alpha_r, t+r)m]\, dr.\end{aligned}$$

By Wald's identity, we also have $\mathbf{E}\langle mx, \eta^{\alpha,t}_{T-t}\rangle = 0$ and $\mathbf{E}[l\cdot\xi^{\alpha}_{T-t} + l_0] = \mathbf{E}[l\cdot(x + \eta^{\alpha,t}_{T-t}) + l_0] = l\cdot x + l_0$. Thus

$$\begin{aligned}\mathbf{E}p(\xi^{\alpha}_{T-t}(t,x)) &= \mathbf{E}\Big[\frac{\xi^{\alpha*}_{T-t}m\xi^{\alpha}_{T-t}}{2} + l\xi^{\alpha}_{T-t} + l_0\Big]\\ &= p(x) + \mathbf{E}\int_0^{T-t}\operatorname{tr}[a(\alpha_r, t+r)m]\, dr.\end{aligned}$$

Therefore,

$$\begin{aligned}v(t,x) &= p(x) + \sup_{\alpha\in U}\mathbf{E}\Big[\int_0^{T-t}\operatorname{tr}[a(\alpha_r, t+r)m] + f^{\alpha_r}(r+t)\, dr\Big]\\ &= p(x) + \int_t^T \sup_{\alpha\in\mathcal{A}}\{\operatorname{tr}[a(\alpha, r)m] + f^{\alpha}(r)\}\, dr.\end{aligned}$$

□

This result is hardly a surprise since the second-order derivatives of any paraboloid are constant. Hence by Lebesgue's differentiation theorem, for any operator $F(b,t)$ for which $\int_0^T |F(b,t)|\, dt < \infty$, the function

$$u(t,x) = p(x) + \int_t^T F(p_{xx}(x), r)\, dr$$

satisfies

$$\begin{cases} F(u_{xx}(t,x), t) + \dfrac{\partial u}{\partial t}(t,x) = 0 \quad \text{a.e. } t\in[0,T)\\ u(T,x) = p(x) \quad x\in E_d\,. \end{cases}$$

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