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NONLINEAR PARALLEL AND PERPENDICULAR DIFFUSION OF CHARGED COSMIC RAYS IN WEAK TURBULENCE

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ABSTRACT

The problem of particle transport perpendicular to a magnetic background field is well known in cosmic-ray astrophysics. Whereas it is widely accepted that quasi-linear theory (QLT) of particle transport does not provide the correct results for perpendicular diffusion, it was assumed for a long time that QLT is the correct theory for parallel diffusion. In the current paper we demonstrate that QLT is in general also incorrect for parallel particle transport if we consider composite turbulence geometry. Motivated through the recent success of the so-called nonlinear guiding center theory of perpendicular diffusion, we present a new theory for parallel and perpendicular diffusion of cosmic rays. This new theory is a nonlinear extension of QLT and provides us with a coupled system of nonlinear Fokker-Planck coefficients. By solving the resulting system of integral equations we obtain new results for the pitch-angle Fokker-Planck coefficient and the Fokker-Planck coefficient of perpendicular diffusion. By integrating over pitch angle we calculate the parallel and perpendicular mean free path. To our knowledge the new theory is the first that can deal with both parallel and perpendicular diffusion in agreement with simulations.

Subject headings: cosmic rays — diffusion — turbulence

1. INTRODUCTION

Recent numerical studies (Qin 2002; Qin et al. 2002a, 2002b) have demonstrated that theoretical results of quasi-linear theory (QLT; Jokipii 1966) for the parallel mean free path do not agree with results from simulations (see Minnie 2002 for a detailed review) if a general turbulence geometry is assumed. In Figure 1 we show QLT results for the parallel mean free path in comparison with simulations (Qin 2002, Fig. 6.6). It is obvious that there is a disagreement for medium and small rigidities. The well-accepted assumption that the parallel mean free path becomes a factor 5 larger if we turn from pure slab geometry to 20% slab/80% two-dimensional composite geometry seems to be incorrect except for special values of the parameters (e.g., high rigidities).

In a recent study of quasi-linear perpendicular diffusion (see Shalchi & Schlickeiser 2004b) it was demonstrated that QLT cannot be the correct theory to describe perpendicular transport in magnetostatic turbulence. So far only the nonlinear guiding center (NLGC) theory (Matthaeus et al. 2003; Shalchi et al. 2004) is able to describe cosmic-ray transport perpendicular to the magnetic background field. Motivated through the success of the NLGC theory we come to the conclusion that nonlinear effects are key input to understanding cosmic-ray perpendicular diffusion and maybe also to understanding parallel diffusion in nonslab geometries.

A further problem of QLT is particle transport in pure two-dimensional geometry. As demonstrated in Appendix A of the current paper, all Fokker-Planck coefficients are equal to zero or infinity if we consider magnetostatic two-dimensional turbulence. Consequently the parallel mean free path is zero and the perpendicular mean free path is equal to infinity (compare with Shalchi & Schlickeiser 2004b). These theoretical results also suggest that QLT is not valid for pure two-dimensional or any nonslab geometry.

In QLT we replace the particle orbit through the unperturbed particle trajectory if Fokker-Planck coefficients and transport parameters are calculated. In this paper we demonstrate that perpendicular diffusion itself is a key input if we calculate Fokker-Planck coefficients in nonslab geometries. Instead of assuming strict unperturbed motion along the magnetic background field, we allow the guiding centers to move also perpendicular to the background field. We demonstrate that nonlinear effects are indispensable if one calculates Fokker-Planck coefficients and transport parameters for general turbulence geometry. We call our new theory weakly nonlinear theory (WNLT) because the theory is based on QLT but contains nonlinear extensions.

The organization of the paper is as follows: in § 2 we discuss the correlation functions of the magnetic field fluctuations. In § 3 we derive general expressions for the nonlinear pitch-angle Fokker-Planck coefficient $D_{\mu\mu}^{\mathrm{NL}}$. This derivation is very similar to the quasi-linear derivation of Teufel & Schlickeiser (2002), but in the current paper we include nonlinear effects. In § 4 we repeat our analysis for the nonlinear Fokker-Planck coefficient of perpendicular diffusion $D_{\perp}^{\mathrm{NL}}$. The new nonlinear effects lead to a Breit-Wigner type resonance function, which is discussed in § 5. In § 6 we apply the new theory to calculate both Fokker-Planck coefficients $D_{\mu\mu}^{\mathrm{NL}}$ and $D_{\perp}^{\mathrm{NL}}$ for composite turbulence geometry. In § 7 we use these results to calculate the parallel and perpendicular mean free path. We discuss the new results and compare them with results from simulations, QLT, and NLGC.

2. THE CORRELATION FUNCTIONS OF THE MAGNETIC FIELD FLUCTUATIONS

To calculate the Fokker-Planck coefficients of spatial diffusion we need a relation between the velocity of the guiding center and the velocity of the particle. In our following discussions we consider the case of weak turbulence amplitudes ($\delta B \ll B_0$) and we assume purely magnetic fluctuations ($\delta \boldsymbol{E} = 0$). Motivated by recent simulations (Qin 2002; Qin et al. 2002a, 2002b) we assume in the current paper that guiding centers

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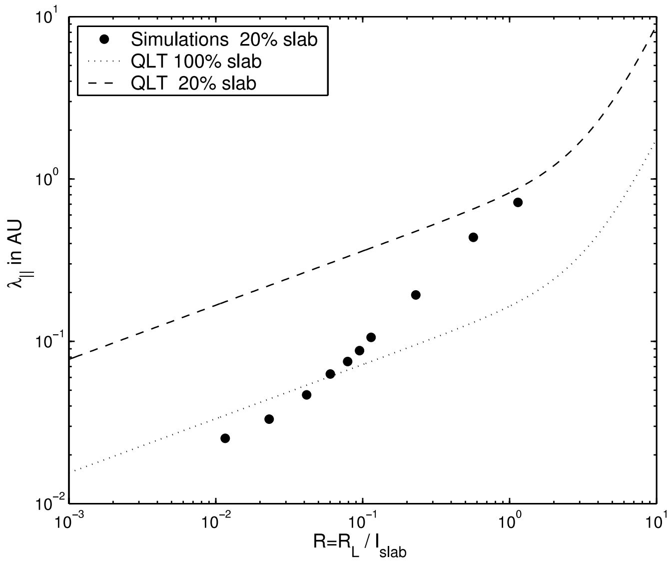

FIG. 1.—Simulations for 20% slab/80% two-dimensional composite geometry (Qin 2002; *dots*) in comparison with QLT results for the parallel mean free path. Shown are QLT results for pure slab (*dotted line*) and 20% slab/80% two-dimensional composite geometry (*dashed line*) as a function of the dimensionless rigidity $R = R_L/l_{\rm slab}$ (R_L is the gyroradius and $l_{\rm slab}$ is the slab bend-over scale).

follow field lines. As derived in Appendix B, the guiding-center velocity can then be related to the particle velocity as follows:

$$\tilde{v}_x = v_z \frac{\delta B_x}{B_0},$$
$$\tilde{v}_y = v_z \frac{\delta B_y}{B_0},$$
$$\tilde{v}_z = v_z + v_x \frac{\delta B_x}{B_0} + v_y \frac{\delta B_y}{B_0} \approx v_z. \tag{1}$$

In Matthaeus et al. (2003) the *Ansatz* $\tilde{v}_x = a v_z \delta B_x / B_0$ was used to derive the NLGC theory. Therefore we come to the conclusion that in the weak-turbulence limit the parameter a should be equal to 1. Equation (1) can be used to calculate the Fokker-Planck coefficients of perpendicular diffusion (D_{XX}, D_{YY}) and drifts (D_{XY}, D_{YX}) of the guiding center. These Fokker-Planck coefficients can be calculated using the TGK formulation (e.g., Kubo 1957):

$$D_{ij} = \mathrm{Re} \int_0^\infty dt \left\langle \tilde{v}_i(t) \tilde{v}_j^*(0) \right\rangle, \quad i, j = X, Y. \tag{2}$$

The Fokker-Planck coefficient of pitch-angle diffusion can be obtained directly from the equation of motion (see Schlickeiser 2002),

$$\dot{\mu} = \frac{\dot{p}_z}{p} = \frac{q}{cp} \left(v_x \delta B_y - v_y \delta B_x \right), \tag{3}$$

where μ is the pitch-angle cosine of the particle. In equation (3) we assume purely magnetic fluctuations, and therefore the electric fluctuations are $\delta \boldsymbol{E} = 0$. The TGK formulation can be applied again:

$$D_{\mu\mu} = \mathrm{Re} \int_0^\infty dt \langle \dot{\mu}(t) \dot{\mu}^*(0) \rangle. \tag{4}$$

All Fokker-Planck coefficients are quadratic forms of the turbulent magnetic fields δB_i. It is usual to replace these fields through Fourier transforms:

$$\delta B_i(\boldsymbol{x}, t) = \int d^3k \, \delta B_i(\boldsymbol{k}, t) e^{i\boldsymbol{k} \cdot \boldsymbol{x}}. \tag{5}$$

Therefore we must calculate the following ensemble-averaged products:

$$\left\langle \delta B_i[\boldsymbol{x}(t), t] \delta B_j^*[\boldsymbol{x}(0), 0] \right\rangle$$
$$= \int d^3k \int d^3k' \left\langle \delta B_i[\boldsymbol{k}(t), t] \delta B_j^*[\boldsymbol{k}'(0), 0] \right.$$
$$\left. \times \exp\{i[\boldsymbol{k} \cdot \boldsymbol{x}(t) - \boldsymbol{k}' \boldsymbol{x}(0)]\} \right\rangle. \tag{6}$$

To proceed it is necessary to specify the position of the particle $\boldsymbol{x}$. One possible approach would be to replace the particle position $\boldsymbol{x}$ through the position of the guiding center $\boldsymbol{x} \approx \boldsymbol{R}$. The motion of the guiding center could be considered as a purely randomized diffusive motion. This *Ansatz* was used in the derivation of NLGC theory (see Matthaeus et al. 2003). In the current paper we use

$$\boldsymbol{x} = \boldsymbol{R} - \frac{1}{\Omega} (\boldsymbol{v} \times \boldsymbol{e}_z), \tag{7}$$

with the gyrofrequency $\Omega = q B_0 / mc\gamma$. For the velocity of the particle we use spherical coordinates

$$v_x = v\sqrt{1 - \mu^2} \cos \Phi,$$
$$v_y = v\sqrt{1 - \mu^2} \sin \Phi,$$
$$v_z = v\mu, \tag{8}$$

with the pitch-angle cosine μ and the gyrophase Φ. To continue we must also specify the time dependence of $\Phi(t)$ and $\boldsymbol{R}(t) = [X(t), Y(t), Z(t)]$. In QLT the unperturbed orbit

$$\Phi_{\rm QLT}(t) = \Phi_0 - \Omega t,$$
$$X_{\rm QLT}(t) = 0,$$
$$Y_{\rm QLT}(t) = 0,$$
$$Z_{\rm QLT}(t) = v\mu t \tag{9}$$

is used to replace the exact position of the particle. In this *Ansatz* there is no motion of the guiding center in the perpendicular direction. In the current paper we use the more general assumption

$$\Phi(t) = \Phi_0 - \Omega t,$$
$$X(t) = X_{\rm GC}(t) \neq 0,$$
$$Y(t) = Y_{\rm GC}(t) \neq 0,$$
$$Z(t) = v\mu t, \tag{10}$$

where $X_{\rm GC}$ and $Y_{\rm GC}$ describe the perpendicular, nonlinear motion of the guiding center. The parameters $X_{\rm GC}(t)$ and $Y_{\rm GC}(t)$ are random variables whereas the other parameters are nonrandomized variables as in QLT. Equation (10) does not contradict any previous assumptions because the vector $\boldsymbol{X}_{\rm GC} = [X_{\rm GC}(t), Y_{\rm GC}(t), 0]$ is of higher order $\delta B / B_0$.

In the next step we substitute the particle trajectory through $\boldsymbol{x} = \boldsymbol{X}_{\rm GC} + \boldsymbol{x}_{\rm QLT}$ with

$$\boldsymbol{x}_{\rm QLT} = v\mu t \boldsymbol{e}_z - \frac{1}{\Omega} (\boldsymbol{v} \times \boldsymbol{e}_z) \tag{11}$$

to get

$$\left\langle \delta B_i[\boldsymbol{x}(t), t]\delta B_j^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$= \int d^3k \int d^3k' \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}_{\rm QLT}(t) - i\boldsymbol{k}'\cdot\boldsymbol{x}_{\rm QLT}(0)\right]$$
$$\times \left\langle \delta B_i(\boldsymbol{k}, t)\delta B_j^*(\boldsymbol{k}', 0)\exp[i\boldsymbol{k}\cdot\boldsymbol{X}_{\rm GC}(t) - i\boldsymbol{k}'\cdot\boldsymbol{X}_{\rm GC}(0)]\right\rangle. \quad (12)$$

Without the loss of generality we can set $\boldsymbol{X}_{\rm GC}(0) = \boldsymbol{0}$. To proceed we can simplify the last equation by applying Corrsin's independence hypothesis (Corrsin 1959):

$$\left\langle \delta B_i(\boldsymbol{k}, t)\delta B_j^*(\boldsymbol{k}', 0)e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{\rm GC}(t)}\right\rangle$$
$$\approx \left\langle \delta B_i(\boldsymbol{k}, t)\delta B_j^*(\boldsymbol{k}', 0)\right\rangle\left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{\rm GC}(t)}\right\rangle. \quad (13)$$

Using the tensor for homogeneous turbulence P_{ij},

$$\left\langle \delta B_i(\boldsymbol{k}, t)\delta B_j^*(\boldsymbol{k}', 0)\right\rangle = P_{ij}(\boldsymbol{k}, t)\cdot\delta(\boldsymbol{k} - \boldsymbol{k}'), \quad (14)$$

and the characteristic function

$$\Gamma_{\rm GC}(\boldsymbol{k}, t) \equiv \left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{\rm GC}(t)}\right\rangle, \quad (15)$$

which describes the nonlinear motion of the guiding center, we obtain

$$\left\langle \delta B_i[\boldsymbol{x}(t), t]\delta B_j^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$= \int d^3k\, P_{ij}(\boldsymbol{k}, t)\Gamma_{\rm GC}(\boldsymbol{k}, t)\exp\left\{i\boldsymbol{k}\cdot\left[\boldsymbol{x}_{\rm QLT}(t) - \boldsymbol{x}_{\rm QLT}(0)\right]\right\}. \quad (16)$$

Now we replace $\boldsymbol{x}_{\rm QLT}$ through (see eqs. [8], [10], and [11])

$$x_{\rm QLT}(t) = -\frac{v}{\Omega}\sqrt{1-\mu^2}\sin(\Phi_0 - \Omega t),$$
$$y_{\rm QLT}(t) = +\frac{v}{\Omega}\sqrt{1-\mu^2}\cos(\Phi_0 - \Omega t),$$
$$z_{\rm QLT}(t) = v_\parallel t, \quad (17)$$

and therefore

$$\exp\left[i\boldsymbol{k}\cdot\boldsymbol{x}_{\rm QLT}(t) - i\boldsymbol{k}\cdot\boldsymbol{x}_{\rm QLT}(0)\right]$$
$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_n(W)J_m(W)$$
$$\times \exp\left[+ik_\parallel v_\parallel t + in(\Psi - \Phi_0 + \Omega t)\right]\exp[-im(\Psi - \Phi_0)], \quad (18)$$

to find for the correlation functions of the magnetic field fluctuations

$$\left\langle \delta B_i[\boldsymbol{x}(t), t]\delta B_j^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$= \int d^3k\, P_{ij}(\boldsymbol{k}, t)\Gamma_{\rm GC}(\boldsymbol{k}, t)\sum_{n,m=-\infty}^{+\infty} J_n(W)J_m(W)$$
$$\times \exp\left[+ik_\parallel v_\parallel t + in(\Psi - \Phi_0 + \Omega t) - im(\Psi - \Phi_0)\right]. \quad (19)$$

In equations (18) and (19) we used

$$W = \frac{v}{\Omega}k_\perp\sqrt{1-\mu^2} \quad (20)$$

and cylindrical coordinates for the wavevector $\boldsymbol{k}$:

$$k_\parallel = k_z,$$
$$k_\perp = \sqrt{k_x^2 + k_y^2},$$
$$\Psi = \mathrm{arccot}(k_x/k_y). \quad (21)$$

In the following sections we use equation (19) to calculate nonlinear Fokker-Planck coefficients.

3. THE NONLINEAR PITCH-ANGLE FOKKER-PLANCK COEFFICIENT $D_{\mu\mu}^{\rm NL}$

To calculate the Fokker-Planck coefficient of pitch-angle diffusion we have to combine equations (3) and (4). The calculation of $D_{\mu\mu}^{\rm NL}$ can be simplified if we use helical coordinates for the turbulent fields:

$$\delta B_L = \frac{1}{\sqrt{2}}\left(\delta B_x + i\delta B_y\right),$$
$$\delta B_R = \frac{1}{\sqrt{2}}\left(\delta B_x - i\delta B_y\right). \quad (22)$$

Then the pitch-angle derivative can be rewritten as (see Teufel & Schlickeiser 2002)

$$\dot\mu(t) = \frac{i\Omega\sqrt{1-\mu^2}}{\sqrt{2}B_0}\left\{e^{+i\Phi(t)}\delta B_R[\boldsymbol{x}(t), t] - e^{-i\Phi(t)}\delta B_L[\boldsymbol{x}(t), t]\right\}. \quad (23)$$

For the ensemble-averaged product we therefore obtain

$$\langle\dot\mu(t)\dot\mu^*(0)\rangle = \frac{\Omega^2(1-\mu^2)}{2B_0^2}$$
$$\times\left(\left\langle\exp\{+i[\Phi(t) - \Phi(0)]\}\delta B_R[\boldsymbol{x}(t), t]\delta B_R^*[\boldsymbol{x}(0), 0]\right\rangle\right.$$
$$- \left\langle\exp\{+i[\Phi(t) + \Phi(0)]\}\delta B_R[\boldsymbol{x}(t), t]\delta B_L^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$- \left\langle\exp\{-i[\Phi(t) + \Phi(0)]\}\delta B_L[\boldsymbol{x}(t), t]\delta B_R^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$\left.+ \left\langle\exp\{-i[\Phi(t) - \Phi(0)]\}\delta B_L[\boldsymbol{x}(t), t]\delta B_L^*[\boldsymbol{x}(0), 0]\right\rangle\right). \quad (24)$$

Now we use the replacement $\Phi(t) = \Phi_0 - \Omega t$ (see eq. [10]) to obtain for the Fokker-Planck coefficient

$$D_{\mu\mu}^{\rm NL} = \frac{\Omega^2(1-\mu^2)}{2B_0^2}\mathrm{Re}\int_0^\infty dt$$
$$\times\left\{e^{-i\Omega t}\left\langle\delta B_R[\boldsymbol{x}(t), t]\delta B_R^*[\boldsymbol{x}(0), 0]\right\rangle\right.$$
$$- e^{-i\Omega t + 2i\Phi_0}\left\langle\delta B_R[\boldsymbol{x}(t), t]\delta B_L^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$- e^{+i\Omega t - 2i\Phi_0}\left\langle\delta B_L[\boldsymbol{x}(t), t]\delta B_R^*[\boldsymbol{x}(0), 0]\right\rangle$$
$$\left.+ e^{+i\Omega t}\left\langle\delta B_L[\boldsymbol{x}(t), t]\delta B_L^*[\boldsymbol{x}(0), 0]\right\rangle\right\}. \quad (25)$$

Applying equation (19) we find for the nonlinear pitch-angle Fokker-Planck coefficient

$$D_{\mu\mu}^{\rm NL} = \frac{\Omega^2(1-\mu^2)}{2B_0^2}\mathrm{Re}\int_0^\infty dt\int d^3k\,\Gamma_{\rm GC}(\boldsymbol{k}, t)$$
$$\times\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\exp\left[ik_\parallel v_\parallel t + in\Omega t + i\Psi(n-m) - i\Phi_0(n-m)\right]$$
$$\times J_n(W)J_m(W)\left[e^{-i\Omega t}P_{RR}(\boldsymbol{k}, t) - e^{-i\Omega t + 2i\Phi_0}P_{RL}(\boldsymbol{k}, t)\right.$$
$$\left.- e^{+i\Omega t - 2i\Phi_0}P_{LR}(\boldsymbol{k}, t) + e^{+i\Omega t}P_{LL}(\boldsymbol{k}, t)\right]. \quad (26)$$

Next we assume that the initial phase Φ_0 of the cosmic-ray particle is a random variable. Therefore we consider the Φ_0-averaged Fokker-Planck coefficient and use

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi_0 e^{i(n-m)\phi_0} = \delta_{n,m} \tag{27}$$

to obtain

$$D_{\mu\mu}^{\rm NL} = \frac{\Omega^2(1-\mu^2)}{2B_0^2}\,{\rm Re}\int_0^\infty dt \int d^3k \sum_{n=-\infty}^{+\infty} \Gamma_{\rm GC}(\boldsymbol{k},t) e^{i(k_\parallel v_\parallel + n\Omega)t}$$
$$\times \Big\{ P_{RR}(\boldsymbol{k},t) J_{n+1}^2(W) + P_{LL}(\boldsymbol{k},t) J_{n-1}^2(W)$$
$$- J_{n-1}(W) J_{n+1}(W) \big[P_{RL}(\boldsymbol{k},t) e^{+2i\Psi} + P_{LR}(\boldsymbol{k},t) e^{-2i\Psi} \big] \Big\}. \tag{28}$$

In the quasi-linear limit we have $\Gamma_{\rm GC} = 1$, and equation (28) is then exactly equal to the equation derived by Teufel & Schlickeiser (2002, eq. [23]).

For the tensor P_{ij} we assume the same temporal behavior for all components:

$$P_{ij}(\boldsymbol{k},t) = P_{ij}^0(\boldsymbol{k}) \cdot \Gamma_{\rm DT}(\boldsymbol{k},t), \tag{29}$$

where $\Gamma_{\rm DT}$ describes dynamical effects. Two prominent models for the function $\Gamma_{\rm DT}$ are the damping model of dynamical turbulence and the random-sweeping model (see Bieber et al. 1994). For the Fokker-Planck coefficient we then obtain

$$D_{\mu\mu}^{\rm NL} = \frac{\Omega^2(1-\mu^2)}{2B_0^2} \int d^3k \sum_{n=-\infty}^{+\infty} R_n(\boldsymbol{k})$$
$$\times \Big\{ P_{RR}^0(\boldsymbol{k}) J_{n+1}^2(W) + P_{LL}^0(\boldsymbol{k}) J_{n-1}^2(W)$$
$$- J_{n-1}(W) J_{n+1}(W) \big[P_{RL}^0(\boldsymbol{k}) e^{+2i\Psi} + P_{LR}^0(\boldsymbol{k}) e^{-2i\Psi} \big] \Big\}, \tag{30}$$

with the resonance function

$$R_n(\boldsymbol{k}) = {\rm Re}\int_0^\infty dt\, \Gamma_{\rm NL}(\boldsymbol{k},t) \Gamma_{\rm DT}(\boldsymbol{k},t) e^{i(k_\parallel v_\parallel + n\Omega)t} \tag{31}$$

and the function of nonlinearity $\Gamma_{\rm NL} = \Gamma_{\rm GC}$, which is discussed in § 5.

4. THE NONLINEAR FOKKER-PLANCK COEFFICIENT OF PERPENDICULAR DIFFUSION $D_\perp^{\rm NL}$

Applying equations (1) and (2), the Fokker-Planck coefficient of perpendicular diffusion can be written as

$$D_\perp^{\rm NL} = \frac{1}{2}\left(D_{XX}^{\rm NL} + D_{YY}^{\rm NL}\right)$$
$$= \frac{1}{2B_0^2}\,{\rm Re}\int_0^\infty dt \big\langle v_z(t) v_z(0)$$
$$\times \left[\delta B_x(t)\delta B_x(0) + \delta B_y(t)\delta B_y(0)\right]\big\rangle. \tag{32}$$

If we replace the fourth-order correlation function by a product of second-order correlation functions, we obtain

$$D_\perp^{\rm NL} = \frac{1}{2}\left(D_{XX}^{\rm NL} + D_{YY}^{\rm NL}\right)$$
$$= \frac{1}{2B_0^2}\,{\rm Re}\int_0^\infty dt \langle v_z(t) v_z(0)\rangle$$
$$\times \left[\langle\delta B_x(t)\delta B_x(0)\rangle + \langle\delta B_y(t)\delta B_y(0)\rangle\right]. \tag{33}$$

If we were to substitute the v_z-correlation function through the unperturbed orbit

$$\langle v_z(t) v_z(0)\rangle_{\rm QLT} = v^2\mu^2, \tag{34}$$

the time integration would yield an infinitely large perpendicular Fokker-Planck coefficient for two-dimensional geometry (compare with Shalchi & Schlickeiser 2004b). Therefore it is necessary to include a second nonlinear effect, namely the initial condition decay of the v_z-correlation function. In the current paper we use the model

$$\langle v_z(t) v_z(0)\rangle = v^2\mu^2\Gamma_{\rm ICD}(t), \tag{35}$$

where the function $\Gamma_{\rm ICD}(t)$, which is discussed in § 5, describes the initial condition decay. Applying equations (19) and (27) one gets

$$D_\perp^{\rm NL} = \frac{v^2\mu^2}{2B_0^2}\,{\rm Re}\int_0^\infty dt \sum_{n=-\infty}^{+\infty} \int d^3k\, \Gamma_{\rm NL}(\boldsymbol{k},t) e^{i(k_\parallel v_\parallel + n\Omega)t}$$
$$\times \left[P_{xx}(\boldsymbol{k},t) + P_{yy}(\boldsymbol{k},t)\right] J_n^2(W), \tag{36}$$

with the function of nonlinearity $\Gamma_{\rm NL} = \Gamma_{\rm GC}\Gamma_{\rm ICD}$. With equation (29) and the resonance function of equation (31), this can be written as

$$D_\perp^{\rm NL} = \frac{v^2\mu^2}{2B_0^2} \sum_{n=-\infty}^{+\infty} \int d^3k\, R_n(\boldsymbol{k}) \Big[P_{xx}^0(\boldsymbol{k}) + P_{yy}^0(\boldsymbol{k})\Big] J_n^2(W). \tag{37}$$

5. THE FUNCTION OF NONLINEARITY $\Gamma_{\rm NL}$

The key input into WNLT (weakly nonlinear theory) is the function

$$\Gamma_{\rm NL}(\boldsymbol{k},t) = \Gamma_{\rm GC}(\boldsymbol{k},t)\Gamma_{\rm ICD}(t), \tag{38}$$

with the characteristic function

$$\Gamma_{\rm GC}(\boldsymbol{k},t) = \big\langle e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{\rm GC}}\big\rangle. \tag{39}$$

If we assume that the guiding centers are distributed like a Gaussian function and that perpendicular transport is diffusive, $\Gamma_{\rm GC}$ becomes (see Appendix C)

$$\Gamma_{\rm GC}(\boldsymbol{k},t) = e^{-D_\perp k_\perp^2 t}. \tag{40}$$

In addition to the diffusive perpendicular motion of the guiding center we also include the effect of the initial condition decay. In WNLT we use the following *Ansatz* for the function $\Gamma_{\rm ICD}$:

$$\Gamma_{\rm ICD}(t) = e^{-\omega t}, \tag{41}$$

with

$$\omega = \begin{cases} \dfrac{2D_{\mu\mu}}{1-\mu^2} & \text{for perpendicular diffusion,} \\ 0 & \text{for pitch-angle diffusion.} \end{cases} \tag{42}$$

If we model the v_z-correlation function as described above, the unperturbed orbit is recovered for small times ($t \to 0$). For infinitely large times ($t \to \infty$) we obtain $\Gamma_{\rm ICD} \to 0$ and the v_z-correlation function goes to zero. Physically this model describes the pitch-angle isotropization process. In principle a similar model could be used to model the v_x- and v_y-correlation functions. In the current paper we still use unperturbed orbits to substitute these correlation functions. One reason for doing this is that in WNLT we consequently treat the gyromotion unperturbed. A second reason is that in general the v_x- and v_y-correlation functions can be dependent on pitch-angle diffusion, gyrophase diffusion, and perpendicular diffusion. Therefore a realistic model for these correlation functions is difficult to derive and could make the application of WNLT complicated.

It should be noted that the decay model of equation (41) is similar to the decay model proposed by Owens (1974). Owens's argument was that the assumption that the unperturbed orbit persists for an infinitely long time is incorrect. In reality a particle can interact resonantly with a given wavelength of the fluctuations only as long as it is not scattered to another region of phase space. In that model ω could be identified with the scattering frequency, which is related to the scattering time $t_S = \omega^{-1}$ in which the decay of the unperturbed orbit occurs.

Together with equation (40) we have

$$\Gamma_{\rm NL}(\boldsymbol{k}, t) = e^{-D_\perp k_\perp^2 t} e^{-\omega t}, \tag{43}$$

and therefore we obtain for the resonance function

$$R_n^{\rm NL,\,DT}(\boldsymbol{k}) = {\rm Re} \int_0^\infty dt\, e^{i(k_\parallel v_\parallel + n\Omega)t} e^{-D_\perp k_\perp^2 t} e^{-\omega t} \Gamma_{\rm DT}(\boldsymbol{k}, t). \tag{44}$$

Because most simulations (e.g., Qin 2002) have been using a magnetostatic turbulence model ($\Gamma_{\rm DT} = 1$), we consider this model in the current paper. For results in dynamical turbulence in the quasi-linear limit ($\Gamma_{\rm NL} = 1$), we refer to Bieber et al. (1994) and Teufel & Schlickeiser (2002, 2003) for pure slab geometry and to Shalchi & Schlickeiser (2004a, 2004b) for pure two-dimensional and composite slab/two-dimensional geometry.

For $R_n^{\rm NL}$ we then find a Breit-Wigner type resonance function:

$$\begin{aligned} R_n^{\rm NL}(\boldsymbol{k}) &= {\rm Re} \int_0^\infty dt\, e^{i(k_\parallel v_\parallel + n\Omega)t} e^{-D_\perp k_\perp^2 t} e^{-\omega t} \\ &= \frac{D_\perp k_\perp^2 + \omega}{(D_\perp k_\perp^2 + \omega)^2 + (k_\parallel v_\parallel + n\Omega)^2}. \end{aligned} \tag{45}$$

In WNLT resonance broadening comes through nonlinear effects. Further reasons for resonance broadening would be dynamical effects (Bieber et al. 1994) or plasma wave damping (Schlickeiser & Achatz 1993). Ng & Reames (1995) obtained resonance broadening by considering medium-scale fluctuations.

6. THE NONLINEAR FOKKER-PLANCK COEFFICIENTS FOR COMPOSITE GEOMETRY

In simulations a composite slab/two-dimensional model for the turbulence geometry was used. Although more general models are available (e.g., the anisotropic model of Lerche & Schlickeiser 2001), we assume composite geometry to compare theory and simulations. If equation (29) holds, the composite model is defined by

$$P_{lm}^0(\boldsymbol{k}) = P_{lm}^{0,{\rm slab}}(\boldsymbol{k}) + P_{lm}^{0,{\rm 2D}}(\boldsymbol{k}), \tag{46}$$

with the magnetostatic correlation tensor for pure slab geometry

$$P_{lm}^{0,{\rm slab}}(\boldsymbol{k}) = g^{\rm slab}(k_\parallel) \frac{\delta(k_\perp)}{k_\perp} \left(\delta_{lm} - \frac{k_l k_m}{k^2} \right) \tag{47}$$

and the magnetostatic correlation tensor for pure two-dimensional geometry

$$P_{lm}^{0,{\rm 2D}}(\boldsymbol{k}) = g^{\rm 2D}(k_\perp) \frac{\delta(k_\parallel)}{k_\perp} \begin{cases} \left(\delta_{lm} - \dfrac{k_l k_m}{k^2} \right), & l, m = x, y, \\ 0, & l \text{ or } m = z. \end{cases} \tag{48}$$

In equations (47) and (48) we assume vanishing magnetic helicity. The tensor for pure two-dimensional geometry was defined in the same way as in Bieber et al. (1994) and Bieber et al. (1996). Such a two-dimensional model assumes that $\delta B_z = 0$. This kind of two-dimensional model is slightly different from the model that was used in Shalchi & Schlickeiser (2004a), where $\delta B_z \neq 0$. The functions $g^{\rm slab}$ and $g^{\rm 2D}$ are the wave spectra for pure slab and pure two-dimensional geometries:

$$g^{\rm slab}(k_\parallel) = \frac{C(\nu)}{2\pi} l_{\rm slab} \delta B_{\rm slab}^2 \left(1 + k_\parallel^2 l_{\rm slab}^2\right)^{-\nu}, \tag{49}$$

$$g^{\rm 2D}(k_\perp) = \frac{2C(\nu)}{\pi} l_{\rm 2D} \delta B_{\rm 2D}^2 \left(1 + k_\perp^2 l_{\rm 2D}^2\right)^{-\nu}, \tag{50}$$

with

$$C(\nu) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\nu)}{\Gamma(\nu - 1/2)}. \tag{51}$$

Here we introduced the spectral index of the inertial range 2ν, the slab bend-over scale $l_{\rm slab}$, and the two-dimensional bend-over scale $l_{\rm 2D}$.

In such a composite slab/two-dimensional model all Fokker-Planck coefficients can be written as a sum of the slab coefficient and the two-dimensional coefficient:

$$\begin{aligned} D_{\mu\mu} &= D_{\mu\mu}^{\rm slab} + D_{\mu\mu}^{\rm 2D}, \\ D_\perp &= D_\perp^{\rm slab} + D_\perp^{\rm 2D}. \end{aligned} \tag{52}$$

In turn we derive an integral representation for each of the four nonlinear Fokker-Planck coefficients.

6.1. *The Fokker-Planck Coefficient* $D_{\mu\mu}^{\rm slab}$

For pitch-angle diffusion in pure slab geometry the resonance function is equal to the δ-function of QLT,

$$R_n^{\rm slab}(\boldsymbol{k}) = \pi\delta(k_\parallel v_\parallel + n\Omega), \tag{53}$$

and the pitch-angle Fokker-Planck coefficient can be written as

$$\begin{aligned} D_{\mu\mu}^{\rm slab} = \frac{2\pi^2\Omega^2(1-\mu^2)}{B_0^2} &\int_0^\infty dk_\parallel g^{\rm slab}(k_\parallel) \\ &\times \left[\delta(k_\parallel v_\parallel + \Omega) + \delta(k_\parallel v_\parallel - \Omega)\right]. \end{aligned} \tag{54}$$

To proceed we use the spectrum of equation (49). For convenience, we define the dimensionless parameters

$$\begin{aligned} R &= \frac{R_L}{l_{\rm slab}}, \\ \xi &= \frac{l_{\rm slab}}{l_{\rm 2D}}, \\ \tilde{\omega} &= \frac{l_{\rm slab}}{v}\omega, \\ \tilde{D}_{\mu\mu} &= \frac{l_{\rm slab}}{v} D_{\mu\mu}, \\ \tilde{D}_\perp &= \frac{1}{l_{\rm slab} v} D_\perp, \end{aligned} \tag{55}$$

with the gyroradius $R_L = v/\Omega$. With these parameters and by solving the $k_\parallel$-integral we find

$$\tilde{D}_{\mu\mu}^{\rm slab} = \pi C(\nu)(1-\mu^2)|\mu|^{2\nu-1} R^{2\nu-2} \frac{\delta B_{\rm slab}^2}{B_0^2} \left(\mu^2 R^2 + 1\right)^{-\nu}, \quad (56)$$

which agrees with earlier results (e.g., Shalchi et. al. 2004). Therefore QLT is recovered if we calculate the pitch-angle Fokker-Planck coefficient for pure slab geometry.

6.2. *The Fokker-Planck Coefficient* $D_{\mu\mu}^{\rm 2D}$

For pitch-angle diffusion in pure two-dimensional geometry the resonance function becomes

$$R_n^{\rm 2D}(\boldsymbol{k}) = \frac{D_\perp k_\perp^2}{(D_\perp k_\perp^2)^2 + (n\Omega)^2}. \quad (57)$$

According to Shalchi & Schlickeiser (2004a, eq. [16]) the pitch-angle Fokker-Planck coefficient for pure two-dimensional geometry has the form

$$D_{\mu\mu}^{\rm 2D} = \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2} \int_0^\infty dk_\perp \sum_{n=-\infty}^{+\infty} R_n^{\rm 2D}(\boldsymbol{k}) g^{\rm 2D}(k_\perp) \frac{n^2 J_n^2(W)}{W^2}, \quad (58)$$

with W given by equation (20). In combination with equation (57) this can be written as

$$D_{\mu\mu}^{\rm 2D} = \frac{4\pi(1-\mu^2)}{B_0^2} \int_0^\infty dk_\perp g^{\rm 2D}(k_\perp) D_\perp k_\perp^2 \times \sum_{n=1}^{\infty} \frac{n^2}{n^2 + (D_\perp k_\perp^2/\Omega)^2} \frac{J_n^2(W)}{W^2}. \quad (59)$$

With the approximation of Shalchi & Schlickeiser (2004a, eq. [19]),

$$\sum_{n=1}^{\infty} \frac{n^2}{n^2+y^2} J_n^2(x) \approx \frac{x^2}{2} \frac{1}{x^2+2y^2+2}, \quad (60)$$

we find

$$D_{\mu\mu}^{\rm 2D} = \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2} \int_0^\infty dk_\perp g^{\rm 2D}(k_\perp) \frac{D_\perp k_\perp^2}{W^2\Omega^2 + 2(D_\perp k_\perp^2)^2 + 2\Omega^2}. \quad (61)$$

Using equation (50), the parameters of equation (55), and the integral transformation $x = k_\perp l_{\rm 2D}$, we find

$$\tilde{D}_{\mu\mu}^{\rm 2D} = 2C(\nu) \frac{1-\mu^2}{R^2} \frac{\delta B_{\rm 2D}^2}{B_0^2} \int_0^\infty dx\,(1+x^2)^{-\nu} \times \frac{\tilde{D}_\perp \xi^2 x^2}{[(1-\mu^2)/2]\xi^2 x^2 + \left(\tilde{D}_\perp \xi^2 x^2\right)^2 + R^{-2}}. \quad (62)$$

The QLT result for this Fokker-Planck coefficient is $\tilde{D}_{\mu\mu}^{\rm 2D} \to 0$ (see Appendix A).

6.3. *The Fokker-Planck Coefficient* $D_\perp^{\rm slab}$

For slab geometry the Fokker-Planck coefficient of perpendicular diffusion becomes

$$D_\perp^{\rm slab} = \frac{4\pi v^2 \mu^2}{B_0^2} \int_0^\infty dk_\parallel\, g^{\rm slab}(k_\parallel) \frac{\omega}{\omega^2 + k_\parallel^2 v_\parallel^2}. \quad (63)$$

With the spectrum of equation (49), the parameters of equation (55), and the integral transformation $x = k_\parallel l_{\rm slab}$, this can be rewritten as

$$\tilde{D}_\perp^{\rm slab} = 2C(\nu)\mu^2 \frac{\delta B_{\rm slab}^2}{B_0^2} \int_0^\infty dx\,(1+x^2)^{-\nu} \frac{\tilde{\omega}}{\tilde{\omega}^2 + \mu^2 x^2}. \quad (64)$$

From this result, QLT (field-line random walk) can be recovered by taking the limit $\tilde{\omega} \to 0$,

$$\tilde{D}_\perp^{\rm FLRW} = \lim_{\tilde{\omega}\to 0} \tilde{D}_\perp^{\rm slab} = \pi C(\nu)|\mu| \frac{\delta B_{\rm slab}^2}{B_0^2}, \quad (65)$$

and we find for the perpendicular mean free path

$$\lambda_\perp^{\rm FLRW} = \frac{3\pi}{2} C(\nu) l_{\rm slab} \frac{\delta B_{\rm slab}^2}{B_0^2} = \frac{3}{4} l_c \frac{\delta B_{\rm slab}^2}{B_0^2}, \quad (66)$$

where we use $\lambda_\perp = (3/2v)\int_{-1}^{+1} d\mu D_\perp(\mu)$ and the correlation length $l_c = 2\pi C(\nu) l_{\rm slab}$. It seems that we can only obtain the FLRW limit if we suppress pitch-angle scattering and therefore parallel diffusion (compare with Qin et al. 2002a, 2002b).

6.4. *The Fokker-Planck Coefficient* $D_\perp^{\rm 2D}$

For two-dimensional geometry the Fokker-Planck coefficient of perpendicular diffusion (eq. [37]) becomes

$$D_\perp^{\rm 2D} = \frac{\pi v^2 \mu^2}{\Omega B_0^2} \int_0^\infty dk_\perp\, g^{\rm 2D}(k_\perp)\, V\, H(V, W), \quad (67)$$

where we use

$$H(V, W) = \sum_{n=-\infty}^{+\infty} \frac{J_n^2(W)}{V^2 + n^2}, \quad (68)$$

$$V = (D_\perp k_\perp^2 + \omega)/\Omega,$$
$$W = R_L k_\perp \sqrt{1-\mu^2}. \quad (69)$$

As demonstrated in Shalchi & Schlickeiser (2004b), the function H can be approximated for different cases:

$$H(V \ll 1,\ W \gg 1) \approx \frac{J_0^2(W)}{V^2} \approx \frac{2}{\pi W V^2} \cos^2\left(W - \frac{\pi}{4}\right) \approx \frac{1}{\pi W V^2},$$
$$H(V \ll 1,\ W \ll 1) \approx \frac{1}{V^2},$$
$$H(V \gg 1,\ V \ll W) \approx \frac{1}{VW},$$
$$H(V \gg 1,\ V \gg W) \approx \frac{1}{V^2}. \quad (70)$$

Using the integral transformation $x = k_\perp l_{\rm 2D}$, the parameters of equation (55), and the spectrum of equation (50), we obtain

$$\tilde{D}_\perp^{\rm 2D} = 2C(\nu)\mu^2 R \frac{\delta B_{\rm 2D}^2}{B_0^2} \int_0^\infty dx\,(1+x^2)^{-\nu}\, \tilde{V}\, H\left(\tilde{V}, \tilde{W}\right), \quad (71)$$

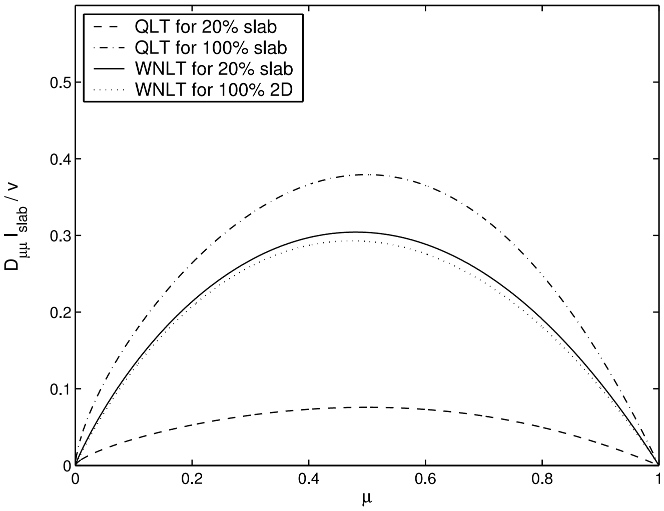

FIG. 2.—Nonlinear pitch-angle Fokker-Planck coefficient for composite geometry (*solid line*) and pure two-dimensional geometry (*dotted line*) in comparison with quasi-linear Fokker-Planck coefficients for composite geometry (*dashed line*) and pure slab geometry (*dash-dotted line*). For this plot we use $R = R_L/l_{\rm slab} = 0.1$ and $l_{\rm slab} = l_{\rm 2D} = 0.03$ AU.

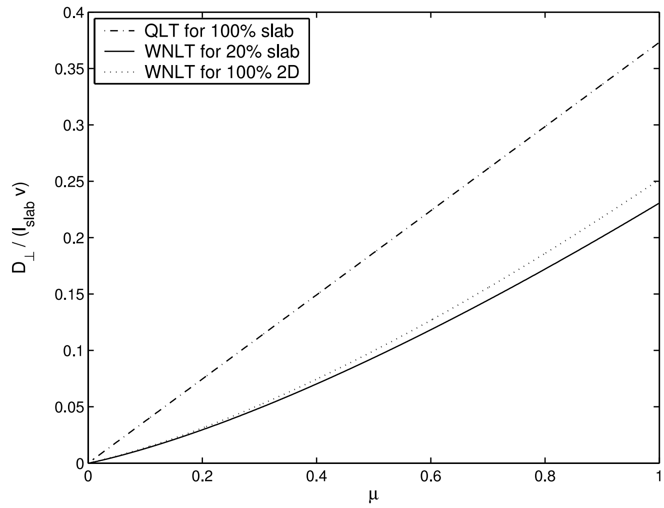

FIG. 3.—Nonlinear Fokker-Planck coefficient of perpendicular diffusion for composite geometry (*solid line*) and for pure two-dimensional geometry (*dotted line*) in comparison with the QLT slab result (FLRW; *dash-dotted line*). For this plot we use $R = R_L/l_{\rm slab} = 0.1$ and $l_{\rm slab} = l_{\bf 2D} = 0.03$ AU.

but now with

$$\tilde{V} = R\tilde{D}_\perp \xi^2 x^2 + R\tilde{\omega},$$
$$\tilde{W} = R\xi x\sqrt{1-\mu^2}. \tag{72}$$

The QLT result for this Fokker-Planck coefficient is $\tilde{D}_\perp^{\rm 2D} \to \infty$ (see Appendix A).

7. RESULTS OF WEAKLY NONLINEAR THEORY FOR THE PARALLEL AND PERPENDICULAR MEAN FREE PATH

By solving the following system of integral equations numerically,

$$\tilde{D}_{\mu\mu}^{\rm slab} = \pi C(\nu)(1-\mu^2)|\mu|^{2\nu-1}R^{2\nu-2}\frac{\delta B_{\rm slab}^2}{B_0^2}\left(\mu^2R^2+1\right)^{-\nu},$$
$$\tilde{D}_{\mu\mu}^{\rm 2D} = 2C(\nu)\frac{1-\mu^2}{R^2}\frac{\delta B_{\rm 2D}^2}{B_0^2}\int_0^\infty dx\,(1+x^2)^{-\nu} \times \frac{\tilde{D}_\perp\xi^2x^2}{[(1-\mu^2)/2]\xi^2x^2+\left(\tilde{D}_\perp\xi^2x^2\right)^2+R^{-2}},$$
$$\tilde{D}_\perp^{\rm slab} = 2C(\nu)\mu^2\frac{\delta B_{\rm slab}^2}{B_0^2}\int_0^\infty dx\,(1+x^2)^{-\nu}\frac{\tilde{\omega}}{\tilde{\omega}^2+\mu^2x^2},$$
$$\tilde{D}_\perp^{\rm 2D} = 2C(\nu)\mu^2R\frac{\delta B_{\rm 2D}^2}{B_0^2}\int_0^\infty dx\,(1+x^2)^{-\nu}\,\tilde{V}\,H\left(\tilde{V},\tilde{W}\right), \tag{73}$$

we calculate the dimensionless nonlinear Fokker-Planck coefficients $\tilde{D}_{\mu\mu}$ and $\tilde{D}_\perp$. With these results we can determine the mean free paths simply by integrating over pitch angle using the following equations (Jokipii 1966; Hasselmann & Wibberenz 1968; Earl 1974):

$$\lambda_\parallel = \frac{3}{4}l_{\rm slab}\int_0^1 d\mu\,\frac{(1-\mu^2)^2}{\tilde{D}_{\mu\mu}}, \tag{74}$$

$$\lambda_\perp = 3l_{\rm slab}\int_0^1 d\mu\,\tilde{D}_\perp. \tag{75}$$

For our numerical calculations of the parallel and perpendicular mean free path we use the following set of parameters to compare our theory with simulations:

$$\delta B = B_0,$$
$$\delta B_{\rm slab}^2 = 0.2\,\delta B^2,$$
$$\delta B_{\rm 2D}^2 = 0.8\,\delta B^2,$$
$$\nu = \frac{5}{6},$$
$$l_{\rm slab} = 0.030 \text{ AU},$$
$$l_{\rm 2D} = \frac{l_{\rm slab}}{10} = 0.003 \text{ AU}. \tag{76}$$

Some figures also show results for different parameters like pure slab geometry or pure two-dimensional geometry. In this case we explicitly write down the parameters that are different from equation (76). Strictly speaking the assumption $\delta B = B_0$ violates our weak-turbulence assumption ($\delta B \ll B_0$). Because simulations have been mostly done for $\delta B/B_0 = 1$, we apply our new theory also for this case but we keep in mind that a possible disagreement between simulations and theory can occur because we assumed the limit of weak turbulence during the derivation of the theory.

7.1. *The Nonlinear Fokker-Planck Coefficients $D_{\mu\mu}^{\rm NL}$ and $D_\perp^{\rm NL}$*

Figure 2 shows the results for the pitch-angle Fokker-Planck coefficient $D_{\mu\mu}^{\rm NL}$. All pitch-angle Fokker-Planck coefficients have the same form, which is a surprising result. Normally the form of the pitch-angle Fokker-Planck coefficient results from gyroresonance. It seems that WNLT provides us with similar results even in pure two-dimensional geometry, where we do not have gyroresonance. The QLT result for composite geometry is exactly a factor of 5 smaller than the QLT result for pure slab geometry, whereas the WNLT result for composite geometry is nearly equal to the QLT slab result. Figure 3 shows the pitch-angle dependence of the Fokker-Planck coefficient of perpendicular diffusion $D_\perp^{\rm NL}$ in comparison with the slab QLT result (FLRW). The QLT result for composite geometry is $D_\perp \to \infty$ (see also Shalchi & Schlickeiser 2004b).

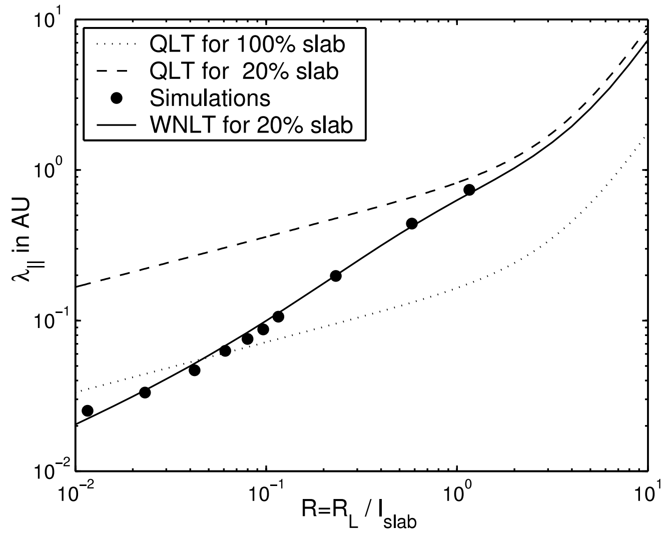

FIG. 4.—Parallel mean free path WNLT results (*solid line*) in comparison with QLT results (*dashed line*) and simulations (Qin 2002; *dots*). All results are for 20% slab/80% two-dimensional geometry. We also show the well known QLT results for pure slab geometry (*dotted line*).

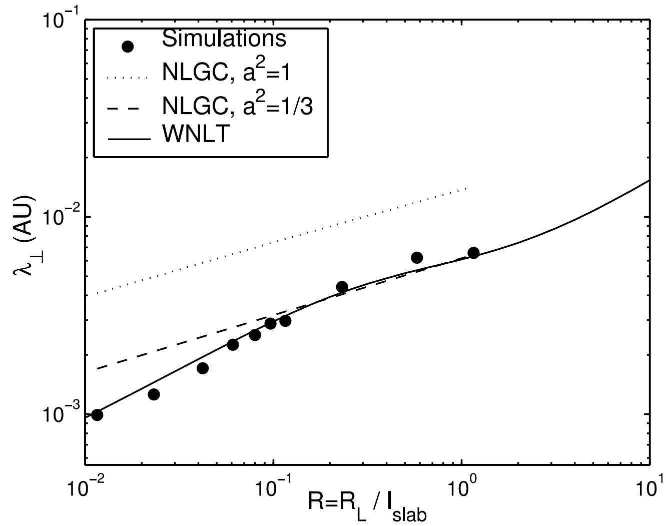

FIG. 5.—Perpendicular mean free path WNLT results (*solid line*) in comparison with NLGC results (*dotted and dashed lines*) and simulations (Qin 2002; *dots*). All results are for 20% slab/80% two-dimensional geometry.

7.2. $\lambda_{\parallel}$, $\lambda_{\perp}$, *and* $\lambda_{\perp}/\lambda_{\parallel}$ *for Composite Geometry*

Figure 4 shows simulations of the parallel mean free path (Qin 2002; Qin et al. 2002a, 2002b; *dots*), magnetostatic QLT results for pure slab geometry (*dotted line*), QLT results for 20% slab geometry (*dashed line*), and the WNLT results for 20% slab geometry (*solid line*). As demonstrated, the WNLT agrees much better with simulations than QLT. The rigidity dependence for small and medium rigidities is approximately $\lambda_{\parallel} \sim R^{0.6}$, which is in contrast with the QLT results ($\lambda_{\parallel} \sim R^{1/3}$) but in agreement with the simulations (see Minnie 2002). For high rigidities ($R \gg 1$), QLT seems to be recovered. Our numerical calculations have also shown that the reason for the difference between QLT and simulations arises from the perpendicular diffusion of the particles. Therefore we come to the conclusion that perpendicular diffusion has a strong influence on the parallel mean free path.

It is a well-known result that the pure slab QLT result gives a too short parallel mean free path compared to the energetic particle observations (Palmer 1982). Therefore the composite slab/two-dimensional model was used in Bieber et al. (1994) to solve this problem. To achieve agreement between QLT and the Palmer observations, dynamical effects (damping model of dynamical turbulence) and a dissipation range were also included. A comparison between our new theoretical results, which suggest that the parallel mean free path is considerably shorter for composite geometry, is however premature because a dissipationless magnetostatic model was used in the current paper. Because of the importance of dissipation and dynamical effects, a detailed comparison between WNLT results and observations will be the subject of future work.

In Figure 5 we show the new results for the perpendicular mean free path. Shown are the results of WNLT (*solid line*) in comparison to simulations (*dots*) and results of the NLGC theory. To calculate $\lambda_{\perp}$ with the NLGC theory we need the parallel mean free path as an input. In the current paper we use the parallel mean free path from simulations (see Fig. 4). The NLGC theory contains a fitting-parameter a. We compare our results with NLGC results for two different values of a: the value $a = 1$ (*dotted line*) is correct in the weak turbulence limit, whereas the value $a = 1/\sqrt{3}$ (*dashed line*) provides the best agreement with simulations.

Figure 6 shows the ratio $\lambda_{\perp}/\lambda_{\parallel}$. From observations, we know (e.g., Palmer 1982) that this ratio should be $\lambda_{\perp}/\lambda_{\parallel} \sim 0.01$. Our new theory can explain these observational results and agrees also with simulations. Except for NLGC theory, no other theory is able to explain the observational results of Palmer (1982) for $\lambda_{\perp}/\lambda_{\parallel}$. There are recent reports that $\lambda_{\perp}/\lambda_{\parallel}$ could approach or exceed unity (Dwyer et al. 1997; Zhang et al. 2003). Like NLGC theory (see Bieber et al. 2004), our new theory can predict rather large values of $\lambda_{\perp}/\lambda_{\parallel}$ in certain parameter regimes. As noted before, a detailed comparison between theoretical results of WNLT and observations will be the subject of future work.

7.3. *The Parallel Mean Free Path as a Function of* $\delta B^2_{\rm slab}/\delta B^2$

In Figure 7 we show the parallel mean free path as a function of the ratio $\delta B^2_{\rm slab}/\delta B^2 = \delta B^2_{\rm slab}/(\delta B^2_{\rm slab} + \delta B^2_{\rm 2D})$. It is easy to

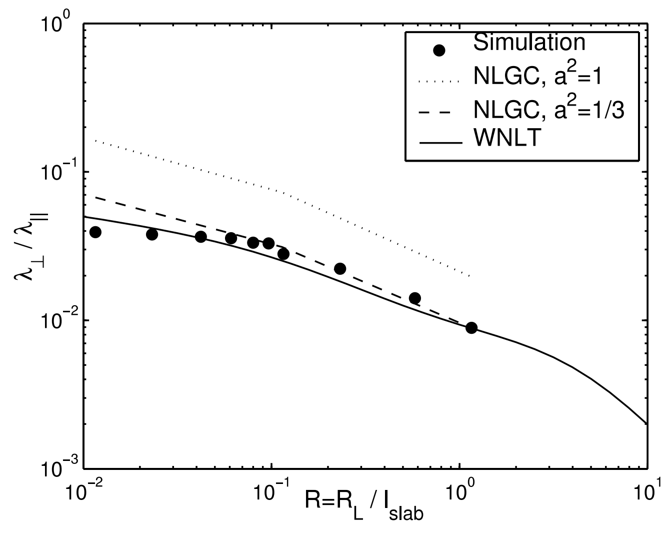

FIG. 6.—Ratio $\lambda_{\perp}/\lambda_{\parallel}$ WNLT results (*solid line*) in comparison with NLGC results (*dotted and dashed lines*) and simulations (Qin 2002; *dots*). All results are for 20% slab/80% two-dimensional geometry.

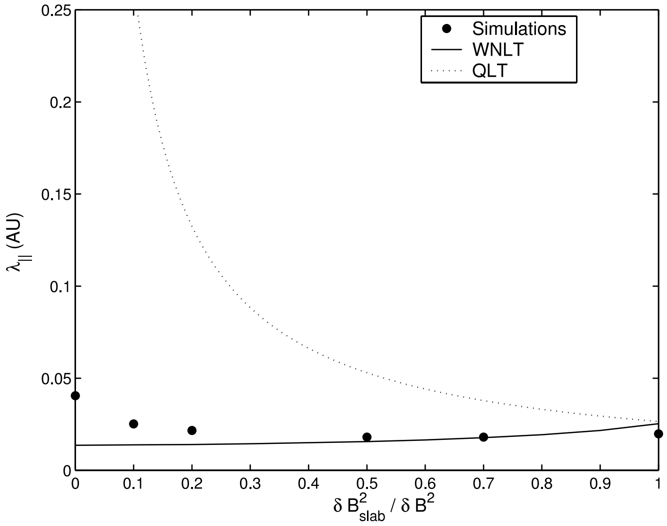

FIG. 7.—Parallel mean free path as a function of the ratio $\delta B^2_{\rm slab}/\delta B^2$ for moderate-amplitude turbulence ($\delta B/B_0 = 1$). Shown are results of our weakly nonlinear theory (*solid line*), simulations (*dots*), and QLT (*dotted line*). For pure two-dimensional geometry ($\delta B^2_{\rm slab}/\delta B^2 = 0$), the QLT prediction is $\lambda_\parallel = \infty$. To obtain these results we used $R = 0.005$.

recognize that QLT disagrees with simulations for nonslab turbulence geometries. The agreement of our nonlinear theory with simulations is much better. Only for $\delta B^2_{\rm slab}/\delta B^2 \leq 0.1$ does WNLT not seem to be very accurate. The key result of Figure 7 is that nonlinear effects are essential to describe parallel particle transport in nonslab turbulence.

7.4. *The Parallel Mean Free Path for Nearly Pure Two-dimensional Geometry*

In Figure 8 we show the parallel mean free path as a function of rigidity. Shown are the results for 1% slab/99% two-dimensional composite geometry. The disagreement between our nonlinear theory and QLT is maximal for pure or nearly pure two-dimensional geometry. The QLT prediction for pure two-dimensional geometry is $\lambda_\parallel = \infty$ because in this case we have $D^{\rm 2D}_{\mu\mu}(\mu) = 0$ if we assume magnetostatic turbulence (see Shalchi & Schlickeiser 2004a and Appendix A of the current paper).

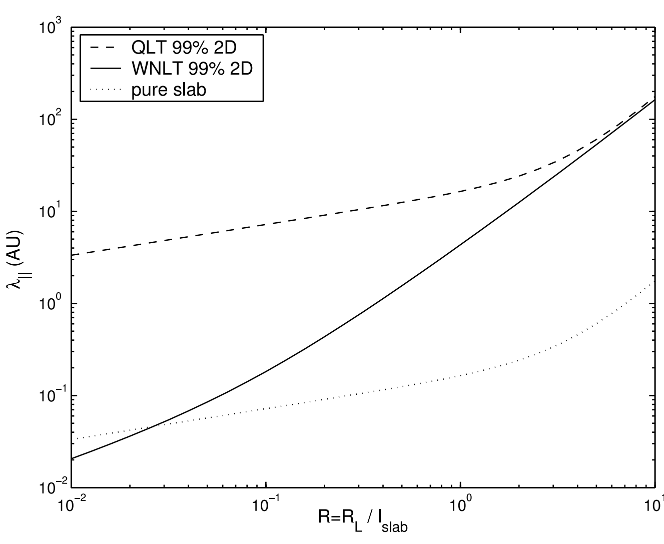

FIG. 8.—Parallel mean free path for 99% two-dimensional geometry. Shown are the weakly nonlinear theory results (*solid line*) in comparison with the QLT prediction (*dashed line*) and pure slab results (*dotted line*).

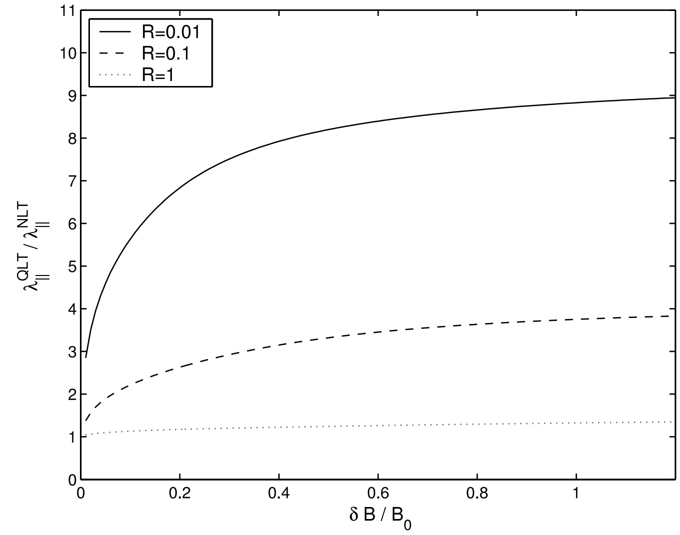

FIG. 9.—Ratio $\lambda_\parallel^{\rm QLT}/\lambda_\parallel^{\rm WNLT}$ as a function of $\delta B/B_0$ for $R = 0.01$, 0.1, 1 and composite geometry. The disagreement between QLT and WNLT becomes smaller for smaller values of $\delta B/B_0$.

Mainly in nearly or pure two-dimensional geometry is QLT not valid. The reason for that is that in pure two-dimensional we no longer have sharp gyroresonance. Nonlinear effects control pitch-angle diffusion for such a turbulence geometry.

7.5. *The Parallel Mean Free Path as a Function of $\delta B/B_0$*

As demonstrated in Figure 4 QLT seems to be incorrect if we assume composite geometry. In this section we calculate the parallel mean free path for different values of $\delta B/B_0$ to find out whether the disagreement between QLT and WNLT becomes smaller for smaller values of $\delta B/B_0$. In Figure 9 we display the ratio $\lambda_\parallel^{\rm QLT}/\lambda_\parallel^{\rm WNLT}$ as a function of $\delta B/B_0$ for different values of the dimensionless rigidity R. It is apparent that even in the limit of very weak turbulence, QLT is still not recovered. Only for extremely weak turbulence ($\delta B/B_0 \ll 0.01$) could QLT be correct, but this limit is not of physical interest.

As demonstrated in the previous subsections, QLT is never valid for pure two-dimensional geometry independent of whether the turbulence is strong or weak. As noted before, QLT is also more accurate for high rigidities. For $\delta B > B_0$ our WNLT could become more and more inaccurate. In such a strong turbulence regime equation (74) could become invalid and strong turbulence theories like NLGC theory might provide a more accurate description of particle transport.

7.6. *The Parallel Mean Free Path for Different Values of the Two-dimensional Bend-over Scale*

In this subsection we calculate the parallel mean free path for different values of the two-dimensional bend-over scale $l_{\rm 2D}$. In QLT and in the magnetostatic limit, the parallel mean free path is independent of $l_{\rm 2D}$. In Figure 10 we demonstrate that in WNLT $\lambda_\parallel$ is strongly dependent on the two-dimensional bend-over scale. For $l_{\rm 2D} = 0.1\, l_{\rm slab}$, QLT seems to be accurate for high rigidities. For larger values of $l_{\rm 2D}/l_{\rm slab}$ QLT is not valid for any values of the rigidity. It seems that in general the two-dimensional bend-over scale has a strong influence on the parallel mean free path if we use WNLT.

8. SUMMARY AND CONCLUSION

With this paper we have presented the weakly nonlinear theory (WNLT) for diffusion of charged particles. This new

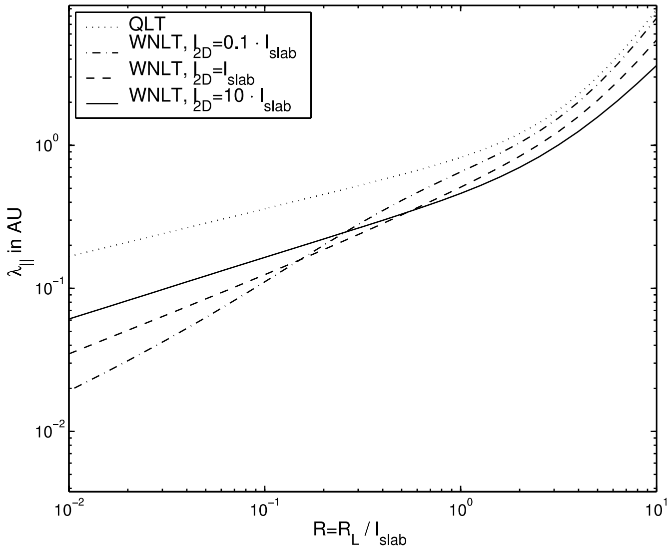

FIG. 10.—Parallel mean free path for different values of the two-dimensional bend-over scale and composite geometry. In WNLT we obtain different results for different values of l_{2D}/l_{slab}. In QLT the parallel mean free path is independent of the two-dimensional bend-over scale if we consider magnetostatic turbulence.

theory includes the effect of perpendicular diffusion and the initial condition decay of the v_z-correlation function. We have demonstrated that these nonlinear effects are indispensable if we calculate Fokker-Planck coefficients and transport parameters for general turbulence geometry. Only in a few cases (e.g., extremely weak turbulence, high rigidities) is QLT recovered. The new theory provides us with a coupled system of Fokker-Planck coefficients. To the best of our knowledge this is the first time that such a theory has been presented. In this paper we have come to the conclusion that for the understanding of cosmic-ray parallel and perpendicular transport, nonlinear effects are a key input and cannot be neglected. In general the quasi-linear approach, which neglects all nonlinear effects, does not provide a reasonable description of particle transport.

We have demonstrated that for 20% slab/80% two-dimensional geometry the parallel mean free path for small and medium rigidities is also controlled by the nonlinear two-dimensional Fokker-Planck coefficient $D_{\mu\mu}^{2D}$. We have shown that the nonlinear parallel mean free path is smaller than the QLT result and that the rigidity dependence is different. The WNLT result ($\lambda_\parallel \sim R^{0.6}$) is in contrast to the well-known QLT result ($\lambda_\parallel \sim R^{1/3}$). In this paper we come to the conclusion that the strength of the turbulence and turbulence geometry determine whether QLT is accurate or not if we calculate the parallel mean free path.

One important conclusion of this paper is that the new theory can solve the problem of perpendicular diffusion. The current theory is the first weak-turbulence approach that provides us with an accurate description of perpendicular cosmic-ray transport. In the past only the NLGC theory (Matthaeus et al. 2003; Shalchi et al. 2004) was able to reproduce perpendicular diffusion coefficients obtained from simulations. The NLGC theory describes particle transport in a highly nonlinear regime where the displacements that describe the statistical motion of the particles are pitch-angle independent. In this approach the decorrelation from unperturbed orbits occurs rapidly. In the current theory we still have a pitch-angle dependent description of particle transport and we still substitute some parameters through the unperturbed orbit. We expect that the current theory is valid in the weak-turbulence regime ($\delta B \ll B_0$) whereas NLGC theory should provide accurate results if the turbulence is strong ($\delta B \gg B_0$). A more detailed discussion of particle transport in these different regimes will be the subject of future work.

Because of the accurate agreement between WNLT and simulations, we also expect completely new and more accurate results for momentum diffusion and drifts. As demonstrated in, e.g., Schlickeiser (2002), momentum diffusion is controlled by the Fokker-Planck coefficients D_{pp}, $D_{\mu p}$, and $D_{\mu\mu}$, whereas drifts are controlled by the coefficients D_{XY} and D_{YX}. It is straightforward to include nonlinear effects if we calculate these Fokker-Planck coefficients. The WNLT can be obtained from QLT by the formal substitution

$$\mathrm{QLT} \quad \rightarrow \quad \mathrm{WNLT},$$

$$\pi\delta\left(k_\parallel v_\parallel + n\Omega\right) \rightarrow \frac{D_\perp k_\perp^2 + \omega}{\left(D_\perp k_\perp^2 + \omega\right)^2 + \left(k_\parallel v_\parallel + n\Omega\right)^2}. \quad (77)$$

Another subject of future work will be to include dynamical effects into WNLT. Then it will be necessary to revisit the Palmer consensus (Palmer 1982; Bieber et al. 1994) in light of these new theoretical results. It will also be a simple matter to include plasma wave propagation effects or plasma wave damping into the new theory. It will also be a subject of future work to include additional nonlinear effects into WNLT in order to achieve further improvement of the new theory.

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APPENDIX A

QUASI-LINEAR FOKKER-PLANCK COEFFICIENTS IN PURE TWO-DIMENSIONAL GEOMETRY

In this appendix we consider particle transport in pure two-dimensional geometry in the QLT approximation. It can easily be demonstrated that all 36 Fokker-Planck coefficients can be written as (e.g., Schlickeiser 2002)

$$D_{ij}^{\mathrm{MS,\,2D}} = \int d^3k \sum_{n=-\infty}^{+\infty} f_{ij}(n, \boldsymbol{k})\, \delta\left(k_\parallel v_\parallel + n\Omega\right), \quad (\mathrm{A1})$$

with $i,j = X, Y, Z, \mu, \Phi, p$, if we consider the special case of magnetostatic (MS) turbulence and if we neglect the plasma wave dispersion relation and any nonlinear effects. For pure two-dimensional geometry, f_{ij} has the form

$$f_{ij}(n, \boldsymbol{k}) = h_{ij}(n, k_\perp)\, \delta\big(k_\parallel\big). \tag{A2}$$

Therefore all Fokker-Planck coefficients can be written as

$$D_{ij}^{\mathrm{MS,\,2D}} = \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{+\infty} dk_\parallel\, \delta(k_\parallel)\delta\left(k_\parallel + \frac{n\Omega}{v_\parallel}\right). \tag{A3}$$

Applying

$$\int_{-\infty}^{+\infty} dx\, \delta(x)\delta(x+a) = \delta(a), \tag{A4}$$

we find

$$D_{ij}^{\mathrm{MS,\,2D}} \sim \sum_{n=-\infty}^{\infty} a_n\, \delta(n\Omega). \tag{A5}$$

For the Fokker-Planck coefficient of perpendicular diffusion we have $a_0 \neq 0$ (see Shalchi & Schlickeiser 2004b) and therefore

$$D_\perp^{\mathrm{MS,\,2D}} \sim \delta(0) \to \infty. \tag{A6}$$

The pitch-angle Fokker-Planck coefficient has the property $a_n \sim n^2$. A rigorous discussion of products such as $n^2\delta(n\Omega)$ is not subject of the current paper, but it can be shown that

$$D_{\mu\mu}^{\mathrm{MS,\,2D}} \to 0 \tag{A7}$$

in magnetostatic two-dimensional turbulence. This result was also obtained by Bieber et al. (1994) and Shalchi & Schlickeiser (2004a). It is straightforward to demonstrate that all 36 Fokker-Planck coefficients $D_{ij}^{\mathrm{MS,\,2D}}$ are equal to zero or infinity. Because of these results we come to the conclusion that QLT cannot describe particle transport in pure two-dimensional geometry. Therefore it seems necessary to include nonlinear effects if we calculate Fokker-Planck coefficients in pure two-dimensional geometry and in the magnetostatic limit.

APPENDIX B

RELATION BETWEEN GUIDING-CENTER VELOCITY AND PARTICLE VELOCITY

In this appendix we derive a relation between the velocity of the guiding center and the velocity of the particle. We need such a relation to calculate Fokker-Planck coefficients of spatial diffusion. Assuming that there is a magnetic background field (mean field) that points into the z-direction of our coordinate system, we derive such a relation under the assumption of weak turbulence ($\delta B/B_0 \ll 1$) and purely magnetic fluctuations ($\delta E = 0$). To lowest order in $\delta B/B_0$ we have

$$\boldsymbol{R} = \boldsymbol{x} + \frac{c}{qB_0}(\boldsymbol{p} \times \boldsymbol{e}_z), \tag{B1}$$

where $\boldsymbol{R}$ is the position of the guiding center and $\boldsymbol{x}$ the position of the particle. This equation can be used to calculate the speed vector of the guiding center:

$$\tilde{\boldsymbol{v}} \equiv \dot{\boldsymbol{R}} = \boldsymbol{v} + \frac{c}{qB_0}(\dot{\boldsymbol{p}} \times \boldsymbol{e}_z) = \boldsymbol{v} + \frac{1}{B_0}[(\boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{e}_z] = v_z \frac{\boldsymbol{B}}{B_0} - \boldsymbol{v}\frac{\delta B_z}{B_0}, \tag{B2}$$

where we use the equation of motion

$$\dot{\boldsymbol{p}} = \frac{q}{c}(\boldsymbol{v} \times \boldsymbol{B}). \tag{B3}$$

Therefore we can express the components $\tilde{v}_i$ of the guiding-center speed vector through the components of the particle speed vector v_i and the magnetic field (see, e.g., Schlickeiser 2002):

$$\begin{aligned} \tilde{v}_x &= v_z \frac{\delta B_x}{B_0} - v_x \frac{\delta B_z}{B_0}, \\ \tilde{v}_y &= v_z \frac{\delta B_y}{B_0} - v_y \frac{\delta B_z}{B_0}, \\ \tilde{v}_z &= v_z. \end{aligned} \tag{B4}$$

This derivation is not accurate because corrections of first-order $\delta B/B_0$ in equation (B1) can change the results (eq. [B4]).

In the current paper we assume that guiding centers follow field lines. Under this assumption we have

$$\tilde{\boldsymbol{v}} \equiv \dot{\boldsymbol{R}} = \alpha \boldsymbol{B}. \tag{B5}$$

The scalar α can be calculated from the projection of the guiding-center velocity onto the total magnetic field:

$$\tilde{\boldsymbol{v}} \cdot \boldsymbol{B} = \alpha B^2. \tag{B6}$$

This projection must be equal to the projection of the particle velocity onto the total magnetic field,

$$\tilde{\boldsymbol{v}} \cdot \boldsymbol{B} = \boldsymbol{v} \cdot \boldsymbol{B} = \alpha B^2, \tag{B7}$$

and we have $\alpha = (\boldsymbol{v} \cdot \boldsymbol{B})/B^2$. Thus the guiding-center velocity can be related to the particle velocity as

$$\tilde{\boldsymbol{v}} = \frac{(\boldsymbol{v} \cdot \boldsymbol{B})}{B^2} \boldsymbol{B}. \tag{B8}$$

Next we write the total field as $\boldsymbol{B} = \delta \boldsymbol{B} + B_0 \boldsymbol{e}_z$ and assume $\delta B \ll B_0$ to find

$$\frac{1}{B^2} \approx \frac{1}{B_0^2} \left(1 - 2\frac{\delta B_z}{B_0}\right). \tag{B9}$$

In lowest order of $\delta B/B_0$ we then find for the components of the guiding-center velocity vector

$$\begin{aligned} \tilde{v}_x &= v_z \frac{\delta B_x}{B_0}, \\ \tilde{v}_y &= v_z \frac{\delta B_y}{B_0}, \\ \tilde{v}_z &= v_z + v_x \frac{\delta B_x}{B_0} + v_y \frac{\delta B_y}{B_0}. \end{aligned} \tag{B10}$$

Analogous to Schlickeiser (2002), we can define the force terms g_i,

$$\begin{aligned} g_X &\equiv \tilde{v}_x = v_z \frac{\delta B_x}{B_0}, \\ g_Y &\equiv \tilde{v}_y = v_z \frac{\delta B_y}{B_0}, \\ g_Z &\equiv \tilde{v}_z - v_z = v_x \frac{\delta B_x}{B_0} + v_y \frac{\delta B_y}{B_0} \neq 0, \end{aligned} \tag{B11}$$

to calculate Fokker-Planck coefficients of spatial diffusion. The other three force terms (g_μ, g_Φ, g_p) can be simply derived from the equation of motion. If we consider equation (B11) we notice that there is also a Fokker-Planck coefficient $D_{ZZ} \neq 0$ that could be defined through

$$D_{ZZ} = \mathrm{Re} \int_0^\infty dt \left\langle g_Z(t) g_Z^*(0) \right\rangle. \tag{B12}$$

In the current paper we neglect D_{ZZ} because we assume that in the limit of weak turbulence this additional Fokker-Planck coefficient is not important.

APPENDIX C

THE CHARACTERISTIC FUNCTION Γ_{GC}

In the current paper we assume that the guiding centers are distributed like a Gaussian function:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma_x^2}\right] \exp\left[-\frac{(y - \langle y \rangle)^2}{2\sigma_y^2}\right]. \tag{C1}$$

With $\langle x \rangle = \langle y \rangle = 0$ the function Γ_{GC} can be written as

$$\begin{aligned} \Gamma_{\mathrm{GC}}(\boldsymbol{k}, t) &= \left\langle e^{i\boldsymbol{k} \cdot \boldsymbol{X}_{\mathrm{GC}}} \right\rangle = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, f(x, y) e^{ik_x x + ik_y y} \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, e^{-\left(x^2/2\sigma_x^2\right)} e^{-\left(y^2/2\sigma_y^2\right)} e^{ik_x x} e^{ik_y y}. \end{aligned} \tag{C2}$$

The width of the Gaussian function can be identified with $\sigma_x^2 = \langle(\Delta x)^2\rangle$ and $\sigma_y^2 = \langle(\Delta y)^2\rangle$. Therefore we find

$$\Gamma_{GC}(\boldsymbol{k}, t) = \frac{1}{2\pi\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta y)^2\rangle}} \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp\left[-\frac{x^2}{2\langle(\Delta x)^2\rangle}\right] e^{ik_x x} \exp\left[-\frac{y^2}{2\langle(\Delta y)^2\rangle}\right] e^{ik_y y}. \quad \text{(C3)}$$

These integrals are elementary and yield

$$\Gamma_{GC}(\boldsymbol{k}, t) = \exp\left[-\frac{\langle(\Delta x)^2\rangle}{2} k_x^2\right] \exp\left[-\frac{\langle(\Delta y)^2\rangle}{2} k_y^2\right]. \quad \text{(C4)}$$

To express $\langle(\Delta x)^2\rangle$ and $\langle(\Delta y)^2\rangle$ through Fokker-Planck coefficients we assume diffusive perpendicular transport. As noted before (Jokipii et al. 1993; Kota & Jokipii 2000; Qin 2002; Qin et al. 2002a, 2002b), there are circumstances in which perpendicular transport is subdiffusive. Because of assuming diffusion perpendicular to the background field, the theory of the current paper becomes incorrect if perpendicular transport is subdiffusive or superdiffusive.

For the present we simply assume diffusion and proceed as follows:

$$\frac{\langle(\Delta x)^2\rangle}{2} \approx D_{xx} t,$$
$$\frac{\langle(\Delta y)^2\rangle}{2} \approx D_{yy} t, \quad \text{(C5)}$$

where we use the Fokker-Planck coefficients of perpendicular diffusion D_{xx} and D_{yy}. Therefore we finally find

$$\Gamma_{GC}(\boldsymbol{k}, t) = \exp\left[-\left(D_{xx} k_x^2 + D_{yy} k_y^2\right) t\right]. \quad \text{(C6)}$$

If we assume axisymmetric turbulence, the function Γ_{GC} becomes

$$\Gamma_{GC}(\boldsymbol{k}, t) = e^{-D_\perp k_\perp^2 t}, \quad \text{(C7)}$$

where we use $D_\perp = D_{xx} = D_{yy}$. Note that QLT is recovered if we use

$$f_{QLT}(x, y) = \delta(x)\delta(y) \quad \text{(C8)}$$

instead of a Gaussian function. Then we would have $\Gamma_{GC} = 1$. As demonstrated, QLT is only recovered for pure slab geometry or if $D_\perp$ can be neglected.

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