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FIXED POINT THEOREMS FOR CONVEX-POWER CONDENSING OPERATORS RELATIVE TO THE WEAK TOPOLOGY AND APPLICATIONS TO VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this paper we present new fixed point theorems for weakly sequentially continuous mappings which are convex-power condensing relative to a measure of weak noncompactness. Our fixed point results extend and improve several earlier works. As an application, we investigate the existence of weak solutions to a Volterra integral equation.

1. Introduction. During the last four decades several interesting studies relating to the existence of weak solutions to the Cauchy differential equation in Banach spaces have been presented. These studies were initiated by Szep [**23**] in 1971 and since then have been addressed by many investigators. We quote the contributions by Cramer, Lakshmikantham and Mitchell [**8**] in 1978 and more recently by Bugajewski [**4**], Cichon [**5, 6**], Cichon and Kubiaczyk [**7**], Mitchell and Smith [**17**], and O'Regan [**18–20**]. Motivated by the paper of Cichon [**6**], O'Regan [**18**] discussed in detail the problem (which was modeled off a first-order differential equation [**6**])

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds, \quad t \in [0, T]; \tag{1.1}$$

here $f\colon [0,T]\times E \to E$ and $x_0 \in E$ with E a real reflexive Banach space. The integral in (1.1) is understood to be the Pettis integral. Our main objective here is to establish existence results for the Volterra integral equation (1.1) in the case where E is nonreflexive. Our approach relies upon the concept of convex-power condensing operators with respect

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to a measure of weak noncompactness which we shall introduce in the present paper. Recall that Sun and Zhang [**22**] introduced the definition of convex-power condensing operator with respect to the Kuratowski measure of noncompactness and proved a fixed point theorem which extended the well-known Sadovskii's fixed point theorem and a fixed point theorem in Liu et al. [**16**]. In [**25**], Zhang et al. established some fixed point theorems of Rothe and Altman types about convex-power condensing operators with respect to the Kuratowski measure of noncompactness. These results were applied to a differential equation of one order with integral boundary conditions. In this paper we shall use the concept of a convex-power condensing operator with respect to a measure of weak noncompactness to prove some fixed point principles which generalize the Arino-Gautier-Penot principle [**2**], Sadovskii's type principle for weakly sequentially continuous mappings [**12, 14**], the Leray-Schauder type principle for weakly sequentially continuous mappings [**21**] and many others. These fixed point principles will be used to derive an existence theory for (1.1) in the case where E is nonreflexive.

For the remainder of this section we gather some notations and preliminary facts. Let X be a Banach space, let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

In our considerations the following definition will play an important role.

Definition 1.1 [**3**]. A function $\psi: \mathcal{B}(X) \to \mathbf{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(1) The family $\ker(\psi) = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{\text{co}}(M)) = \psi(M)$, where $\overline{\text{co}}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty} \psi(M_n) = 0$, then $M_\infty := \cap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n and $\lim_{n\to\infty} \psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi(\overline{M^w}) = \psi(M) \tag{1.2}$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \text{ if and only if } M \text{ is relatively weakly compact,} \tag{1.3}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.4}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbf{R}, \tag{1.5}$$

and *set additive* if

$$\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)). \tag{1.6}$$

The first important example of a measure of weak noncompactness has been defined by De Blasi [**9**] as follows:

$$w(M) = \inf\{r > 0 : \text{ there exists a } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}, \tag{1.7}$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(.)$ is regular, homogeneous, subadditive and set additive (see [**9**]).

The following results are crucial for our purposes. We first state a theorem of Ambrosetti type (see [**15, 17**] for a proof).

Theorem 1.2. *Let E be a Banach space, and let $H \subseteq C([0,T],E)$ be bounded and equicontinuous. Then the map $t \to w(H(t))$ is continuous on $[0,T]$ and*

$$w(H) = \sup_{t\in[0,T]} w(H(t)) = w(H[0,T]),$$

where $H(t) = \{h(t): h \in H\}$ *and* $H[0,T] = \cup_{t\in[0,T]}\{h(t): h \in H\}$.

The following lemma is well-known (see for example [**22**]).

Lemma 1.3. *If* $H \subseteq C([0,T],E)$ *is equicontinuous and* $x_0 \in C([0,T],E)$, *then* $\overline{\text{co}}\,(H \cup \{x_0\})$ *is also equicontinuous in* $C([0,T],E)$.

In what follows, let X be a Banach space, C a nonempty closed convex subset of X, $F: C \to C$ a mapping and $x_0 \in C$. For any $M \subseteq C$, we set

$$\begin{aligned} &F^{(1,x_0)}(M) = F(M), \\ &F^{(n,x_0)}(M) = F\left(\overline{\text{co}}\left(F^{(n-1,x_0)}(M) \cup \{x_0\}\right)\right), \end{aligned} \tag{1.8}$$

for $n = 2, 3, \ldots$.

Definition 1.4. Let X be a Banach space, C a nonempty closed convex subset of X and ψ a measure of weak noncompactness on X. Let $F: C \to C$ be a bounded mapping (that is, it takes bounded sets into bounded ones), $x_0 \in C$ and n_0 a positive integer. We say that F is a ψ- convex-power condensing operator about x_0 and n_0 if, for any bounded set $M \subseteq C$ with $\psi(M) > 0$, we have

$$\psi(F^{(n_0,x_0)}(M)) < \psi(M). \tag{1.9}$$

Obviously, $F: C \to C$ is ψ-condensing if and only if it is ψ- convex-power condensing operator about x_0 and 1.

2. Fixed point theorems.

Theorem 2.1. *Let* X *be a Banach space and* ψ *a regular, and set additive measure of weak noncompactness on* X. *Let* C *be a nonempty closed convex subset of* X, $x_0 \in C$ *and* n_0 *a positive integer. Suppose* $F: C \to C$ *is* ψ*-convex-power condensing about* x_0 *and* n_0. *If* F *is* *weakly sequentially continuous and $F(C)$ is bounded, then F has a fixed point in C.*

Proof. Let

$$\mathcal{F} = \{A \subseteq C, \overline{\text{co}}\,(A) = A,\ x_0 \in A \text{ and } F(A) \subseteq A\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set $M = \cap_{A\in\mathcal{F}} A$. Now we show that, for any positive integer n we have

$$\mathcal{P}(n) \qquad M = \overline{\text{co}}\left(F^{(n,x_0)}(M) \cup \{x_0\}\right).$$

To see this, we proceed by induction. Clearly M is a closed convex subset of C and $F(M) \subseteq M$. Thus, $M \in \mathcal{F}$. This implies $\overline{\text{co}}\,(F(M) \cup \{x_0\}) \subseteq M$. Hence, $F(\overline{\text{co}}\,(F(M)\cup\{x_0\})) \subseteq F(M) \subseteq \overline{\text{co}}\,(F(M)\cup\{x_0\})$. Consequently, $\overline{\text{co}}\,(F(M) \cup \{x_0\}) \in \mathcal{F}$. Hence, $M \subseteq \overline{\text{co}}\,(F(M) \cup \{x_0\})$. As a result $\overline{\text{co}}\,(F(M) \cup \{x_0\}) = M$. This shows that $\mathcal{P}(1)$ holds. Let n be fixed, and suppose $\mathcal{P}(n)$ holds. This implies $F^{(n+1,x_0)}(M) = F(\overline{\text{co}}\,(F^{(n,x_0)}(M) \cup \{x_0\})) = F(M)$. Consequently,

$$\overline{\text{co}}\left(F^{(n+1,x_0)}(M) \cup \{x_0\}\right) = \overline{\text{co}}\,(F(M) \cup \{x_0\}) = M. \tag{2.1}$$

As a result,

$$\overline{\text{co}}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right) = M. \tag{2.2}$$

Using the properties of the measure of weak noncompactness we get

$$\psi(M) = \psi\left(\overline{\text{co}}\left(F^{(n_0,x_0)}(M) \cup \{x_0\}\right)\right) = \psi(F^{(n_0,x_0)}(M)),$$

which yields that M is weakly compact. Since $F\colon M \to M$ is weakly sequentially continuous, the result follows from the Arino-Gautier-Penot fixed point theorem [**2**]. □

As an easy consequence of Theorem 2.1 we obtain the following sharpening of [**12**, Theorem 12] and [**14**, Theorem 2].

Corollary 2.2. *Let X be a Banach space and ψ a regular, set additive measure of weak noncompactness on X. Let C be a nonempty* *closed convex subset of X. Assume $F\colon C \to C$ is a sequentially weakly continuous map with $F(C)$ bounded. If F is ψ-condensing, i.e., $\psi(F(M)) < \psi(M)$, whenever M is a bounded non-weakly compact subset of C, then F has a fixed point.*

Remark 2.3. Theorem 2.1 is also an extension of [**21**, Theorem 2.2] and the Arino-Gautier-Penot fixed point theorem [**2**].

Lemma 2.4. *Let $F\colon X \to X$ be convex-power condensing about x_0 and n_0 (n_0 is a positive integer) with respect to a regular and set additive measure of weak noncompactness ψ. Let $\widetilde{F}\colon X \to X$ be the operator defined on X by $\widetilde{F}(x) = F(x + x_0) - x_0$. Then, $\widetilde{F}$ is convex-power condensing about 0 and n_0 with respect to ψ. Moreover, F has a fixed point if $\widetilde{F}$ does.*

Proof. Let M be a bounded subset of X with $\psi(M) > 0$. We claim that, for all integers $n \geq 1$, we have

$$\widetilde{F}^{(n,0)}(M) \subseteq F^{(n,x_0)}(M + x_0) - x_0. \tag{2.3}$$

To see this, we shall proceed by induction. Clearly,

$$\widetilde{F}^{(1,0)}(M) = \widetilde{F}(M) = F(M + x_0) - x_0 = F^{(1,x_0)}(M + x_0) - x_0. \tag{2.4}$$

Fix an integer $n \geq 1$ and suppose (2.3) holds. Then

$$\widetilde{F}^{(n,0)}(M) \cup \{0\} \subseteq \overline{\text{co}}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0. \tag{2.5}$$

Hence,

$$\overline{\text{co}}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right) \subseteq \overline{\text{co}}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0. \tag{2.6}$$

As a result,

$$\begin{aligned}
\widetilde{F}^{(n+1,0)}(M) &= \widetilde{F}\left(\overline{\text{co}}\left(\widetilde{F}^{(n,0)}(M) \cup \{0\}\right)\right) \\
&\subseteq \widetilde{F}\left(\overline{\text{co}}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0\right) \\
&= F\left(\overline{\text{co}}\left(F^{(n,x_0)}(M + x_0) \cup \{x_0\}\right) - x_0\right) \\
&= F^{(n+1,x_0)}(M + x_0) - x_0.
\end{aligned}$$

This proves our claim. Consequently,

$$\begin{aligned}\psi(\widetilde{F}^{(n_0,0)}(M))) &\le \psi(F^{(n_0,x_0)}(M+x_0)-x_0)\\ &\le \psi(F^{(n_0,x_0)}(M+x_0))\\ &< \psi(M+x_0) \le \psi(M).\end{aligned}$$

This proves the first statement. The second statement is straightforward.

Theorem 2.5. *Let X be a Banach space, and let ψ be a regular and set additive measure of weak noncompactness on X. Let Q and C be closed, convex subsets of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $x_0 \in U$ and such that U is weakly open in C. Suppose $F\colon X \to X$ is a weakly sequentially continuous and ψ-power-convex condensing map about x_0 and n_0 (n_0 is a positive integer). If, moreover, $F(\overline{U^w})$ is bounded and $F(\overline{U^w}) \subseteq C$, then either*

$$\text{(2.7)} \qquad F \text{ has a fixed point,}$$

or

$$\text{(2.8)} \qquad \text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u = \lambda F u;$$

here $\partial_Q U$ is the weak boundary of U in Q.

Proof. By replacing F, Q, C and U by $\widetilde{F}, Q - x_0, C - x_0$ and $U - x_0$, respectively, and using Lemma 2.4, we may assume that $0 \in U$ and F is ψ-power-convex condensing about 0 and n_0. Now suppose (2.8) does not occur and F does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.7) occurs). Let

$$M = \{x \in \overline{U^w} : x = \lambda F x \text{ for some } \lambda \in [0,1]\}.$$

The set M is nonempty since $0 \in U$. Also, M is weakly sequentially closed. Indeed, let (x_n) be a sequence of M which converges weakly to some $x \in \overline{U^w}$, and let (λ_n) be a sequence of $[0,1]$ satisfying $x_n = \lambda_n F x_n$. By passing to a subsequence, if necessary, we may assume that (λ_n) converges to some $\lambda \in [0,1]$. Since F is weakly sequentially continuous, then $Fx_n \rightharpoonup Fx$. Consequently, $\lambda_n Fx_n \rightharpoonup \lambda Fx$. Hence, $x = \lambda Fx$ and therefore $x \in M$. Thus, M is weakly sequentially closed. We now claim that M is relatively weakly compact. Suppose $\psi(M) > 0$. Clearly,

$$M \subseteq \mathrm{co}\,(F(M) \cup \{0\}). \tag{2.9}$$

By induction, note for all positive integers n we have

$$M \subseteq \mathrm{co}\,(F^{(n,0)}(M) \cup \{0\}). \tag{2.10}$$

Indeed, fix an integer $n \geq 1$ and suppose (2.10) holds. Then

$$F(M) \subseteq F(\overline{\mathrm{co}}\,(F^{(n,0)}(M) \cup \{0\}) = F^{(n+1,0)}(M). \tag{2.11}$$

Hence,

$$\mathrm{co}\,(F(M) \cup \{0\}) \subseteq \mathrm{co}\,(F^{(n+1,0)}(M) \cup \{0\}). \tag{2.12}$$

Combining (2.9) and (2.12), we arrive at

$$M \subseteq \mathrm{co}\,(F^{(n+1,0)}(M) \cup \{0\}). \tag{2.13}$$

This proves (2.10). In particular, we have

$$M \subseteq \mathrm{co}\,(F^{(n_0,0)}(M) \cup \{0\}). \tag{2.14}$$

Thus,

$$\psi(M) \leq \psi(\mathrm{co}\,(F^{(n_0,0)}(M) \cup \{0\})) = \psi(F(M)) < \psi(M), \tag{2.15}$$

which is a contradiction. Hence, $\psi(M) = 0$, and therefore $\overline{M^w}$ is compact. This proves our claim. Now let $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact, then there is a sequence (x_n) in M which converges weakly to x. Since M is weakly sequentially closed, we have $x \in M$. Thus, $\overline{M^w} = M$. Hence, M is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \varnothing$. Since X endowed with the weak topology is a locally convex space then there exists a weakly continuous mapping $\rho\colon \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$ (see [**11**]). Let

$$T(x) = \begin{cases} \rho(x)F(x) & x \in \overline{U^w}, \\ 0 & x \in X \setminus \overline{U^w}. \end{cases}$$

Clearly $T: X \to X$ is weakly sequentially continuous since F is weakly sequentially continuous. Moreover, for any $S \subseteq C$ we have

$$T(S) \subseteq \mathrm{co}\,(F(S) \cup \{0\}).$$

This implies that

$$\begin{aligned} T^{(2,0)}(S) &= T\,(\overline{\mathrm{co}}\,(T(S) \cup \{0\})) \subseteq T\,(\overline{\mathrm{co}}\,(F(S) \cup \{0\})) \\ &\subseteq \overline{\mathrm{co}}\,\left(F(\overline{\mathrm{co}}\,(F(S) \cup \{0\}) \cup \{0\})\right) \\ &= \overline{\mathrm{co}}\,(F^{(2,0)}(S) \cup \{0\}). \end{aligned}$$

By induction,

$$\begin{aligned} T^{(n,0)}(S) &= T\left(\overline{\mathrm{co}}\,(T^{(n-1,0)}(S) \cup \{0\})\right) \\ &\subseteq T\left(\overline{\mathrm{co}}\,(F^{(n-1,0)}(S) \cup \{0\})\right) \\ &\subseteq \overline{\mathrm{co}}\,\left(F(\overline{\mathrm{co}}\,(F^{(n-1,0)}(S) \cup \{0\}) \cup \{0\})\right) \\ &= \overline{\mathrm{co}}\,(F^{(n,0)}(S) \cup \{0\}), \end{aligned}$$

for each integer $n \geq 1$. Using the properties of the measure of weak noncompactness we get

(2.16)
$$\psi(T^{(n_0,0)}(S)) \leq \psi(\overline{\mathrm{co}}\,(F^{(n_0,0)}(S) \cup \{0\})) = \psi(F^{(n_0,0)}(S)) < \psi(S),$$

if $\psi(S) > 0$. Thus, $T: X \to X$ is weakly sequentially continuous, $T(C) \subseteq C$ and T is ψ-power-convex condensing about 0 and n_0. By Theorem 2.1 there exists an $x \in C$ such that $Tx = x$. Now $x \in U$ since $0 \in U$. Consequently, $x = \rho(x)F(x)$ and so $x \in M$. This implies that $\rho(x) = 1$ and so $x = F(x)$. ◻

Remark 2.6. Theorem 2.5 is a sharpening of [**21**, Theorem 2.3] and [**1**, Theorem 2.3]. In Theorem 2.5, notice that $\partial_Q U = \partial_C U$.

3. Existence results. In this section we shall discuss the existence of weak solutions to the Volterra integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds, \quad t \in [0, T]; \tag{3.1}$$

here $f\colon [0,T]\times E \to E$ and $x_0 \in E$ with E is a real Banach space. The integral in (3.1) is understood to be the Pettis integral and solutions to (3.1) will be sought in $C([0,T],E)$.

This equation will be studied under the following assumptions:

(i) for each $t \in [0,T]$, $f_t = f(t,.)$ is weakly sequentially continuous (i.e., for each $t \in [0,T]$, for each weakly convergent sequence (x_n), the sequence $f_t(x_n)$ is weakly convergent),

(ii) for each continuous $x\colon [0,T] \to E$, $f(.,x(.))$ is Pettis integrable on $[0,T]$,

(iii) there exists an $\alpha \in L^1[0,T]$ and $\theta\colon [0,+\infty) \to (0,+\infty)$ a nondecreasing continuous function such that $|f(s,u)| \leq \alpha(s)\theta(|u|)$ for almost every $s \in [0,t]$ and all $u \in E$, with

$$\int_0^T \alpha(s)\,ds < \int_{|x_0|}^{\infty} \frac{dx}{\theta(x)},$$

(iv) there is a constant $\tau \geq 0$ such that, for any bounded subset S of E and for any $t \in [0,T]$, we have

$$w(f([0,t]\times S)) \leq \tau w(S).$$

Theorem 3.1. *Let E be a Banach space and suppose that* (i)–(iv) *hold. Then* (3.1) *has a solution in* $C([0,T],E)$.

Proof. Let

$$C = \{x \in C([0,T],E) : |x(t)| \leq b(t) \text{ for } t \in [0,T] \text{ and } |x(t)-x(s)| \leq |b(t)-b(s)| \text{ for } t,s \in [0,T]\},$$

where

$$b(t) = I^{-1}\bigg(\int_0^t \alpha(s)\,ds\bigg) \quad \text{and} \quad I(z) = \int_{|x_0|}^{z} \frac{dx}{\theta(x)}.$$

Notice that C is a closed, convex, bounded, equicontinuous subset of $C([0,T],E)$ with $0 \in C$. Define the operator F on C by

$$Fx(t) = x_0 + \int_0^t f(s,x(s))\,ds. \tag{3.2}$$

Arguing exactly as in [**18**], we see that F is weakly sequentially continuous and maps C into C. Now we show that there is an integer n_0 such that F is w-power-convex condensing about 0 and n_0, where w is the De Blasi measure of weak noncompactness. To see this, notice that, for each bounded set $H \subseteq C$ and for each $t \in [0, T]$,

$$\begin{aligned} w(F^{(1,0)}(H)(t)) &= w(F(H)(t)) \\ &= w\biggl(\biggl\{x_0 + \int_0^t f(s, x(s))\, ds \colon x \in H\biggr\}\biggr) \\ &\le w(t\overline{\mathrm{co}}\,\{f(s, x(s)) \colon x \in H,\ s \in [0, t]\}) \\ &= tw(\overline{\mathrm{co}}\,\{f(s, x(s)) \colon x \in H,\ s \in [0, t]\}) \\ &\le tw(f([0, t] \times H[0, t])) \\ &\le t\tau w(H[0, t]). \end{aligned}$$

Theorem 1.2 implies (since H is equicontinuous) that

$$w(F^{(1,0)}(H)(t)) \le t\tau w(H). \tag{3.3}$$

Since $F^{(1,0)}(H)$ is equicontinuous, it follows from Lemma 1.3 that $F^{(2,0)}(H)$ is equicontinuous. Using (3.3), we get

$$\begin{aligned} &w(F^{(2,0)}(H)(t)) \\ &\quad = w\biggl(\biggl\{x_0 + \int_0^t f(s, x(s))\, ds \colon \\ &\qquad\qquad x \in \overline{\mathrm{co}}\,(F^{(1,0)}(H) \cup \{0\})\biggr\}\biggr) \\ &\quad \le w\biggl(\biggl\{\int_0^t f(s, x(s))\, ds \colon x \in \overline{\mathrm{co}}\,(F^{(1,0)}(H) \cup \{0\})\biggr\}\biggr) \\ &\quad = w\biggl(\biggl\{\int_0^t f(s, x(s))\, ds \colon x \in V\biggr\}\biggr), \end{aligned}$$

where $V = \overline{\mathrm{co}}\,(F^{(1,0)}(H) \cup \{0\})$. Fix $t \in [0, T]$. We divide the interval $[0, t]$ into m parts $0 = t_0 < t_1 < \cdots < t_m = t$ in such a way that

$\Delta t_i = t_i - t_{i-1} = t/m$, $i = 1, \dots, m$. For each $x \in V$, we have

$$\begin{aligned}\int_0^t f(s, x(s))\, ds &= \sum_{i=1}^m \int_{t_{i-1}}^{t_i} f(s, x(s))\, ds \\ &\in \sum_{i=1}^m \Delta t_i \overline{\text{co}}\,\{f(s, x(s)) : x \in V,\ s \in [t_{i-1}, t_i]\} \\ &\subseteq \sum_{i=1}^m \Delta t_i \overline{\text{co}}\,(f([t_{i-1}, t_i] \times V([t_{i-1}, t_i]))).\end{aligned}$$

Using again Theorem 1.2, we infer that, for each $i = 2, \dots, m$, there is an $s_i \in [t_{i-1}, t_i]$ such that

$$\sup_{s \in [t_{i-1}, t_i]} w(V(s)) = w(V[t_{i-1}, t_i]) = w(V(s_i)). \tag{3.4}$$

Consequently,

$$\begin{aligned}&w\bigg(\bigg\{\int_0^t f(s, x(s))\, ds : x \in V\bigg\}\bigg) \\ &\qquad\le \sum_{i=1}^m \Delta t_i w(\overline{\text{co}}\,(f([t_{i-1}, t_i] \times V([t_{i-1}, t_i])))) \\ &\qquad\le \tau \sum_{i=1}^m \Delta t_i w(\overline{\text{co}}\,(V([t_{i-1}, t_i]))) \\ &\qquad\le \tau \sum_{i=1}^m \Delta t_i w(V((s_i))).\end{aligned}$$

On the other hand, if $m \to \infty$, then

$$\sum_{i=1}^m \Delta t_i w(V((s_i)) \longrightarrow \int_0^t w(V(s))\, ds. \tag{3.5}$$

Using regularity, set additivity and convex closure invariance of the De Blasi measure of weak noncompactness together with (3.3), we obtain

$$w(V(s)) = w(F^{(1,0)}(H)(s)) \le s\tau w(H), \tag{3.6}$$

and therefore,

$$\int_0^t w(V(s))\,ds \le s\tau \frac{t^2}{2} w(H). \tag{3.7}$$

As a result,

$$w(F^{(2,0)}(H)(t)) \le \frac{(\tau t)^2}{2} w(H). \tag{3.8}$$

By induction, we get

$$w(F^{(n,0)}(H)(t)) \le \frac{(\tau t)^n}{n!} w(H). \tag{3.9}$$

Invoking Theorem 1.2, we obtain

$$w(F^{(n,0)}(H)) \le \frac{(\tau T)^n}{n!} w(H). \tag{3.10}$$

Since $\lim_{n\to\infty} (\tau T)^n/n! = 0$, then there is an n_0 with $(\tau T)^{n_0}/n_0! < 1$. This implies

$$w(F^{(n_0,0)}(H)) < w(H). \tag{3.11}$$

Consequently, F is w-power-convex condensing about 0 and n_0. The result follows from Theorem 2.1.

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