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Research Article

On the Convergence of an Implicit Iterative Process for Generalized Asymptotically Quasi-Nonexpansive Mappings

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The purpose of this paper is to introduce and consider a general implicit iterative process which includes Schu's explicit iterative processes and Sun's implicit iterative processes as special cases for a finite family of generalized asymptotically quasi-nonexpansive mappings. Strong convergence of the purposed iterative process is obtained in the framework of real Banach spaces.

1. Introduction and Preliminaries

Let E be a real Banach space and $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (1.1)$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Let C be a nonempty closed and convex subset of a Banach space E . Let $T : C \rightarrow C$ be a mapping. Denote by $F(T)$ the fixed point set of T .

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.2)$$

T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \quad \forall x \in C, y \in F(T). \quad (1.3)$$

A nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive; however, the inverse may be not true. See the following example [1].

Example 1.1. Let $E = \mathbb{R}^1$ and define a mapping by $T : E \rightarrow E$ by

$$Tx = \begin{cases} \frac{x}{2} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad (1.4)$$

Then T is quasi-nonexpansive but not nonexpansive.

T is said to be *asymptotically nonexpansive* if there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \geq 1. \quad (1.5)$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the asymptotical sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. It is known that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E , then every asymptotically nonexpansive mapping on C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied weak and strong convergence problems of implicit iterative processes for such a class of mappings.

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exists a positive sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - y\| \leq k_n \|x - y\|, \quad \forall x \in C, y \in F(T), n \geq 1. \quad (1.6)$$

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \quad (1.7)$$

Putting $\xi_n = \max\{0, \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|)\}$, we see that $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. Then (1.7) is reduced to the following:

$$\|T^n x - T^n y\| \leq \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1. \quad (1.8)$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Kirk [3] (see also Bruck et al. [4]) as a generalization of the class of asymptotically nonexpansive mappings. It is known that if C is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self-mapping in the intermediate sense has a fixed point; see [5] more details.

T is said to be *asymptotically quasi-nonexpansive in the intermediate sense* if it is continuous, $F(T) \neq \emptyset$, and the following inequality holds:

$$\limsup_{n \rightarrow \infty} \sup_{x \in C, y \in F(T)} (\|T^n x - y\| - \|x - y\|) \leq 0. \quad (1.9)$$

Putting $\xi_n = \max\{0, \sup_{x \in C, y \in F(T)} (\|T^n x - y\| - \|x - y\|)\}$, we see that $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. Then (1.9) is reduced to the following:

$$\|T^n x - y\| \leq \|x - y\| + \xi_n, \quad \forall x \in C, y \in F(T), n \geq 1. \quad (1.10)$$

T is said to be *generalized asymptotically nonexpansive* if there exist two positive sequences $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1. \quad (1.11)$$

It is easy to see that the class of generalized asymptotically nonexpansive includes the class of asymptotically nonexpansive as a special case.

T is said to be *generalized asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exist two positive sequences $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\|T^n x - y\| \leq k_n \|x - y\| + \xi_n, \quad \forall x \in C, y \in F(T), n \geq 1. \quad (1.12)$$

The class of generalized asymptotically quasi-nonexpansive was considered by Shahzad and Zegeye [6]; see [6, 7] for more details.

Recall that the modified Mann iteration which was introduced by Schu [8] generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1, \quad (1.13)$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$ and $T: C \rightarrow C$ is an asymptotically nonexpansive mapping.

In 1991, Schu [8] obtained the following results.

Theorem Schu 1. *Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T: C \rightarrow C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\{\alpha_n\} \in [0, 1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in (1.13). Then $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$.*

Theorem Schu 2. Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T : C \rightarrow C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\{\alpha_n\} \in [0, 1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in (1.13). Suppose that T^m is compact for some positive integer $m \geq 1$. Then the sequence $\{x_n\}$ converges strongly to some fixed point of T .

Theorem Schu 3. Let E be a uniformly convex Banach space, $\emptyset \neq C \subset E$ closed bounded and convex, and $T : C \rightarrow C$ asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$ for which $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\{\alpha_n\} \in [0, 1]$ is bounded away. Let $\{x_n\}$ be a sequence generated in (1.13). Suppose that there exists a nonempty compact and convex subset K of E and $\lambda \in (0, 1)$ such that

$$d(Tx, K) \leq \lambda d(x, K), \quad \forall x \in C. \quad (1.14)$$

Then the sequence $\{x_n\}$ converges strongly to some fixed point of T .

In 2007, Shahzad and Zegeye [6] considered the following implicit iterative process for a finite family of generalized asymptotically quasi-nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\ &\vdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1}, \\ &\vdots, \end{aligned} \quad (1.15)$$

where x_0 is the initial value and $\{\alpha_n\}$ is a sequence $(0, 1)$. Since for each $n \geq 1$, it can be written as $n = (h - 1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\}$, $h = h(n) \geq 1$ is a positive integer, and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + \alpha_n T_{i(n)}^{h(n)} x_n, \quad \forall n \geq 1. \quad (1.16)$$

We remark that the implicit iterative process (1.16) was first considered by Sun [9]; see [9] for more details.

Shahzad and Zegeye [6] obtained the following results.

Theorem SZ 1. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $\{T_i : i \in J\}$, where $J = \{1, 2, \dots, N\}$, be N uniformly Lipschitz, generalized asymptotically quasi-nonexpansive self-mappings of C with $\{k_{in}\} \subset [1, \infty)$, $\{\xi_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{in} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{in} < \infty$ for all $i \in J$. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i : i \in J\}$ which is either semicompact or satisfies condition (\overline{C}) . Let $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.16). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$.*

Theorem SZ 2. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $\{T_i : i \in J\}$, where $J = \{1, 2, \dots, N\}$, be N generalized asymptotically quasi-nonexpansive self-mappings of C with $\{k_{in}\} \subset [1, \infty)$, $\{\xi_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{in} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{in} < \infty$ for all $i \in J$. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is closed. Let $\{\alpha_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.16). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in J\}$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.*

In this paper, motivated by the above results, we consider the following implicit iterative process for two finite families of generalized asymptotically quasi-nonexpansive mappings $\{S_1, S_2, \dots, S_N\}$ and $\{T_1, T_2, \dots, T_N\}$:

$$\begin{aligned}
 x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1 + \delta_1 u_1, \\
 x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_1 + \gamma_2 T_2 x_2 + \delta_2 u_2, \\
 &\vdots \\
 x_N &= \alpha_N x_{N-1} + \beta_N S_N x_{N-1} + \gamma_N T_N x_N + \delta_N u_N, \\
 x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_1^2 x_N + \gamma_{N+1} T_1^2 x_{N+1} + \delta_{N+1} u_{N+1}, \\
 &\vdots \\
 x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} S_N^2 x_{2N-1} + \gamma_{2N} T_N^2 x_{2N} + \delta_{2N} u_{2N}, \\
 x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} S_1^3 x_{2N} + \gamma_{2N+1} T_1^3 x_{2N+1} + \delta_{2N+1} u_{2N+1}, \\
 &\vdots,
 \end{aligned} \tag{1.17}$$

where x_0 is the initial value, $\{u_n\}$ is a bounded sequence in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are sequences $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Since for each $n \geq 1$, it can be written as $n = (h - 1)N + i$, where $i = i(n) \in \{1, 2, \dots, N\}$, $h = h(n) \geq 1$ is a positive integer and $h(n) \rightarrow \infty$ as $n \rightarrow \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1. \tag{1.18}$$

We remark that our implicit iterative process (1.18) which includes the explicit iterative process (1.13) and the implicit iterative process (1.16) as special cases is general.

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then the implicit iterative process (1.18) is reduced to the following implicit iterative process:

$$x_n = (\alpha_n + \beta_n)x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x_n + \delta_n u_n, \quad \forall n \geq 1. \quad (1.19)$$

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then the implicit iterative process (1.18) is reduced to the following explicit iterative process:

$$x_n = \frac{\alpha_n}{1 - \gamma_n} x_{n-1} + \frac{\beta_n}{1 - \gamma_n} S_{i(n)}^{h(n)} x_{n-1} + \frac{\delta_n}{1 - \gamma_n} u_n, \quad \forall n \geq 1. \quad (1.20)$$

The purpose of this paper is to study the convergence of the implicit iteration process (1.18) for two finite families of generalized asymptotically quasi-nonexpansive mappings. Strong convergence theorems are obtained in the framework of real Banach spaces. The results presented in this paper improve and extend the corresponding results in Shahzad and Zegeye [6], Sun [9], Chang et al. [10], Chidume and Shahzad [11], Guo and Cho [12], Kim et al. [13], Qin et al. [14], Thianwan and Suantai [15], Xu and Ori [16], and Zhou and Chang [17].

In order to prove our main results, we also need the following lemmas.

Lemma 1.2 (see [18]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \geq n_0, \quad (1.21)$$

where n_0 is some positive integer. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \rightarrow \infty} r_n$ exists.

Lemma 1.3 (see [19]). *Let E be a real uniformly convex Banach space, $s > 0$ a positive number, and $B_s(0)$ a closed ball of E . Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|ax + by + cz + dw\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - abg(\|x - y\|) \quad (1.22)$$

for all $x, y, z, w \in B_s(0) = \{x \in E : \|x\| \leq s\}$ and $a, b, c, d \in [0, 1]$ such that $a + b + c + d = 1$.

2. Main Results

Lemma 2.1. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$ and $S_i : C \rightarrow C$ a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F =$*

$\bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.18). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c, d \in (0,1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then

$$\lim_{n \rightarrow \infty} \|x_n - T_r x_n\| = \lim_{n \rightarrow \infty} \|x_n - S_r x_n\| = 0, \quad \forall r \in \{1, 2, \dots, N\}. \quad (2.1)$$

Proof. First, we show that the sequence $\{x_n\}$ generated in (1.18) is well defined. For each $n \geq 1$, define a mapping $C_n : C \rightarrow C$ as follows:

$$C_n x = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{h(n)} x_{n-1} + \gamma_n T_{i(n)}^{h(n)} x + \delta_n u_n, \quad \forall x \in C. \quad (2.2)$$

Notice that

$$\begin{aligned} \|C_n x - C_n y\| &\leq \gamma_n \|T_{i(n)}^{h(n)} x - T_{i(n)}^{h(n)} y\| \\ &\leq dL_t \|x - y\|, \quad \forall x, y \in C. \end{aligned} \quad (2.3)$$

From the restriction (a), we see that C_n is a contraction for each $n \geq 1$. From Banach contraction mapping principle, we can prove that the sequence $\{x_n\}$ generated in (1.18) is well defined.

Fixing $p \in F$, we see that

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|S_{i(n)}^{h(n)} x_{n-1} - p\| + \gamma_n \|T_{i(n)}^{h(n)} x_n - p\| + \delta_n \|u_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n (k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)}) + \gamma_n (k_{h(n)} \|x_n - p\| + \xi_{h(n)}) \\ &\quad + \delta_n \|u_n - p\| \\ &\leq (\alpha_n + \beta_n k_{h(n)}) \|x_{n-1} - p\| + (1 - \alpha_n - \beta_n) k_{h(n)} \|x_n - p\| + 2\xi_{h(n)} \\ &\quad + \delta_n \|u_n - p\|. \end{aligned} \quad (2.4)$$

Notice that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. We see from the restrictions (a) and (b) that there exists a positive integer n_0 such that

$$(1 - \alpha_n - \beta_n) k_{h(n)} \leq R < 1, \quad \forall n \geq n_0, \quad (2.5)$$

where $R = (1 - (a + b))(1 + (a + b)/(2 - 2(a + b)))$. It follows from (2.4) that

$$\begin{aligned}
\|x_n - p\| &\leq \frac{\alpha_n + \beta_n k_{h(n)}}{1 - (1 - \alpha_n - \beta_n)k_{h(n)}} \|x_{n-1} - p\| + \frac{\delta_n}{1 - (1 - \alpha_n - \beta_n)k_{h(n)}} \|u_n - p\| \\
&\quad + \frac{2\xi_{h(n)}}{1 - (1 - \alpha_n - \beta_n)k_{h(n)}} \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + \frac{\delta_n}{1 - R} \|u_n - p\| + \frac{2\xi_{h(n)}}{1 - R} \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + M_1(\delta_n + \xi_{h(n)}), \quad \forall n \geq n_0,
\end{aligned} \tag{2.6}$$

where M_1 is an appropriate constant such that $M_1 = \max\{\sup_{n \geq 1}\{\|u_n - p\|/(1 - R)\}, 2/(1 - R)\}$. In view of the restrictions (a) and (b), we obtain from Lemma 1.2 that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. It follows that the sequence $\{x_n\}$ is bounded. In view of Lemma 1.3, we see that

$$\begin{aligned}
\|x_n - p\|^2 &\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n \left\| S_{i(n)}^{h(n)} x_{n-1} - p \right\|^2 + \gamma_n \left\| T_{i(n)}^{h(n)} x_n - p \right\|^2 \\
&\quad + \delta_n \|u_n - p\|^2 - \alpha_n \beta_n \mathcal{G} \left(\left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \right) \\
&\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n (k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)})^2 + \gamma_n (k_{h(n)} \|x_n - p\| + \xi_{h(n)})^2 \\
&\quad + \delta_n \|u_n - p\|^2 - \alpha_n \beta_n \mathcal{G} \left(\left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \right) \\
&\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n \left(k_{h(n)}^2 \|x_{n-1} - p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} \|x_{n-1} - p\| \right) \\
&\quad + \gamma_n \left(k_{h(n)}^2 \|x_n - p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} \|x_n - p\| \right) \\
&\quad + \delta_n \|u_n - p\|^2 - \alpha_n \beta_n \mathcal{G} \left(\left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \right) \\
&\leq \left(\alpha_n + \beta_n k_{h(n)}^2 \right) \|x_{n-1} - p\|^2 + \gamma_n k_{h(n)}^2 \|x_n - p\|^2 + 2\xi_{h(n)}^2 \\
&\quad + 2k_{h(n)} \xi_{h(n)} M_2 + \delta_n M_3 - \alpha_n \beta_n \mathcal{G} \left(\left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \right),
\end{aligned} \tag{2.7}$$

where M_2 and M_3 are appropriate constants such that $M_2 = \sup_{n \geq 1} \{\|x_n - p\| + \|x_{n-1} - p\|\}$ and $M_3 = \sup_{n \geq 1} \{\|u_n - p\|^2\}$. This implies that

$$\begin{aligned}
&\alpha_n \beta_n \mathcal{G} \left(\left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \right) \\
&\leq \left(\alpha_n + \beta_n k_{h(n)}^2 \right) \left(\|x_{n-1} - p\|^2 - \|x_n - p\|^2 \right) + \left(k_{h(n)}^2 - 1 \right) \|x_n - p\|^2 \\
&\quad + 2\xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} M_2 + \delta_n M_3.
\end{aligned} \tag{2.8}$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n \rightarrow \infty} g\left(\left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\|\right) = 0. \quad (2.9)$$

Since $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, we obtain that

$$\lim_{n \rightarrow \infty} \left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\| = 0. \quad (2.10)$$

Next, we show that

$$\lim_{n \rightarrow \infty} \left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| = 0. \quad (2.11)$$

From Lemma 1.3, we also see that

$$\begin{aligned} \|x_n - p\|^2 &\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n \left\|S_{i(n)}^{h(n)} x_{n-1} - p\right\|^2 + \gamma_n \left\|T_{i(n)}^{h(n)} x_n - p\right\|^2 \\ &\quad + \delta_n \|u_n - p\|^2 - \alpha_n \gamma_n g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right) \\ &\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n (k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)})^2 + \gamma_n (k_{h(n)} \|x_n - p\| + \xi_{h(n)})^2 \\ &\quad + \delta_n \|u_n - p\|^2 - \alpha_n \gamma_n g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right) \\ &\leq \alpha_n \|x_{n-1} - p\|^2 + \beta_n \left(k_{h(n)}^2 \|x_{n-1} - p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} \|x_{n-1} - p\|\right) \\ &\quad + \gamma_n \left(k_{h(n)}^2 \|x_n - p\|^2 + \xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} \|x_n - p\|\right) \\ &\quad + \delta_n \|u_n - p\|^2 - \alpha_n \gamma_n g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right) \\ &\leq \left(\alpha_n + \beta_n k_{h(n)}^2\right) \|x_{n-1} - p\|^2 + \gamma_n k_{h(n)}^2 \|x_n - p\|^2 + 2\xi_{h(n)}^2 \\ &\quad + 2k_{h(n)} \xi_{h(n)} M_2 + \delta_n M_3 - \alpha_n \gamma_n g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right). \end{aligned} \quad (2.12)$$

This implies that

$$\begin{aligned} &\alpha_n \gamma_n g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right) \\ &\leq \left(\alpha_n + \beta_n k_{h(n)}^2\right) \left(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\right) + \left(k_{h(n)}^2 - 1\right) \|x_n - p\|^2 \\ &\quad + 2\xi_{h(n)}^2 + 2k_{h(n)} \xi_{h(n)} M_2 + \delta_n M_3. \end{aligned} \quad (2.13)$$

In view of the restrictions (a) and (b), we obtain that

$$\lim_{n \rightarrow \infty} g\left(\left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\|\right) = 0. \quad (2.14)$$

Since $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, we obtain that (2.11) holds. Notice that

$$\|x_n - x_{n-1}\| \leq \left\|S_{i(n)}^{h(n)} x_{n-1} - x_{n-1}\right\| + \left\|T_{i(n)}^{h(n)} x_n - x_{n-1}\right\| + \delta_n \|u_n - x_{n-1}\|. \quad (2.15)$$

In view of (2.10) and (2.11), we see from the restriction (b) that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0, \quad (2.16)$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+j}\| = 0, \quad \forall j \in \{1, 2, \dots, N\}. \quad (2.17)$$

Since for any positive integer $n > N$, it can be written as $n = (h(n) - 1)N + i(n)$, where $i(n) \in \{1, 2, \dots, N\}$, observe that

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| + \left\|T_{i(n)}^{h(n)} x_n - T_n x_n\right\| \\ &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| + L_t \left\|T_{i(n)}^{h(n)-1} x_n - x_n\right\| \\ &\leq \left\|x_{n-1} - T_{i(n)}^{h(n)} x_n\right\| \\ &\quad + L_t \left(\left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\| + \left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\|\right. \\ &\quad \left. + \left\|x_{(n-N)-1} - x_n\right\|\right). \end{aligned} \quad (2.18)$$

Since for each $n > N$, $n = (n - N)(\text{mod } N)$, on the other hand, we obtain from $n = (h(n) - 1)N + i(n)$ that $n - N = ((h(n) - 1) - 1)N + i(n) = (h(n - N) - 1)N + i(n - N)$. That is,

$$h(n - N) = h(n) - 1, \quad i(n - N) = i(n). \quad (2.19)$$

Notice that

$$\begin{aligned} \left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n)}^{h(n)-1} x_{n-N}\right\| \\ &\leq L_t \|x_n - x_{n-N}\|, \\ \left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\right\| &= \left\|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\right\|. \end{aligned} \quad (2.20)$$

Substituting (2.20) into (2.18), we arrive at

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \left\| x_{n-1} - T_{i(n)}^{h(n)} x_n \right\| \\ &\quad + L_t \left(L_t \|x_n - x_{n-N}\| + \left\| T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1} \right\| + \|x_{(n-N)-1} - x_n\| \right). \end{aligned} \quad (2.21)$$

In view of (2.11) and (2.17), we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0. \quad (2.22)$$

Notice that

$$\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|. \quad (2.23)$$

It follows from (2.16) and (2.22) that

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \quad (2.24)$$

Notice that

$$\begin{aligned} \|x_n - T_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\| + \|T_{n+j} x_{n+j} - T_{n+j} x_n\| \\ &\leq (1 + L_t) \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j} x_{n+j}\|, \quad \forall j \in \{1, 2, \dots, N\}. \end{aligned} \quad (2.25)$$

From (2.17) and (2.24), we arrive at

$$\lim_{n \rightarrow \infty} \|x_n - T_{n+j} x_n\| = 0, \quad \forall j \in \{1, 2, \dots, N\}. \quad (2.26)$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n \rightarrow \infty} \|x_n - T_r x_n\| = 0, \quad \forall r \in \{1, 2, \dots, N\}. \quad (2.27)$$

Letting $L_s = \max\{L_{s,i} : 1 \leq i \leq N\}$, we have

$$\begin{aligned} \left\| S_{i(n)}^{h(n)} x_n - x_{n-1} \right\| &\leq \left\| S_{i(n)}^{h(n)} x_n - S_{i(n)}^{h(n)} x_{n-1} \right\| + \left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\| \\ &\leq L_s \|x_n - x_{n-1}\| + \left\| S_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\|. \end{aligned} \quad (2.28)$$

In view of (2.10) and (2.16), we see that

$$\lim_{n \rightarrow \infty} \left\| S_{i(n)}^{h(n)} x_n - x_{n-1} \right\| = 0. \quad (2.29)$$

Observe that

$$\begin{aligned}
\|x_{n-1} - S_n x_{n-1}\| &\leq \|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\| + \|S_{i(n)}^{h(n)} x_{n-1} - S_n x_{n-1}\| \\
&\leq \|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\| + L_s \|S_{i(n)}^{h(n)-1} x_{n-1} - x_{n-1}\| \\
&\leq \|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\| \\
&\quad + L_s \left(\|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n-N)}^{h(n)-1} x_{n-N}\| + \|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| \right. \\
&\quad \left. + \|x_{(n-N)-1} - x_{n-1}\| \right).
\end{aligned} \tag{2.30}$$

In view of

$$\begin{aligned}
\|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n-N)}^{h(n)-1} x_{n-N}\| &= \|S_{i(n)}^{h(n)-1} x_{n-1} - S_{i(n)}^{h(n)-1} x_{n-N}\| \\
&\leq L_s \|x_{n-1} - x_{n-N}\|, \\
\|S_{i(n-N)}^{h(n)-1} x_{n-N} - x_{(n-N)-1}\| &= \|S_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\|,
\end{aligned} \tag{2.31}$$

we arrive at

$$\begin{aligned}
\|x_{n-1} - S_n x_{n-1}\| &\leq \|x_{n-1} - S_{i(n)}^{h(n)} x_{n-1}\| \\
&\quad + L_s \left(L_s \|x_{n-1} - x_{n-N}\| + \|S_{i(n-N)}^{h(n-N)} x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\| \right).
\end{aligned} \tag{2.32}$$

In view of (2.10), (2.17), and (2.29), we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n-1} - S_n x_{n-1}\| = 0. \tag{2.33}$$

Notice that

$$\begin{aligned}
\|x_n - S_n x_n\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\| + \|S_n x_{n-1} - S_n x_n\| \\
&\leq (1 + L_s) \|x_n - x_{n-1}\| + \|x_{n-1} - S_n x_{n-1}\|.
\end{aligned} \tag{2.34}$$

From (2.16) and (2.33), we see that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{2.35}$$

On the other hand, we have

$$\begin{aligned}
\|x_n - S_{n+j} x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j} x_{n+j}\| + \|S_{n+j} x_{n+j} - S_{n+j} x_n\| \\
&\leq (1 + L_s) \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j} x_{n+j}\|, \quad \forall j \in \{1, 2, \dots, N\}.
\end{aligned} \tag{2.36}$$

It follows from (2.17) and (2.35) that

$$\lim_{n \rightarrow \infty} \|x_n - S_{n+j}x_n\| = 0, \quad \forall j \in \{1, 2, \dots, N\}. \quad (2.37)$$

Note that any subsequence of a convergent number sequence converges to the same limit. It follows that

$$\lim_{n \rightarrow \infty} \|x_n - S_r x_n\| = 0, \quad \forall r \in \{1, 2, \dots, N\}. \quad (2.38)$$

This completes the proof. \square

Recall that a mapping $T : C \rightarrow C$ is said to be *semicompact* if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightarrow x \in C$.

Next, we give strong convergence theorems with the help of the semicompactness.

Theorem 2.2. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$, and let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.18). Assume that the following restrictions are satisfied:*

- (a) *there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;*
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \dots, S_N\}$ or one of $\{T_1, T_2, \dots, T_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F .

Proof. Without loss of generality, we may assume that S_1 is semicompact. From (2.38), we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging strongly to $x \in C$. For each $r \in \{1, 2, \dots, N\}$, we get that

$$\|x - S_r x\| \leq \|x - x_{n_i}\| + \|x_{n_i} - S_r x_{n_i}\| + \|S_r x_{n_i} - S_r x\|. \quad (2.39)$$

Since S_r is Lipschitz continuous, we obtain from (2.38) that $x \in \bigcap_{r=1}^N F(S_r)$. Notice that

$$\|x - T_r x\| \leq \|x - x_{n_i}\| + \|x_{n_i} - T_r x_{n_i}\| + \|T_r x_{n_i} - T_r x\|. \quad (2.40)$$

Since T_r is Lipschitz continuous, we obtain from (2.27) that $x \in \bigcap_{r=1}^N F(T_r)$. This means that $x \in F$. In view of Lemma 2.1, we obtain that $\lim_{n \rightarrow \infty} \|x_n - x\|$ exists. Therefore, we can obtain the desired conclusion immediately. \square

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.3. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$, and $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.19). Assume that the following restrictions are satisfied:*

- (a) *there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;*
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{T_1, T_2, \dots, T_N\}$ is semicompact, then the sequence converges $\{x_n\}$ strongly to some point in F .

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.4. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.20). Assume that the following restrictions are satisfied:*

- (a) *there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n$, for all $n \geq 1$;*
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If one of $\{S_1, S_2, \dots, S_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in F .

In 2005, Chidume and Shahzad [11] introduced the following conception. Recall that a family $\{T_i\}_{i=1}^N : C \rightarrow C$ with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy *Condition (B)* on C if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(m) > 0$ for all $m \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, F)). \quad (2.41)$$

Based on Condition (B), we introduced the following conception for two finite families of mappings. Recall that two families $\{S_i\}_{i=1}^N : C \rightarrow C$ and $\{T_i\}_{i=1}^N : C \rightarrow C$ with $F = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$ are said to satisfy *Condition (B')* on C if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(m) > 0$ for all $m \in (0, \infty)$ such that for all $x \in C$

$$\max_{1 \leq i \leq N} \{\|x - S_i x\|\} + \max_{1 \leq i \leq N} \{\|x - T_i x\|\} \geq f(d(x, F)). \quad (2.42)$$

Next, we give strong convergence theorems with the help of Condition (B').

Theorem 2.5. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$, and let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.18). Assume that the following restrictions are satisfied:*

- (a) *there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;*
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \dots, S_N\}$ and $\{T_1, T_2, \dots, T_N\}$ satisfy Condition (B'), then the sequence converges strongly to some point in F .

Proof. In view of Condition (B'), we obtain from (2.27) and (2.38) that $f(d(x_n, F)) \rightarrow 0$, which implies $d(x_n, F) \rightarrow 0$. Next, we show that the sequence $\{x_n\}$ is Cauchy. In view of (2.6), for any positive integers m, n , where $m > n > n_0$, we see that

$$\|x_m - p\| \leq B\|x_n - p\| + B \sum_{i=n+1}^{\infty} M_1(\delta_i + \xi_{h(i)}) + M_1(\delta_m + \xi_{h(m)}), \quad (2.43)$$

where $B = \exp\{\sum_{n=1}^{\infty} (k_{h(n)} - 1)/(1 - R)\}$. It follows that

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq (1 + B)\|x_n - p\| + B \sum_{i=n+1}^{\infty} M_1(\delta_i + \xi_{h(i)}) + M_1(\delta_m + \xi_{h(m)}). \end{aligned} \quad (2.44)$$

It follows that $\{x_n\}$ is a Cauchy sequence in C and so $\{x_n\}$ converges strongly to some $\bar{q} \in C$. Since T_r and S_r are Lipschitz for each $r \in \{1, 2, \dots, N\}$, we see that F is closed. This in turn implies that $\bar{q} \in F$. This completes the proof. \square

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.6. *Let E be a real uniformly convex uniformly convex Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that*

$\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.19). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{T_1, T_2, \dots, T_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in F .

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.7. Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E . Let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$, and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.20). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$ and $c \leq \gamma_n$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

If $\{S_1, S_2, \dots, S_N\}$ satisfies Condition (B), then the sequence $\{x_n\}$ converges strongly to some point in F .

Finally, we give a strong convergence theorem criterion.

Theorem 2.8. Let E be a real Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$, and let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_n = \max\{k_{n,t}, k_{n,s}\}$, where $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$ and $\xi_n = \max\{\xi_{n,t}, \xi_{n,s}\}$, where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$ and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.18). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n \leq d < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Proof. The necessity is obvious. We only show the sufficiency. Assume that

$$\liminf_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0. \quad (2.45)$$

For each $p \in F$, we see that

$$\begin{aligned}
\|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|S_{i(n)}^{h(n)} x_{n-1} - p\| + \gamma_n \|T_{i(n)}^{h(n)} x_n - p\| + \delta_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n (k_{h(n)} \|x_{n-1} - p\| + \xi_{h(n)}) + \gamma_n (k_{h(n)} \|x_n - p\| + \xi_{h(n)}) \\
&\quad + \delta_n \|u_n - p\| \\
&\leq (\alpha_n + \beta_n k_{h(n)}) \|x_{n-1} - p\| + (1 - \alpha_n - \beta_n) k_{h(n)} \|x_n - p\| + 2\xi_{h(n)} \\
&\quad + \delta_n \|u_n - x_n\|.
\end{aligned} \tag{2.46}$$

Notice that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. We see from the restrictions (a) and (b) that there exists a positive integer n_0 such that

$$(1 - \alpha_n - \beta_n) k_{h(n)} \leq R < 1, \quad \forall n \geq n_0, \tag{2.47}$$

where $R = (1 - (a + b))(1 + (a + b)/(2 - 2(a + b)))$. Notice that the sequence $\{x_n\}$ is bounded. It follows from (2.46) that

$$\begin{aligned}
\|x_n - p\| &\leq \frac{\alpha_n + \beta_n k_{h(n)}}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \|x_{n-1} - p\| + \frac{\delta_n}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \|u_n - x_n\| \\
&\quad + \frac{2\xi_{h(n)}}{1 - (1 - \alpha_n - \beta_n) k_{h(n)}} \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + \frac{\delta_n}{1 - R} \|u_n - x_n\| + \frac{2\xi_{h(n)}}{1 - R} \\
&\leq \left(1 + \frac{k_{h(n)} - 1}{1 - R}\right) \|x_{n-1} - p\| + M_4 (\delta_n + \xi_{h(n)}), \quad \forall n \geq n_0,
\end{aligned} \tag{2.48}$$

where M_4 is an appropriate constant such that $M_4 = \max\{\sup_{n \geq 1} \{\|u_n - x_n\|/(1 - R)\}, 2/(1 - R)\}$. In view of the restrictions (a) and (b), we obtain from Lemma 1.2 that $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F})$ exists. This implies that

$$\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0. \tag{2.49}$$

In view of Theorem 2.5, we can conclude the desired conclusion easily. \square

If $S_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.9. *Let E be a real Banach space and C a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly $L_{t,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,t,i}\} \subset [1, \infty)$ and $\{\xi_{n,t,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,t,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,t,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,t} = \max\{k_{n,t,i} : 1 \leq i \leq N\}$ and where $\xi_{n,t} = \max\{\xi_{n,t,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$*

and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.19). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c \in (0, 1)$ such that $a \leq \alpha_n + \beta_n$ and $b \leq \gamma_n \leq c < 1/L_t$, where $L_t = \max\{L_{t,i} : 1 \leq i \leq N\}$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

If $T_i = I$, where I denotes the identity mapping, for each $i \in \{1, 2, \dots, N\}$, then Theorem 2.2 is reduced to the following.

Corollary 2.10. Let E be a real Banach space and C a nonempty closed convex subset of E . Let $S_i : C \rightarrow C$ be a uniformly $L_{s,i}$ -Lipschitz and generalized asymptotically quasi-nonexpansive mapping with sequences $\{k_{n,s,i}\} \subset [1, \infty)$ and $\{\xi_{n,s,i}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_{n,s,i} - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_{n,s,i} < \infty$ for each $1 \leq i \leq N$. Assume that $F = \bigcap_{i=1}^N F(S_i)$ is nonempty. Let $\{u_n\}$ be a bounded sequence in C , $k_{n,s} = \max\{k_{n,s,i} : 1 \leq i \leq N\}$, and $\xi_{n,s} = \max\{\xi_{n,s,i} : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (1.20). Assume that the following restrictions are satisfied:

- (a) there exist constants $a, b, c, d \in (0, 1)$ such that $a \leq \alpha_n$, $b \leq \beta_n$, and $c \leq \gamma_n$, for all $n \geq 1$;
- (b) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\{x_n\}$ converges strongly to some point in F if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

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