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On a class of elliptic free boundary problems with multiple solutions

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Abstract

We prove that a certain class of elliptic free boundary problems, which includes the Prandtl-Batchelor problem from fluid dynamics as a special case, has two distinct nontrivial solutions for large values of a parameter. The first solution is a global minimizer of the energy. The energy functional is nondifferentiable, so standard variational arguments cannot be used directly to obtain a second nontrivial solution. We obtain our second solution as the limit of mountain pass points of a sequence of C^1-functionals approximating the energy. We use careful estimates of the corresponding energy levels to show that this limit is neither trivial nor a minimizer.

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1 Introduction

Consider the class of sublinear elliptic free boundary problems

$$\begin{cases} -\Delta u = \lambda\, \chi_{\{u>1\}}(x)\, g(x, (u-1)_+) & \text{in } \Omega \setminus F(u)\\[10pt] |\nabla u^+|^2 - |\nabla u^-|^2 = 2 & \text{on } F(u)\\[10pt] u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 2$ with $C^{2,\alpha}$-boundary $\partial\Omega$,

$$F(u) = \partial\, \{u > 1\}$$

is the free boundary of u, $\lambda > 0$ is a parameter, $\chi_{\{u>1\}}$ is the characteristic function of the set $\{u > 1\}$, $(u-1)_+ = \max\,(u-1, 0)$ is the positive part of $u-1$, $\nabla u^\pm$ are the limits of ∇u from the sets $\{u > 1\}$ and $\{u \le 1\}^\circ$, respectively, and $g : \Omega \times [0, \infty) \to [0, \infty)$ is a locally Hölder continuous function satisfying

(g_1) for some $a_1, a_2 > 0$ and $1 < p < 2$,

$$|g(x,s)| \le a_1 + a_2\, s^{p-1} \quad \forall (x,s) \in \Omega \times [0, \infty),$$

(g_2) $g(x,s) > 0$ for all $x \in \Omega$ and $s > 0$.

The purpose of this paper is to prove that this problem has two distinct nontrivial (suitably generalized) solutions for all sufficiently large λ.

The special case $g(x,s) \equiv 1$ is the well-known Prandtl-Batchelor free boundary problem, where the phase $\{u > 1\}$ represents a vortex patch bounded by the vortex line $u = 1$ in a steady-state fluid flow when $N = 2$ (see Batchelor [5, 6]). This particular case has been studied in Caflisch [11], Elcrat and Miller [12], Acker [1, 2], and Jerison and Perera [14]. Problem (1.1) also arises in the confinement of a plasma by a magnetic field, where the region $\{u > 1\}$ represents the plasma and the boundary of the plasma is the free boundary (see, e.g., Temam [16, 17], Caffarelli and Friedman [9], Friedman and Liu [13], and Jerison and Perera [15]).

The solutions of problem (1.1) that we construct here are Lipschitz continuous functions of class $H^1_0(\Omega) \cap C^2(\overline{\Omega} \setminus F(u))$ that satisfy the equation $-\Delta u = \lambda\, \chi_{\{u>1\}}(x)\, g(x, (u-1)_+)$ in the classical sense in $\Omega \setminus F(u)$ and vanish continuously on $\partial\Omega$. They satisfy the free boundary condition in the following generalized sense: for all $\Phi \in C^1_0(\Omega, \mathbb{R}^N)$ such that $u \ne 1$ a.e. on the support of Φ,

$$\lim_{\delta^+ \searrow 0} \int_{\{u=1+\delta^+\}} \left(2 - |\nabla u|^2\right) \Phi \cdot n\, d\sigma - \lim_{\delta^- \searrow 0} \int_{\{u=1-\delta^-\}} |\nabla u|^2\, \Phi \cdot n\, d\sigma = 0,$$

where n is the outward unit normal to $\{1-\delta^- < u < 1+\delta^+\}$ (the sets $\{u = 1 \pm \delta^\pm\}$ are smooth hypersurfaces for a.a. $\delta^\pm > 0$ by Sard's theorem and the above limits are taken through such $\delta^\pm$). In particular, the free boundary condition is satisfied in the classical sense on any smooth portion of $F(u)$.

The variational functional associated with problem (1.1) is given by

$$J(u) = \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 + \chi_{\{u>1\}}(x) - \lambda\, G(x,(u-1)_+)\right] dx, \quad u \in H^1_0(\Omega),$$

where

$$G(x,s) = \int_0^s g(x,t)\,dt, \quad s \geq 0.$$

We will prove the following multiplicity result.

Theorem 1.1. *Assume (g_1) and (g_2). Then there exists a $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, problem (1.1) has two Lipschitz continuous solutions $u_0, u_1 \in H^1_0(\Omega) \cap C^2(\overline{\Omega} \setminus F(u))$ that satisfy the equation $-\Delta u = \lambda\, \chi_{\{u>1\}}(x)\, g(x,(u-1)_+)$ in the classical sense in $\Omega \setminus F(u)$, the free boundary condition in the generalized sense, and vanish continuously on $\partial\Omega$. Moreover,*

(i) $J(u_0) < -\mathcal{L}(\Omega) \leq -\mathcal{L}(\{u_1 = 1\}) < J(u_1)$, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^N$, and hence u_0 and u_1 are nontrivial and distinct;

(ii) $0 < u_1 \leq u_0$, the sets $\{u_0 < 1\} \subset \{u_1 < 1\}$ are connected if $\partial\Omega$ is connected, and the sets $\{u_0 > 1\} \supset \{u_1 > 1\}$ are nonempty;

(iii) u_0 is a minimizer of J, but u_1 is not a minimizer of J.

This theorem will be proved in the next section. Since u_0 is a minimizer of J, it follows from standard arguments that it satisfies the free boundary condition in the viscosity sense and its free boundary $F(u_0)$ has finite $(N-1)$-dimensional Hausdorff measure and is a smooth hypersurface except on a closed set of Hausdorff dimension at most $N-3$. Near the smooth subset of $F(u_0)$, $(u_0-1)_\pm$ are smooth and the free boundary condition is satisfied in the classical sense (see, e.g., Caffarelli and Salsa [8]). The nondegeneracy and regularity of u_1 is presently an open problem.

2 Proof of Theorem 1.1

Since the functional J is nondifferentiable, we approximate it by C^1-functionals as follows. Let $\beta : \mathbb{R} \to [0,2]$ be a smooth function such that $\beta(s) = 0$ for $s \leq 0$,

$\beta(s) > 0$ for $0 < s < 1$, $\beta(s) = 0$ for $s \geq 1$, and $\int_0^1 \beta(s)\, ds = 1$. Then let

$$\mathcal{B}(s) = \int_0^s \beta(t)\, dt$$

and note that $\mathcal{B} : \mathbb{R} \to [0,1]$ is a smooth nondecreasing function such that $\mathcal{B}(s) = 0$ for $s \leq 0$, $0 < \mathcal{B}(s) < 1$ for $0 < s < 1$, and $\mathcal{B}(s) = 1$ for $s \geq 1$. For $\varepsilon > 0$, let

$$g_\varepsilon(x,s) = \mathcal{B}\left(\frac{s}{\varepsilon}\right) g(x,s), \quad G_\varepsilon(x,s) = \int_0^s g_\varepsilon(x,t)\, dt, \quad s \geq 0$$

and set

$$J_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}\, |\nabla u|^2 + \mathcal{B}\left(\frac{u-1}{\varepsilon}\right) - \lambda\, G_\varepsilon(x,(u-1)_+)\right] dx, \quad u \in H^1_0(\Omega).$$

The functional J_ε is of class C^1 and its critical points coincide with weak solutions of the problem

$$\begin{cases} -\Delta u = -\dfrac{1}{\varepsilon}\, \beta\left(\dfrac{u-1}{\varepsilon}\right) + \lambda\, g_\varepsilon(x,(u-1)_+) & \text{in } \Omega \\[2ex] u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

If $u \in H^1_0(\Omega)$ is a weak solution of this problem, then $u \in C^{2,\alpha}(\overline{\Omega})$ and is a classical solution by elliptic regularity theory. If u is not identically zero, then it is nontrivial in a stronger sense, namely, $u > 0$ in Ω and $u > 1$ in a nonempty open set. Indeed, if $u \leq 1$ everywhere, then u is harmonic in Ω and hence vanishes identically since $u = 0$ on $\partial\Omega$. Furthermore, in the set $\{u < 1\}$, u is the harmonic function with boundary values 0 on $\partial\Omega$ and 1 on $\partial\{u \geq 1\}$, and hence strictly positive since Ω is connected.

First we prove the following convergence result.

Lemma 2.1. *Assume (g_1) and (g_2). Let $\varepsilon_j \searrow 0$ and let u_j be a critical point of J_{ε_j}. If the sequence (u_j) is bounded in $H^1_0(\Omega) \cap L^\infty(\Omega)$, then there exists a Lipschitz continuous function u on $\overline{\Omega}$ such that $u \in H^1_0(\Omega) \cap C^2(\overline{\Omega} \setminus F(u))$ and, for a renamed subsequence,*

- *(i)* $u_j \to u$ *uniformly on* $\overline{\Omega}$,
- *(ii)* $u_j \to u$ *locally in* $C^1(\overline{\Omega} \setminus \{u = 1\})$,
- *(iii)* $u_j \to u$ *strongly in* $H^1_0(\Omega)$,

(*iv*) $J(u) \le \liminf J_{\varepsilon_j}(u_j) \le \limsup J_{\varepsilon_j}(u_j) \le J(u) + \mathcal{L}(\{u = 1\})$, *in particular, u is nontrivial if* $\liminf J_{\varepsilon_j}(u_j) < 0$ *or* $\limsup J_{\varepsilon_j}(u_j) > 0$.

Moreover, u satisfies the equation $-\Delta u = \lambda\, \chi_{\{u>1\}}(x)\, g(x,(u-1)_+)$ in the classical sense in $\Omega \setminus F(u)$, the free boundary condition in the generalized sense, and vanishes continuously on $\partial\Omega$. If u is nontrivial, then $u > 0$ in Ω, the set $\{u < 1\}$ is connected if $\partial\Omega$ is connected, and the set $\{u > 1\}$ is nonempty.

The crucial ingredient in the passage to the limit in the proof of this lemma is the following uniform Lipschitz continuity result of Caffarelli et al. [10].

Lemma 2.2 ([10, Theorem 5.1])**.** *Let u be a Lipschitz continuous function on $B_1(0) \subset \mathbb{R}^N$ satisfying the distributional inequalities*

$$\pm\Delta u \le A \left(\frac{1}{\varepsilon}\, \chi_{\{|u-1|<\varepsilon\}}(x) + 1\right)$$

for some constants $A > 0$ and $0 < \varepsilon \le 1$. Then there exists a constant $C > 0$, depending on N, A, and $\displaystyle\int_{B_1(0)} u^2\, dx$, but not on ε, such that

$$\sup_{x \in B_{1/2}(0)} |\nabla u(x)| \le C.$$

Proof of Lemma 2.1. We may assume that $0 < \varepsilon_j \le 1$. The function u_j is a solution of

$$\begin{cases} -\Delta u_j = -\dfrac{1}{\varepsilon_j}\, \beta\left(\dfrac{u_j - 1}{\varepsilon_j}\right) + \lambda\, g_{\varepsilon_j}(x,(u_j-1)_+) & \text{in } \Omega \\[2ex] \quad u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Since (u_j) is bounded in $L^\infty(\Omega)$, $0 \le g_{\varepsilon_j}(x,(u_j-1)_+) \le A_0$ for some constant $A_0 > 0$ by (g_1). Let $\varphi_0 > 0$ be the solution of

$$\begin{cases} -\Delta \varphi_0 = \lambda A_0 & \text{in } \Omega \\[2ex] \quad \varphi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\beta \ge 0$, $-\Delta u_j \le \lambda A_0$ in Ω, and hence

$$0 \le u_j(x) \le \varphi_0(x) \quad \forall x \in \Omega$$

by the maximum principle. The majorant φ_0 gives a uniform lower bound $\delta_0 > 0$ on the distance from the set $\{u_j \geq 1\}$ to $\partial\Omega$. Since u_j is positive, harmonic, and bounded by 1 in a δ_0-neighborhood of $\partial\Omega$, it follows from standard boundary regularity theory that the sequence (u_j) is bounded in the $C^{2,\alpha}$ norm, and hence compact in the C^2 norm, in a $\delta_0/2$-neighborhood.

Since $0 \leq \beta \leq 2\,\chi_{(-1,1)}$,

$$\pm\Delta u_j = \pm\frac{1}{\varepsilon_j}\,\beta\left(\frac{u_j - 1}{\varepsilon_j}\right) \mp \lambda\, g_{\varepsilon_j}(x, (u_j - 1)_+) \leq \frac{2}{\varepsilon_j}\,\chi_{\{|u_j - 1| < \varepsilon_j\}}(x) + \lambda A_0.$$

Since (u_j) is bounded in $L^2(\Omega)$, it follows from this and Lemma 2.2 that there exists a constant $C > 0$ such that

$$\max_{x \in B_{r/2}(x_0)} |\nabla u_j(x)| \leq \frac{C}{r}$$

whenever $r > 0$ and $B_r(x_0) \subset \Omega$. Hence u_j is uniformly Lipschitz continuous on the compact subset of Ω at distance greater or equal to $\delta_0/2$ from $\partial\Omega$.

Thus, a renamed subsequence of (u_j) converges uniformly on $\overline{\Omega}$ to a Lipschitz continuous function u with zero boundary values, with strong convergence in C^2 on a $\delta_0/2$-neighborhood of $\partial\Omega$. Since (u_j) is bounded in $H^1_0(\Omega)$, a further subsequence converges weakly in $H^1_0(\Omega)$ to u.

Next we show that u satisfies the equation $-\Delta u = \lambda\,\chi_{\{u>1\}}(x)\, g(x, (u-1)_+)$ in the set $\{u \neq 1\}$. Let $\varphi \in C^\infty_0(\{u > 1\})$. Then $u \geq 1 + 2\,\varepsilon$ on the support of φ for some $\varepsilon > 0$. For all sufficiently large j, $\varepsilon_j < \varepsilon$ and $|u_j - u| < \varepsilon$ in Ω, so $u_j \geq 1 + \varepsilon_j$ on the support of φ. So testing (2.2) with φ gives

$$\int_\Omega \nabla u_j \cdot \nabla\varphi\, dx = \int_\Omega \lambda\, g(x, u_j - 1)\,\varphi\, dx,$$

and passing to the limit gives

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega \lambda\, g(x, u - 1)\,\varphi\, dx$$

since u_j converges to u weakly in $H^1_0(\Omega)$ and uniformly on Ω. Hence u is a distributional, and hence a classical, solution of $-\Delta u = \lambda\, g(x, u-1)$ in the set $\{u > 1\}$. A similar argument shows that u satisfies $\Delta u = 0$ in the set $\{u < 1\}$.

Now we show that u is also harmonic in the possibly larger set $\{u \leq 1\}^\circ$. Since $\beta \geq 0$ and $\mathcal{B} \leq 1$, testing (2.2) with any nonnegative test function and passing to the limit shows that

$$-\Delta u \leq \lambda\, g(x, (u-1)_+) \quad \text{in } \Omega \tag{2.3}$$

in the distributional sense. On the other hand, since u is harmonic in $\{u < 1\}$, $\mu := \Delta(u-1)_-$ is a nonnegative Radon measure supported on $\Omega \cap \partial\{u < 1\}$ by Alt and Caffarelli [3, Remark 4.2], so

$$-\Delta u = \mu \geq 0 \quad \text{in } \{u \leq 1\}. \tag{2.4}$$

It follows from (2.3) and (2.4) that $u \in W^{2,p}_{\text{loc}}(\{u \leq 1\}^\circ)$, $1 < p < \infty$ and hence μ is actually supported on $\Omega \cap \partial\{u < 1\} \cap \partial\{u > 1\}$, so u is harmonic in $\{u \leq 1\}^\circ$.

Since u_j converges in the C^2 norm to u in a neighborhood of $\partial\Omega$ in $\overline{\Omega}$, it suffices to show that $u_j \to u$ locally in $C^1(\Omega \setminus \{u = 1\})$ to prove (ii). Let $U \subset\subset \{u > 1\}$. Then $u \geq 1 + 2\,\varepsilon$ in U for some $\varepsilon > 0$. For all sufficiently large j, $\varepsilon_j < \varepsilon$ and $|u_j - u| < \varepsilon$ in Ω, so $u_j \geq 1 + \varepsilon_j$ in U. So (2.2) gives $-\Delta u_j = \lambda\, g(x, u_j - 1)$ in U. Since g is locally Hölder continuous and $u_j \to u$ uniformly, $g(x, u_j - 1) \to g(x, u - 1)$ in $L^p(U)$ for $1 < p < \infty$. Since $-\Delta u = \lambda\, g(x, u - 1)$ in U, then $u_j \to u$ in $W^{2,p}(U)$. Since $W^{2,p}(U) \hookrightarrow C^1(U)$ for $p > 2$, it follows that $u_j \to u$ in $C^1(U)$. A similar argument shows that $u_j \to u$ locally in $C^1(\{u < 1\})$ also.

Since $u_j \rightharpoonup u$ in $H^1_0(\Omega)$, $\|u\| \leq \liminf \|u_j\|$, so it suffices to show that $\limsup \|u_j\| \leq \|u\|$ to prove (iii). Multiplying the first equation in (2.2) by $u_j - 1$, integrating by parts, and noting that $\beta(s/\varepsilon_j)\, s \geq 0$ for all s gives

$$\int_\Omega |\nabla u_j|^2\, dx \leq \int_\Omega \lambda\, g(x, (u_j - 1)_+)\, (u_j - 1)_+\, dx - \int_{\partial\Omega} \frac{\partial u_j}{\partial n}\, d\sigma$$

$$\to \int_\Omega \lambda\, g(x, (u - 1)_+)\, (u - 1)_+\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma, \tag{2.5}$$

where n is the outward unit normal to $\partial\Omega$. Fix $0 < \varepsilon < 1$. Recall that u is a solution of $-\Delta u = \lambda\, g(x, u-1)$ in $\{u > 1\}$. Testing this equation with $\varphi = (u - 1 - \varepsilon)_+$ gives

$$\int_{\{u > 1 + \varepsilon\}} |\nabla u|^2\, dx = \int_\Omega \lambda\, g(x, (u - 1)_+)\, (u - 1 - \varepsilon)_+\, dx. \tag{2.6}$$

Integrating $(u - 1 + \varepsilon)_-\, \Delta u = 0$ over Ω gives

$$\int_{\{u < 1 - \varepsilon\}} |\nabla u|^2\, dx = -(1 - \varepsilon) \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma. \tag{2.7}$$

Adding (2.6) and (2.7), and letting $\varepsilon \searrow 0$ gives

$$\int_\Omega |\nabla u|^2\, dx = \int_\Omega \lambda\, g(x, (u - 1)_+)\, (u - 1)_+\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma$$

since $\displaystyle\int_{\{u=1\}} |\nabla u|^2\,dx = 0$. This together with (2.5) gives

$$\limsup \int_\Omega |\nabla u_j|^2\,dx \le \int_\Omega |\nabla u|^2\,dx$$

as desired.

To prove (iv), write

$$J_{\varepsilon_j}(u_j) = \int_\Omega \left[\frac{1}{2}\,|\nabla u_j|^2 + \mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)\chi_{\{u\neq 1\}}(x) - \lambda\,G_{\varepsilon_j}(x,(u_j-1)_+)\right]dx \\ + \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)dx.$$

Since $u_j \to u$ in $H^1_0(\Omega)$, and $\mathcal{B}((u_j-1)/\varepsilon_j)\,\chi_{\{u\neq 1\}}$ and $G_{\varepsilon_j}(x,(u_j-1)_+)$ are bounded and converge pointwise to $\chi_{\{u>1\}}$ and $G(x,(u-1)_+)$, respectively, the first integral converges to $J(u)$. Since

$$0 \le \int_{\{u=1\}} \mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)dx \le \mathcal{L}(\{u=1\}),$$

the desired conclusion follows.

Finally we show that u satisfies the free boundary condition in the generalized sense. Let $\Phi \in C^1_0(\Omega,\mathbb{R}^N)$ be such that $u \neq 1$ a.e. on the support of Φ. Multiplying the first equation in (2.2) by $\nabla u_j \cdot \Phi$ and integrating over the set $\{1-\delta^- < u < 1+\delta^+\}$ gives

$$\int_{\{1-\delta^-<u<1+\delta^+\}} \left[-\Delta u_j + \frac{1}{\varepsilon_j}\,\beta\left(\frac{u_j-1}{\varepsilon_j}\right)\right]\nabla u_j\cdot\Phi\,dx \\ = \int_{\{1-\delta^-<u<1+\delta^+\}} \lambda\,g_{\varepsilon_j}(x,(u_j-1)_+)\,\nabla u_j\cdot\Phi\,dx.$$

Noting that the integrand on the left-hand side is equal to

$$\operatorname{div}\left(\frac{1}{2}\,|\nabla u_j|^2\,\Phi - (\nabla u_j\cdot\Phi)\,\nabla u_j\right) + \nabla u_j\,D\Phi\cdot\nabla u_j - \frac{1}{2}\,|\nabla u_j|^2\,\operatorname{div}\Phi + \nabla\mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)\cdot\Phi$$

and integrating by parts gives

$$\int_{\{u=1+\delta^+\}\cup\{u=1-\delta^-\}} \left[\frac{1}{2}\,|\nabla u_j|^2\,\Phi - (\nabla u_j\cdot\Phi)\,\nabla u_j + \mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)\Phi\right]\cdot n\,d\sigma$$

$$= \int_{\{1-\delta^-<u<1+\delta^+\}} \left(\frac{1}{2}\,|\nabla u_j|^2\,\operatorname{div}\Phi - \nabla u_j\,D\Phi\cdot\nabla u_j\right)dx$$

$$+ \int_{\{1-\delta^-<u<1+\delta^+\}} \left[\mathcal{B}\left(\frac{u_j-1}{\varepsilon_j}\right)\operatorname{div}\Phi + \lambda\,g_{\varepsilon_j}(x,(u_j-1)_+)\,\nabla u_j\cdot\Phi\right]dx. \quad (2.8)$$

By (ii), the integral on the left-hand side converges to

$$\int_{\{u=1+\delta^+\}\cup\{u=1-\delta^-\}} \left(\frac{1}{2}\,|\nabla u|^2\,\Phi - (\nabla u\cdot\Phi)\,\nabla u\right)\cdot n\,d\sigma + \int_{\{u=1+\delta^+\}} \Phi\cdot n\,d\sigma,$$

which is equal to

$$\int_{\{u=1+\delta^+\}} \left(1-\frac{1}{2}\,|\nabla u|^2\right)\Phi\cdot n\,d\sigma - \int_{\{u=1-\delta^-\}} \frac{1}{2}\,|\nabla u|^2\,\Phi\cdot n\,d\sigma$$

since $n = \pm\nabla u/|\nabla u|$ on $\{u = 1\pm\delta^\pm\}$. The first integral on the right-hand side of (2.8) converges to

$$\int_{\{1-\delta^-<u<1+\delta^+\}} \left(\frac{1}{2}\,|\nabla u|^2\,\operatorname{div}\Phi - \nabla u\,D\Phi\cdot\nabla u\right)dx$$

by (iii), and the second integral is bounded by

$$\int_{\{1-\delta^-<u<1+\delta^+\}} \big(|\operatorname{div}\Phi| + a_3\,|\Phi|\big)\,dx$$

for some constant $a_3 > 0$. Since $\mathcal{L}(\{u=1\}\cap\operatorname{supp}\Phi) = 0$, the last two integrals go to zero as $\delta^\pm \searrow 0$. So first letting $j\to\infty$ and then letting $\delta^\pm\searrow 0$ in (2.8) gives the desired conclusion. □

By (g_1),

$$J_\varepsilon(u) \ge \int_\Omega \left(\frac{1}{2}\,|\nabla u|^2 - \lambda\left[a_1\,(u-1)_+ + \frac{a_2}{p}\,(u-1)_+^p\right]\right)dx,$$

and since $1 < p < 2$, this implies that J_ε is bounded from below and coercive. Hence J_ε satisfies the (PS) condition, i.e., every sequence $(u_j)\subset H^1_0(\Omega)$ such that $J_\varepsilon(u_j)$ is bounded and $J'_\varepsilon(u_j) \to 0$ has a convergent subsequence. Indeed, every such sequence is bounded by coercivity and hence contains a convergent subsequence by a standard argument. First we show that J_ε has a minimizer u_0^ε. Note that J is also bounded from below. By (g_2), there exists a $\lambda^* > 0$ such that for all $\lambda > \lambda^*$,

$$c_1(\lambda) := \inf_{u \in H_0^1(\Omega)} J(u) < -\mathcal{L}(\Omega). \tag{2.9}$$

For $\lambda > \lambda^*$, set

$$\varepsilon_0(\lambda) = \min \left\{ \frac{|c_1(\lambda)|}{2\lambda a_1 \mathcal{L}(\Omega)}, \left(\frac{p a_1}{a_2} \right)^{1/(p-1)} \right\}.$$

Lemma 2.3. *For all $\lambda > \lambda^*$ and $\varepsilon < \varepsilon_0(\lambda)$, J_ε has a minimizer $u_0^\varepsilon > 0$ satisfying*

$$J_\varepsilon(u_0^\varepsilon) \le c_1(\lambda) + 2\lambda\varepsilon a_1 \mathcal{L}(\Omega) < 0. \tag{2.10}$$

Proof. Since J_ε is bounded from below and satisfies the (PS) condition, it has a minimizer u_0^ε. Since $\mathcal{B}((t-1)/\varepsilon) \le \chi_{(1,\infty)}(t)$ for all t,

$$\begin{aligned} J_\varepsilon(u) - J(u) &\le \lambda \int_\Omega \left[G(x,(u-1)_+) - G_\varepsilon(x,(u-1)_+) \right] dx \\ &= \lambda \int_\Omega \int_0^{(u-1)_+} \left[1 - B\left(\frac{t}{\varepsilon}\right) \right] g(x,t)\, dt\, dx \\ &\le \lambda \int_\Omega \int_0^\varepsilon g(x,t)\, dt\, dx \\ &\le \lambda \left(a_1 \varepsilon + \frac{a_2}{p}\, \varepsilon^p \right) \mathcal{L}(\Omega) \end{aligned}$$

by (g_1), and (2.10) follows from this for $\varepsilon < \varepsilon_0(\lambda)$. Since $J_\varepsilon(u_0^\varepsilon) < 0 = J_\varepsilon(0)$, u_0^ε is nontrivial and hence positive. □

Next we show that J_ε has a second nontrivial critical point u_1^ε using the mountain pass lemma of Ambrosetti and Rabinowitz [4], which we now recall.

Lemma 2.4 ([4, Theorem 2.1])**.** *Let I be a C^1-functional defined on a Banach space X. Assume that I satisfies the* (PS) *condition and that there exist an open set $U \subset X$, $u_0 \in U$, and $u_1 \in X \setminus \overline{U}$ such that*

$$\inf_{u \in \partial U} I(u) > \max\left\{ I(u_0), I(u_1) \right\}.$$

Then I has a critical point at the level

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u) \geq \inf_{u \in \partial U} I(u),$$

where $\Gamma = \big\{ \gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1 \big\}$ *is the class of paths in X joining u_0 and u_1.*

Lemma 2.5. *For all $\lambda > \lambda^*$, there exists a constant $c_2(\lambda) > 0$ such that for all $\varepsilon < \varepsilon_0(\lambda)$, J_ε has a second critical point $0 < u_1^\varepsilon \leq u_0^\varepsilon$ satisfying*

$$c_2(\lambda) \leq J_\varepsilon(u_1^\varepsilon) \leq \frac{1}{2} \|u_0^\varepsilon\|^2 + \mathcal{L}(\Omega).$$

In particular, $\{u_0^\varepsilon > 1\} \supset \{u_1^\varepsilon > 1\} \neq \emptyset$.

Proof. For $\varepsilon < \varepsilon_0(\lambda)$, let

$$\beta_\varepsilon(x,s) = \frac{1}{\varepsilon}\, \beta\left(\frac{\min\{s, u_0^\varepsilon(x)\} - 1}{\varepsilon}\right), \qquad \mathcal{B}_\varepsilon(x,s) = \int_0^s \beta_\varepsilon(x,t)\, dt,$$

$$\widetilde{g}_\varepsilon(x,s) = g_\varepsilon(x, (\min\{s, u_0^\varepsilon(x)\} - 1)_+), \qquad \widetilde{G}_\varepsilon(x,s) = \int_0^s \widetilde{g}_\varepsilon(x,t)\, dt$$

and set

$$\widetilde{J}_\varepsilon(u) = \int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \mathcal{B}_\varepsilon(x,u) - \lambda\, \widetilde{G}_\varepsilon(x,u)\right] dx, \quad u \in H_0^1(\Omega).$$

The functional $\widetilde{J}_\varepsilon$ is of class C^1 and its critical points coincide with weak solutions of the problem

$$\begin{cases} -\Delta u = -\beta_\varepsilon(x,u) + \lambda\, \widetilde{g}_\varepsilon(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

If u is a weak solution of this problem, then u is also a classical solution by elliptic regularity theory and $u \leq u_0^\varepsilon$ by the maximum principle. So u is a solution of problem (2.1), and hence a critical point of J_ε, with $J_\varepsilon(u) = \widetilde{J}_\varepsilon(u)$. We will show that $\widetilde{J}_\varepsilon$ has a critical point u_1^ε satisfying

$$c_2(\lambda) \leq \widetilde{J}_\varepsilon(u_1^\varepsilon) \leq \frac{1}{2} \|u_0^\varepsilon\|^2 + \mathcal{L}(\Omega)$$

for some constant $c_2(\lambda) > 0$. This will prove the lemma since it follows from $J_\varepsilon(u_1^\varepsilon) = \widetilde{J}_\varepsilon(u_1^\varepsilon) > 0 > J_\varepsilon(u_0^\varepsilon)$ that u_1^ε is positive and distinct from u_0^ε.

We apply Lemma 2.4 to the functional $\widetilde{J}_\varepsilon$, which is also coercive and hence satisfies the (PS) condition. Since $\widetilde{g}_\varepsilon(x,s) = g_\varepsilon(x,0) = 0$ for $s \le 1$ and

$$\widetilde{g}_\varepsilon(x,s) \le a_1 + a_2 \left(\min\left\{s, u_0^\varepsilon(x)\right\} - 1\right)_+^{p-1} \le a_1 + a_2 \,(s-1)^{p-1}$$

for $s > 1$ by (g_1),

$$\widetilde{G}_\varepsilon(x,s) \le a_1\,(s-1)_+ + \frac{a_2}{p}\,(s-1)_+^p \le \left(a_1 + \frac{a_2}{p}\right)|s|^q$$

for all s, where $q > 2$ if $N = 2$ and $2 < q \le 2N/(N-2)$ if $N \ge 3$. Since $\mathcal{B}_\varepsilon(x,s) \ge 0$ for all s, then

$$\widetilde{J}_\varepsilon(u) \ge \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 - \lambda\left(a_1 + \frac{a_2}{p}\right)|u|^q\right] dx.$$

Since $L^q(\Omega) \hookrightarrow H_0^1(\Omega)$ and $q > 2$, the infimum $c_2(\lambda)$ of the last integral on $\partial B_\rho(0)$ is positive for all sufficiently small $\rho > 0$, where $B_\rho(0) = \{u \in H_0^1(\Omega) : \|u\| < \rho\}$. Since $\widetilde{J}_\varepsilon(u_0^\varepsilon) = J_\varepsilon(u_0^\varepsilon) < 0 = \widetilde{J}_\varepsilon(0)$, taking $\rho < \|u_0^\varepsilon\|$ and applying Lemma 2.4 now gives a critical point u_1^ε of $\widetilde{J}_\varepsilon$ with

$$\widetilde{J}_\varepsilon(u_1^\varepsilon) = \inf_{\gamma \in \Gamma}\ \max_{u \in \gamma([0,1])} \widetilde{J}_\varepsilon(u) \ge \inf_{u \in \partial B_\rho(0)} \widetilde{J}_\varepsilon(u) \ge c_2(\lambda),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_0^\varepsilon\}$ is the class of paths joining 0 and u_0^ε. For the path $\gamma_0(t) = tu_0^\varepsilon$, $t \in [0,1]$,

$$\widetilde{J}_\varepsilon(\gamma_0(t)) \le \int_\Omega \left(\frac{1}{2}\,|\nabla u_0^\varepsilon|^2 + \mathcal{B}_\varepsilon(x, u_0^\varepsilon)\right) dx$$

since $\mathcal{B}_\varepsilon(x,s)$ is nondecreasing in s and $\widetilde{G}_\varepsilon(x,s) \ge 0$ for all s by (g_2). Since

$$\mathcal{B}_\varepsilon(x, u_0^\varepsilon(x)) = \int_0^{u_0^\varepsilon(x)} \frac{1}{\varepsilon}\,\beta\left(\frac{t-1}{\varepsilon}\right) dt = \mathcal{B}\left(\frac{u_0^\varepsilon(x) - 1}{\varepsilon}\right) \le 1,$$

then

$$\widetilde{J}_\varepsilon(u_1^\varepsilon) \le \max_{u \in \gamma_0([0,1])} \widetilde{J}_\varepsilon(u) \le \int_\Omega \left(\frac{1}{2}\,|\nabla u_0^\varepsilon|^2 + 1\right) dx = \frac{1}{2}\,\|u_0^\varepsilon\|^2 + \mathcal{L}(\Omega). \quad \square$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\lambda > \lambda^*$ and take a sequence $\varepsilon_j \searrow 0$ with $\varepsilon_j < \varepsilon_0(\lambda)$. For each j, Lemma 2.3 gives a minimizer $u_0^{\varepsilon_j} > 0$ of J_{ε_j} satisfying

$$J_{\varepsilon_j}(u_0^{\varepsilon_j}) \le c_1(\lambda) + 2\lambda\varepsilon_j\, a_1 \mathcal{L}(\Omega) < 0 \tag{2.11}$$

and Lemma 2.5 gives a second critical point $0 < u_1^{\varepsilon_j} \le u_0^{\varepsilon_j}$ satisfying

$$c_2(\lambda) \le J_{\varepsilon_j}(u_1^{\varepsilon_j}) \le \frac{1}{2} \left\|u_0^{\varepsilon_j}\right\|^2 + \mathcal{L}(\Omega). \tag{2.12}$$

We will show that the sequences $\left(u_0^{\varepsilon_j}\right)$ and $\left(u_1^{\varepsilon_j}\right)$ are bounded in $H^1_0(\Omega) \cap L^\infty(\Omega)$ and apply Lemma 2.1.

Since $\mathcal{B} \ge 0$ and

$$G_\varepsilon(x, (s-1)_+) \le a_1\, (s-1)_+ + \frac{a_2}{p}\, (s-1)_+^p \le \left(a_1 + \frac{a_2}{p}\right) |s|^p$$

for all s by (g_1),

$$\frac{1}{2} \left\|u_0^\varepsilon\right\|^2 \le J_\varepsilon(u_0^\varepsilon) + \lambda \left(a_1 + \frac{a_2}{p}\right) \int_\Omega (u_0^\varepsilon)^p\, dx.$$

Since $J_{\varepsilon_j}(u_0^{\varepsilon_j}) < 0$ by (2.11) and $p < 2$, it follows from this that $\left(u_0^{\varepsilon_j}\right)$ is bounded in $H^1_0(\Omega)$. Then $J_{\varepsilon_j}(u_1^{\varepsilon_j})$ is bounded by (2.12), so a similar argument shows that $\left(u_1^{\varepsilon_j}\right)$ is also bounded in $H^1_0(\Omega)$.

Since $g_\varepsilon(x, (s-1)_+) = g_\varepsilon(x, 0) = 0$ for $s \le 1$ and

$$g_\varepsilon(x, (s-1)_+) \le a_1 + a_2\, (s-1)^{p-1} \le (a_1 + a_2)\, s^{p-1}$$

for $s > 1$ by (g_1),

$$-\Delta u_0^{\varepsilon_j} = -\frac{1}{\varepsilon_j}\, \beta\left(\frac{u_0^{\varepsilon_j} - 1}{\varepsilon_j}\right) + \lambda\, g_{\varepsilon_j}(x, (u_0^{\varepsilon_j} - 1)_+) \le \lambda\, (a_1 + a_2)\, (u_0^{\varepsilon_j})^{p-1}.$$

This together with the fact that $\left(u_0^{\varepsilon_j}\right)$ is bounded in $H^1_0(\Omega)$ implies that $\left(u_0^{\varepsilon_j}\right)$ is also bounded in $L^\infty(\Omega)$ (see, e.g., Bonforte et al. [7, Theorem 3.1]). Then so is $\left(u_1^{\varepsilon_j}\right)$ since $0 < u_1^{\varepsilon_j} \le u_0^{\varepsilon_j}$.

By Lemma 2.1, for a renamed subsequence of (ε_j), the sequences $\left(u_0^{\varepsilon_j}\right)$ and $\left(u_1^{\varepsilon_j}\right)$ converge uniformly to Lipschitz continuous solutions $u_0, u_1 \in H^1_0(\Omega) \cap C^2(\overline{\Omega} \setminus F(u))$ of problem (1.1) that satisfy the equation $-\Delta u = \lambda\, \chi_{\{u>1\}}(x)\, g(x, (u-1)_+)$ in the classical sense in $\Omega \setminus F(u)$, the free boundary condition in the generalized sense, and vanish continuously on $\partial\Omega$. Moreover,

$$J(u_0) \leq \liminf J_{\varepsilon_j}(u_0^{\varepsilon_j}) \leq \limsup J_{\varepsilon_j}(u_0^{\varepsilon_j}) \leq J(u_0) + \mathcal{L}(\{u_0 = 1\}) \tag{2.13}$$

and

$$J(u_1) \leq \liminf J_{\varepsilon_j}(u_1^{\varepsilon_j}) \leq \limsup J_{\varepsilon_j}(u_1^{\varepsilon_j}) \leq J(u_1) + \mathcal{L}(\{u_1 = 1\}). \tag{2.14}$$

Combining (2.13) with (2.11) and (2.9) gives $J(u_0) \leq \limsup J_{\varepsilon_j}(u_0^{\varepsilon_j}) \leq c_1(\lambda) \leq J(u_0)$, so

$$J(u_0) = c_1(\lambda) < -\mathcal{L}(\Omega). \tag{2.15}$$

On the other hand, combining (2.14) with (2.12) gives $J(u_1) + \mathcal{L}(\{u_1 = 1\}) \geq \liminf J_{\varepsilon_j}(u_1^{\varepsilon_j}) \geq c_2(\lambda) > 0$, so

$$J(u_1) > -\mathcal{L}(\{u_1 = 1\}) \geq -\mathcal{L}(\Omega). \tag{2.16}$$

It follows from (2.15) and (2.16) that u_0 and u_1 are nontrivial and distinct, u_0 is a minimizer of J, and u_1 is not a minimizer. Since $u_1^{\varepsilon_j} \leq u_0^{\varepsilon_j}$ for all j, $u_1 \leq u_0$. Since u_1 is nontrivial, then $0 < u_1 \leq u_0$, the sets $\{u_0 < 1\} \subset \{u_1 < 1\}$ are connected if $\partial\Omega$ is connected, and the sets $\{u_0 > 1\} \supset \{u_1 > 1\}$ are nonempty. □

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