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**MULTIPLICITY RESULTS FOR p -SUBLINEAR p -LAPLACIAN
PROBLEMS INVOLVING INDEFINITE EIGENVALUE
PROBLEMS VIA MORSE THEORY**

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ABSTRACT. We establish some multiplicity results for a class of p -sublinear p -Laplacian problems involving indefinite eigenvalue problems using Morse theory.

1. INTRODUCTION

The purpose of this note is to establish some multiplicity results for a class of p -sublinear p -Laplacian problems involving indefinite eigenvalue problems using Morse theory.

As motivation, we begin by recalling a well-known result for the semilinear elliptic boundary value problem

$$\begin{aligned} -\Delta u &= f(x, u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in \mathbb{R}^n , $n \geq 1$, f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the sublinear growth condition

$$|f(x, t)| \leq C(|t|^{r-1} + 1) \tag{1.2}$$

for some $r \in (1, 2)$, and C denotes a generic positive constant. Weak solutions of (1.1) coincide with the critical points of the C^1 -functional

$$\Phi(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - F(x, u), \quad u \in H_0^1(\Omega)$$

where $F(x, t) = \int_0^t f(x, s) ds$ is the primitive of f . By (1.2), Φ is bounded from below and satisfies the (PS) condition.

Assume that

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{t} = \lambda, \quad \text{uniformly a.e.}, \tag{1.3}$$

which implies $f(x, 0) = 0$ a.e. and hence (1.1) has the trivial solution $u(x) \equiv 0$. Let $\lambda_1 < \lambda_2 \leq \dots$ denote the Dirichlet eigenvalues of the negative Laplacian on Ω . If $\lambda > \lambda_1$ and is not an eigenvalue, then (1.1) has at least two nontrivial solutions.

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Indeed, if $\lambda_k < \lambda < \lambda_{k+1}$, then the (cohomological) critical groups of Φ at zero are given by

$$C^q(\Phi, 0) \approx \delta_{qk} \mathcal{G}$$

where \mathcal{G} is the coefficient group and $\delta_{\cdot, \cdot}$ denotes the Kronecker delta (see, e.g., Chang [3] or Mawhin and Willem [10]), so Φ has two nontrivial critical points by the following “three critical points theorem” of Chang [2] and Liu and Li [9].

Proposition 1.1. *Let Φ be a C^1 -functional defined on a Banach space. If Φ is bounded from below, satisfies (PS), and $C^k(\Phi, 0) \neq 0$ for some $k \geq 1$, then Φ has two nontrivial critical points.*

Remark 1.2. Li and Willem [6] used a local linking to obtain a similar result when λ is an eigenvalue and f satisfies a suitable sign condition near zero.

The above result can be extended to the corresponding p -sublinear p -Laplacian problem

$$\begin{aligned} -\Delta_p u &= f(x, u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian of u , $p \in (1, \infty)$, and f now satisfies (1.2) with $r \in (1, p)$. Then the associated variational functional

$$\Phi(u) = \int_{\Omega} \frac{1}{p} |\nabla u|^p - F(x, u), \quad u \in W_0^{1,p}(\Omega)$$

is bounded from below and satisfies (PS). Assume that

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2} t} = \lambda, \quad \text{uniformly a.e.} \tag{1.5}$$

The associated quasilinear eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

is far more complicated. It is known that the first eigenvalue λ_1 is positive, simple, and has an associated eigenfunction φ_1 that is positive in Ω (see Anane [1] and Lindqvist [7, 8]). Moreover, λ_1 is isolated in the spectrum $\sigma(-\Delta_p)$, so the second eigenvalue $\lambda_2 = \inf \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined. In the ODE case $n = 1$, where Ω is an interval, the spectrum consists of a sequence of simple eigenvalues $\lambda_k \nearrow \infty$, and the eigenfunction φ_k associated with λ_k has exactly $k - 1$ interior zeroes (see, e.g., Drábek [4]). In the PDE case $n \geq 2$, an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselskii's genus, but it is not known whether this gives a complete list of the eigenvalues.

Perera [11] used a minimax scheme involving the \mathbb{Z}_2 -cohomological index of Fadell and Rabinowitz [5] to construct a new sequence of eigenvalues $\lambda_k \nearrow \infty$ such that if $\lambda_k < \lambda < \lambda_{k+1}$ in (1.5), then

$$C^k(\Phi, 0) \neq 0$$

and hence Φ has two nontrivial critical points by Proposition 1.1. Thus, problem (1.4) has at least two nontrivial solutions when $\lambda > \lambda_1$ is not an eigenvalue from this particular sequence.

Note that (1.5) implies $tf(x, t) > 0$ for $t \neq 0$ near zero when $\lambda > 0$. Naturally we may ask whether these results hold without such a sign condition. More specifically, can we replace (1.5) with

$$\lim_{t \rightarrow 0} \frac{f(x, t)}{|t|^{p-2}t} = \lambda V(x), \quad \text{uniformly a.e.} \quad (1.6)$$

and let V change sign?

This leads us to the indefinite eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda V(x) |u|^{p-2} u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.7)$$

We assume that the weight function $V \in L^s(\Omega)$ for some

$$s \begin{cases} > n/p, & p \leq n \\ = 1, & p > n. \end{cases} \quad (1.8)$$

Then the smallest positive and largest negative eigenvalues of (1.7) are given by

$$\lambda_1^+ = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_{\Omega} V(x) |u|^p > 0}} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} V(x) |u|^p}, \quad \lambda_1^- = \sup_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_{\Omega} V(x) |u|^p < 0}} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} V(x) |u|^p},$$

respectively. Noting that (1.6) implies

$$F(x, t) = \frac{\lambda}{p} V(x) |t|^p + o(|t|^p) \quad \text{as } t \rightarrow 0, \text{ uniformly a.e.}, \quad (1.9)$$

we shall prove the following result.

Theorem 1.3. *Assume (1.2) with $r \in (1, p)$, $V \in L^s(\Omega)$ with s satisfying (1.8), and (1.9). If $\lambda \notin (\lambda_1^-, \lambda_1^+)$ and is not an eigenvalue of (1.7), then problem (1.4) has at least two nontrivial solutions.*

Since $\lambda_1^- = -\infty$ when $V \geq 0$ a.e. and $\lambda_1^+ = +\infty$ when $V \leq 0$ a.e., this theorem applies in all possible cases:

- (i) V changes sign: $\dots < \lambda_1^- < 0 < \lambda_1^+ < \dots$,
- (ii) $V \geq 0$ a.e. and $\not\equiv 0$: $-\infty = \lambda_1^- < 0 < \lambda_1^+ < \dots$,
- (iii) $V \leq 0$ a.e. and $\not\equiv 0$: $\dots < \lambda_1^- < 0 < \lambda_1^+ = +\infty$,
- (iv) $V \equiv 0$: $-\infty = \lambda_1^- < \lambda_1^+ = +\infty$ (in this case the theorem is vacuously true).

Our proof will be based on an abstract framework for indefinite eigenvalue problems introduced in Perera, Agarwal, and O'Regan [12], which we will recall in the next section.

2. PRELIMINARIES

In this section we recall an abstract framework for indefinite eigenvalue problems introduced in Perera, Agarwal, and O'Regan [12].

Let $(W, \|\cdot\|)$ be a real reflexive Banach space with the dual $(W^*, \|\cdot\|)$ and the duality pairing (\cdot, \cdot) . We consider the nonlinear eigenvalue problem

$$A_p u = \lambda B_p u \quad (2.1)$$

in W^* , where $A_p \in C(W, W^*)$ is

(A1) $(p - 1)$ -homogeneous and odd for some $p \in (1, \infty)$:

$$A_p(\alpha u) = |\alpha|^{p-2} \alpha A_p u \quad \forall u \in W, \alpha \in \mathbb{R},$$

(A2) uniformly positive: $\exists c_0 > 0$ such that

$$(A_p u, u) \geq c_0 \|u\|^p \quad \forall u \in W,$$

(A3) a potential operator: there is a functional $I_p \in C^1(W, \mathbb{R})$, called a potential for A_p , such that

$$I'_p(u) = A_p u \quad \forall u \in W,$$

(A4) of type (S): for any sequence $\{u_j\} \subset W$,

$$u_j \rightharpoonup u, \quad (A_p u_j, u_j - u) \rightarrow 0 \implies u_j \rightarrow u,$$

and $B_p \in C(W, W^*)$ is

(B1) $(p - 1)$ -homogeneous and odd,

(B2) a compact potential operator.

The following proposition is often useful for verifying (A4).

Proposition 2.1 ([12, Proposition 1.0.3]). *If W is uniformly convex and*

$$(A_p u, v) \leq r \|u\|^{p-1} \|v\|, \quad (A_p u, u) = r \|u\|^p \quad \forall u, v \in W$$

for some $r > 0$, then (A4) holds.

By [12, Proposition 1.0.2], the potentials I_p and J_p of A_p and B_p satisfying $I_p(0) = 0 = J_p(0)$ are given by

$$I_p(u) = \frac{1}{p} (A_p u, u), \quad J_p(u) = \frac{1}{p} (B_p u, u),$$

respectively, and are p -homogeneous and even. Let

$$\mathcal{M} = \{u \in W : I_p(u) = 1\}, \quad \mathcal{M}^\pm = \{u \in \mathcal{M} : J_p(u) \gtrless 0\}.$$

Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1 -Finsler manifold radially homeomorphic to the unit sphere in W , \mathcal{M}^\pm are symmetric open submanifolds of \mathcal{M} , and the positive (resp. negative) eigenvalues of (2.1) coincide with the critical values of the even C^1 -functionals

$$\Psi^\pm(u) = \frac{1}{J_p(u)}, \quad u \in \mathcal{M}^\pm$$

(see [12, Sections 9.1 and 9.2]).

Denote by \mathcal{F}^\pm the classes of symmetric subsets of \mathcal{M}^\pm and by $i(M)$ the Fadell-Rabinowitz cohomological index of $M \in \mathcal{F}^\pm$. Then

$$\lambda_k^+ := \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geq k}} \sup_{u \in M} \Psi^+(u), \quad 1 \leq k \leq i(\mathcal{M}^+),$$

$$\lambda_k^- := \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geq k}} \inf_{u \in M} \Psi^-(u), \quad 1 \leq k \leq i(\mathcal{M}^-)$$

define nondecreasing (resp. nonincreasing) sequences of positive (resp. negative) eigenvalues of (2.1) that are unbounded when $i(\mathcal{M}^\pm) = \infty$ (see [12, Theorems 9.1.2 and 9.2.1]). When $\mathcal{M}^\pm = \emptyset$, we set $\lambda_1^\pm = \pm\infty$ for convenience.

Now we consider the operator equation

$$A_p u = F'(u) \tag{2.2}$$

where $F \in C^1(W, \mathbb{R})$ with F' compact, whose solutions coincide with the critical points of the functional

$$\Phi(u) = I_p(u) - F(u), \quad u \in W.$$

The following proposition is useful for verifying the (PS) condition for Φ .

Proposition 2.2 ([12, Lemma 3.1.3]). *Every bounded (PS) sequence of Φ has a convergent subsequence.*

Suppose that $u = 0$ is a solution of (2.2) and the asymptotic behavior of F near zero is given by

$$F(u) = \lambda J_p(u) + o(\|u\|^p) \quad \text{as } u \rightarrow 0. \quad (2.3)$$

Proposition 2.3 ([12, Proposition 9.4.1]). *Assume (A1) - (A4), (B1), (B2), and (2.3) hold, F' is compact, and zero is an isolated critical point of Φ .*

- (i) *If $\lambda_1^- < \lambda < \lambda_1^+$, then $C^q(\Phi, 0) \approx \delta_{q0} \mathbb{Z}_2$.*
- (ii) *If $\lambda_{k+1}^- < \lambda < \lambda_k^-$ or $\lambda_k^+ < \lambda < \lambda_{k+1}^+$, then $C^k(\Phi, 0) \neq 0$.*

3. PROOF OF THEOREM 1.3

First let us verify that our problem fits into the abstract framework of the previous section. Let $W = W_0^{1,p}(\Omega)$,

$$(A_p u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v, \quad (B_p u, v) = \int_{\Omega} V(x) |u|^{p-2} uv,$$

and

$$F(u) = \int_{\Omega} F(x, u).$$

Then (A1) and (B1) are clear, $(A_p u, u) = \|u\|^p$ in (A2), and (A3) and (B2) hold with

$$I_p(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p, \quad J_p(u) = \frac{1}{p} \int_{\Omega} V(x) |u|^p,$$

respectively. By the Hölder inequality,

$$(A_p u, v) \leq \left(\int_{\Omega} |\nabla u|^p \right)^{1-\frac{1}{p}} \left(\int_{\Omega} |\nabla v|^p \right)^{1/p} = \|u\|^{p-1} \|v\|,$$

so (A4) follows from Proposition 2.1. By (1.2) and (1.9), (2.3) also holds.

Since $\lambda \notin (\lambda_1^-, \lambda_1^+)$ and is not an eigenvalue of (1.7), it now follows from Proposition 2.3 that $C^k(\Phi, 0) \neq 0$ for some $k \geq 1$. By (1.2),

$$|F(x, t)| \leq C(|t|^r + 1),$$

so by the Sobolev imbedding,

$$\Phi(u) \geq \frac{1}{p} \|u\|^p - C(\|u\|^r + 1) \quad \forall u \in W_0^{1,p}(\Omega).$$

Since $p > r$, it follows that Φ is bounded from below and coercive. Then every (PS) sequence of Φ is bounded and hence Φ satisfies the (PS) condition by Proposition 2.2. Thus, Φ has two nontrivial critical points by Proposition 1.1.

Remark 3.1. Note that it suffices to assume $\lambda \notin (\lambda_1^-, \lambda_1^+)$ is not an eigenvalue from the particular sequences (λ_k^{\pm}) .

REFERENCES

- [1] Aomar Anane. Simplicité et isolation de la première valeur propre du p -laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.
- [2] Kung Ching Chang. Solutions of asymptotically linear operator equations via Morse theory. *Comm. Pure Appl. Math.*, 34(5):693–712, 1981.
- [3] Kung-ching Chang. *Infinite-dimensional Morse theory and multiple solution problems*, volume 6 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston Inc., Boston, MA, 1993.
- [4] P. Drábek. *Solvability and bifurcations of nonlinear equations*, volume 264 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1992.
- [5] Edward R. Fadell and Paul H. Rabinowitz. Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.*, 45(2):139–174, 1978.
- [6] Shu Jie Li and Michel Willem. Applications of local linking to critical point theory. *J. Math. Anal. Appl.*, 189(1):6–32, 1995.
- [7] Peter Lindqvist. On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.*, 109(1):157–164, 1990.
- [8] Peter Lindqvist. Addendum: “On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ ” [Proc. Amer. Math. Soc. 109 (1990), no. 1, 157–164; MR 90h:35088]. *Proc. Amer. Math. Soc.*, 116(2):583–584, 1992.
- [9] Jia Quan Liu and Shu Jie Li. An existence theorem for multiple critical points and its application. *Kexue Tongbao (Chinese)*, 29(17):1025–1027, 1984.
- [10] Jean Mawhin and Michel Willem. *Critical point theory and Hamiltonian systems*, volume 74 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.
- [11] Kanishka Perera. Nontrivial critical groups in p -Laplacian problems via the Yang index. *Topol. Methods Nonlinear Anal.*, 21(2):301–309, 2003.
- [12] Kanishka Perera, Ravi P. Agarwal, and Donal O’Regan. *Morse Theoretic Aspects of p -Laplacian Type Operators*, volume 161 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

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