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Research Article

Set-Theoretic Inequalities in Stochastic Noncooperative Games with Coalition

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We model and analyze antagonistic stochastic games of three players, two of whom form a coalition against the third one. The actions of the players are modeled by random walk processes recording the cumulative damages to each player at any moment of time. The game continues until the single player or the coalition is defeated. The defeat of any particular player takes place when the associated process (representing the collateral damage) crosses a fixed threshold. Once the threshold is exceeded at some time, the associated player exits the game. All involved processes are being "observed by a third party process" so that the information regarding the status of all players is restricted to those special epochs. Furthermore, all processes are modulated (with their parameters being modified in due course of the game). We obtain a closed form joint functional of the named processes at key reference points.

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1. Introduction

Antagonistic games are those with two players A and B having totally opposite interests. In our game-theoretic setting, we have a group of three players, A, B, and C, of whom A and B form a coalition against C. The game is modeled by three stochastic processes, namely, generalized random walk processes with drifts that at random times strike each other causing random casualties. Each of the three players "accumulates" damages totaling successful strikes of random magnitudes and each player is assumed to have his own threshold of tolerance. Once the total damage to player A, B, or C exceeds its respective threshold, the inflicted player is defeated and exits the game. In the beginning of the game, players A and B strike player C, each at different times. Correspondingly, player C strikes players A and B at different times. Player C wins the game if each of his adversaries is defeated and then the game is over. The game also ends if player C is defeated by A or B. In this special case, one of them, say A, can be defeated by player C followed by the defeat of player C by player B.

In this particular game setting, we are interested in the event that player C defeats the coalition of players A and B. At the core of the game are four marked modulated Poisson random measures

$$\Pi_A = \sum_{i=0}^{\chi} x_i \varepsilon_{s_i}, \quad \Pi_B = \sum_{j=0}^{\xi} y_j \varepsilon_{t_j}, \quad \Pi_{CA} = \sum_{k=0}^{\zeta} w_k^A \varepsilon_{u_k}, \quad \Pi_{CB} = \sum_{n=0}^{\eta} w_n^B \varepsilon_{v_n} \quad (1.1)$$

on a probability space $(\Omega, \mathcal{F}(\Omega), P)$ (ε_a is the point mass) that will describe the respective casualties to players A and B by player C and to player C by players A and B. Before we turn to specifics of (1.1), we assume that there is a third-party point process

$$\mathcal{T} := \sum_{i \geq 0} \varepsilon_{\tau_i}, \quad \tau_0 \geq 0, \quad (1.2)$$

over which the game is observed, and thus the information on the status of the players will not be continuously available but upon the epochs of time from \mathcal{T} only. The Poisson processes of (1.1) are conditionally independent in each interval $[\tau_{j-1}, \tau_j)$ given the status of the players at τ_{j-1} and, furthermore, their respective parameters (i.e., intensities of marks) will depend on the game status at τ_{j-1} . This makes perfect sense because if one of the players, say A, is defeated at some point known at τ_k , his upcoming actions against player C can be reduced or completely halted. Perhaps player C will care little about player A either and concentrate his attention on player B. Consequently, we will say that the parameters of (1.1) are as follows. In interval $[\tau_{j-1}, \tau_j)$, the r.v.'s χ, ξ, ζ, η are independent Poisson with parameters $\lambda_A^j, \lambda_B^j, \lambda_{CA}^j, \lambda_{CB}^j$ and measures (1.1) are specified by their respective transforms

$$\begin{aligned} Ee^{-u\Pi_A(\cdot)} &= e^{\lambda_A^j | \cdot | [g_A^j(u)-1]}, & g_A^j(u) &= Ee^{-ux_1}, \quad \operatorname{Re}(u) \geq 0, \\ Ee^{-u\Pi_B(\cdot)} &= e^{\lambda_B^j | \cdot | [g_B^j(u)-1]}, & g_B^j(u) &= Ee^{-uy_1}, \quad \operatorname{Re}(u) \geq 0, \\ Ee^{-u\Pi_{CA}(\cdot)} &= e^{\lambda_{CA}^j | \cdot | [g_{CA}^j(u)-1]}, & g_{CA}^j(u) &= Ee^{-uw_1^A}, \quad \operatorname{Re}(u) \geq 0, \\ Ee^{-u\Pi_{CB}(\cdot)} &= e^{\lambda_{CB}^j | \cdot | [g_{CB}^j(u)-1]}, & g_{CB}^j(u) &= Ee^{-uw_1^B}, \quad \operatorname{Re}(u) \geq 0, \end{aligned} \quad (1.3)$$

where $|\cdot|$ is the Borel-Lebesgue measure. We set $s_0 = t_0 = u_0 = v_0 = 0$. A more rigorous formalism on modulated measures as per Dshalalow [1] is not mandatory, because, as we will see it, the "automodulation" of $(\Pi_A, \Pi_B, \Pi_{CA}, \Pi_{CB})$ will be restricted to a few reference points from \mathcal{T} .

The assumption is that random measures (1.1) are positive and the game natural settings require the respective marks in (1.1) to be nonnegative. The interpretation of (1.1) is as follows. Player A receives strikes of magnitudes x_0, x_1, \dots upon times s_0, s_1, \dots , respectively, from player C. Player B received strikes y_0, y_1, \dots upon t_0, t_1, \dots from player C. Similarly, Π_{CA} and Π_{CB} formalize the damages exerted to C by players A and B.

Now, relative to \mathcal{T} of (1.2), we form the four-dimensional "embedded" marked random measure

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T}) = \sum_{j \geq 0} \varepsilon_{\tau_j} (X_j, Y_j, W_j^A, W_j^B) \quad (1.4)$$

with modulated position dependence, where

$$\begin{aligned} X_j &= \Pi_A((\tau_{j-1}, \tau_j]), & Y_j &= \Pi_B((\tau_{j-1}, \tau_j]), \\ W_j^A &= \Pi_{CA}((\tau_{j-1}, \tau_j]), & W_j^B &= \Pi_{CB}((\tau_{j-1}, \tau_j]), \\ Z_j &= W_j^A + W_j^B, & j &= 0, 1, \dots, \tau_{-1} = 0. \end{aligned} \quad (1.5)$$

Now, given thresholds M, N, R (positive reals), we define the exit indices

$$\begin{aligned} \mu &:= \inf \{m \geq 0 : A_m = X_0 + \dots + X_m > M\}, \\ \nu &:= \inf \{n \geq 0 : B_n = Y_0 + \dots + Y_n > N\}, \\ \rho &:= \inf \{k \geq 0 : C_k = Z_0 + \dots + Z_k > R\}. \end{aligned} \quad (1.6)$$

Hence, players A, B, C are doomed to exit the game at $\tau_\mu, \tau_\nu, \tau_\rho$ (exit times or first passage times), respectively; of course, it is not yet specified in which order. Informally, the game in our case, is the stochastic process $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T})$ of (1.1)-(1.6) on the trace σ -algebra $\mathcal{F}(\Omega) \cap (\{\mu \leq \nu < \rho\} \cup \{\nu < \mu < \rho\})$, implying that only those paths of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T})$ will be included which lead player C to defeat the coalition of players A and B. Consequently, we are interested in the joint functional

$$\begin{aligned} \Phi_{\mu\nu\rho} &:= E \left[e^{-a_0 A_{\mu-1} - a_1 A_\mu - a_2 A_{\nu-1} - a_3 A_\nu} e^{-b_0 B_{\mu-1} - b_1 B_\mu - b_2 B_{\nu-1} - b_3 B_\nu} \right. \\ &\quad \times \left. e^{-c_0 C_{\mu-1} - c_1 C_\mu - c_2 C_{\nu-1} - c_3 C_\nu - c_4 C_{\rho-1} - c_5 C_\rho} e^{-\theta_0 \tau_{\mu-1} - \theta_1 \tau_\mu - \theta_2 \tau_{\nu-1} - \theta_3 \tau_\nu - \theta_4 \tau_{\rho-1} - \theta_5 \tau_\rho} \mathbf{1}_{\{\mu \leq \nu < \rho\} \cup \{\nu < \mu < \rho\}} \right], \end{aligned} \quad (1.7)$$

providing the key information upon major reference points of the game lasting until player C wins. The way how the ‘‘modulated’’ point process \mathcal{T} functions will be described in the upcoming sections. Section 3 contains the main result of the paper, Theorem 3.3, which provides a closed form expression for functional (1.7).

It is observed that in this paper we focus entirely on probabilistic analysis of the conflict rather than on an optimal strategy for winning the game. The tools for the investigation we use are also different from those in most literature, concentrating on fluctuation analysis specifically designed and embellished for these types of games. We also notice that the game we formalize and study is strictly antagonistic as far as the parties of players A and B and player C. Consequently, the primary application of our game will be economics and warfare, with the main emphasis on economics. We believe that most actions among competitors in any specific branch of industry, banking, recreation, and so forth are hostile in nature, which agrees with the free market principles. Among them, we mention acquiring or selling large quantities of shares of stocks of a competitor, merging with another competitor, hostile commercials, political lobbying, and outsourcing the labor.

All pertinent work related to antagonistic games are concerned with two players with opposite interests. Of course, each player can represent a group of other players, but they are not noticeably distinguishable within the game [2, 3]. In this paper, we make a first attempt to bring a coalition into the game, where the actions of the players differ for each player, and defeating the coalition means to defeat each player, possibly at different times. We also plan another work for coalition games where a coalition ruins a single player and a coalition defeats another coalition.

Game-theoretical work most commonly applies to economics, although it stemmed from warfare during the second world war. The literature on games is vast and a good portion of

it is on cooperative games [4–6]. Others are on noncooperative (antagonistic) games [2, 3, 7–12] of which many relate to economics [5, 7, 10, 11] and some to warfare [8, 12]. The primary tools explored in this paper are on the theory of fluctuations related to the random walk and occurring in economics [13, 14], physics [15], and other areas of engineering and technology. Article [2] by the first author contains a more detailed bibliography regarding fluctuations and games. Article [3] is somewhat related to our present paper, as it also models a noncooperative game by random walk processes, but with two active players only. The literature on coalition games is quite populous. We mention a few related papers [8, 16–21] which all use different techniques and settings.

2. Preliminaries

We will begin with the description of the observation process \mathcal{T} . We assume that \mathcal{T} is a delayed renewal process with

$$\begin{aligned} Ee^{-\theta\tau_0} &= \delta_0(\theta), \quad \operatorname{Re}(\theta) \geq 0, \\ Ee^{-\theta\Delta_j} &= \delta_1(\theta), \quad \operatorname{Re}(\theta) \geq 0, \quad j = 1, 2, \dots, \end{aligned} \quad (2.1)$$

where

$$\Delta_j = \tau_j - \tau_{j-1}. \quad (2.2)$$

Next, from (1.5) followed by (1.3), we find

$$\begin{aligned} \gamma_j(x, y, z, \theta) &:= Ee^{-xX_j - yY_j - zZ_j - \theta\Delta_j} \\ &= E[e^{-\theta\Delta_j} E[e^{-x\Pi_A((\tau_{j-1}, \tau_j])} \mid \Delta_j] E[e^{-y\Pi_B((\tau_{j-1}, \tau_j])} \mid \Delta_j] E[e^{-z\{\Pi_{CA} + \Pi_{CB}\}((\tau_{j-1}, \tau_j])} \mid \Delta_j]]] \\ &= E[e^{-\theta\Delta_j} e^{\Delta_j \{\lambda_A^j [g_A^j(x) - 1] + \lambda_B^j [g_B^j(y) - 1] + \lambda_{CA}^j [g_{CA}^j(z) - 1] + \lambda_{CB}^j [g_{CB}^j(z) - 1]\}}] \\ &= \delta_1 \{ \theta + \lambda_A^j [1 - g_A^j(x)] + \lambda_B^j [1 - g_B^j(y)] + \lambda_{CA}^j [1 - g_{CA}^j(z)] + \lambda_{CB}^j [1 - g_{CB}^j(z)] \}, \quad j = 1, 2, \dots, \end{aligned} \quad (2.3)$$

and analogously,

$$\begin{aligned} \gamma_0(x, y, z, \theta) &:= Ee^{-xX_0 - yY_0 - zZ_0 - \theta\tau_0} \\ &= \delta_0 \{ \theta + \lambda_A^0 [1 - g_A^0(x)] + \lambda_B^0 [1 - g_B^0(y)] + \lambda_{CA}^0 [1 - g_{CA}^0(z)] + \lambda_{CB}^0 [1 - g_{CB}^0(z)] \}. \end{aligned} \quad (2.4)$$

Throughout the rest of the paper, we will be using the following abbreviations:

$$\gamma_i = \gamma_i(a_0 + \dots + a_3 + x, b_0 + \dots + b_3 + y, c_0 + \dots + c_5 + z, \theta_0 + \dots + \theta_5), \quad i = 0, 1, \quad (2.5)$$

$$\Gamma_i = \gamma_i(a_1 + a_2 + a_3 + x, b_1 + b_2 + b_3 + y, c_1 + \dots + c_5 + z, \theta_1 + \dots + \theta_5), \quad i = 0, 1, \quad (2.6)$$

$$\Gamma_i^1 = \gamma_i(a_1 + a_2 + a_3, b_1 + b_2 + b_3 + y, c_1 + \dots + c_5 + z, \theta_1 + \dots + \theta_5), \quad i = 0, 1, \quad (2.7)$$

$$\Gamma_i^2 = \gamma_i(a_1 + a_2 + a_3 + x, b_1 + b_2 + b_3, c_1 + \dots + c_5 + z, \theta_1 + \dots + \theta_5), \quad i = 0, 1, \quad (2.8)$$

$$g_i = \gamma_i(a_0 + a_1 + x, b_0 + b_1 + y, c_0 + \dots + c_5 + z, \theta_0 + \dots + \theta_5), \quad i = 0, 1, \quad (2.9)$$

$$G_i = \gamma_i(a_1 + x, b_1 + y, c_1 + c_3 + c_4 + c_5 + z, \theta_1 + \theta_3 + \theta_4 + \theta_5), \quad i = 0, 1, \quad (2.10)$$

$$G_i^{12} = \gamma_i(a_1, b_1, c_1 + c_3 + c_4 + c_5 + z, \theta_1 + \theta_3 + \theta_4 + \theta_5), \quad i = 0, 1, \quad (2.11)$$

$$G_i^1 = \gamma_i(a_1, b_1 + y, c_1 + c_3 + c_4 + c_5 + z, \theta_1 + \theta_3 + \theta_4 + \theta_5), \quad i = 0, 1, \quad (2.12)$$

$$G_i^2 = \gamma_i(a_1 + x, b_1, c_1 + c_3 + c_4 + c_5 + z, \theta_1 + \theta_3 + \theta_4 + \theta_5), \quad i = 0, 1. \quad (2.13)$$

3. Main results

We recall that the game is over when player C beats the coalition of players A and B by defeating both of them. The σ -algebra of the paths of the process $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T})$ can be analogously rewritten as $\mathcal{F}(\Omega) \cap \{\mu < \nu < \rho\} \cup \{\mu = \nu < \rho\} \cup \{\nu < \mu < \rho\}$. We break the functional $\Phi_{\mu\nu\rho}$ of (1.7) accordingly in three parts below.

Part 1. Player C knocks down player A and then player B. Before we further proceed, we would now like to focus on the specifics of the automodulation mentioned in Section 1. Consider the σ -subalgebra $\mathcal{F}(\Omega) \cap \{\mu < \nu < \rho\}$ of the paths of the game, with player A losing first, followed by player B. Subdividing the time axis into five intervals

$$[0, \tau_0] \cup (\tau_0, \tau_\mu] \cup (\tau_\mu, \tau_\nu] \cup (\tau_\nu, \tau_\rho] \cup (\tau_\rho, \infty), \quad (3.1)$$

we specify the parameters of the Poisson processes in (1.1) as follows. We assume they will not alter within these five intervals, only upon transition from one to another. Counting these intervals as I_0, I_1, I_2, I_3, I_4 , we will assign the corresponding parameters and functionals in (1.3), now indexed from 0 through 4.

The corresponding subfunctional will look as

$$\begin{aligned} \Phi_{\mu\nu\rho}^1 &= E \left[e^{-a_0 A_{\mu-1} - a_1 A_\mu - a_2 A_{\nu-1} - a_3 A_\nu - b_0 B_{\mu-1} - b_1 B_\mu - b_2 B_{\nu-1} - b_3 B_\nu} \right. \\ &\quad \times e^{-c_0 C_{\mu-1} - c_1 C_\mu - c_2 C_{\nu-1} - c_3 C_\nu - c_4 C_{\rho-1} - c_5 C_\rho} \\ &\quad \left. \times e^{-\theta_0 \tau_{\mu-1} - \theta_1 \tau_\mu - \theta_2 \tau_{\nu-1} - \theta_3 \tau_\nu - \theta_4 \tau_{\rho-1} - \theta_5 \tau_\rho} \mathbf{1}_{\{\mu < \nu < \rho\}} \right]. \end{aligned} \quad (3.2)$$

Now, we extend the indices of (1.6) to the random families:

$$\begin{aligned} \mu(p) &:= \inf \{ m \geq 0 : A_m = X_0 + \cdots + X_m > p \}, \quad p \geq 0, \\ \nu(q) &:= \inf \{ n \geq 0 : B_n = Y_0 + \cdots + Y_n > q \}, \quad q \geq 0, \\ \rho(r) &:= \inf \{ k \geq 0 : C_k = Z_0 + \cdots + Z_k > r \}, \quad r \geq 0. \end{aligned} \quad (3.3)$$

Correspondingly, the functional (3.2) will turn to a parametric family of functionals:

$$\begin{aligned} (p, q, r) &\longmapsto \Phi_{\mu(p)\nu(q)\rho(r)}^1 = E \left[e^{-a_0 A_{\mu(p)-1} - a_1 A_{\mu(p)} - a_2 A_{\nu(q)-1} - a_3 A_{\nu(q)}} \right. \\ &\quad \times e^{-b_0 B_{\mu(p)-1} - b_1 B_{\mu(p)} - b_2 B_{\nu(q)-1} - b_3 B_{\nu(q)}} \\ &\quad \times e^{-c_0 C_{\mu(p)-1} - c_1 C_{\mu(p)} - c_2 C_{\nu(q)-1} - c_3 C_{\nu(q)} - c_4 C_{\rho(r)-1} - c_5 C_{\rho(r)}} \\ &\quad \left. \times e^{-\theta_0 \tau_{\mu(p)-1} - \theta_1 \tau_{\mu(p)} - \theta_2 \tau_{\nu(q)-1} - \theta_3 \tau_{\nu(q)} - \theta_4 \tau_{\rho(r)-1} - \theta_5 \tau_{\rho(r)}} \mathbf{1}_{\{\mu(p) < \nu(q) < \rho(r)\}} \right]. \end{aligned} \quad (3.4)$$

Now, we define a Laplace-type operator

$$\begin{aligned} \mathcal{L}_{pqr}(\cdot)(x, y, z) &:= xyz \int_{p=0}^{\infty} \int_{q=0}^{\infty} \int_{r=0}^{\infty} e^{-xp-yq-zr}(\cdot) d(p, q, r), \\ \operatorname{Re}(x) &> 0, \quad \operatorname{Re}(y) > 0, \quad \operatorname{Re}(z) > 0, \end{aligned} \quad (3.5)$$

which we apply to $\{\mathbf{1}_{\{\mu(p)=j, \nu(q)=k, \rho(r)=n\}}; p \geq 0, q \geq 0, r \geq 0\}$ arriving at

$$\mathcal{L}_{pqr}(\mathbf{1}_{\{\mu(p)=j, \nu(q)=k, \rho(r)=n\}})(x, y, z) = (e^{-xA_{j-1}} - e^{-xA_j})(e^{-yB_{k-1}} - e^{-yB_k})(e^{-zC_{n-1}} - e^{-zC_n}). \quad (3.6)$$

Equation (3.6) is not difficult to prove (cf. [3] for a related formula). With the notation

$$\Psi^i(x, y, z) = \mathcal{L}_{pqr}(\Phi_{\mu(p), \nu(q), \rho(r)}^i)(x, y, z), \quad i = 1, 2, 3, \quad (3.7)$$

we have, after the use of Fubini's theorem,

$$\begin{aligned} \Psi^1(x, y, z) = & \sum_{j \geq 0} \sum_{k > j} \sum_{n > k} E \left[e^{-a_0 A_{j-1} - a_1 A_j - a_2 A_{k-1} - a_3 A_k - b_0 B_{j-1} - b_1 B_j - b_2 B_{k-1} - b_3 B_k} \right. \\ & \times e^{-c_0 C_{j-1} - c_1 C_j - c_2 C_{k-1} - c_3 C_k - c_4 C_{n-1} - c_5 C_n} e^{-\theta_0 \tau_{j-1} - \theta_1 \tau_j - \theta_2 \tau_{k-1} - \theta_3 \tau_k - \theta_4 \tau_{n-1} - \theta_5 \tau_n} \\ & \left. \times (e^{-xA_{j-1}} - e^{-xA_j})(e^{-yB_{k-1}} - e^{-yB_k})(e^{-zC_{n-1}} - e^{-zC_n}) \right]. \end{aligned} \quad (3.8)$$

A further regrouping of the random factors yields

$$\begin{aligned} \Psi^1(x, y, z) = & \sum_{j \geq 0} \sum_{k > j} \sum_{n > k} E \left[e^{-(a_0 + \dots + a_3) A_{j-1} - (a_1 + a_2 + a_3) X_j - (a_2 + a_3)(X_{j+1} + \dots + X_{k-1}) - a_3 X_k} \right. \\ & \times e^{-(b_0 + \dots + b_3) B_{j-1} - (b_1 + b_2 + b_3) Y_j - (b_2 + b_3)(Y_{j+1} + \dots + Y_{k-1}) - b_3 Y_k} \\ & \times e^{-(c_0 + \dots + c_5) C_{j-1} - (c_1 + \dots + c_5) Z_j - (c_2 + \dots + c_5)(Z_{j+1} + \dots + Z_{k-1}) - (c_3 + c_4 + c_5) Z_k} \\ & \times e^{-(c_4 + c_5)(Z_{k+1} + \dots + Z_{n-1}) - c_5 Z_n} \\ & \times e^{-(\theta_0 + \dots + \theta_5) \tau_{j-1} - (\theta_1 + \dots + \theta_5) \Delta_j - (\theta_2 + \dots + \theta_5)(\Delta_{j+1} + \dots + \Delta_{k-1}) - (\theta_3 + \theta_4 + \theta_5) \Delta_k} \\ & \times e^{-(\theta_4 + \theta_5)(\Delta_{k+1} + \dots + \Delta_{n-1}) - \theta_5 \Delta_n} \\ & \times e^{-xA_{j-1}} (1 - e^{-xX_j}) e^{-yB_{j-1}} e^{-yY_j} e^{-y(Y_{j+1} + \dots + Y_{k-1})} (1 - e^{-yY_k}) \\ & \left. \times e^{-zC_{j-1}} e^{-zZ_j} e^{-z(Z_{j+1} + \dots + Z_{k-1})} e^{-zZ_k} e^{-z(Z_{k+1} + \dots + Z_{n-1})} (1 - e^{-zZ_n}) \right]. \end{aligned} \quad (3.9)$$

Under our assumptions on independence of the increments of the process $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T})$ in intervals I_0, \dots, I_4 , we have after straightforward arguments,

$$\begin{aligned} \Psi^1(x, y, z) = & \sum_{j \geq 0} \sum_{k > j} \sum_{n > k} E \left[e^{-(a_0 + \dots + a_3 + x) A_{j-1} - (b_0 + \dots + b_3 + y) B_{j-1} - (c_0 + \dots + c_5 + z) C_{j-1} - (\theta_0 + \dots + \theta_5) \tau_{j-1}} \right. \\ & \times E \left[e^{-(a_1 + a_2 + a_3) X_j} (1 - e^{-xX_j}) e^{-(b_1 + b_2 + b_3 + y) Y_j - (c_1 + \dots + c_5 + z) Z_j - (\theta_1 + \dots + \theta_5) \Delta_j} \right] \\ & \times E \left[e^{-(a_2 + a_3)(X_{j+1} + \dots + X_{k-1}) - (b_2 + b_3 + y)(Y_{j+1} + \dots + Y_{k-1})} \right. \\ & \quad \left. \times e^{-(c_2 + \dots + c_5 + z)(Z_{j+1} + \dots + Z_{k-1}) - (\theta_2 + \dots + \theta_5)(\Delta_{j+1} + \dots + \Delta_{k-1})} \right] \\ & \times E \left[e^{-a_3 X_k - b_3 Y_k} (1 - e^{-yY_k}) e^{-(c_3 + c_4 + c_5 + z) Z_k - (\theta_3 + \theta_4 + \theta_5) \Delta_k} \right] \\ & \times E \left[e^{-(c_4 + c_5 + z)(Z_{k+1} + \dots + Z_{n-1}) - (\theta_4 + \theta_5)(\Delta_{k+1} + \dots + \Delta_{n-1})} \right] \\ & \times E \left[e^{-c_5 Z_n} (1 - e^{-zZ_n}) e^{-\theta_5 \Delta_n} \right] \\ = & \sum_{j \geq 0} \sum_{k > j} \sum_{n > k} R_{1j} R_{2j} R_{3jk} R_{4k} R_{5kn} R_{6n}, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned}
R_{1j} &= E\left[e^{-(a_0+\dots+a_3+x)A_{j-1}-(b_0+\dots+b_3+y)B_{j-1}-(c_0+\dots+c_5+z)C_{j-1}-(\theta_0+\dots+\theta_5)\tau_{j-1}}\right] \\
&= \begin{cases} 1, & j = 0 \\ \gamma_0\gamma_1^{j-1}, & j > 0, \end{cases} \\
R_{2j} &= E\left[e^{-(a_1+a_2+a_3)X_j}(1 - e^{-xX_j})e^{-(b_1+b_2+b_3+y)Y_j-(c_1+\dots+c_5+z)Z_j-(\theta_1+\dots+\theta_5)\Delta_j}\right] \\
&= \begin{cases} \Gamma_0^1 - \Gamma_0, & j = 0 \\ \Gamma_1^1 - \Gamma_1, & j > 0, \end{cases}
\end{aligned} \tag{3.11}$$

γ 's and Γ 's are defined in (2.5)–(2.7), and

$$\begin{aligned}
R_{3jk} &= E\left[e^{-(a_2+a_3)(X_{j+1}+\dots+X_{k-1})-(b_2+b_3+y)(Y_{j+1}+\dots+Y_{k-1})}e^{-(c_2+\dots+c_5+z)(Z_{j+1}+\dots+Z_{k-1})-(\theta_2+\dots+\theta_5)(\Delta_{j+1}+\dots+\Delta_{k-1})}\right] \\
&= \gamma_2^{k-1-j}(a_2 + a_3, b_2 + b_3 + y, c_2 + \dots + c_5 + z, \theta_2 + \dots + \theta_5), \quad k > j \geq 0, \\
R_{4k} &= E\left[e^{-a_3X_k-b_3Y_k}(1 - e^{-yY_k})e^{-(c_3+c_4+c_5+z)Z_k-(\theta_3+\theta_4+\theta_5)\Delta_k}\right] \\
&= \gamma_3(a_3, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5) - \gamma_3(a_3, b_3 + y, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5), \\
R_{5kn} &= E\left[e^{-(c_4+c_5+z)(Z_{k+1}+\dots+Z_{n-1})-(\theta_4+\theta_5)(\Delta_{k+1}+\dots+\Delta_{n-1})}\right] \\
&= \gamma_4^{n-1-k}(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5), \quad n > k, \\
R_{6n} &= E\left[e^{-c_5Z_n}(1 - e^{-zZ_n})e^{-\theta_5\Delta_n}\right] = \gamma_4(0, 0, c_5, \theta_5) - \gamma_4(0, 0, c_5 + z, \theta_5).
\end{aligned} \tag{3.12}$$

The summation $\sum_{k>j} R_{3jk}R_{4k}$ yields the expression

$$\frac{\gamma_3(a_3, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5) - \gamma_3(a_3, b_3 + y, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5)}{1 - \gamma_2(a_2 + a_3, b_2 + b_3 + y, c_2 + \dots + c_5 + z, \theta_2 + \dots + \theta_5)}. \tag{3.13}$$

The summation $\sum_{n>k} R_{5kn}R_{6n}$ yields the expression $(\gamma_4(0, 0, c_5, \theta_5) - \gamma_4(0, 0, c_5 + z, \theta_5)) / (1 - \gamma_4(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5))$. Therefore, completing the rest of the summations, we have altogether

$$\begin{aligned}
\Psi^1(x, y, z) &= \left[\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^1 - \Gamma_1) \right] \\
&\quad \times \frac{\gamma_3(a_3, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5) - \gamma_3(a_3, b_3 + y, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5)}{1 - \gamma_2(a_2 + a_3, b_2 + b_3 + y, c_2 + \dots + c_5 + z, \theta_2 + \dots + \theta_5)} \\
&\quad \times \frac{\gamma_4(0, 0, c_5, \theta_5) - \gamma_4(0, 0, c_5 + z, \theta_5)}{1 - \gamma_4(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5)}.
\end{aligned} \tag{3.14}$$

Finally, for the convergence of the above series, we need the following.

Lemma 3.1. Let $h(x, y, z, \theta) = Ee^{-xX_j - yY_j - zZ_j - \theta\Delta_j}$ with $\operatorname{Re}(x) \geq 0$, and $\operatorname{Re}(y) \geq 0, \operatorname{Re}(z) \geq 0$, and $\operatorname{Re}(\theta) \geq 0$. Then, the norm $\|h\|$ is strictly less than 1 for all $(x, y, z, \theta) \in \mathbb{C}^4$ such that

$$\operatorname{Re}(x) \geq 0, \quad \operatorname{Re}(y) \geq 0, \quad \operatorname{Re}(z) \geq 0, \quad \operatorname{Re}(\theta) \geq 0, \quad (\text{L.1})$$

$$[\operatorname{Re}(x)]^2 + [\operatorname{Re}(y)]^2 + [\operatorname{Re}(z)]^2 + [\operatorname{Re}(\theta)]^2 > 0. \quad (\text{L.2})$$

For the proof of Lemma 3.1, see the appendix.

Remark 3.2. The convergence of $\sum_{j>0} \gamma_i^{j-1}$ (and other similar series) is valid in light of Lemma 3.1 applied to the norm of the functional

$$\gamma_i(a_0 + a_1 + x, b_0 + \cdots + b_3 + y, c_0 + \cdots + c_5 + z, \theta_0 + \cdots + \theta_5), \quad (3.15)$$

which is strictly less than 1 if $\operatorname{Re}(a_0 + \cdots + a_3 + x) > 0$, $\operatorname{Re}(b_0 + \cdots + b_3 + y) > 0$, $\operatorname{Re}(c_0 + \cdots + c_5 + z) > 0$, and $\operatorname{Re}(\theta_0 + \cdots + \theta_5) > 0$, or even replacing one of the four inequalities with \geq . The rest of the series can be adjusted correspondingly.

Part 2. The automodulation of Part 1 is modified as follows. Consider the σ -subalgebra $\mathcal{F}(\Omega) \cap \{\mu = \nu < \rho\}$ of the paths of the game, where players A and B perish simultaneously before player C does. Subdividing the time axis into four intervals

$$I_0 \cup I_1 \cup I_2 \cup I_3 = [0, \tau_0] \cup (\tau_0, \tau_\mu] \cup (\tau_\mu, \tau_\rho] \cup (\tau_\rho, \infty), \quad (3.16)$$

we specify the parameters of the Poisson processes in (1.1) as follows. We assume they will not alter within these five intervals, only upon transition from one to another. The corresponding subfunctional will read as

$$\begin{aligned} \Phi_{\mu\nu\rho}^2 = E & \left[e^{-a_0 A_{\mu-1} - a_1 A_\mu - b_0 B_{\mu-1} - b_1 B_\mu - c_0 C_{\mu-1} - c_1 C_\mu - c_2 C_{\nu-1} - c_3 C_\nu - c_4 C_{\rho-1} - c_5 C_\rho} \right. \\ & \left. \times e^{-\theta_0 \tau_{\mu-1} - \theta_1 \tau_\mu - \theta_2 \tau_{\nu-1} - \theta_3 \tau_\nu - \theta_4 \tau_{\rho-1} - \theta_5 \tau_\rho} \mathbf{1}_{\{\mu=\nu<\rho\}} \right]. \end{aligned} \quad (3.17)$$

Proceeding further analogously, we again have the extended family of functionals

$$\begin{aligned} \Phi_{\mu(p)\nu(q)\rho(r)}^2 = E & \left[e^{-a_0 A_{\mu(p)-1} - a_1 A_{\mu(p)} - b_0 B_{\mu(p)-1} - b_1 B_{\mu(p)}} \right. \\ & \times e^{-c_0 C_{\mu(p)-1} - c_1 C_{\mu(p)} - c_2 C_{\nu(q)-1} - c_3 C_{\nu(q)} - c_4 C_{\rho(r)-1} - c_5 C_{\rho(r)}} \\ & \left. \times e^{-\theta_0 \tau_{\mu(p)-1} - \theta_1 \tau_{\mu(p)} - \theta_2 \tau_{\nu(q)-1} - \theta_3 \tau_{\nu(q)} - \theta_4 \tau_{\rho(r)-1} - \theta_5 \tau_{\rho(r)}} \mathbf{1}_{\{\mu(p)=\nu(q)<\rho(r)\}} \right]. \end{aligned} \quad (3.18)$$

Applying operator \mathcal{L}_{pqr} to the family $\{\mathbf{1}_{\{\mu(p)=j=\nu(q)=k,\rho(r)=n\}}; p \geq 0, q \geq 0, r \geq 0\}$, we arrive at

$$\mathcal{L}_{pqr}(\mathbf{1}_{\{\mu(p)=j=\nu(q)=k,\rho(r)=n\}})(x, y, z) = (e^{-xA_{j-1}} - e^{-xA_j})(e^{-yB_{j-1}} - e^{-yB_j})(e^{-zC_{n-1}} - e^{-zC_n}). \quad (3.19)$$

Then, by Fubini's theorem and (3.19), it holds that

$$\begin{aligned} \Psi^2(x, y, z) = \mathcal{L}_{pqr} & \left(\sum_{j \geq 0} \sum_{n > j} E \left[e^{-a_0 A_{j-1} - a_1 A_j - b_0 B_{j-1} - b_1 B_j - c_0 C_{j-1} - c_1 C_j - c_2 C_{j-1} - c_3 C_j - c_4 C_{n-1} - c_5 C_n} \right. \right. \\ & \left. \left. \times e^{-\theta_0 \tau_{j-1} - \theta_1 \tau_j - \theta_2 \tau_{j-1} - \theta_3 \tau_j - \theta_4 \tau_{n-1} - \theta_5 \tau_n} \mathbf{1}_{\{\mu(p)=j=\nu(q)=k,\rho(r)=n\}} \right] \right) (x, y, z) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j \geq 0} \sum_{n > j} E \left[e^{-a_0 A_{j-1} - a_1 A_j - b_0 B_{j-1} - b_1 B_j - c_0 C_{j-1} - c_1 C_j - c_2 C_{j-1} - c_3 C_j - c_4 C_{n-1} - c_5 C_n} \right. \\
&\quad \times e^{-\theta_0 \tau_{j-1} - \theta_1 \tau_j - \theta_2 \tau_{j-1} - \theta_3 \tau_j - \theta_4 \tau_{n-1} - \theta_5 \tau_n} \\
&\quad \left. \times (e^{-x A_{j-1}} - e^{-x A_j}) (e^{-y B_{j-1}} - e^{-y B_j}) (e^{-z C_{n-1}} - e^{-z C_n}) \right]. \tag{3.20}
\end{aligned}$$

After a straightforward algebra and using similar independence arguments regarding the automodulation through intervals I_0, I_1, I_2, I_3 , we have

$$\begin{aligned}
\Psi^2(x, y, z) &= \sum_{j \geq 0} \sum_{n > j} E \left[e^{-(a_0 + a_1 + x) A_{j-1} - (b_0 + b_1 + y) B_{j-1} - (c_0 + \dots + c_5 + z) C_{j-1} - (\theta_0 + \dots + \theta_5) \tau_{j-1}} \right] \\
&\quad \times E \left[e^{-a_1 X_j} (1 - e^{-x X_j}) e^{-b_1 Y_j} (1 - e^{-y Y_j}) e^{-(c_1 + c_3 + c_4 + c_5 + z) Z_j - (\theta_1 + \theta_3 + \theta_4 + \theta_5) \Delta_j} \right] \\
&\quad \times E \left[e^{-(c_4 + c_5 + z) (Z_{j+1} + \dots + Z_{n-1}) - (\theta_4 + \theta_5) (\Delta_{j+1} + \dots + \Delta_{n-1})} \right] \\
&\quad \times E \left[e^{-c_5 Z_n} (1 - e^{-z Z_n}) e^{-\theta_5 \Delta_n} \right] \\
&= \sum_{j \geq 0} \sum_{n > j} R_{1j} R_{2j} R_{3jn} R_{4n}, \tag{3.21}
\end{aligned}$$

where

$$\begin{aligned}
R_{1j} &= E \left[e^{-(a_0 + a_1 + x) A_{j-1} - (b_0 + b_1 + y) B_{j-1} - (c_0 + \dots + c_5 + z) C_{j-1} - (\theta_0 + \dots + \theta_5) \tau_{j-1}} \right] \\
&= \begin{cases} 1, & j = 0 \\ g_0 g_1^{j-1}, & j > 0, \end{cases} \\
R_{2j} &= E \left[e^{-a_1 X_j} (1 - e^{-x X_j}) e^{-b_1 Y_j} (1 - e^{-y Y_j}) e^{-(c_1 + c_3 + c_4 + c_5 + z) Z_j - (\theta_1 + \theta_3 + \theta_4 + \theta_5) \Delta_j} \right] \\
&= \begin{cases} G_0^{12} - G_0^1 - G_0^2 + G_0, & j = 0 \\ G_1^{12} - G_1^1 - G_1^2 + G_1, & j > 0, \end{cases} \tag{3.22}
\end{aligned}$$

g 's and G 's are defined in (2.9)–(2.13), and

$$\begin{aligned}
R_{3jn} &= E \left[e^{-(c_4 + c_5 + z) (Z_{j+1} + \dots + Z_{n-1}) - (\theta_4 + \theta_5) (\Delta_{j+1} + \dots + \Delta_{n-1})} \right] \\
&= \gamma_2^{n-1-j} (0, 0, c_4 + c_5 + z, \theta_4 + \theta_5), \quad n > j, \\
R_{4n} &= E \left[e^{-c_5 Z_n} (1 - e^{-z Z_n}) e^{-\theta_5 \Delta_n} \right] \\
&= \gamma_3(0, 0, c_5, \theta_5) - \gamma_3(0, 0, c_5 + z, \theta_5). \tag{3.23}
\end{aligned}$$

The summation $\sum_{n > j} R_{3jn} R_{4n}$ yields the expression $(\gamma_3(0, 0, c_5, \theta_5) - \gamma_3(0, 0, c_5 + z, \theta_5)) / (1 - \gamma_2(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5))$. Therefore, completing the rest of the summations, we finally have

$$\begin{aligned}
\Psi^2(x, y, z) &= \left[G_0^{12} - G_0^1 - G_0^2 + G_0 + \frac{g_0}{1 - g_1} (G_1^{12} - G_1^1 - G_1^2 + G_1) \right] \\
&\quad \times \frac{\gamma_3(0, 0, c_5, \theta_5) - \gamma_3(0, 0, c_5 + z, \theta_5)}{1 - \gamma_2(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5)}. \tag{3.24}
\end{aligned}$$

The needed convergence everywhere we go is due to Lemma 3.1.

Part 3. Player C defeats player B and then player A, being expressed through the functional

$$\begin{aligned} \Phi_{\mu\nu\rho}^3 &= E \left[e^{-a_0 A_{\nu-1} - a_1 A_{\nu} - a_2 A_{\mu-1} - a_3 A_{\mu} - b_0 B_{\nu-1} - b_1 B_{\nu} - b_2 B_{\mu-1} - b_3 B_{\mu}} \right. \\ &\quad \times e^{-c_0 C_{\nu-1} - c_1 C_{\nu} - c_2 C_{\mu-1} - c_3 C_{\mu} - c_4 C_{\rho-1} - c_5 C_{\rho}} \\ &\quad \left. \times e^{-\theta_0 \tau_{\nu-1} - \theta_1 \tau_{\nu} - \theta_2 \tau_{\mu-1} - \theta_3 \tau_{\mu} - \theta_4 \tau_{\rho-1} - \theta_5 \tau_{\rho}} \mathbf{1}_{\{\nu < \mu < \rho\}} \right]. \end{aligned} \quad (3.25)$$

This case is literally identical to Part 1 by interchanging the roles of A and B yielding the transform of $\Phi_{\mu\nu\rho}^3$:

$$\begin{aligned} \Psi^3(x, y, z) &= \left[\Gamma_0^2 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^2 - \Gamma_1) \right] \\ &\quad \times \frac{\gamma_3(a_3, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5) - \gamma_3(a_3 + x, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5)}{1 - \gamma_2(a_2 + a_3 + x, b_2 + b_3, c_2 + \dots + c_5 + z, \theta_2 + \dots + \theta_5)} \\ &\quad \times \frac{\gamma_4(0, 0, c_5, \theta_5) - \gamma_4(0, 0, c_5 + z, \theta_5)}{1 - \gamma_4(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5)}, \end{aligned} \quad (3.26)$$

where Γ_i^2 's are due to (2.8).

Finally,

$$\begin{aligned} \Psi(x, y, z) &:= \Psi^1(x, y, z) + \Psi^2(x, y, z) + \Psi^3(x, y, z) \\ &= \frac{\gamma_3(a_3, b_3, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5) - \gamma_3(a_3, b_3 + y, c_3 + c_4 + c_5 + z, \theta_3 + \theta_4 + \theta_5)}{1 - \gamma_2(a_2 + a_3, b_2 + b_3 + y, c_2 + \dots + c_5 + z, \theta_2 + \dots + \theta_5)} \\ &\quad \times \frac{\gamma_4(0, 0, c_5, \theta_5) - \gamma_4(0, 0, c_5 + z, \theta_5)}{1 - \gamma_4(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5)} \\ &\quad \times \left[\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^1 - \Gamma_1) + \Gamma_0^2 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^2 - \Gamma_1) \right] \\ &\quad + \left[G_0^{12} - G_0^1 - G_0^2 + G_0 + \frac{g_0}{1 - g_1} (G_1^{12} - G_1^1 - G_1^2 + G_1) \right] \\ &\quad \times \frac{\gamma_3(0, 0, c_5, \theta_5) - \gamma_3(0, 0, c_5 + z, \theta_5)}{1 - \gamma_2(0, 0, c_4 + c_5 + z, \theta_4 + \theta_5)}. \end{aligned} \quad (3.27)$$

Setting

$$c_4 = c_5 = \theta_4 = \theta_5 = 0, \quad (3.28)$$

(i.e., dropping the information past the exit of the second defeated player) and relaxing some of modulation with

$$\gamma_2 = \gamma_3 \quad (3.29)$$

simplifies (3.27) to

$$\begin{aligned} \Psi(x, y, z) &= \left[\Gamma_0^1 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^1 - \Gamma_1) + \Gamma_0^2 - \Gamma_0 + \frac{\gamma_0}{1 - \gamma_1} (\Gamma_1^2 - \Gamma_1) \right] \\ &\quad \times \frac{\gamma_2(a_3, b_3, c_3 + z, \theta_3) - \gamma_2(a_3, b_3 + y, c_3 + z, \theta_3)}{1 - \gamma_2(a_2 + a_3, b_2 + b_3 + y, c_2 + c_3 + z, \theta_2 + \theta_3)} \\ &\quad + G_0^{12} - G_0^1 - G_0^2 + G_0 + \frac{g_0}{1 - g_1} (G_1^{12} - G_1^1 - G_1^2 + G_1). \end{aligned} \quad (3.30)$$

Using the tensor-like index operators

$$\begin{aligned} \mathcal{J}^i(R, r) &:= R_0^i - R_0 + \frac{r_0}{1 - r_1} (R_1^i - R_1), \\ \mathcal{J}^i(R^s, r) &:= R_0^{si} - R_0^s + \frac{r_0}{1 - r_1} (R_1^{si} - R_1^s), \end{aligned} \quad (3.31)$$

and specifying them as linear, we can rewrite (3.28) in a more compact form:

$$\begin{aligned} \Psi(x, y, z) &= (\mathcal{J}^1 + \mathcal{J}^2)(\Gamma, \gamma) \frac{\gamma_2(a_3, b_3, c_3 + z, \theta_3) - \gamma_2(a_3, b_3 + y, c_3 + z, \theta_3)}{1 - \gamma_2(a_2 + a_3, b_2 + b_3 + y, c_2 + c_3 + z, \theta_2 + \theta_3)} \mathcal{J}^1[(G^2, g) - (G, g)]. \end{aligned} \quad (3.32)$$

The targeted functional $\Phi_{\mu\nu\rho}$ can be restored from Ψ by applying the inverse of \mathcal{L} to Ψ , which is the inverse Laplace operator, after the division by xyz . The main result of this section can be summarized in the following theorem.

Theorem 3.3. *In the stochastic game with coalition of players A and B against player C on the σ -subalgebra $\mathcal{F}(\Omega) \cap (\{\mu \leq \nu < \rho\} \cup \{\nu < \mu < \rho\})$ (i.e., with player C defeating the coalition), the functional*

$$\begin{aligned} \Phi_{\mu\nu\rho} &:= E \left[e^{-a_0 A_{\mu-1} - a_1 A_{\mu} - a_2 A_{\nu-1} - a_3 A_{\nu}} e^{-b_0 B_{\mu-1} - b_1 B_{\mu} - b_2 B_{\nu-1} - b_3 B_{\nu}} \right. \\ &\quad \times e^{-c_0 C_{\mu-1} - c_1 C_{\mu} - c_2 C_{\nu-1} - c_3 C_{\nu}} e^{-c_4 C_{\rho-1} - c_5 C_{\rho}} e^{-\theta_0 \tau_{\mu-1} - \theta_1 \tau_{\mu} - \theta_2 \tau_{\nu-1} - \theta_3 \tau_{\nu} - \theta_4 \tau_{\rho-1} - \theta_5 \tau_{\rho}} \\ &\quad \left. \times \mathbf{1}_{\{\mu \leq \nu < \rho\} \cup \{\nu < \mu < \rho\}} \right], \end{aligned} \quad (3.33)$$

relative to the key reference points of the process $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{T})$, satisfies the formulas

$$\Phi_{\mu\nu\rho} = \mathcal{L}_{xyz}^{-1}(\Psi(x, y, z))(M, N, R) \quad (3.34)$$

and (3.27). For special case (3.28)-(3.29), the functional $\Psi(x, y, z)$ in (3.27) and (3.34) takes a compact form (3.32) with index operators (3.31) and, for this special $\Phi_{\mu\nu\rho}$ in (3.33), is

$$\begin{aligned} \Phi_{\mu\nu\rho} &:= E \left[e^{-a_0 A_{\mu-1} - a_1 A_{\mu} - a_2 A_{\nu-1} - a_3 A_{\nu}} e^{-b_0 B_{\mu-1} - b_1 B_{\mu} - b_2 B_{\nu-1} - b_3 B_{\nu}} \right. \\ &\quad \left. \times e^{-c_0 C_{\mu-1} - c_1 C_{\mu} - c_2 C_{\nu-1} - c_3 C_{\nu}} e^{-\theta_0 \tau_{\mu-1} - \theta_1 \tau_{\mu} - \theta_2 \tau_{\nu-1} - \theta_3 \tau_{\nu}} \mathbf{1}_{\{\mu \leq \nu < \rho\} \cup \{\nu < \mu < \rho\}} \right]. \end{aligned} \quad (3.35)$$

Remark 3.4. Expressions (2.3) and (2.4) are more explicit when specifying γ 's, g 's, Γ 's, and G 's.

Appendix

Proof of Lemma 3.1. Since

$$h(x, y, z, \theta) = \int_{u \geq 0} \int_{v \geq 0} \int_{w \geq 0} \int_{t \geq 0} e^{-ux-vy-wz-\theta t} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \quad (\text{A.1})$$

we have

$$\begin{aligned} \|h(x, y, z, \theta)\| &\leq \int_{u \geq 0} \int_{v \geq 0} \int_{w \geq 0} \int_{t \geq 0} \|e^{-ux-vy-wz-\theta t}\| P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt) \\ &\leq \int_{u \geq 0} \int_{v \geq 0} \int_{w \geq 0} \int_{t \geq 0} e^{-\operatorname{Re}(x)u - \operatorname{Re}(y)v - \operatorname{Re}(z)w - \operatorname{Re}(\theta)t} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt) \\ &\leq \alpha_0 + e^{-\operatorname{Re}(\theta)} \alpha_1 + e^{-\operatorname{Re}(z)} \alpha_2 + e^{-\operatorname{Re}(y)} \alpha_3 + e^{-\operatorname{Re}(x)} \alpha_4 + e^{-\operatorname{Re}(z) - \operatorname{Re}(\theta)} \alpha_5 \\ &\quad + e^{-\operatorname{Re}(y) - \operatorname{Re}(\theta)} \alpha_6 + e^{-\operatorname{Re}(x) - \operatorname{Re}(\theta)} \alpha_7 + e^{-\operatorname{Re}(y) - \operatorname{Re}(z)} \alpha_8 + e^{-\operatorname{Re}(x) - \operatorname{Re}(z)} \alpha_9 \\ &\quad + e^{-\operatorname{Re}(x) - \operatorname{Re}(y)} \alpha_{10} + e^{-\operatorname{Re}(y) - \operatorname{Re}(z) - \operatorname{Re}(\theta)} \alpha_{11} + e^{-\operatorname{Re}(x) - \operatorname{Re}(z) - \operatorname{Re}(\theta)} \alpha_{12} \\ &\quad + e^{-\operatorname{Re}(x) - \operatorname{Re}(y) - \operatorname{Re}(\theta)} \alpha_{13} + e^{-\operatorname{Re}(x) - \operatorname{Re}(y) - \operatorname{Re}(z)} \alpha_{14} + e^{-\operatorname{Re}(x) - \operatorname{Re}(y) - \operatorname{Re}(z) - \operatorname{Re}(\theta)} \alpha_{15}, \end{aligned} \quad (\text{A.2})$$

where

$$\begin{aligned} \alpha_0 &= \int_{u=0}^1 \int_{v=0}^1 \int_{w=0}^1 \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_1 &= \int_{u=0}^1 \int_{v=0}^1 \int_{w=0}^1 \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_2 &= \int_{u=0}^1 \int_{v=0}^1 \int_{w \geq 1} \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_3 &= \int_{u=0}^1 \int_{v \geq 1} \int_{w=0}^1 \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_4 &= \int_{u \geq 1} \int_{v=0}^1 \int_{w=0}^1 \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_5 &= \int_{u=0}^1 \int_{v=0}^1 \int_{w \geq 1} \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_6 &= \int_{u=0}^1 \int_{v \geq 1} \int_{w=0}^1 \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_7 &= \int_{u \geq 1} \int_{v=0}^1 \int_{w=0}^1 \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\ \alpha_8 &= \int_{u=0}^1 \int_{v \geq 1} \int_{w \geq 1} \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \end{aligned}$$

$$\begin{aligned}
 \alpha_9 &= \int_{u \geq 1} \int_{v=0}^1 \int_{w \geq 1} \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{10} &= \int_{u \geq 1} \int_{v \geq 1} \int_{w=0}^1 \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{11} &= \int_{u=0}^1 \int_{v \geq 1} \int_{w \geq 1} \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{12} &= \int_{u \geq 1} \int_{v=0}^1 \int_{w \geq 1} \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{13} &= \int_{u \geq 1} \int_{v \geq 1} \int_{w=0}^1 \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{14} &= \int_{u \geq 1} \int_{v \geq 1} \int_{w \geq 1} \int_{t=0}^1 P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt), \\
 \alpha_{15} &= \int_{u \geq 1} \int_{v \geq 1} \int_{w \geq 1} \int_{t \geq 1} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt).
 \end{aligned} \tag{A.3}$$

Thus, if $e^{-\text{Re}(x)} < 1, e^{-\text{Re}(y)} < 1, e^{-\text{Re}(z)} < 1, e^{-\text{Re}(\theta)} < 1,$

$$\int_{u \geq 0} \int_{v \geq 0} \int_{w \geq 0} \int_{t \geq 0} e^{-\text{Re}(x)u - \text{Re}(y)v - \text{Re}(z)w - \text{Re}(\theta)t} P_{X_j \otimes Y_j \otimes Z_j \otimes \Delta_j}(du, dv, dw, dt) < \alpha_0 + \dots + \alpha_{15} = 1. \tag{A.4}$$

The conditions $e^{-\text{Re}(x)} < 1, e^{-\text{Re}(y)} < 1, e^{-\text{Re}(z)} < 1, e^{-\text{Re}(\theta)} < 1$ are equivalent to

$$\text{Re}(x) > 0, \quad \text{Re}(y) > 0, \quad \text{Re}(z) > 0, \quad \text{Re}(\theta) > 0. \tag{A.5}$$

The latter also permits a weaker form allowing us to replace one of the strict inequalities with \geq and still have (A.3) being held. \square

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