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PHILOS-TYPE OSCILLATION CRITERIA FOR SECOND ORDER HALF-LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. In this paper we establish some oscillation theorems for the second order half-linear dynamic equation

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0, \quad t \in [a, b],$$

on time scales. Special cases of our results include some well-known oscillation results for second-order differential and half-linear differential equations. Our results are new for difference, generalized difference and q difference half-linear equations.

1. Introduction. The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis, see [**16**]. The theory of "dynamic equations" unifies the theories of differential equations and difference equations and it also extends these classical cases to cases "in between," e.g., to the so-called $q-$difference equations. A time scale $\mathbf{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications, see [**5**]. Since Stefen Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of this new theory, see the paper by Agarwal et al. [**1**] and the references cited therein. A book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [**5**] summarizes and organizes much of time scale calculus, and in the next section, we recall some of the main tools used in the subsequent sections of this paper.

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In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of some different equations on time scales; we refer the reader to the papers [**2, 3, 4, 6, 7–14, 20, 24**].

Dosly and Hilger [**8**] considered the second order linear dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\sigma} = 0, \quad t \in [a, b], \tag{1.1}$$

and established necessary and sufficient conditions for oscillation of all solutions on unbounded time scales.

Erbe and Peterson [**12**] considered (1.1) and supposed that $r(t)$ is bounded above on $[t_0, \infty)$, $t_0 \in \mathbf{T}$, $h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$, and used the Riccati transformation and proved that (Wintener-type) if

$$\int_{t_0}^{\infty} p(t)\Delta t = \infty, \tag{1.2}$$

then every solution is oscillatory in $[t_0, \infty)$. It is clear that the results given in [**8, 12**] cannot be applied when $p(t)$ is unbounded, $\mu(t) = 0$ and $p(t) = t^{-\alpha}$ when $\alpha > 1$.

Recently Saker [**24**] and Bohner and Saker [**7**] used the Riccati substitution and provided several oscillation criteria for the equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)(f \circ x^{\sigma}) = 0, \quad t \in [a, b], \tag{1.3}$$

when

$$\int_{a}^{\infty} \frac{\Delta t}{r(t)} = \infty, \tag{1.4}$$

or

$$\int_{a}^{\infty} \frac{\Delta t}{r(t)} < \infty, \tag{1.5}$$

holds, and improved the results established in [**8, 12**].

Erbe, Peterson and Saker [**14**], used generalized Riccati transformation techniques and the generalized exponential function and obtained some different oscillation criteria for (1.3) on time scales, and applied these results to the linear dynamic equations with damping terms to give some sufficient conditions for oscillations of all their solutions.

Recently, Sun and Li [**26**] considered the half-linear second order dynamic equation

$$\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta} + p(t)x^{\gamma}(t) = 0, \quad t \in [a,b], \tag{1.6}$$

where γ is an odd positive integer, and r and p are positive real-valued rd-continuous functions such that

$$\int_a^{\infty} \frac{\Delta t}{\big(r(t)\big)^{1/\gamma}} = \infty, \tag{1.7}$$

and established some sufficient conditions for existence of positive solutions. They also extended the results to dynamic equations of advanced type, namely

$$\big(r(t)(x^{\Delta}(t))^{\gamma}\big)^{\Delta} + p(t)x^{\gamma}(t+\tau) = 0, \quad t \in [a,b]. \tag{1.8}$$

In this paper we develop a qualitative theory of dynamic equations on time scales which complement the results in [**26**]. We will establish some sufficient conditions for oscillation of (1.6) on time scales where $\gamma \geq 1$ is a quotient of odd positive integers, r and p are positive, real-valued rd-continuous functions defined on the time scale interval $[a,b]$ (throughout $a, b \in \mathbf{T}$ with $a < b$) with (1.7) or

$$\int_a^{\infty} \frac{\Delta t}{\big(r(t)\big)^{1/\gamma}} < \infty, \tag{1.9}$$

holding.

Recall that a solution of (1.6) is a nontrivial real function $x(t) \in C_{rd}^1[t_x,\infty)$, $t_x \geq t_0 \geq a$, which has the property $r(t)(x^{\Delta}(t))^{\gamma} \in C_{rd}^1[t_x,\infty)$ and satisfies equation (1.6) for $t \geq t_x$. Our attention is restricted to those solutions of (1.6) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geq t_x$. A solution $x(t)$ of (1.6) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.6) is said to be oscillatory if all its solutions are oscillatory.

We note that, if $\mathbf{T} = \mathbf{R}$, $x^{\Delta}(t) = x'(t)$ and (1.6) becomes the second order half-linear differential equation

$$(r(t)(x'(t))^{\gamma})' + p(t)x^{\gamma}(t) = 0, \quad t \in [t_0, \infty). \tag{1.10}$$

For oscillation of (1.10) we refer the reader to the paper by Li [**17**] and the paper by Manojlovic [**19**] and also to the references cited therein. Recall the oscillation criteria of Philos-type for the second order linear differential equation

$$x''(t) + p(t)x(t) = 0, \quad t \in [t_0, \infty),$$

is

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)q(s) - \frac{1}{4}\, h^2(t,s) \Big]\, ds = \infty;$$

here H and h will be defined in Section 3. However, it is known that when $p(t) = \mu/t^2$, this equation reduces to the well-known Euler equation where the results of Philos [**21**], Li [**17**] and Manojlovic [**19**] cannot be applied. In fact, the Euler equation is oscillatory if $\mu > 1/4$ and nonoscillatory if $\mu < 1/4$, see [**18, 25**]. Also we note that the results of Philos-type [**21**] cannot be derived from the results of Li [**17**] and Manojlovic [**19**], since the inequality

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1)B^{\lambda} \geq 0, \quad \lambda > 1,$$

they have used in the proof is true only if $\lambda > 1$.

If $\mathbf{T} = \mathbf{Z}$, then $x^{\Delta}(t) = \Delta x(t)$ and (1.6) becomes the second order half-linear difference equation

$$\Delta\left(r(t)(\Delta x(t))^{\gamma}\right) + p(t)x^{\gamma}(t) = 0, \quad t \in [t_0, \infty). \tag{1.11}$$

For oscillation of the half-linear difference equation (1.11), we refer the reader to the papers [**15, 22, 23, 27**–**30**] and the reference cited therein.

If $\mathbf{T} = h\mathbf{Z}$, $h > 0$, then $x^{\Delta}(t) = \Delta_h x(t)$ and (1.6) becomes the more general half-linear difference equation

$$\Delta_h\left(r(t)(\Delta_h x(t))^{\gamma}\right) + p(t)x^{\gamma}(t) = 0, \quad t \in [t_0, \infty). \tag{1.12}$$

If $\mathbf{T} = q^{\mathbf{N}} = \{q^k,\ k \in \mathbf{N},\ q > 1\}$, then $x^{\Delta}(t) = \Delta_q x(t)$ and (1.1) becomes the q-half-linear difference equation

$$\Delta_q \left(r(t)(\Delta_q x(t))^{\gamma}\right) + p(t)x^{\gamma}(t) = 0, \quad t \in [t_0, \infty). \tag{1.13}$$

Our aim in this paper is to give some oscillation criteria for (1.6). Our results include the results of Li [**17**], Manojlovic [**19**] and Philos [**21**] and we also improve their results for the second order differential equation (1.10). Also, our results are new for the second order half-linear difference equation, see [**15, 22, 23, 27**–**30**] and also for the equations (1.12) and (1.13).

The paper is organized as follows: In Section 3, we use Riccati transformation techniques to obtain some new oscillation criteria of Philos-type for (1.6) when (1.7) holds. In the case when $\mathbf{T} = \mathbf{R}$ our results reduce to Philos-type oscillation criteria of Li [**17**] and Manojlovic [**19**], and our results improve the oscillation of second order differential equations given by Philos [**21**], Li [**17**] and Manojlovic [**19**] since these referenced results cannot be applied to the Euler equation when $\gamma = 1$ and $p(t) = \mu/t^{2\cdot}$, see [**18, 25**]. When $\mathbf{T} = \mathbf{Z}$, our results will give oscillation results for second order difference equations which are new, and in the case when $\mathbf{T} = q^{\mathbf{N}}$, for $q > 1$, we obtain new oscillation results for q-difference equations. Finally, in Section 4, we consider equations that do not satisfy (1.7) and present some conditions that ensure that all solutions are either oscillatory or converge to zero.

2. Some preliminaries on time scales. On any time scale $\mathbf{T}$ we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbf{T},\ s > t\}, \quad \rho(t) = \sup\{s \in \mathbf{T},\ s < t\}. \tag{2.1}$$

A point $t \in \mathbf{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale $\mathbf{T}$ is defined by $\mu(t) = \sigma(t) - t$.

For a function $f : \mathbf{T} \to \mathbf{R}$, the (delta) derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{for} \quad s \in \mathbf{T} \backslash \{\sigma(t)\}.$$

A function $f : [a, b] \to \mathbf{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points, and f is said to be differentiable if its derivative exists. The derivative and the shift operator σ are related by the formula

$$f^\sigma = f + \mu f^\Delta \quad \text{where} \quad f^\sigma := f \circ \sigma = f(\sigma). \tag{2.2}$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively.

Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]$ if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \geq 0$ and $f^\Delta(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g where $gg^\sigma > 0$ of two differentiable functions f and g:

$$(fg)^\Delta = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)) \tag{2.3}$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. \tag{2.4}$$

For $a, b \in \mathbf{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a). \tag{2.5}$$

The integration by parts formula reads

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\sigma(t)g^\Delta(t)\Delta t, \tag{2.6}$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s,$$

and

$$\left(\int_a^t f(s)\Delta s\right)^{\Delta} = f(t). \tag{2.7}$$

In the case $\mathbf{T} = \mathbf{R}$ we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) \equiv 0, \quad f^{\Delta} = f',$$

and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt,$$

and in the case $\mathbf{T} = \mathbf{Z}$ we have

$$\sigma(t) = t+1, \quad \rho(t) = t-1, \quad \mu(t) \equiv 1, \quad f^{\Delta} = \Delta f,$$

and

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

whereas in the case $\mathbf{T} = h\mathbf{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$ and

$$f^{\Delta} = \Delta_h f = \frac{f(t+h) - f(t)}{h},$$

and

$$\int_a^b f(t)\Delta t = \sum_{i=a/h}^{(b/h)-1} f(i),$$

and in the case $\mathbf{T} = q^{\mathbf{N}} = \{t : t = q^k,\ k \in \mathbf{N},\ q > 1\}$ we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$ and

$$x_q^{\Delta}(t) = \frac{x(qt) - x(t)}{(q-1)t} \quad \text{and} \quad \int_a^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} \mu(q^k) f(q^k).$$

3. Oscillation criteria. In this section we give some new oscillation criteria of Philos-type for (1.6) which includes as a special case the results of Li [**17**], Manojlovic [**19**] and Philos [**21**].

First, let us introduce the class of functions $\Re$ which will be extensively used in the sequel. Let $\mathbf{D}_0 \equiv \{(t,s) : t > s \geq t_0\}$ and $\mathbf{D} \equiv \{(t,s) : t \geq s \geq t_0\}$.

The function $H \in C_{rd}(\mathbf{D}, \mathbf{R})$ is said to belong to the class $\Re$ if

(H1) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$, on $\mathbf{D}_0$,

(H2) H has a continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathbf{D}_0$ with respect to the second variable. (H is an rd-continuous function if H is an rd-continuous function in t and s).

In the sequel, we assume that:

$(h)_1$ $\gamma \geq 1$ is a quotient of odd positive integers and r and p are positive real-valued rd-continuous functions.

Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. In order to prove our theorems, we need the following auxiliary lemmas.

Lemma 3.1. *Assume that* $(h)_1$ *and* (1.7) *hold, and suppose* x *solves* (1.6) *with* $x(t) > 0$ *for all* $t \geq t_0 \geq a$. *Then*

$$(3.1) \quad x^{\Delta}(t) \geq 0 \quad \textit{and} \quad x^{\Delta}(t) \geq \left(\frac{r^{\sigma}}{r}\right)^{1/\gamma} x^{\Delta}(\sigma(t)) \quad \textit{for all} \quad t \geq t_0.$$

Proof. Let $x(t) > 0$ for $t \geq t_0$. Now (1.6) implies

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} = -\, p(t)x^{\gamma}(t) < 0,$$

so $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing. Now, we prove that $x^{\Delta}(t) \geq 0$. Suppose not. Without loss of generality, assume there exists $t_1 \geq t_0$ such that $r(t_1)(x^{\Delta}(t_1))^{\gamma} = c < 0$. Then

$$r(s)\left(x^{\Delta}(t)\right)^{\gamma} \leq r(t_1)\left(x^{\Delta}(t_1)\right)^{\gamma} = c \quad \text{for all} \quad s \geq t_1,$$

and therefore

$$x^{\Delta}(s) \leq \left(\frac{c}{r(s)}\right)^{1/\gamma} \quad \text{for all} \quad s \geq t_1.$$

For $t \geq t_1$, we have

$$x(t) = x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \leq x(t_1) + c^{1/\gamma} \int_{t_1}^{t} \frac{\Delta s}{(r(s))^{1/\gamma}} \longrightarrow -\infty \quad \text{as} \quad t \to \infty, \tag{3.2}$$

a contradiction. Thus, $x^{\Delta}(t) \geq 0$. The proof of the second part of (3.1) follows from the fact that $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing. This completes the proof.

Lemma 3.2. *Let* $f(u) = bu - au^{(\gamma+1)/\gamma}$*, where* $a > 0$ *and* b *are constants,* γ *is a quotient of odd positive integers. Then* f *attends its maximum value on* $\mathbf{R}$ *at* $u^* = ((b\gamma)/(a(\gamma+1)))^{\gamma}$*, and*

$$\max_{u \in \mathbf{R}} f(u) = f(u^*) = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{b^{\gamma+1}}{a^{\gamma}}. \tag{3.3}$$

Now, we are ready to state and prove our main results.

Theorem 3.1. *Assume that* $(h)_1$ *and* (1.7) *hold, and let* $H : \mathbf{D} \to \mathbf{R}$ *be an rd-continuous function belonging to the class* $\Re$*, and suppose there exists a positive rd-continuous function* δ *such that*

$$\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} K(t, s)\, \Delta s = \infty, \tag{3.4}$$

where $(\delta^{\Delta}(s))_+ = \max\{0, (\delta^{\Delta}(s))\}$ *and*

$$K(t,s) = H(t,s)\delta(s)p(s) - \frac{(\delta^{\sigma})^{\gamma+1} r(s) \left[H(t,s)[\left(\delta^{\Delta}(s)\right)_+/\delta^{\sigma}] - H^{\Delta_s}(t,s)\right]^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1} H^{\gamma}(t,s)}.$$

Then every solution of equation (1.6) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.6), and let $t_0 \geq a$ be such that $x(t) \neq 0$ for all $t \geq t_0$, so, without loss of generality, we may assume that x is an eventually positive solution of (1.6) with $x(t) > 0$ for all $t \geq t_0 \geq a$. Define the function $w(t)$ by

$$w(t) = \delta(t)\,\frac{\left(r(t)x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}, \quad \text{for} \quad t \geq t_0. \tag{3.5}$$

Then by Lemma 3.1 we have $w(t) > 0$, and using (2.3) we obtain

$$w^{\Delta}(t) = \left((rx^{\Delta})^{\sigma}\right)^{\gamma} \left(\frac{\delta(t)}{x^{\gamma}(t)}\right)^{\Delta} + \frac{\delta(t)}{x^{\gamma}(t)} \left(\left(r(t)\,x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}.$$

This implies by (2.4) that

$$\begin{aligned} w^{\Delta}(t) = {} & \left((rx^{\Delta})^{\sigma}\right)^{\gamma} \left(\frac{x^{\gamma}(t)\delta^{\Delta}(t) - \delta(t)\,(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\,(x^{\sigma})^{\gamma}}\right) \\ & + \frac{\delta(t)}{x^{\gamma}(t)} \left(\left(r(t)\,x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}. \end{aligned} \tag{3.6}$$

Now (1.6) and (3.6) imply

$$w^{\Delta}(t) \leq -\,\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}}\, w^{\sigma} - \left((rx^{\Delta})^{\sigma}\right)^{\gamma}\, \frac{\delta(t)\,(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\,(x^{\sigma})^{\gamma}}. \tag{3.7}$$

Now using the Chain rule [**4**, Theorem 1.87], we have

$$(x^{\gamma}(t))^{\Delta} = \gamma\,\eta^{\gamma-1}x^{\Delta}(t), \quad \eta \in [x(t), x(\sigma(t))].$$

Using the last equality and Lemma 3.1 in (3.7), we have

$$\begin{aligned} w^{\Delta}(t) \leq {} & -\,\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}}\, w^{\sigma} \\ & - \left((rx^{\Delta})^{\sigma}\right)^{\gamma}\, \frac{\gamma\delta(t)\eta^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)\,(x^{\sigma})^{\gamma}} \\ \leq {} & -\,\delta(t)p(t) + \frac{\left(\delta^{\Delta}(t)\right)_{+}}{\delta^{\sigma}}\, w^{\sigma} \\ & - \frac{\gamma\delta(t)}{(\delta^{\sigma})^{\lambda} r^{\lambda-1}}\, (w^{\sigma})^{\lambda} \end{aligned}$$

where $\lambda = (\gamma+1)/\gamma$. Thus, we have

$$w^{\Delta}(t) \le -\delta(t)p(t) + \frac{\big(\delta^{\Delta}(t)\big)_{+}}{\delta^{\sigma}}\, w^{\sigma} - \frac{\gamma\delta(t)}{(\delta^{\sigma})^{\lambda} r^{\lambda-1}}\,(w^{\sigma})^{\lambda}. \tag{3.8}$$

From (3.8), it follows that

$$\begin{aligned}\int_{t_0}^{t} H(t,s)\delta(s)p(s)\Delta s \le & -\int_{t_0}^{t} H(t,s)w^{\Delta}(s)\Delta s \\ & + \int_{t_0}^{t} H(t,s)\,\frac{\big(\delta^{\Delta}(s)\big)_{+}}{\delta^{\sigma}}\, w^{\sigma}\Delta s \\ & - \int_{t_0}^{t} H(t,s)\,\frac{\gamma\delta(t)}{(\delta^{\sigma})^{\lambda} r^{\lambda-1}}\,(w^{\sigma})^{\lambda}\Delta s.\end{aligned} \tag{3.9}$$

Using the integration by parts formula (2.6), we have

$$\begin{aligned}\int_{t_0}^{t} H(t,s)w^{\Delta}(s)\Delta s &= H(t,s)w(s)\Big|_{t_0}^{t} - \int_{t_0}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s \\ &= -H(t,t_0)w(t_0) - \int_{t_0}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s\end{aligned} \tag{3.10}$$

since $H(t,t)=0$. Substituting (3.10) in (3.9) we get

$$\begin{aligned}\int_{t_0}^{t} H(t,s)\delta(s)p(s)\Delta s \le & H(t,t_0)w(t_0) - \int_{t_0}^{t} H^{\Delta_s}(t,s)w^{\sigma}\Delta s \\ & + \int_{t_0}^{t} H(t,s)\,\frac{\big(\delta^{\Delta}(s)\big)_{+}}{\delta^{\sigma}}\, w^{\sigma}\Delta s \\ & - \int_{t_0}^{t} H(t,s)\,\frac{\gamma\delta(t)}{(\delta^{\sigma})^{\lambda} r^{\lambda-1}}\,(w^{\sigma})^{\lambda}\Delta s.\end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}&\int_{t_0}^{t} H(t,s)\delta(s)p(s)\Delta s \\ &\quad \le H(t,t_0)w(t_0) + \int_{t_0}^{t}\left[H(t,s)\,\frac{\big(\delta^{\Delta}(s)\big)_{+}}{\delta^{\sigma}} - H^{\Delta_s}(t,s)\right] w^{\sigma}\Delta s \\ &\qquad - \int_{t_0}^{t} H(t,s)\,\frac{\gamma\delta(t)}{(\delta^{\sigma})^{\lambda} r^{\lambda-1}}\,(w^{\sigma})^{\lambda}\Delta s.\end{aligned} \tag{3.12}$$

Now use Lemma 3.2 with

$$a = \frac{\gamma\delta(t)H(t,s)}{(\delta^\sigma)^\lambda r^{\lambda-1}}, \quad b = H(t,s)\,\frac{\left(\delta^\Delta(s)\right)_+}{\delta^\sigma} - H^{\Delta_s}(t,s) \quad \text{and} \quad u = w^\sigma$$

to obtain

(3.13)

$$\begin{aligned}\int_{t_0}^{t} & H(t,s)\delta(s)p(s)\,\Delta s \\ &\leq H(t,t_0)w(t_0) \\ &\quad + \int_{t_0}^{t} \frac{(\delta^\sigma)^{\gamma+1} r(s)\left[H(t,s)\left((\delta^\Delta(s))_+/\delta^\sigma\right) - H^{\Delta_s}(t,s)\right]^{\gamma+1}}{\delta^\gamma(s)(\gamma+1)^{\gamma+1}\,(H(t,s))^\gamma}\,\Delta s.\end{aligned}$$

Then, for all $t \geq t_0$, we have

$$\int_{t_0}^{t} K(t,s)\,\Delta s \leq H(t,t_0)w(t_0), \tag{3.14}$$

and this implies that

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t} K(t,s)\,\Delta s \leq w(t_0), \tag{3.15}$$

for all large t, which contradicts (3.4). Therefore, every solution of (1.6) oscillates on $[a,\infty)$.

As an immediate consequence of Theorem 3.1 we get the following.

Corollary 3.1. *Let assumption* (3.4) *in Theorem* 3.1 *be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\delta(s)p(s)\,\Delta s = \infty, \tag{3.16}$$

and

(3.17)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \cdot \int_{t_0}^{t} \frac{(\delta^\sigma)^{\gamma+1} r(s) \left[H(t,s)\left(\delta^\Delta(s)/\delta^\sigma\right) - H^{\Delta_s}(t,s)\right]^{\gamma+1}}{\delta^\gamma(s)(\gamma+1)^{\gamma+1} H^\gamma(t,s)} \Delta s < \infty.$$

Then every solution of equation (1.6) *is oscillatory on* $[a,\infty)$.

Corollary 3.2. *Assume that* $(h)_1$ *and* (1.7) *hold,* $\delta(t) \equiv 1$, $r(t) \equiv 1$, *and we write* $H^{\Delta_s}(t,s) = -h(t,s)$. *If*

(3.18)
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)p(s) - \frac{r(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^\gamma(t,s)}\right] \Delta s = \infty,$$

then every solution of (1.6) *is oscillatory on* $[a,\infty)$.

Corollary 3.3. *Assume that* $(h)_1$ *and* (1.7) *hold,* $r(t) = 1$, $\delta(t) \equiv 1$, $\gamma = p-1$, *and suppose* $H^{\Delta_s}(t,s) = -h(t,s)[H(t,s)]^{1/q}$ *for some function* h; *here* $1/q + 1/p = 1$. *If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)p(s) - \frac{h^p(t,s)}{p^p}\right] \Delta s = \infty, \tag{3.19}$$

then every solution of (1.6) *is oscillatory on* $[a,\infty)$.

As a special case of Theorem 3.1, if $\mathbf{T} = \mathbf{R}$, then $\sigma(t) = t$, $\mu(t) \equiv 0$, $\delta^\Delta = \delta'$ and $H^{\Delta_s}(t,s) = \partial H(t,s)/\partial s$, so (3.4) becomes

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} A(t,s)\, ds = \infty, \tag{3.20}$$

where

(3.21)
$$A(t,s) = H(t,s)\delta(s)p(s) - \frac{\delta(s)r(s)\left[H(t,s)\left((\delta'(s))_+/\delta(s)\right) - \partial H(t,s)/\partial s\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^\gamma(t,s)} = \infty,$$

and

$$(\delta'(s))_+ = \max\{0, (\delta'(s))\}.$$

Then we have the following oscillation criteria for (1.10).

Corollary 3.4. *Assume that* $(h)_1$ *and*

$$\int_a^\infty \frac{1}{r^{1/\gamma}(s)}\, ds = \infty$$

hold. Furthermore, let $H : \mathbf{D} \to \mathbf{R}$ *be a continuous function belonging to the class* $\Re$*, and suppose there exists a positive continuous function* δ *such that* (3.20) *holds. Then every solution of equation* (1.10) *is oscillatory.*

From Corollary 3.2 and Corollary 3.3, if $\mathbf{T} = \mathbf{R}$ we have the following well-known oscillation criteria.

Corollary 3.5 (Philos's theorem). *Assume that* $(h)_1$ *holds,* $\delta(t) \equiv 1$*,* $\gamma = 1$*,* $r(t) \equiv 1$ *and* $\partial H(t,s)/\partial s = -h(t,s)\sqrt{H(t,s)}$ *for some function* h*. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[p(s)H(t,s) - \frac{h^2(t,s)}{4} \right] ds = \infty,$$

then every solution of

$$x''(t) + p(t)x(t) = 0, \quad t \in [t_0, \infty)$$

is oscillatory.

Corollary 3.6 (Li's theorem). *Assume that* $(h)_1$ *holds,* $r(t) = 1$*,* $\delta(t) \equiv 1$*,* $\gamma = p - 1$ *and* $\partial H(t,s)/\partial s = -h(t,s)[H(t,s)]^{1/q}$ *for some function* h*; here* $1/q + 1/p = 1$*. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)p(s) - \frac{h^p(t,s)}{p^p} \right] ds = \infty,$$

then every solution of

$$\Big((x'(t))^{\gamma-1}\Big)' + p(t)x^{\gamma-1}(t) = 0, \quad t \in [t_0, \infty)$$

is oscillatory.

Corollary 3.7 (Manojlovic's theorem). *Assume that* $(h)_1$ *and* (1.7) *hold,* $\delta(t) \equiv 1$*, and we write* $\partial H(t,s)/\partial s = -h(t,s)$*. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)p(s) - \frac{r(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)} \right] ds = \infty,$$

then every solution of (1.10) *is oscillatory.*

If $\mathbf{T} = \mathbf{Z}$, then $\delta^{\Delta}(n) = \Delta\delta(n) = \delta(n+1) - \delta(n)$, $H^{\Delta_s}(m,n) = \Delta_2 H(m,n) = H(m,n+1) - H(m,n)$ and (3.4) becomes

$$\limsup_{m\to\infty} \frac{1}{H(m,n_0)} \sum_{n=n_0}^{m-1} L(m,n) = \infty, \tag{3.22}$$

where

$$L(m,n) = H(m,n)\delta(n)p(n) - \frac{\delta^{\gamma+1}(n+1)r(n)}{(\gamma+1)^{\gamma+1}\delta(n)H^{\gamma}(m,n)} B^{\gamma+1}(m,n)$$

and

$$B(m,n) = \left(\frac{(\Delta\delta(n))_+}{\delta(n+1)} - (H(m,n+1) - H(m,n)) \right).$$

If $\mathbf{T} = h\mathbf{Z}$, $h > 0$ then $\sigma(t) = t + h$, $\mu(t) = h$, $\delta^{\Delta}(t) = \Delta_h\delta(t) = (\delta(t+h) - \delta(t))/h$, $H^{\Delta_s}(t,s) = \Delta_2 H(t,s) = H(t,s+h) - H(t,s)$ and (3.4) becomes

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{(t/h)-1} Q(t,s) = \infty, \tag{3.23}$$

where

$$Q(t,s) = H(t,s)\delta(s)p(s) - \frac{\delta^{\gamma+1}(s+h)r(s)}{(\gamma+1)^{\gamma+1}\delta(s)H^{\gamma}(t,s)} C^{\gamma+1}(t,s),$$

$$C(t,s) = \left(\frac{(\Delta_h\delta(s))_+}{\delta(s+h)} - (H(t,s+h) - H(t,s)) \right),$$

and

$$(\Delta_h \delta(s))_+ = \max\{0, (\Delta_h \delta(s))\}.$$

If $\mathbf{T} = q^{\mathbf{N}} = \{q^k,\ k \in \mathbf{N},\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $\delta_q^{\Delta}(t) = (\delta(qt) - \delta(t))/((q-1)t)$, $H^{\Delta_s}(t,s) = (H(t,qs) - H(t,s))/((q-1)s)$, and from (3.4) and the definition of the integration on $q^{\mathbf{N}}$, one can deduce oscillation conditions for (1.13); the details are left to the reader.

For the oscillation of half-linear difference and half-linear generalized difference equations, we have the following.

Corollary 3.8. *Assume* $(h)_1$ *holds and that*

$$\sum_{n=a}^{\infty} \frac{1}{(r(n))^{1/\gamma}} = \infty.$$

Let $H : \mathbf{D} \to \mathbf{R}$ *be a sequence belonging to the class* $\Re$. *If there exists a positive sequence* $\delta(n)$ *such that* (3.22) *holds, then every solution of equation* (1.11) *is oscillatory.*

Corollary 3.9. *Assume* $(h)_1$ *holds and that*

$$\sum_{n=a}^{\infty} \frac{1}{(r(n))^{1/\gamma}} = \infty.$$

Let $H : \mathbf{D} \to \mathbf{R}$ *be a sequence belonging to the class* $\Re$. *If there exists a positive sequence* δ *such that* (3.23) *holds, then every solution of equation* (1.12) *is oscillatory.*

Remark 3.1. With an appropriate choice of a function H, one can derive a number of oscillation criteria for (1.6) on different types of time scales. Consider, for example, the function $H(t,s) = (t-s)^{\lambda}$, $(t,s) \in \mathbf{D}$, with $\lambda \geq 1$ an odd integer. Evidently, H belongs to the class $\Re$. Then (3.4) reduces to an oscillation criterion of Kamenev-type.

Example 3.1. Consider the half-linear dynamic equation

$$\left(t^{\gamma-1}\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}+\frac{\beta}{t^2}\,(x)^{\gamma}=0, \tag{3.24}$$

for $t\in[1,\infty)$, where β is a positive constant and $\gamma>1$ is a positive integer (and a quotient of odd positive integers). Let $p(t)=\beta/t^2$ and $r(t)=t^{\gamma-1}$. Note that (1.7) is satisfied since

$$\int_1^{\infty}\left(\frac{1}{t}\right)^{(\gamma-1)/\gamma}\Delta t=\infty \quad \text{for} \quad \gamma>1.$$

We will apply Theorem 3.1 with $\delta=t$ and $H(t,s)=1$. Now

$$\begin{aligned}\limsup_{t\to\infty}\int_1^t K(t,s)\,\Delta s &= \limsup_{t\to\infty}\int_1^t\left[sp(s)-\frac{s^{\gamma-1}}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s\\ &= \limsup_{t\to\infty}\int_1^t\left[\frac{\beta}{s}-\frac{1}{(\gamma+1)^{\gamma+1}s}\right]\Delta s=\infty\end{aligned}$$

when $\beta>1/((\gamma+1)^{\gamma+1})$. Then every solution of (3.24) is oscillatory when $\beta>1/((\gamma+1)^{\gamma+1})$. We also note that when $\gamma=1$ the condition is $\beta>1/4$ which is the sufficient condition for the Euler equation. Also, one can see that the results of Philos [**21**], Li [**17**] and Manojlovic [**19**] cannot be applied to (3.24) when $\mathbf{T}=\mathbf{R}$. So our results not only include as special cases the results of Philos [**21**], Li [**17**] and Manojlovic [**19**] but also improve these results and can be applied to the Euler equations provided that $\beta>1/4$ (see [**18, 25**]).

4. Other criteria. In this section we consider (1.6), where r does not satisfy (1.7), i.e.,

$$\int_a^{\infty}\frac{1}{(r(t))^{1/\gamma}}\Delta t<\infty. \tag{4.1}$$

We start with the following auxiliary lemma, whose proof is similar to the proof of Theorem 3.3 in [**14**] and hence is omitted.

Lemma 4.1. *Assume* $(h)_1$ *and* (4.1) *hold, and suppose*

$$\int_a^{\infty}\left[\frac{1}{r(t)}\int_a^t p(s)\Delta s\right]^{1/\gamma}\Delta t=\infty. \tag{4.2}$$

Suppose that x *is a nonoscillatory solution of* (1.6) *such that there exists* $t_1 \in \mathbf{T}$ *with*

$$x(t)\left(x^{\Delta}(t)\right)^{\gamma} < 0 \quad \textit{for all} \quad t \geq t_1. \tag{4.3}$$

Then

$$\lim_{t\to\infty} x(t) \quad \textit{exists and is zero.}$$

Using Lemma 4.1, we can derive the following criteria.

Theorem 4.1. *Assume* $(h)_1$, (4.1) *and* (4.2) *hold. Let* $H : \mathbf{D} \to \mathbf{R}$ *be an rd-continuous function belonging to the class* $\Re$. *If there exists a positive differentiable function* $\delta(t)$ *such that* (3.4) *holds, then every solution of* (1.6) *is oscillatory or converges to zero.*

Proof. Assume that x is a nonoscillatory solution of (1.6). Hence, x is either eventually positive or eventually negative, i.e., there exists $t_0 \geq a$ with $x(t) > 0$ for all $t \geq \mathrm{t}_0$ or $x(t) < 0$ for all $t \geq t_0$. Without loss of generality, we assume that $x(t)$ is eventually positive. From (1.6) we have

$$\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta} = -\,p(t)x^{\gamma}(t) < 0,$$

and so $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually decreasing function and either $x^{\Delta}(t)$ is eventually positive or eventually negative. If $x^{\Delta}(t)$ is eventually positive we can derive a contradiction as in Theorem 3.1.

If $x^{\Delta}(t)$ is eventually negative, we see from Lemma 4.1 that $x(t)$ converges to zero as $t \to \infty$. This completes the proof. □

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