Computational Method to Calculate Output Current for Gated SPAD to Determine DCR and PDE

Ahmad Salih Azzahrani

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Computational Method to Calculate Output Current for Gated SPAD to Determine DCR and PDE

by
Ahmad Salih Azzahrani

A dissertation submitted to the College of Engineering and Science at Florida Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering

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We the undersigned committee hereby recommend that the attached document be accepted as fulfilling in part the requirements for the degree of Doctor of philosophy of Electrical Engineering.

“Computational Method to Calculate Output Current for Gated SPAD to Determine DCR and PDE”
a dissertation by Ahmad Salih Azzahrani

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Abstract

Title: Computational Method to Calculate Output Current for Gated SPAD to Determine DCR and PDE.

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The problem of detecting a single-photon is receiving considerable attention with the development of new single-photon detectors in stressed free-space optical communication, low probability of intercept (LPI) optical communication, quantum communication, and remote sensing. The theoretical modeling of the waveforms of the single-photon detector output produced in particular systems will expedite the search for and analysis of detected signals.

A mathematical model of the output current of a single-photon avalanche detector (SPAD) gated with sinusoidal and Gaussian signals is developed and discussed. The mathematical model is combined with existing circuit and physical models to produce a compact model of the SPAD. Compact modeling of single-photon detectors allows the performance of single-photon detectors to be further improved. This compact model eliminates the ambiguous calculation of the output
current, dark count rate, and photon detection efficiency produced by the single-photon avalanche detector.
# Table of Contents

List of Figures ............................................................................................................. vii

1. **Introduction** ......................................................................................................... 1
   1.1. Single-photon Detector Types ........................................................................... 2
       1.1.1. First Generation: Photomultiplier Tubes (PMTs) ..................................... 2
       1.1.2. Second Generation: Avalanche Photodiode (APD) and Single-photon Avalanche Detectors (SPADs) ............................................................. 4
       1.1.3. Third Generation: Superconducting Nanowire Single-photon Detectors (SNSPD) .................................................................................................. 5
   1.2. SPAD Performance Parameters ........................................................................ 6
       1.2.1. Photon Detection Efficiency (PDE) ......................................................... 6
       1.2.2. Dark Count Rate (DCR) .......................................................................... 7
       1.2.3. Afterpulsing Probability (AP) ................................................................. 8
       1.2.4. Jitter Time ($t_j$) ....................................................................................... 9
       1.2.5. Operating Temperature (T) ................................................................. 11

2. **Background** ........................................................................................................ 13

3. **Proposed Method** ............................................................................................. 23
   3.1. Sinusoidal- Zero input ..................................................................................... 29
       3.1.1. Dark count rate modeling ........................................................................ 34
3.1.2. Photon detection efficiency modeling ......................................................... 34

3.2. Sinusoidal- Nonzero input .................................................................................. 35

3.2.1. Dark count rate modeling ............................................................................. 42

3.2.2. Photon detection efficiency modeling ......................................................... 43

3.3. Gaussian- Zero input .......................................................................................... 44

3.3.1. Dark count rate modeling ............................................................................. 53

3.3.2. Photon detection efficiency modeling ......................................................... 54

3.4. Gaussian- Nonzero input .................................................................................... 56

3.4.1. Dark count rate modeling ............................................................................. 76

3.4.2. Photon detection efficiency modeling ......................................................... 77

3.5. Comparing mathematical results with experimental ....................................... 78

4. Conclusion ........................................................................................................... 86

5. References ........................................................................................................... 87
List of Figures

Fig. 1.1 PMT structure (Midgall 2013) ........................................................................... 3
Fig. 1.2 APD and SPAD operating regions ....................................................................... 4
Fig. 1.3 Superconducting nanowire single-photon detector principle of operation.
   (a) a photon hit the device, (b) a resistive hotspot is created, (c)
   preventing the current from flowing, (d) and a voltage pulse is appeared
   on the output of the device (Nanowire 2016). ......................................................... 5
Fig. 1.4 Photon detection efficiency description .................................................................. 7
Fig. 1.5 A process to reduce afterpulsing ......................................................................... 9
Fig. 1.6 Jitter time (Midgall 2013) .................................................................................. 10
Fig. 2.1 Schematic cross section of a planar SPAD (Pellegrini 2006) ......................... 13
Fig. 2.2 The output of the receiver (Campbell 2006) ....................................................... 14
Fig. 2.3 Square pulse gated circuit (Liu 2007) ............................................................... 15
Fig. 2.4 Self-differencing and low-pass circuit (Liang 2011) ........................................ 17
Fig. 2.5 Setup of experiment block diagram (Scarcella 2015) ...................................... 21
Fig. 3.1 A schematic of SPAD detection circuit ............................................................... 23
Fig. 3.2 Schematic of the proposed method ...................................................................... 29
Fig. 3.3. DCR as a function of excess voltage when $R_s$ is varied, and compared to
   experimental results (Tosi 2014) .............................................................................. 79
Fig. 3.4 DCR as a function of excess voltage when $C_i$ is varied, and compared to experimental results (Tosi 2014).................................................................80

Fig. 3.5 DCR as a function of excess voltage when $R_{dc}$ is varied, and compared to experimental results (Tosi 2014).................................................................81

Fig. 3.6. DCR VS excess voltage, mathematical model and experimental data (Tosi 2014) ...........................................................................................................82

Fig. 3.7. DCR versus time when CIP off ...............................................................83

Fig. 3.8. DCR versus time when CIP on .................................................................84

Fig. 3.9 DCR versus $V_{ex}$ with CIP on .................................................................84

Fig. 3.10 Trade off between DCR and PDE with different excess voltages ..........85
1. Introduction

Single-photon detectors are optoelectronic devices capable of detecting very weak signals of light, which push researchers to develop these devices to detect a single-photon, the light unit (Midgall 2013 and Rochas 2002). Photomultiplier tubes (PMTs) are the first generation of avalanche photon detectors (APDs) invented in 1930's. PMTs work over the visible wavelength range between 300 – 1000 nm. Depending on materials of photocathode, there are several types of PMTs: bialkali, ultra-bialkali, multi-alkali, GaAs, and GaAsP and performance of PMTs vary from type to another type (Bulter 2014). Like the PMTs, many APDs have been developed from the release of Ge-APD to the SNSPDs depending on wavelength of operation. The goal of these developments is to construct new optical quantum devices that can use in many optical quantum information and remote sensing applications. The second generation of photodetectors uses single-photon avalanche photodiodes developed for single-photon counting (Cova 1996). In The last decade, the third generation of single-photon detectors has been developed by Gol’tsman et al. This generation introduces two types of single-photon detectors: superconducting nanowire single-photon detector (SNSPD) and quantum dot-based detector (Richards 1994 and Natarajan 2012).
This work will focus on the second generation, SPADs made of InGaAs/InP. First, a brief review of photon detectors is given. Then, a review of the parameters (photon detection efficiency, dark count rate, afterpulsing probability, and jitter time) that affect the SPAD performance as assessed via photon detection efficiency.

1.1. Single-photon Detector Types

The main single-photon detector categories are photomultiplier tubes (PMTs), single-photon avalanche detectors (SPADs), and superconducting single-photon detectors (SSPDs). Each category has different types of detectors.

1.1.1. First Generation: Photomultiplier Tubes (PMTs)

Light in a single-photon level was detected initially by a photomultiplier tube (PMT), which was invented in 1935 by Iams et al. (Midgall 2013, and Hamamatsu 2006). That tube contains a photocathode and only single stage dynode. A year later, Zworykin et al. enhanced a PMT making more than one dynode stage. In 1939, Zworykin and Rajchman improved a PMT using an Ag-O-Cs photocathode, which is used these days. The reason of naming this type a multiplier is that the total gain $M$ is achieved by multiplying the gain of the first dynode by the gain of the second one, etc. using the following equation (Midgall 2013):
\[ M = \prod_{i=1}^{N} g_i \]  

(1.1)

Where \( N \) is the number of stages and \( g \) is the gain of the dynode and it can be determined by (Midgall 2013):

\[ g_i = \delta_i n_i \]  

(1.2)

Where \( \delta_i \) is the secondary emission factor of dynode \( i \), and \( n_i \) is collection efficiency of dynode \( i \) (Flyckt 2002). Fig. 1.1 shows this type of photodetectors.

Mostly, PMTs are classified based on the materials the photocathode, dynodes and the anode made of. More than ten materials, commercially available, are used in the fabrication of PMTs. Most popular materials used to fabricate PMTs are Cs-I, Cs-Te, Sb-Cs, alkali (bialkali and multialkali), Ag-O-Cs, GaAsP, InGaAs, ... etc.
Every material has a wavelength range that works over and its sensitivity varies over that window of wavelength

1.1.2. Second Generation: Avalanche Photodiode (APD) and Single-photon Avalanche Detectors (SPADs)

An avalanche photodiode (APD) is a p-i-n photodiode with very large width of intrinsic area (i) compared to other areas (p and n). If an external voltage, called excess voltage ($V_E$), is applied to this APD, it became a single-photon avalanche detector (SPAD). Excess voltage makes the detector works in the region after the breakdown voltage and this called Geiger mode. Fig. 1.2 below shows APD and SPAD operation modes.

![Fig. 1.2 APD and SPAD operating regions](image-url)
1.1.3. Third Generation: Superconducting Nanowire Single-photon Detectors (SNSPD)

This type of detectors is made of Niobium Nitride (NbN) and consists of several superconducting nanowires made of cooper. The device is operated below the critical temperature and biased below the critical current. When a photon hits the device, a resistive hotspot is created as shown in Fig. 1.3. The hotspot breaks the nanowires and prevents the current from flowing which creates a voltage pulse in the output of the device.

Fig. 1.3 Superconducting nanowire single-photon detector principle of operation. (a) a photon hit the device, (b) a resistive hotspot is created, (c) preventing the current from flowing, (d) and a voltage pulse is appeared on the output of the device (Nanowire 2016).
1.2. SPAD Performance Parameters

There are several parameters that affect the performance of a single-photon detector. Those parameters vary from type to another type and depend on the physical material, device structure, and the circuitry connected to the detector.

1.2.1. Photon detection efficiency

All devices are assessed based on the functioning efficiency. Simply, the definition of efficiency is output power divided by input power, and in the ideal case, where there are no losses, the result should be one. In single-photon detectors, output power is the photons that are detected (electrical pulses) and input power is the power of the incident light that hits sensing area of the detector. Photon detection efficiency (PDE) as shown in Fig. 1.4 mainly depends on three probabilities: i) the probability that an incoming photon generates electron-hole pairs, ii) the probability that electron-hole pairs generated are injected to the multiplication layer, and iii) the probability that injected electron-hole pairs trigger an avalanche at output of the device.
When avalanche probability increases, PDE increases. However, at the same time, dark count rate (DCR), which must be low for the device to have good performance, increased as well. In addition, PDE is affected by electric field; increasing electric field result in increasing in photon detection efficiency (Jiang 2007).

### 1.2.2. Dark count rate

The reading on the output of the SPAD when there is no light hitting the device is called dark count rate (DCR). One reason for DCR to occur increase is carriers created due to heating of the device in the junction of the single-photon detector. However, DCR is mainly affected by excess bias voltage, $V_{EX}$ where
increasing $V_{EX}$ causes an increase in DCR. When the circuit is operated in gated mode, the DCR can be calculated by the following equation (Tosi 2009):

$$DCR = -\frac{1}{T_{ON}} \ln \left( 1 - \frac{T_{counter}}{f_{Gate}} \right)$$  \hspace{1cm} (1.3)

Where, $T_{counter}$ is the avalanche rate, and $f_{Gate}$ is the gate frequency and it is calculated by (Tosi 2009):

$$f_{Gate} = \frac{1}{(T_{ON} + T_{OFF})}$$  \hspace{1cm} (1.4)

Aside from being able to vary the excess voltage applied, reducing the operating temperature can effectively reduce dark count rate. One effect of cooling the device is a reduction in the carriers generated inside the junction and therefore the resulting currents that may cause DCR in absence of incident light.

1.2.3. Afterpulsing probability

After incident light triggers an avalanche on the output of a SPAD, some carriers in the multiplication region or between layers are trapped for some time and then trigger a new avalanche. This unwanted new avalanche is called
Afterpulsing is the main reason for using gated-mode circuits instead of free-running mode and reducing it is a challenge for researchers in terms of improving SPAD performance. First, when optimizing DCR, temperature needs to be reduced. However, reducing temperature increases afterpulsing because the rate carriers are release from trapped levels become slower. Second, reducing the population of trapped carriers can reduce afterpulsing. The trapped carriers can be reduced by reducing the flow of avalanche charge which requires reducing excess voltage applied to the circuit. Conversely, reducing the excess voltage leads to lower photon detection efficiency of the detector. To summarize that, look at the following figure (Korzh 2015).

Fig. 1.5 A process to reduce afterpulsing

1.2.4. Jitter time

Jitter time (t_j) is calculated using the full width at half maximum (FWHM) of the distribution of the photon arrival time. Jitter time is the time between the arrival of a photon and the electrical output, which is effectively the processing time of the
detector. Fig. 1.6 shows the variation in picoseconds between 1% of jitter time and 50%, where 1% takes time of about 800 ps and 50% less than 100 ps (Midgall 2003). Some authors refer timing resolution to jitter time (Bulter 2014). Jitter timing is affected by a number of factors. First, photons absorption locations inside the absorption layer make carriers excited in different times which leads to increasing in jitter time. Second, the same result can be occurred due to trapped carriers between layers and inside the multiplication region. Third, on the other hand, applying excess voltage to the circuit reduces jitter time.

Fig. 1.6 Jitter time (Midgall 2013)
1.2.5. Operating temperature

The temperature is an important parameter that can determine the efficiency of detectors. Dark count rate is primarily correlated with temperature. By increasing temperature, DCR is increased, and the performance of the detector is affected. Some detectors operate at very low temperatures (a few degrees in Kelvin) and some others work at room temperature (300 K). This variation of operating temperatures is based on the material used to make the detectors (Warburton 2009).

SPADs become the key technology to improve many applications such as quantum communications and optical communications because the efficiency of those applications is a function of SPAD parameters. Most of the previous work to improve SPAD performance is based on experimental studies and there is a lack of mathematical models to evaluate the SPAD performance. However, in this work, mathematical models are presented to represent the output current of the SPAD operated in Geiger-mode when the circuit is gated first with a sinusoidal signal and second with a Gaussian signal. The circuit is modeled under two conditions: zero photon input and non-zero photon input. The output current models are used to determine dark count rate and photon detection efficiency. Using the proposed mathematical models, circuits parameters can be evaluated to achieve best performance of the SPAD.
Chapter 2 provides a review of the second generation of single-photon detectors (SPADs) made using InGaAs/InP. All of the review provided focuses on experimental studies.

Chapter 3 illustrates the proposed method of calculating the SPAD output current and then the dark count rate and photon detection efficiency based on a gated circuit operated at Geiger mode.
2. Background

In 2006, Sara Pellegrini et al. studied a single-photon avalanche device with planar geometry operating at wavelength of 1550 nm and at a temperature of 200 K. They considered some changes in the design to reach the desired results of photon detection efficiency, jitter time, and afterpulsing probability. Using grading layer between the absorption layer (InGaAs) and the multiplication layer (InP) enhanced the electric field and ensured that holes transfer across the valence band. Fig. 2.1 shows the layers and overall structure.

![Fig. 2.1 Schematic cross section of a planar SPAD (Pellegrini 2006)](image)

Pellegrini’s design resulted in a Photon detection efficiency of 10 % with jitter time of 460 ps at 200 K (best jitter time was 450 ps at 150 K). Acceptable noise equivalent power is also achieved which is $10^{-15} W \cdot Hz^{-1/2}$. 
Guang Wu et al. in 2006 designed a model of InGaAs/InP SPADs using a square gate mode quenching circuit. This circuit works at a very low temperature and at a repetition rate of 100 KHz. The circuit was biased 5 V above the breakdown voltage to allow the APD to work in Geiger Mode, which make it suitable for detecting single-photon. Photon detection efficiency of 10% was obtained at dark rate probability of $1.3 \times 10^{-5}$, and 20 % at $1.6 \times 10^{-5}$ (Wu 2006).

In the same year, 2006, Joe Campbell at al. used a sinusoidal gated mode on InGaAs/InP SPAD to gain high photon detection efficiency and low dark count rate. The method used is to have two APDs to provide two opposite phase signals to eliminate the total noise; Fig. 2.2 shows the receiver output.

![Diagram](image)

Fig. 2.2 The output of the receiver (Campbell 2006)

This experiment was implemented at temperature of 240 K and at the wavelength of 1310 nm. The repetition rate of the laser source is 1 MHz. Jitter time
was obtained is 240 ps, with photon detection efficiency of 43%, and dark count rate of 58 KHz. Afterpulsing probability was lower than that in single sinusoidal-gated SPAD. Photon detection efficiency of about 10% was attained at DCR of 9.6 KHz and dark count probability of $2.8 \times 10^{-5}$ (Campbell 2006).

In 2007, Mingguo Liu et al. described high photon detection efficiency (PDE) with low dark current of indium gallium arsenide/indium phosphide single-photon avalanche detectors (InGaAs/InP SPAD). They operated their circuit on square pulse gated mode. The circuit was biased with 0.6 V below the breakdown point and a capacitor of 30 ns was connected to the AC pulse generator to make it biased above the breakdown point. Fig. 2.3 below shows that circuit.

![Fig. 2.3 Square pulse gated circuit (Liu 2007)]
Results obtained from this circuit showed good performance over the wavelength range of 1310 nm at acceptable operating temperature (200 K). The DCR was 12 KHz with a good timing resolution of 140 ps. The photon detection efficiency (PDE) achieved was 45% compared to the PDE calculated by

\[
PDE = \frac{1}{n} \ln \left( \frac{1 - P_d}{1 - P_a} \right)
\]  

(2.1)

Where, \(n\) is average photons per pulse, \(P_d\) is dark count probability, and \(P_a\) is avalanche probability (Liu 2007).

A new observation of high speed InGaAs/InP detector with high photon detection efficiency was realized by Lilin Xu et al. in 2009. They experimentally implemented a square pulse gated circuit with the self-differencing method. This experiment operated over the wavelength of 1550 nm and at temperature of -30 °C (243 K). The authors obtained great photon detection efficiency, 30%, and low afterpulsing probability, error of 6% (Xu 2009).

In July 2011, a team of researchers led by Yan Liang demonstrated a new way to achieve as low as possible of jitter time using sin-wave gated InGaAs/InP single-photon avalanche photodiode. Two techniques were implemented as shown in Fig. 2.4: differencing and low-pass filtering. This detector works at the wavelength
of 1550 nm and temperature of -25 °C (248 K). Best jitter time observed was 60 ps, with photon detection efficiency of 10.4%. Dark count rate was $6.1 \times 10^{-6} \text{Hz}$, and afterpulsing probability was 3%. They also realized the fast recovery time of 10 ns (Liang 2011)

![Diagram](image)

**Fig. 2.4** Self-differencing and low-pass circuit (Liang 2011)

Fabio Acerbi et al. in 2011 characterized an InGaAs/InP SPAD that has photon detection efficiency up to 30% and few thousands of counts per second. This detector operated at 1550 nm wavelength and at temperature of 225 K, with excess voltage of 5 V. Jitter time calculated as full width half maximum was 47 ps, and the circuit was biased at 7 V over breakdown voltage (Acerbi 2011).
In 2012, Albert Tosi et al. designed and fabricated a new InGaAs/InP single-photon avalanche detector that can be operated over up to 1700 nm of wavelength. They used gated mode with passive quenching circuit, which can bring the detector back to its original state in very short time. To define gating values, they biased the circuit 0.5 V below the breakdown voltage for the OFF mode and 5 V above the breakdown voltage for the ON mode. This design achieved a very low afterpulsing probability and high photon-counting rate (up to 1 Mcps). The photon detection efficiency was 25% over the wavelength range of 1550 nm at a temperature of 225K (Tosi a 2012).

In 2012, Alberto Tosi et al. designed and fabricated an InGaAs/InP SPAD that has high photon detection efficiency. They built their device based on 25 μm active area diameter hetero-structure design adding some layers between the absorption layer (InGaAS) and the multiplication layer (InP) to insure the maximum electric field in the multiplication region and the minimum in the absorption layer. They operated the circuit at temperature of 225 K, and over the wavelength of 1550 nm and biased it at 5 V. This experiment gave the following results: 25 % photon detection efficiency, jitter time of 87 ps (57 ps at excess voltage of 9 V, and fast decaying tail of 30 ps), and 100 Kcps dark count rate (less than 10 Kcps at temperature of 200 K). The maximum count rate was 1 Mcps (Tosi b 2012).
In 2012, Fabio Acerbi et al. characterized an InGaAs/InP SPAD with low dark count rate. The circuit was biased at 5 V and operated 200 K temperature. The device designed with 25 μm active area diameter and operated in gated mode with passive quenching circuit at 1550 nm of wavelength. Photon detection efficiency obtained was 25 % and jitter time was 90 ps. Dark count rate was about 10 Kcps (100 Kcps at 225 K).

In 2013, Alberto Tosi et al. designed and fabricated a new InGaAs/InP SPAD with low afterpulsing and narrow timing response. To obtain a very narrow timing response, they added a 100 nm-graded layer of InGaAsP between absorption and multiplication layers. This inserted layer can shrink the barrier transit time of photogenerated carriers and avoid carriers’ pile-up at interferences between layers. 25 μm active area diameter device used for this experiment operated over wavelength of 1550 nm at 225 K temperature with excess voltage of 5 V. Photon detection efficiency of 25 % was obtained at this operating temperature with 100 Kcps dark count rate and jitter time of 90 ps. It has also a fast tail time of 30 ps. The authors increased the gate off-time ($T_{\text{OFF}}$) and lowered the concentration of deep levels in the multiplication layer in order to get low afterpulsing signals (Tosi a 2013).
Alberto Tosi is one of researchers who has done many experiments and tests to gain the best results from single-photon avalanche detectors. In 2014, he came up with a design that attained high photon detection efficiency, low jitter time, and low noise. They worked on an InGaAs/InP SPAD operated in gated mode with excess voltage of 5 V at the wavelength of 1550 nm at 225 K temperature. The design was made by adding two layers between the absorption layer and the multiplication layer to lower the electric field at the absorbing region and limit tunneling and thermal carrier generation. Photon detection efficiency obtained reaches 28% with dark count rate of only few thousands counts per second and low afterpulsing probability. Jitter time, full width half maximum (FWHM), was less than 87 ps, and the decaying tail was 60 ps (Tosi a 2014).

In 2014, Alberto Tosi et al. optimized an InGaAs/InP SPAD to have low dark count rate and low jitter time. They fabricated a 25 μm active area diameter detector and design it to be able to sense at temperature of 225 K at wavelength of 1550 nm. The key of improving this detector is zinc diffusion, which used to maintain electric field over the multiplication region. Two layers, deep and shallow, were used to make the electric field uniformly distributed over the whole active area and to prevent the electric field from peaking at the edge of that area, respectively. It is operated at gated mode with passive quenching circuit at biased voltage of 5 V over breakdown point. Photon detection efficiency of 30% was
obtained with few K cps dark count rate. Jitter time is as low as 90 ps with fast exponential tail time of about 60 second (Tosi b 2012).

In 2015, Carmelo Scarcella et al. use the same material (InGaAs/InP) to present their SPAD using sinusoidal gated mode (with 1.3 GHz gating frequency) operated at temperature of 240 K and wavelength of 1550 nm. The differencing method and low-pass filtering are used to demonstrate this experiment. Fig. 2.5 shows the setup of the experiment implemented. The main aim they worked on in this work is to reduce the avalanche charge, which leads to reducing afterpulsing, and keep low-jitter observation and high photon detection efficiency unchanged. Results obtained from this experiment are 30% photon detection efficiency, dark count rate of $2.2 \times 10^{-5}$, afterpulsing probability of 1.5%, less than 70 ps jitter time, and count rate of 650 Mcps (Scarcella 2015).

![Fig. 2.5 Setup of experiment block diagram (Scarcella 2015).](image)
In 2015, Mirko Sanzaro et al. demonstrated an InGaAs/InP SPAD that shows high photon detection efficiency and low noise. This was done by using sinusoidal gate mode for getting good readings of afterpulsing and count rate. They optimized their previous work adjusting Zn diffusion and its effect on other layers. A device of 25 μm active area diameter was operated at temperature of 225 K, and at wavelength of 1550 nm. Maximum count rate observed was 650 Mcps, and 30% photon detection efficiency when biased at 5 V with afterpulsing probability of 1.5%. Jitter time of less than 90 ps achieved but at excess voltage of 7 V. Zinc diffusion conditions and vertical layers structure helped in gaining those results (Sanzaro 2015).
3. Proposed Method

The schematic circuit diagram in Fig. 3.1 is used to produce the mathematical model shown in equation (3.1). The mathematical model is based on gating the SPAD with a sinusoidal gating signal. The gating frequency is set to 1 GHz with \( 10 V_{p-p} \) gating voltage. The time domain analysis is applied to find out the circuit response for a very short gating time.

![Schematic Circuit Diagram](image1)

**Fig. 3.1 A schematic of SPAD detection circuit**

First, we apply KVL to the circuit:

\[
\frac{1}{C} \int i(t) dt = V_g(t) - V_d(t) - i(t)R_s - V_o(t)
\]

\[3.1\]

Then, take the derivative of both sides:
\[ i_i(t) = C_i \frac{dV_g(t)}{dt} - C_i \frac{dV_a(t)}{dt} - C_i R_s \frac{di_i(t)}{dt} - C_i \frac{dV_o(t)}{dt} \]  \hspace{1cm} 3.2

Where:

\[ i_i(t) = I_{dc} + i_d(t) \]  \hspace{1cm} 3.3

and

\[ V_o(t) = i_o(t) R_o \]  \hspace{1cm} 3.4

Where, \( V_o(t), i_o(t), \) and \( R_o \) are the output voltage, current, and resistor respectively. On the other hand, \( i_d(t) \) is the diode current at the cathode side, and \( I_{dc} \) is the DC source current and given by \( I_{dc} = \frac{V_{dc}}{R_{dc}} \). Currents \( i_d(t) \) and \( i_o(t) \) can be determined based on the equivalent circuit of the SPAD (Giustolisi 2011).

\[ i_d(t) = I_{SPAD} + \frac{dQ_j}{dt} + \frac{dQ_{ks}}{dt} \]  \hspace{1cm} 3.5

\[ i_o(t) = I_{SPAD} + \frac{dQ_j}{dt} - \frac{dQ_{as}}{dt} \]  \hspace{1cm} 3.6

From Equation (3.6), \( I_{SPAD} \) is given by:

\[ I_{SPAD} = i_o(t) - \frac{dQ_j}{dt} + \frac{dQ_{as}}{dt} \]  \hspace{1cm} 3.7

Substitute Equation (3.7) into (3.5) results in:
\[ i_d(t) = i_o(t) + \frac{dQ_{as}}{dt} + \frac{dQ_{ks}}{dt} \]  

3.8

Since \( i_i(t) = I_{dc} + i_d(t) \), then:

\[ i_i(t) = I_{dc} + i_o(t) + \frac{dQ_{as}}{dt} + \frac{dQ_{ks}}{dt} \]  

3.9

Substitute Equation (3.9) into (3.2) results in:

\[
i_o(t) + \frac{dQ_{as}}{dt} + \frac{dQ_{ks}}{dt} + I_{dc} \\
= C_i \frac{dV_g(t)}{dt} - C_i \frac{dV_A(t)}{dt} - C_i R_s \left( \frac{di_o(t)}{dt} + \frac{d^2 Q_{as}}{dt^2} + \frac{d^2 Q_{ks}}{dt^2} \right) \\
- C_i R_o \frac{di_o(t)}{dt}
\]  

3.10

Now, \( Q_{as} \) and \( Q_{ks} \) can be determined from (Giustolisi 2011).

\[ Q_{as} = \frac{A_D \varphi_i C_{as}}{1 - m_i} \left[ 1 + \frac{V_A}{\varphi_i} \right]^{1-m_i} \]  

3.11

\[ Q_{ks} = \frac{A_D \varphi_i C_{ks}}{1 - m_i} \left[ 1 + \frac{V_K}{\varphi_i} \right]^{1-m_i} \]  

3.12

However, for linearity, we use equations (3.13) and (3.14) below to represent

\[ Q_{as} \text{ and } Q_{ks} \] (Giustolisi 2011):

\[ Q_{as} = C_{as} V_A \]  

3.13
\[ Q_{ks} = C_{ks} V_K \quad 3.14 \]

From the circuit in Fig. 3.1, \( V_A \) and \( V_K \) are determined:

\[ V_A = i_o(t) R_o \quad 3.15 \]

\[ V_K = i_o(t) R_o + V_d(t) \quad 3.16 \]

Now, substituting equations (3.15) and (3.16) into equations (3.13) and (3.14), respectively, and finding the first and second derivative of both equations:

\[ Q_{as} = R_o C_{as} i_o(t) \]

\[ \frac{dQ_{as}}{dt} = R_o C_{as} \frac{di_o(t)}{dt} \quad 3.17 \]

\[ \frac{d^2Q_{as}}{dt^2} = R_o C_{as} \frac{d^2i_o(t)}{dt^2} \quad 3.18 \]

\[ Q_{ks} = R_o C_{ks} i_o(t) + C_{ks} V_d(t) \]

\[ \frac{dQ_{ks}}{dt} = R_o C_{ks} \frac{di_o(t)}{dt} + C_{ks} \frac{dv_d(t)}{dt} \quad 3.19 \]

\[ \frac{d^2Q_{ks}}{dt^2} = R_o C_{ks} \frac{d^2i_o(t)}{dt^2} + C_{ks} \frac{d^2v_d(t)}{dt^2} \quad 3.20 \]

Use equations (3.17), (3.18), (3.19), and (3.20) to replace relative terms in equation (3.10):
\[ i_o(t) + R_o C_{as} \frac{di_o(t)}{dt} + R_o C_{ks} \frac{di_o(t)}{dt} + C_{ks} \frac{dv_o(t)}{dt} + I_{dc} \]
\[ = C_i \frac{dV_g(t)}{dt} - C_i \frac{dV_d(t)}{dt} - R_s C_i \frac{di_o(t)}{dt} \]
\[ - R_s R_o C_i C_{as} \frac{d^2i_o(t)}{dt^2} - R_s R_o C_i C_{ks} \frac{d^2i_o(t)}{dt^2} \]
\[ - R_s C_i C_{ks} \frac{d^2v_d(t)}{dt^2} - R_o C_i \frac{di_o(t)}{dt} \]

Rearrange:
\[ (R_s R_o C_i C_{ks} + R_s R_o C_i C_{as}) \frac{d^2i_o(t)}{dt^2} \]
\[ + (R_o C_{as} + R_o C_{ks} + R_s C_i + R_o C_i) \frac{di_o(t)}{dt} + i_o(t) \]
\[ = C_i \frac{dV_g(t)}{dt} - \frac{V_{dc}}{R_{dc}} - R_s C_i C_{ks} \frac{d^2v_d(t)}{dt^2} \]
\[ - (C_i + C_{ks}) \frac{dV_d(t)}{dt} \]

Let:
\[ K_1 = C_i \]
\[ K_2 = -\frac{V_{dc}}{R_{dc}} \]
\[ K_3 = R_s R_o C_i (C_{as} + C_{ks}) \]
\[ K_4 = R_o (C_{as} + C_{ks} + C_i) + R_s C_i \]
\[ K_5 = 1 \]
\[ K_6 = -(R_s C_i C_{ks}) \]
\[ K_7 = -(C_i + C_{ks}) \]

Produces the second-order equation, equation (3.23):

\[
K_3 \frac{d^2 i_o(t)}{dt^2} + K_4 \frac{d i_o(t)}{dt} + K_5 i_o(t) = K_1 \frac{dv_d(t)}{dt} + K_2 + K_6 \frac{d^2 v_d(t)}{dt^2} + K_7 \frac{dv_d(t)}{dt}
\]

\[ 3.23 \]

Circuit response Device response

In equation (3.23), a second-order, linear, nonhomogeneous, ordinary differential equation (NHODE) is introduced, which describes the SPAD output response. The first part as shown in the equation (3.22) represents the circuit response including gating and dc circuits and the second part represents the SPAD response.

The circuit is gated first with a sinusoidal signal to calculate output current of the SPAD. SPAD, in the middle box of schematic shown in Fig. 3.2, is mathematically modeled at two cases: 1) When CIP (Controlled Incident Photon) is off, 2) When CIP is on, where SPAD receives an incident photon from CIP. Then dark count rate and photon detection efficiency is modeled for all four cases.
3.1. Sinusoidal- Zero input

For the zero-input response, the effect of SPAD is eliminated in the equation above since there is no incident photon hit the device. The equation becomes:

\[ K_3 \frac{d^2 i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_1 \frac{dv_g(t)}{dt} + K_2 \]  \hspace{1cm} (3.24)

Substituting gating voltage with \( \sin(\omega t) \) and \( i_o(t) \) with \( y(t) \):

\[ K_3 y''(t) + K_4 y'(t) + K_5 y(t) = K_1 \frac{d}{dt} \sin(\omega t) + K_2 \]  \hspace{1cm} (3.25)

\[ K_3 y''(t) + K_4 y'(t) + K_5 y(t) = K_2 - K_1 \cos(\omega t) \]  \hspace{1cm} (3.26)

The solution of the second order nonhomogeneous differential equation is the sum of general (natural) solution and particular (forced) solution.

Find the general solution by solving:

\[ K_3 y''(t) + K_4 y'(t) + K_5 y(t) = 0 \]
Assume solution is proportional to \( e^{st} \) and then substitute \( y(t) = e^{st} \) into the differential equation:

\[
K_3 \frac{d^2}{dt^2}(e^{st}) + K_4 \frac{d}{dt}(e^{st}) + K_5(e^{st}) = 0
\]

Substitute \( \frac{d^2}{dt^2}(e^{st}) = s^2 e^{st} \), and \( \frac{d}{dt}(e^{st}) = s e^{st} \):

\[
K_3s^2 e^{st} + K_4s e^{st} + K_5 e^{st} = 0
\]

Factor out \( e^{st} \):

\[
(K_3 s^2, +K_4 s + K_5) e^{st} = 0
\]

Since \( e^{st} \neq 0 \) for any positive \( s \), the zeros must come from the polynomial:

\[
K_3 s^2, +K_4 s + K_5 = 0
\]

Solve for \( s \):

\[
s = -\frac{K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \quad \text{or} \quad s = -\frac{K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}
\]

Then, the general solution:

\[
y_g(t) = C_1 e\left(-\frac{K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t + C_2 e\left(-\frac{K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t
\]
Determine the particular solution for: \( K_3y''(t) + K_4y'(t) + K_5y(t) = K_2 - K_1\cos(\omega t) \) by the method of undetermined coefficients. The particular solution is the sum of the particular solutions for: \( K_3y''(t) + K_4y'(t) + K_5y(t) = K_2 \) and let’s call that \( y_{p1}(t) \) and \( K_3y''(t) + K_4y'(t) + K_5y(t) = -K_1\cos(\omega t) \) and let’s call that \( y_{p2}(t) \).

Since \( K_2 \) is a constant, it follows that \( \frac{d}{dt}(K_2) \) and \( \frac{d^2}{dt^2}(K_2) \) are both equal zero, so it is obvious that the solution takes the form:

\[
y_{p1}(t) = a_1
\]

The particular solution to \( K_3y''(t) + K_4y'(t) + K_5y(t) = -K_1\cos(\omega t) \) is of the form:

\[
y_{p2}(t) = a_2 \cos(\omega t) + a_3 \sin(\omega t)
\]

Since \( y_p(t) = y_{p1}(t) + y_{p2}(t) \), then:

\[
y_p(t) = a_1 + a_2 \cos(\omega t) + a_3 \sin(\omega t)
\]

Solve for the unknown constants \( a_1, a_2, \) and \( a_3 \):

First compute \( \frac{d}{dt}(y_p(t)) \):
\[
\frac{d}{dt}(y_p(t)) = \frac{d}{dt}(a_1 + a_2 \cos(\omega t) + a_3 \sin(\omega t)) \\
= -\omega a_2 \sin(\omega t) + \omega a_3 \cos(\omega t)
\]

Second compute \( \frac{d^2}{dt^2}(y_p(t)) \):

\[
\frac{d^2}{dt^2}(y_p(t)) = \frac{d^2}{dt^2}(a_1 + a_2 \cos(\omega t) + a_3 \sin(\omega t)) \\
= -\omega^2 a_2 \cos(\omega t) - \omega^2 a_3 \sin(\omega t)
\]

Substitute the particular solution \( y_p(t) \) into the differential equation:

\[
K_3(-\omega^2 a_2 \cos(\omega t) - \omega^2 a_3 \sin(\omega t)) + K_4(-\omega a_2 \sin(\omega t) + \omega a_3 \cos(\omega t)) \\
+ K_5(a_1 + a_2 \cos(\omega t) + a_3 \sin(\omega t)) = K_2 - K_1 \cos(\omega t)
\]

Simplify:

\[
K_5 a_1 + (K_5 a_2 - K_3 \omega^2 a_2 + K_4 \omega a_3) \cos(\omega t) + (-K_4 \omega a_2 + K_5 a_3) \\
- K_3 \omega^2 a_3 \sin(\omega t) = K_2 - K_1 \cos(\omega t)
\]

Equate the coefficients of 1 on both sides of the equations:

\[K_5 a_1 = K_2\]

Equate the coefficients of \( \cos(\omega t) \) on both sides of the equations:
\[ K_5 a_2 - K_3 \omega^2 a_2 + K_4 \omega a_3 = -K_1 \]

Equate the coefficients of \( \sin(\omega t) \) on both sides of the equations:

\[ -K_4 \omega a_2 + K_5 a_3 - K_3 \omega^2 a_3 = 0 \]

Solve the system of equations above for \( a_1, a_2, \) and \( a_3 \):

\[ a_1 = \frac{K_2}{K_5} \]

\[ a_2 = \frac{-K_1 (K_5 - K_3 \omega^2)}{K_5^2 + K_3^2 + \omega^4 + K_4^2 \omega^2 - 2K_3 K_5 \omega^2} \]

\[ a_3 = \frac{-K_1 K_4 \omega}{K_5^2 + K_3^2 + \omega^4 + K_4^2 \omega^2 - 2K_3 K_5 \omega^2} \]

Substitute \( a_1, a_2, \) and \( a_3 \) into \( y_p(t) = a_1 + a_2 \cos(\omega t) + a_3 \sin(\omega t) \):

\[ y_p(t) = \frac{K_2}{K_5} - \frac{K_1 (K_5 - K_3 \omega^2) \cos(\omega t) - K_1 K_4 \omega \cdot \sin(\omega t)}{K_5^2 + K_3^2 + \omega^4 + K_4^2 \omega^2 - 2K_3 K_5 \omega^2} \]

The final solution is:

\[ y(t) = y_g(t) + y_p(t) \]
\[
    i_o(t) = C_1 e^{\left(\frac{-K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t} + C_2 e^{\left(\frac{-K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t} + \frac{K_2}{K_5}
\]

\[
    - \frac{K_1(K_5 - K_3\omega^2) \cos(\omega t) - K_1K_4\omega \cdot \sin(\omega t)}{K_5^2 + K_3^2 + \omega^4 + K_4^2\omega^2 - 2K_3K_5\omega^2}
\]

3.1.1. Dark count rate modeling

To calculate dark count rate, the formula below is used (Tosi 2009):

\[
    DCR = - \frac{1}{T_{ON}} \ln\left(1 - \frac{\text{Actual counts}}{f_{Gate}}\right)
\]

Where Actual counts is given by: Actual counts = \( \frac{i_o(t) \cdot t}{q} \)

\[
    DCR = - \frac{1}{T_{ON}} \ln\left(1 - \frac{t}{q \cdot f_{Gate}} \left( C_1 e^{\left(\frac{-K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t} + C_2 e^{\left(\frac{-K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\right)t} + \frac{K_2}{K_5} \right)
\]

\[
    - \frac{K_1(K_5 - K_3\omega^2) \cos(\omega t) - K_1K_4\omega \cdot \sin(\omega t)}{K_5^2 + K_3^2 + \omega^4 + K_4^2\omega^2 - 2K_3K_5\omega^2}
\]

3.1.2. Photon detection efficiency modeling

Photon detection efficiency is calculated based on the actual counts detected and dark count rates:

\[
    PDE = \frac{\text{Actual counts} - DCR}{\text{Actual counts}}
\]
\[ PDE = 1 - q \cdot DCR \]

\[
\frac{1}{t} \left( C_1 e^{-K_4 \sqrt{K_4^2 - 4K_3 K_5} t} \frac{t}{2K_3} + C_2 e^{-K_4 \sqrt{K_4^2 - 4K_3 K_5} t} \frac{t}{2K_3} + \frac{K_2}{K_5} \right)
\]

3.33

3.2. Sinusoidal- Nonzero input

For the nonzero-input response, the effect of the incident photon hitting the SPAD contributes to the output of the circuit and the equation takes the original form described in Equation 3.23

\[
K_3 \frac{d^2 i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_1 \frac{dv_o(t)}{dt} + K_2 + K_6 \frac{d^2 v_d(t)}{dt^2} + K_7 \frac{dv_d(t)}{dt}
\]

3.34

\[
K_3 \frac{d^2 i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_1 \sin(\omega t) + K_2 + a_1 \cos(\omega_o t) + a_2 \sin(\omega_o t)
\]

3.35

Where \( a_1 = -\omega_0^2 K_6 \) and \( a_2 = \omega_o K_7 \).

Find the general solution by solving:

\[ K_3 y''(t) + K_4 y'(t) + K_5 y(t) = 0 \]
Assume solution is proportional to $e^{st}$ and then substitute $y(t) = e^{st}$ into the differential equation:

$$K_3 \frac{d^2}{dt^2}(e^{st}) + K_4 \frac{d}{dt}(e^{st}) + K_5 (e^{st}) = 0$$

Substitute $\frac{d^2}{dt^2}(e^{st}) = s^2 e^{st}$, and $\frac{d}{dt}(e^{st}) = s e^{st}$:

$$K_3 s^2 e^{st} + K_4 s e^{st} + K_5 e^{st} = 0$$

Factor out $e^{st}$:

$$(K_3 s^2, +K_4 s + K_5) e^{st} = 0$$

Since $e^{st} \neq 0$ for any positive $s$, the zeros must come from the polynomial:

$$K_3 s^2 + K_4 s + K_5 = 0$$

Solve for $s$:

$$s = -\frac{K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \quad \text{or} \quad s = -\frac{K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}$$

Then, the general solution:
\[ y_g(t) = C_1 e^{\left( \frac{-K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \right)t} + C_2 e^{\left( \frac{-K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \right)t} \quad 3.36 \]

Now the general solution is calculated, particular solution needs to be estimated.

The particular solution is the sum of the particular solutions to:

- \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_2 \)
- \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = -a_1 \cos (\omega_o t) + a_2 \sin (\omega_o t) \)
- \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_1 \sin(\omega t) \)

The particular solution to \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_2 \) is:

\[ y_{p1}(t) = s_1 \quad 3.37 \]

The particular solution to \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = -a_1 \cos (\omega_o t) + a_2 \sin (\omega_o t) \) is:

\[ y_{p2}(t) = s_2 \cos (\omega_o t) + s_3 \sin (\omega_o t) \quad 3.38 \]

The particular solution to \( K_3 \frac{d^2i_o(t)}{dt^2} + K_4 \frac{di_o(t)}{dt} + K_5 i_o(t) = K_1 \sin(\omega t) \) is:

\[ y_{p3}(t) = s_4 \cos (\omega t) + s_5 \sin (\omega t) \quad 3.39 \]

The total particular solution is:
\[ y_p(t) = y_{p1}(t) + y_{p2}(t) + y_{p3}(t) \quad 3.40 \]

Substitute equations 3.37, 3.38 and 3.39 into equation 3.40

\[ y_p(t) = s_1 + s_2 \cos(\omega_0 t) + s_3 \sin(\omega_0 t) + s_4 \cos(\omega t) + s_5 \sin(\omega t) \quad 3.41 \]

Now, solve for unknowns \( s_1, s_2, s_3, s_4, \) and \( s_5. \) First, we need to derive equation 3.41 two times:

\[
\frac{dy_p(t)}{dt} = -\omega_0 s_2 \sin(\omega_0 t) + \omega_0 s_3 \cos(\omega_0 t) + \omega s_4 \cos(\omega t) \\
+ \omega s_5 \sin(\omega t) \quad 3.42
\]

\[
\frac{d^2y_p(t)}{dt^2} = -\omega_0^2 s_2 \cos(\omega_0 t) - \omega_0^2 s_3 \sin(\omega_0 t) - \omega^2 s_4 \cos(\omega t) \\
- \omega^2 s_5 \sin(\omega t) \quad 3.43
\]

Substitute equations 3.41, 3.42, and 3.43 into equation 3.44

\[
K_3 \frac{d^2y_p(t)}{dt^2} + K_4 \frac{dy_p(t)}{dt} + K_5 y_p(t) \\
= K_1 \sin(\omega t) + K_2 + a_1 \cos(\omega_0 t) + a_2 \sin(\omega_0 t) \quad 3.44
\]

\[
3.45
\]
\[ K_3(-\omega_o^2 s_2 \cos(\omega_o t) - \omega_o^2 s_3 \sin(\omega_o t) - \omega^2 s_4 \cos(\omega t) - \omega^2 s_5 \sin(\omega t)) \]
\[ + K_4(-\omega_o s_2 \sin(\omega_o t) + \omega_o s_3 \cos(\omega_o t) + \omega s_4 \cos(\omega t) + \omega s_5 \sin(\omega t)) \]
\[ + \omega s_5 \sin(\omega t) \]
\[ + K_5(s_1 + s_2 \cos(\omega_o t) + s_3 \sin(\omega_o t) + s_4 \cos(\omega t) + s_5 \sin(\omega t)) \]
\[ = K_1 \sin(\omega t) + K_2 + a_1 \cos(\omega_o t) + a_2 \sin(\omega_o t) \]

Simplify:

\[ K_5 s_1 + [K_5 s_2 - K_3 \omega_o^2 s_2 + K_4 \omega_o s_4] \cos(\omega_o t) \]
\[ + [K_5 s_3 - K_3 \omega_o^2 s_3 + K_4 \omega s_5] \cos(\omega t) \]
\[ + [-K_4 \omega_o s_2 + K_5 s_4 - K_3 \omega_o^2 s_4] \sin(\omega_o t) \]
\[ + [-K_4 \omega s_3 + K_5 s_5 - K_3 \omega s_5] \sin(\omega t) \]
\[ = K_1 \sin(\omega t) + K_2 + a_1 \cos(\omega_o t) + a_2 \sin(\omega_o t) \]

Equate coefficients of each term on both sides:

1. \( K_5 s_1 = K_2 \)
2. \( K_5 s_2 - K_3 \omega_o^2 s_2 + K_4 \omega_o s_4 = -a_1 \)
3. \( K_5 s_3 - K_3 \omega_o^2 s_3 + K_4 \omega s_5 = 0 \)
4. \( -K_4 \omega_o s_2 + K_5 s_4 - K_3 \omega_o^2 s_4 = a_2 \)
5. \( -K_4 \omega s_3 + K_5 s_5 - K_3 \omega s_5 = K_1 \)
Solve the system in order to find $s_1$, $s_2$, $s_3$, $s_4$, and $s_5$:

$$s_1 = \frac{K_2}{K_5}$$

$$s_2 = -\frac{a_1K_5 - a_1K_3\omega_o^2 + a_2K_4\omega_o}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}$$

$$s_3 = -\frac{K_1K_4\omega}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}$$

$$s_4 = -\frac{-a_2K_5 - a_2K_3\omega_o^2 + a_1K_4\omega_o}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}$$

$$s_5 = \frac{K_1(K_5 - K_3\omega^2)}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}$$

Substitute values of $s_1$, $s_2$, $s_3$, $s_4$, and $s_5$ into equation 3.41:

$$y_p(t) = \frac{K_2}{K_5} + \left(-\frac{a_1K_5 - a_1K_3\omega_o^2 + a_2K_4\omega_o}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}\right)\cos(\omega_o t)$$

$$+ \left(-\frac{K_1K_4\omega}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}\right)\sin(\omega_o t)$$

$$+ \left(-\frac{-a_2K_5 - a_2K_3\omega_o^2 + a_1K_4\omega_o}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}\right)\cos(\omega t)$$

$$+ \left(\frac{K_1(K_5 - K_3\omega^2)}{K_5^2 + K_3^2\omega_o^4 + K_4^2\omega_o^2 + 2K_3K_5\omega_o^2}\right)\sin(\omega t)$$

The final solution to equation 3.35 is $y(t) = y_g(t) + y_p(t)$:
\[ y(t) = \frac{K_2}{K_5} + \left( -\frac{a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o}{K_5^2 + K_3^2 \omega_o^4 + K_4^2 \omega_o^2 + 2 K_3 K_5 \omega_o^2} \right) \cos(\omega_o t) \]
\[ + \left( -\frac{K_1 K_4 \omega}{K_5^2 + K_3^2 \omega_o^4 + K_4^2 \omega_o^2 + 2 K_3 K_5 \omega_o^2} \right) \sin(\omega_o t) \]
\[ + \left( -\frac{-a_2 K_5 - a_2 K_3 \omega_o^2 + a_1 K_4 \omega_o}{K_5^2 + K_3^2 \omega_o^4 + K_4^2 \omega_o^2 + 2 K_3 K_5 \omega_o^2} \right) \cos(\omega t) \]
\[ + \left( \frac{K_1 (K_5 - K_3 \omega^2)}{K_5^2 + K_3^2 \omega_o^4 + K_4^2 \omega_o^2 + 2 K_3 K_5 \omega_o^2} \right) \sin(\omega t) \]
\[ + C_1 e^{\left( \frac{-K_4 - \sqrt{K_4^2 - 4 K_3 K_5}}{2K_3} \right) t} + C_2 e^{\left( \frac{-K_4 + \sqrt{K_4^2 - 4 K_3 K_5}}{2K_3} \right) t} \]

Simplify:

\[ y(t) = i_o(t) \]
\[ = \frac{K_2}{K_5} \]
\[ \left( a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o \right) \cos(\omega_o t) + \left( -a_2 K_5 - a_2 K_3 \omega_o^2 + a_1 K_4 \omega_o \right) \sin(\omega_o t) \]
\[ \left( a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o \right) \cos(\omega t) - \left( a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o \right) \sin(\omega_o t) \]
\[ \left( a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o \right) \cos(\omega t) - \left( a_1 K_5 - a_1 K_3 \omega_o^2 + a_2 K_4 \omega_o \right) \sin(\omega_o t) \]
\[ + C_1 e^{\left( \frac{-K_4 - \sqrt{K_4^2 - 4 K_3 K_5}}{2K_3} \right) t} + C_2 e^{\left( \frac{-K_4 + \sqrt{K_4^2 - 4 K_3 K_5}}{2K_3} \right) t} \]
\[ i_o(t) = \frac{K_2}{K_5} + C_1 e^{\left(\frac{-K_4 - \sqrt{K_4^2 - 4K_3 K_5}}{2K_3}\right)t} + C_2 e^{\left(\frac{-K_4 + \sqrt{K_4^2 - 4K_3 K_5}}{2K_3}\right)t} \]

\[
- \left( a_1 K_5 - a_1 K_3 \omega_0^2 + a_2 K_3 \omega_0 \cos(\omega_0 t) + (-a_2 K_5 - a_2 K_3 \omega_0^2 + a_1 K_4 \omega_0) \sin(\omega_0 t) \right) \\
- \frac{K_1 K_4 \omega \cos(\omega t) - [K_1 (K_5 - K_3 \omega^2)] \sin(\omega t)}{K_5^2 + K_3^2 \omega^4 + K_4^2 \omega^2 + 2 K_3 K_5 \omega^2}
\]

**3.2.1. Dark count rate modeling**

To calculate dark count rate, the formula below is used (Tosi 2009):

\[
DCR = -\frac{1}{T_{ON}} \ln \left( 1 - \frac{Actual \ counts}{f_{Gate}} \right)
\]

Where *Actual counts* is given by: \[
Actual \ counts = \frac{i_o(t) \cdot t}{q}\]
3.2.2. Photon detection efficiency modeling

Photon detection efficiency is calculated based on the actual counts detected and dark count rates:

\[
PDE = \frac{\text{Actual counts} - \text{DCR}}{\text{Actual counts}}
\]

\[
PDE = 1 - \frac{\text{DCR} \cdot q}{t} \left( \frac{K_2}{K_5} e^{-\frac{K_1 - \sqrt{K_1^2 - 4K_3K_5}}{2K_3}} + \frac{K_3}{K_5} e^{\frac{K_1 + \sqrt{K_1^2 - 4K_3K_5}}{2K_3}} \right)\]

\[
\frac{(a_1K_5 - a_2K_3\omega_2^2 + a_3K_2\omega_1\cos(\omega_2 t)) + (-a_2K_5 - a_2K_3\omega_2^2 + a_3K_2\omega_1\sin(\omega_2 t))}{K_5^2 + K_3^2\omega_2^4 + K_2^2\omega_1^2 + 2K_3K_4\omega_2^2}
\]

\[
\frac{K_3K_4\omega_1\cos(\omega t) - [K_1(K_5 - K_2\omega_1^2)]\sin(\omega t)}{K_5^2 + K_3^2\omega_2^4 + K_2^2\omega_1^2 + 2K_3K_4\omega_2^2} \right)^{-1}
\]
3.3. Gaussian-Zero input

The solution of the second order nonhomogeneous differential equation is the sum of general (natural) solution and particular (forced) solution.

Find the general solution by solving:

\[ K_3 y''(t) + K_4 y'(t) + K_5 y(t) = 0 \]

Assume solution is proportional to \( e^{st} \) and then substitute \( y(t) = e^{st} \) into the differential equation:

\[ K_3 \frac{d^2}{dt^2}(e^{st}) + K_4 \frac{d}{dt}(e^{st}) + K_5 (e^{st}) = 0 \]

Substitute \( \frac{d^2}{dt^2}(e^{st}) = s^2 e^{st} \), and \( \frac{d}{dt}(e^{st}) = s e^{st} \):

\[ K_3 s^2 e^{st} + K_4 s e^{st} + K_5 e^{st} = 0 \]

Factor out \( e^{st} \):

\[ (K_3 s^2 + K_4 s + K_5) e^{st} = 0 \]

Since \( e^{st} \neq 0 \) for any positive \( s \), the zeros must come from the polynomial:

\[ K_3 s^2 + K_4 s + K_5 = 0 \]

Solve for \( s \):
\[ s = -\frac{K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \quad \text{or} \quad s = -\frac{K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \]

Then, the general solution:

\[ y_g(t) = C_1 e^{\left(-K_4 - \sqrt{K_4^2 - 4K_3K_5}\right) \frac{t}{2K_3}} + C_2 e^{\left(-K_4 + \sqrt{K_4^2 - 4K_3K_5}\right) \frac{t}{2K_3}} \]

Now, determine the particular solution to \( K_3y''(t) + K_4y'(t) + K_5y(t) = K_2 - 2K_1t e^{-t^2} \) using method of variation of parameters:

\[ y_{p1}(t) = e^{\left(-K_4 - \sqrt{K_4^2 - 4K_3K_5}\right) \frac{t}{2K_3}} \quad \text{and} \quad y_{p2}(t) = e^{\left(-K_4 + \sqrt{K_4^2 - 4K_3K_5}\right) \frac{t}{2K_3}} \]

Find the Wronskian of \( y_{p1}(t) \) and \( y_{p2}(t) \):

\[ \mathcal{W}(t) = \begin{vmatrix} y_{p1}(t) & y_{p2}(t) \\ y'_{p1}(t) & y'_{p2}(t) \end{vmatrix} \]

\[ \mathcal{W}(t) = \frac{1}{K_3} \sqrt{K_4^2 - 4K_3K_5} \cdot e^{-\left(\frac{K_4}{K_3}\right)t} \]

45
Divide main equation by $K_3$:

$$y''(t) + \frac{K_4}{K_3}y'(t) + \frac{K_5}{K_3}y(t) = \frac{K_2 - 2K_1 t e^{-t^2}}{K_3}$$

Let $f(t) = \frac{K_2 - 2K_1 t e^{-t^2}}{K_3}$

And let $v_1(t) = \int \frac{f(t)y_p(t)}{w(t)} \, dt$ and $v_2(t) = \int \frac{f(t)y_p(t)}{w(t)} \, dt$

Then the particular solution is given by:

$$y_p(t) = v_1(t)y_{p1}(t) + v_2(t)y_{p2}(t)$$

Calculate $v_1(t)$

$$v_1(t) = -\int \frac{K_2 - 2K_1 t e^{-t^2}}{K_3} \cdot e^{-\frac{K_4 + \sqrt{K_4^2 - 4K_3 K_5}}{2K_3} t} \, dt$$

$$v_1(t) = -\frac{1}{\sqrt{K_4^2 - 4K_3 K_5}} \int e^{\frac{K_4}{K_3} t + \frac{(-K_4 + \sqrt{K_4^2 - 4K_3 K_5}) t}{2K_3}} \left( K_2 - 2K_1 t e^{-t^2} \right) dt$$

Let: $A = \frac{K_4}{K_3}$

$$B = \sqrt{K_4^2 - 4K_3 K_5}$$
\[ v_1(t) = -\frac{1}{B} \int e^{At + \frac{1}{2}(-A + B) t} \* (K_2 - 2K_1 te^{-t^2}) dt \]

\[ v_1(t) = -\frac{1}{B} \int e^{\frac{1}{2}(A + B) t} \* (K_2 - 2K_1 te^{-t^2}) dt \]

Let \( \alpha = \frac{1}{2} \left( A + \frac{B}{K_3} \right) \)

\[ v_1(t) = -\frac{1}{B} \int e^{\alpha t} \* (K_2 - 2K_1 te^{-t^2}) dt \]

\[ v_1(t) = -\frac{1}{B} \left[ K_2 \int e^{\alpha t} dt - 2K_1 \int te^{-t^2} e^{\alpha t} dt \right] \]

\[ v_1(t) = -\frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \int te^{-t^2} e^{\alpha t} dt \right] \]

\[ v_1(t) = -\frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \int te^{-t^2} e^{\alpha t} dt \right] \]

\[ \int te^{\alpha t - t^2} dt \]

Write \( t \) as \( \frac{\alpha}{2} - \frac{1}{2} (\alpha - 2t) \)

\[ \int \left( \frac{\alpha}{2} - \frac{1}{2} (\alpha - 2t) \right) e^{\alpha t - t^2} dt \]

\[ \left[ \frac{\alpha}{2} \int e^{\alpha t - t^2} dt - \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} dt \right] \]
1 \[\frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} \, dt\]

Let \( u = \alpha t - t^2, \quad \frac{du}{dt} = \alpha - 2t, \quad dt = \frac{1}{\alpha - 2t} \, du \)

\[\frac{1}{2} \int e^u \, du = \frac{1}{2} e^u\]

Undo substitution

\[\frac{1}{2} e^{\frac{1}{2}(B-A)t-t^2}\]

2 \[\frac{\alpha}{2} \int e^{\alpha t - t^2} \, dt\]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - (t - \frac{\alpha}{2})^2 \)

\[\frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - (t - \frac{\alpha}{2})^2} \, dt\]

Let \( u = t - \frac{\alpha}{2}, \quad \frac{du}{dt} = 1, \quad dt = du \)

\[\frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - u^2} \, du\]

\[\frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - u^2} \, du = \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4}} e^{-u^2} \, du\]

\[\frac{\alpha}{2} \int e^{\frac{\alpha^2}{4}} e^{-u^2} \, du = \frac{\alpha}{2} \int \frac{\alpha}{2} e^{-u^2} \, du = \frac{\alpha}{2} \int \frac{\alpha}{4} e^{-u^2} \, du\]

\[\frac{\alpha}{4} e^{\frac{\alpha^2}{4}} \text{erf}(u) = \frac{\alpha}{4} e^{\frac{\alpha^2}{4}} \text{erf}(t - \frac{\alpha}{2})\]
\[
\frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2}
\]

Plug this result into \( v_1(t) \)

\[
v_1(t) = -\frac{1}{B} \left[ K_2 e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right]
\]

Where \( \alpha = \frac{1}{2} \left( A + \frac{B}{K_3} \right) \)

Calculate \( v_2(t) \)

\[
v_2(t) = \int \frac{K_2 - 2K_1 t e^{-t^2}}{K_3} e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3K_5} \right) t} \frac{1}{K_3 \sqrt{K_4^2 - 4K_3K_5}} e^{-\left( K_2 \right) t} dt
\]

\[
v_2(t) = \frac{1}{\sqrt{K_4^2 - 4K_3K_5}} \int \frac{K_4 t}{e^{\frac{K_4^2}{K_3} t + \left( -K_4 - \sqrt{K_4^2 - 4K_3K_5} \right) t}} \frac{1}{2K_3} \left( K_2 - 2K_1 t e^{-t^2} \right) dt
\]

Let: \( A = \frac{K_4}{K_3} \)

\[
B = \sqrt{K_4^2 - 4K_3K_5}
\]
\[ v_2(t) = \frac{1}{B} \int e^{\frac{1}{2}(A+\frac{B}{K_3})t} (K_2 - 2K_1 t e^{-t^2}) dt \]

\[ v_2(t) = \frac{1}{B} \int e^{\frac{1}{2}(A-\frac{B}{K_3})t} (K_2 - 2K_1 t e^{-t^2}) dt \]

Let \( \alpha = \frac{1}{2} \left( A - \frac{B}{K_3} \right) \)

\[ v_2(t) = \frac{1}{B} \int e^{\alpha t} (K_2 - 2K_1 t e^{-t^2}) dt \]

\[ v_2(t) = \frac{1}{B} \left[ K_2 \int e^{\alpha t} dt - 2K_1 \int t e^{-t^2} e^{\alpha t} dt \right] \]

\[ v_2(t) = \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \int t e^{-t^2} e^{\alpha t} dt \right] \]

\[ v_2(t) = \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \int t e^{-t^2} e^{\alpha t} dt \right] \]

\[ \int t e^{\alpha t - t^2} dt \]

Write \( t \) as \( \frac{\alpha}{2} - \frac{1}{2} (\alpha - 2t) \)

\[ \int \left( \frac{\alpha}{2} - \frac{1}{2} (\alpha - 2t) \right) e^{\alpha t - t^2} dt \]

\[ \left[ \frac{\alpha}{2} \int e^{\alpha t - t^2} dt - \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} dt \right] \]
1 \[ \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} dt \]

Let \( u = \alpha t - t^2 \), \( \frac{du}{dt} = \alpha - 2t \), \( dt = \frac{1}{\alpha - 2t} du \)

\[ \frac{1}{2} \int e^u du = \frac{1}{2} e^u \]

Undo substitution

\[ \frac{1}{2} e^{\frac{1}{2} (B-A)t - t^2} \]

2 \[ \frac{\alpha}{2} \int e^{\alpha t - t^2} dt \]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - (t - \frac{\alpha}{2})^2 \)

\[ \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - (t - \frac{\alpha}{2})^2} dt \]

Let \( u = t - \frac{\alpha}{2} \), \( \frac{du}{dt} = 1 \), \( dt = du \)

\[ \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - u^2} du \]

\[ \frac{\alpha \sqrt{\pi}}{2 \times e^{\frac{\alpha^2}{4}}} \int \frac{2}{\sqrt{\pi}} e^{-u^2} du \]

\[ \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf}(u) \]

\[ \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf}\left(t - \frac{\alpha}{2}\right) \]
\[ \frac{\alpha \sqrt{\pi} \frac{\alpha^2}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} }{4} \]

Plug this result into \( v_2(t) \)

\[ v_2(t) = \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi} \frac{\alpha^2}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

Where \( \alpha = \frac{1}{2} \left( A - \frac{B}{k_3} \right) \)

Then the particular solution \( y_p(t) = v_1(t)y_{p1}(t) + v_2(t)y_{p2}(t) \)

\[ y_p(t) = -\frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi} \frac{\alpha^2}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

\[ \ast e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3 K_5} \right) t \frac{1}{2K_3}} \]

\[ + \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi} \frac{\alpha^2}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

\[ \ast e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3 K_5} \right) t \frac{1}{2K_3}} \]

The final solution

\[ y(t) = i_0(t) = y_g(t) + y_p(t) \]
\[ i_0(t) = C_1 e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} + C_2 e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} - \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \\
* e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} + \frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \\
* e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} \]

3.3.1. Dark count rate modeling

To calculate dark count rate, the formula below is used (Tosi 2009):

\[ DCR = - \frac{1}{T_{ON}} \ln \left( 1 - \frac{\text{Actual counts}}{f_{Gate}} \right) \]

Where \text{Actual counts} is given by: \text{Actual counts} = \frac{i_0(t)\cdot t}{q}
\[ DCR = -\frac{1}{T_{ON}} \ln \left( 1 - \frac{t}{q \cdot f_{\text{Gate}}} \left( C_1 e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3 K_5} \frac{t}{2K_3} \right)} + C_2 e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3 K_5} \frac{t}{2K_3} \right)} \right) \right) \]

\[ -\frac{1}{B} \left[ \frac{K_2}{\alpha} e^{\alpha t} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

3.3.2. Photon detection efficiency modeling

Photon detection efficiency is calculated based on the actual counts detected and dark count rates:

\[ PDE = \frac{\text{Actual counts} - DCR}{\text{Actual counts}} \]
\[ PDE = 1 - \frac{DCR}{q} \left( C_1 e^{\left(-K_4 - \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}} + C_2 e^{\left(-K_4 + \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}} \right) \]

\[ - \frac{1}{B} \left[ K_2 \frac{e^{\alpha t}}{\alpha} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

\[ \ast e^{\left(-K_4 - \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}} \]

\[ + \frac{1}{B} \left[ K_2 \frac{e^{\alpha t}}{\alpha} - 2K_1 \left( \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{\alpha}{2} \right) - \frac{1}{2} e^{\alpha t - t^2} \right) \right] \]

\[ \ast e^{\left(-K_4 + \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}} \]
3.4. Gaussian- Nonzero input

The output response when applying a pulse of photons on the SPAD gated with a Gaussian signal is given by the equation below

\[ K_3y''(t) + K_4y'(t) + K_5y(t) = K_2 - a_1te^{-t^2} + 4K_6t^2e^{-t^2} - 2K_6e^{-t^2} \quad 3.57 \]

Where \( a_1 = 2K_1 + 2K_7 \)

The solution of the second order nonhomogeneous differential equation is the sum of general (natural) solution and particular (forced) solution.

Find the general solution by solving:

\[ K_3y''(t) + K_4y'(t) + K_5y(t) = 0 \]

Assume solution is proportional to \( e^{st} \) and then substitute \( y(t) = e^{st} \) into the differential equation:

\[ K_3\frac{d^2}{dt^2}(e^{st}) + K_4\frac{d}{dt}(e^{st}) + K_5(e^{st}) = 0 \]

Substitute \( \frac{d^2}{dt^2}(e^{st}) = s^2e^{st} \), and \( \frac{d}{dt}(e^{st}) = se^{st} \):

\[ K_3s^2e^{st} + K_4se^{st} + K_5e^{st} = 0 \]

Factor out \( e^{st} \):
\[(K_3 s^2, +K_4 s + K_5) e^{st} = 0\]

Since \(e^{st} \neq 0\) for any positive \(s\), the zeros must come from the polynomial:

\[K_3 s^2, +K_4 s + K_5 = 0\]

Solve for \(s\):

\[s = -\frac{K_4 + \sqrt{K_4^2 - 4K_3K_5}}{2K_3} \quad \text{or} \quad s = -\frac{K_4 - \sqrt{K_4^2 - 4K_3K_5}}{2K_3}\]

Then, the general solution:

\[y_g(t) = C_1 e^{\left(-K_4 - \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}} + C_2 e^{\left(-K_4 + \sqrt{K_4^2 - 4K_3K_5}\right)\frac{t}{2K_3}}\]

Now, determine the particular solution to

\[K_3y''(t) + K_4y'(t) + K_5y(t) = K_2 - a_1 t e^{-t^2} + 4K_6 t^2 e^{-t^2} - 2K_6 e^{-t^2}\]

using method of variation of parameters.

First, we need to make the leading coefficient equal to 1:

\[\frac{y''}{K_3} + \frac{K_4}{K_3}y'(t) + \frac{K_5}{K_3}y(t) = \frac{K_2}{K_3} - \frac{a_1}{K_3} t e^{-t^2} + \frac{4K_6}{K_3} t^2 e^{-t^2} - \frac{2K_6}{K_3} e^{-t^2}\]

Then, considering the roots of the equation, we find \(y_{p1}(t)\) and \(y_{p2}(t)\):
\( y_{p1}(t) = e^{\frac{1}{2} \left( \frac{K_4}{K_3} - \sqrt{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \right) t} \) and \( y_{p2}(t) = e^{\frac{1}{2} \left( \frac{K_4}{K_3} + \sqrt{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \right) t} \)

Find the Wronskian of \( y_{p1}(t) \) and \( y_{p2}(t) \):

\[
\mathcal{W}(t) = \begin{vmatrix} y_{p1}(t) & y_{p2}(t) \\ y'_{p1}(t) & y'_{p2}(t) \end{vmatrix} = \left( \frac{K_4}{K_3} - \sqrt{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \right) * e^{\frac{1}{2} \left( \frac{K_4}{K_3} - \sqrt{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \right) t} \right) \]

\[
\mathcal{W}(t) = -\frac{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}}{\sqrt{\frac{K_3}{K_3}}} * e^{\frac{K_4}{K_3} t}
\]

Let \( f(t) = \frac{K_2}{K_3} \frac{a_1}{K_3} t e^{-t^2} + \frac{4K_6}{K_3} t e^{-t^2} - \frac{2K_6}{K_3} e^{-t^2} \)

And let \( v_1(t) = \int f(t) y_{p2}(t) \mathcal{W}(t) \ dt \) and \( v_2(t) = \int f(t) y_{p1}(t) \mathcal{W}(t) \ dt \)

Then the particular solution is given by:

\( y_{p}(t) = v_1(t) y_{p1}(t) + v_2(t) y_{p2}(t) \)

Calculate \( v_1(t) \)
\[ v_1(t) = \int \left( \frac{K_2}{K_3} - \frac{a_1}{K_3} e^{-t^2} + \frac{4K_6}{K_3} t^2 e^{-t^2} - \frac{2K_6}{K_3} e^{-t^2} \right) \cdot e^{\frac{1}{2} \left( -\frac{K_4}{K_3} - \sqrt{\left( \frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \right) t} \] 

Now, let:

\( A = -\frac{K_4}{K_3} \)

\[ B = \sqrt{\left( -\frac{K_4}{K_3} \right)^2 - \frac{4K_5}{K_3}} \]

\[ C = \frac{K_2}{K_3} \]

\[ D = \frac{a_1}{K_3} \]

\[ E = \frac{2K_6}{K_3} \]

\[ v_1(t) = \int \left( C - D t e^{-t^2} + 2E t^2 e^{-t^2} - E e^{-t^2} \right) \cdot e^{\frac{1}{2}(A-B)t} \cdot e^B e^{At} \] 

\[ v_1(t) = -\frac{1}{B} \int \left( C - D t e^{-t^2} + 2E t^2 e^{-t^2} - E e^{-t^2} \right) \cdot e^{\frac{1}{2}(A-B)t} dt \]

\[ v_1(t) = -\frac{1}{B} \int \left( C e^{\frac{1}{2}(A-B)t} - D t e^{-t^2} e^{\frac{1}{2}(A-B)t} + 2E t^2 e^{-t^2} e^{\frac{1}{2}(A-B)t} \right. \]

\[ \left. - E e^{-t^2} e^{\frac{1}{2}(A-B)t} \right) dt \]
\[ v_1(t) = -\frac{1}{B} \int \left( Ce^{\frac{1}{2}(A-B)t} - Dte^{-t^2}e^{\frac{1}{2}(A-B)t} + 2Et^2e^{-t^2}e^{\frac{1}{2}(A-B)t} \right) dt \]

\[ v_1(t) = -\frac{1}{B} \left[ C \int e^{\frac{1}{2}(A-B)t} \, dt - D \int te^{-t^2}e^{\frac{1}{2}(A-B)t} \, dt + 2E \int t^2e^{-t^2}e^{\frac{1}{2}(A-B)t} \, dt \right. \]

\[ \left. - E \int e^{-t^2}e^{\frac{1}{2}(A-B)t} \, dt \right] \]

I \quad C \int e^{\frac{\alpha}{2}(A-B)t} \, dt \]

\[ = \frac{2C}{A-B}e^{\frac{\alpha}{2}(A-B)t} \]

II \quad D \int te^{-t^2}e^{\frac{\alpha}{2}(A-B)t} \, dt \]

\[ D \int te^{-t^2}e^{\frac{\alpha}{2}(A-B)t} \, dt \]

Let \( \alpha = \frac{1}{2} (A - B) \)

Write \( t \) as \( \frac{\alpha}{2} - \frac{1}{2 } (\alpha - 2t) \)
\[ D \int \left( \frac{\alpha}{2} - \frac{1}{2}(\alpha - 2t) \right) e^{\alpha t - t^2} dt \]

\[ D \left[ \frac{\alpha}{2} \int e^{\alpha t - t^2} dt - \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} dt \right] \]

1 \[ \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} dt \]

Let \( u = \alpha t - t^2 \), \( \frac{du}{dt} = \alpha - 2t \), \( dt = \frac{1}{\alpha - 2t} du \)

\[ \frac{1}{2} \int e^u du = \frac{1}{2} e^u \]

Undo substitution

\[ \frac{1}{2} e^{\frac{1}{2}(\alpha - 2t)(\alpha - 2t)} \]

2 \[ \frac{\alpha}{2} \int e^{\alpha t - t^2} dt \]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2 \)

\[ \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2} dt \]

Let \( u = t - \frac{\alpha}{2} \), \( \frac{du}{dt} = 1 \) \( dt = du \)
\[
\frac{\alpha}{2} \int e^{\frac{-u^2}{4}} du
\]

\[
\frac{\alpha \sqrt{\pi}}{2^\frac{\alpha^2}{4}} e^{\frac{\alpha^2}{4}} \int \frac{2}{\sqrt{\pi}} e^{-u^2} du
\]

\[
\frac{\alpha \sqrt{\pi}}{4} e^{\frac{-u^2}{4}} \text{erf} (u) \quad \frac{\alpha \sqrt{\pi}}{4} e^{\frac{-u^2}{4}} \text{erf} \left( \frac{t - (A - B)}{4} \right)
\]

Then the integral II is

\[
\frac{D \alpha \sqrt{\pi}}{4} e^{\frac{-u^2}{4}} \text{erf} \left( \frac{t - (A - B)}{4} \right) - \frac{D}{2} e^{\frac{1}{4}(A - B)t - t^2}
\]

Where \( \alpha = \frac{1}{2} (A - B) \)

III

\[
2E \int t^2 e^{-t^2} e^{\frac{1}{2}(A - B)t} dt
\]

Let \( \alpha = \frac{1}{2} (A - B) \)

\[
2E \int t^2 e^{-t^2} e^{\alpha t} dt
\]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2 \)

\[
2E \int t^2 e^{\frac{-\alpha^2}{4} - (t - \frac{\alpha}{2})^2} dt
\]

\[
2E e^{\frac{-\alpha^2}{4}} \int t^2 e^{-(t - \frac{\alpha}{2})^2} dt
\]

Let \( u = 2t - \alpha \)

\[
\frac{du}{dt} = 2 \quad dt = \frac{1}{2} du \quad t^2 = \frac{(u + \alpha)^2}{4}
\]
Substitute

\[
\frac{1}{4} E e^{\frac{\alpha^2}{4}} \int (u+\alpha)^2 e^{-\frac{u^2}{4}} \, du
\]

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

\[
\frac{1}{4} E e^{\frac{\alpha^2}{4}} \left[ \int u^2 e^{-\frac{u^2}{4}} \, du + \int 2u \alpha e^{-\frac{u^2}{4}} \, du + \int \alpha^2 e^{-\frac{u^2}{4}} \, du \right]
\]

1  \[ \int u^2 e^{-\frac{u^2}{4}} \, du \]

Integrate by parts: \[ \int f(t)g'(t) \, dt = f(t)g(t) - \int f'(t)g(t) \, dt \]

Let \[ f = u \quad g' = ue^{-\frac{u^2}{4}} \]

\[ f' = 1 \quad g = -2e^{-\frac{u^2}{4}} \]

\[ \int u^2 e^{-\frac{u^2}{4}} \, du = -2ue^{-\frac{u^2}{4}} - \int -2e^{-\frac{u^2}{4}} \, du \]

Let \[ v = \frac{u}{2} \quad \frac{dv}{du} = \frac{1}{2} \quad du = 2dv \]

\[ = -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \int \frac{2}{\sqrt{\pi}} e^{-v^2} \, dv \]

\[ = -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( \frac{u}{2} \right) \]

\[ = -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \]
\[ 2 \int 2u \propto e^{-\frac{u^2}{4}} \, du \]

\[ 2 \propto \int u e^{-\frac{u^2}{4}} \, du \]

Let \( v = -\frac{u^2}{4} \quad \frac{dv}{du} = -\frac{u}{2} \quad du = -\frac{2}{u} \, dv \)

\[ 2 \propto \int -ue^v \frac{2}{u} \, dv \]

\[ -4 \propto \int e^v \, dv = -4 \propto e^v = -4 \propto e^{-\frac{u^2}{4}} \]

\[ \alpha^2 \int e^{-\frac{u^2}{4}} \, du \]

Let \( v = \frac{u}{2} \quad \frac{dv}{du} = \frac{1}{2} \quad du = 2 \, dv \)

\[ \sqrt{\pi} \alpha^2 \int \frac{2}{\sqrt{\pi}} e^{-v^2} \, dv \]

\[ \sqrt{\pi} \alpha^2 \, \text{erf} \left( \frac{u}{2} \right) = \sqrt{\pi} \alpha^2 \, \text{erf} \left( t - \frac{\alpha}{2} \right) \]

The integral III is:

\[ \frac{1}{4} \, E^{\frac{\alpha^2}{4}} \left[ \left(-2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \, \text{erf} \left( t - \frac{\alpha}{2} \right) \right) + \left(-4 \propto e^{-\frac{u^2}{4}} \right) + \sqrt{\pi} \alpha^2 \, \text{erf} \left( t - \frac{\alpha}{2} \right) \right] \]
Let \( \alpha = \frac{1}{2} (A - B) \)

\[
\int e^{-t^2} e^\frac{1}{2}(A-B)t \, dt
\]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left(t - \frac{\alpha}{2}\right)^2 \)

\[
\int e^\frac{\alpha^2}{4}(t - \frac{\alpha}{2})^2 \, dt
\]

Let \( u = t - \frac{\alpha}{2} \quad \frac{du}{dt} = 1 \quad dt = du \)

\[
\int e^\frac{\alpha^2}{4}u^2 \, du
\]

\[
\frac{e^{\sqrt{\pi}}}{2} e^{\frac{\alpha^2}{4}} \int \frac{2}{\sqrt{\pi}} e^{-u^2} \, du
\]

\[
\frac{e^{\sqrt{\pi}}}{2} e^{\frac{\alpha^2}{4}} \, \operatorname{erf}(u) + \frac{e^{\sqrt{\pi}}}{2} e^{\frac{\alpha^2}{4}} \, \operatorname{erf}\left(t - \frac{(A-B)}{4}\right)
\]

Then the integral IV is \( \frac{e^{\sqrt{\pi}}}{2} e^{\frac{\alpha^2}{4}} \, \operatorname{erf}\left(t - \frac{(A-B)}{4}\right) \)

\[
v_1(t) = -\frac{1}{B} [I - II + III - IV]
\]
\[ v_1(t) = -\frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2} (B - A) t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2} (B - A) t - t^2} \right) \right. \]
\[ \left. + \frac{1}{4} E e^{\frac{\alpha^2}{4}} \left[ \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf}\left( t - \frac{\alpha}{2} \right) \right) + \left( -4 \alpha e^{-\frac{u^2}{4}} \right) \right] \right. \]
\[ \left. + \sqrt{\pi} \alpha e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{\alpha}{2} \right) - \left( \frac{E}{2} e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B - A)}{4} \right) \right) \right] \]

Where \( \alpha = \frac{1}{2} (A - B) \)

Calculate \( v_2(t) \)

\[ v_2(t) = \int \left( \frac{K_2}{K_3} - \frac{a_1}{K_3} t e^{-t^2} + \frac{4K_5}{K_3} t^2 e^{-t^2} - \frac{2K_6}{K_3} e^{-t^2} \right) e^{\frac{1}{2} \left( -\frac{K_4}{K_3} \sqrt{\frac{K_4}{K_3} - \frac{4K_5}{K_3}} \right) t} dt \]

Now, let: \( A = -\frac{K_4}{K_3} \)

\( B = \sqrt{\frac{K_4}{K_3} - \frac{4K_5}{K_3}} \)

\( C = \frac{K_2}{K_3} \)

\( D = \frac{a_1}{K_3} \)
\[ E = \frac{2K_6}{K_3} \]

\[ v_2(t) = \int \frac{(C - Dte^{-t^2} + 2Et^2e^{-t^2} - Ee^{-t^2}) \cdot e^{\frac{1}{2}(A+B)t}}{-B \cdot e^{At}} \, dt \]

\[ v_2(t) = -\frac{1}{B} \int (C - Dte^{-t^2} + 2Et^2e^{-t^2} - Ee^{-t^2}) \cdot e^{\frac{1}{2}(B-A)t} \, dt \]

\[ v_2(t) = -\frac{1}{B} \int \left( Ce^{\frac{1}{2}(B-A)t} - Dte^{-t^2}e^{\frac{1}{2}(B-A)t} + 2Et^2e^{-t^2}e^{\frac{1}{2}(B-A)t} \right. \]

\[ \left. - Ee^{-t^2}e^{\frac{1}{2}(B-A)t} \right) dt \]

\[ v_2(t) = -\frac{1}{B} \left[ C \int e^{\frac{1}{2}(B-A)t} \, dt - D \int te^{-t^2}e^{\frac{1}{2}(B-A)t} \, dt + 2E \int t^2e^{-t^2}e^{\frac{1}{2}(B-A)t} \, dt \right. \]

\[ \left. - E \int e^{-t^2}e^{\frac{1}{2}(B-A)t} \, dt \right] \]

I \[ C \int e^{\frac{1}{2}(B-A)t} \, dt \]

\[ = \frac{2C}{B-A} e^{\frac{1}{2}(B-A)t} \]

II \[ D \int te^{-t^2}e^{\frac{1}{2}(B-A)t} \, dt \]

Let \( \alpha = \frac{1}{2}(B - A) \)
\[ D \int te^{-t^2} e^{\alpha t} \, dt \Rightarrow D \int te^{\alpha t - t^2} \, dt \]

Write \( t \) as \( \frac{\alpha}{2} - \frac{1}{2}(\alpha - 2t) \)

\[ D \int \left( \frac{\alpha}{2} - \frac{1}{2}(\alpha - 2t) \right) e^{\alpha t - t^2} \, dt \]

\[ D \left[ \frac{\alpha}{2} \int e^{\alpha t - t^2} \, dt - \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} \, dt \right] \]

1: \[ \frac{1}{2} \int (\alpha - 2t) e^{\alpha t - t^2} \, dt \]

Let \( u = \alpha t - t^2 \), \( \frac{du}{dt} = \alpha - 2t \), \( dt = \frac{1}{\alpha - 2t} \, du \)

\[ \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u \]

Undo substitution

\[ \frac{1}{2} e^{\frac{1}{2}(B-A)t - t^2} \]

2: \[ \frac{\alpha}{2} \int e^{\alpha t - t^2} \, dt \]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2 \)
\[ \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} \left( t - \frac{\alpha}{2} \right)^2} dt \]

Let \( u = t - \frac{\alpha}{2} \)
\[ \frac{du}{dt} = 1 \quad dt = du \]

\[ \frac{\alpha}{2} \int e^{\frac{\alpha^2}{4} u^2} du \]

\[ \frac{\alpha \sqrt{\pi}}{2^\alpha} e^{\frac{\alpha^2}{4}} \int \frac{2}{\sqrt{\pi}} e^{-u^2} du \]

\[ \frac{\alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf}(u) \]

Then the integral II is
\[ \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B-A)}{4} \right) - \frac{D}{2} e^{\frac{\alpha^2}{4} (B-A)t - t^2} \]

III \[ E \int t^2 e^{-t^2} e^{\frac{1}{2} (B-A)t} dt \]

Let \( \alpha = \frac{1}{2} (B - A) \)

\[ E \int t^2 e^{-t^2} e^{\alpha t} dt \]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2 \)

\[ E \int t^2 e^{\frac{\alpha^2}{4} \left( t - \frac{\alpha}{2} \right)^2} dt \]

\[ E e^{\frac{\alpha^2}{4}} \int t^2 e^{-\left( t - \frac{\alpha}{2} \right)^2} dt \]
Let \( u = 2t - \alpha \) 
\[ \frac{du}{dt} = 2 \quad dt = \frac{1}{2} \, du \quad t^2 = \frac{(u+\alpha)^2}{4} \]

Substitute

\[ \frac{1}{8} e^{\frac{\alpha^2}{4}} \int (u+\alpha)^2 e^{-\frac{u^2}{4}} \, du \]

\[ \frac{1}{8} e^{\frac{\alpha^2}{4}} \left[ \int u^2 e^{-\frac{u^2}{4}} \, du + \int 2u \, \alpha e^{-\frac{u^2}{4}} \, du + \int \alpha^2 e^{-\frac{u^2}{4}} \, du \right] \]

1. \( \int u^2 e^{-\frac{u^2}{4}} \, du \)

Integrate by parts: \[ \int f(t)g'(t) \, dt = f(t)g(t) - \int f'(t)g(t) \, dt \]

Let \( f = u \quad g' = ue^{-\frac{u^2}{4}} \)

\[ f' = 1 \quad g = -2e^{-\frac{u^2}{4}} \]

\[ \int u^2 e^{-\frac{u^2}{4}} \, du = -2ue^{-\frac{u^2}{4}} - \int -2e^{-\frac{u^2}{4}} \, du \]

Let \( v = \frac{u}{2} \quad \frac{dv}{du} = \frac{1}{2} \quad du = 2 \, dv \)

\[ = -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \int \frac{2}{\sqrt{\pi}} e^{-v^2} \, dv \]
\[= -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi}\ \text{erf}\left(\frac{u}{2}\right)\]

\[= -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi}\ \text{erf}\left(t - \frac{\alpha}{2}\right)\]

1. \(\int 2u \propto e^{-\frac{u^2}{4}} \, du\)

2. \(2 \propto \int ue^{-\frac{u^2}{4}} \, du\)

Let \(v = -\frac{u^2}{4}\)  \(\frac{dv}{du} = -\frac{u}{2}\)  \(du = -\frac{2}{u} \, dv\)

\[2 \propto \int -ue^{-v} \frac{2}{u} \, dv\]

\[-4 \propto \int e^v \, dv = -4 \propto e^v = -4 \propto e^{-\frac{u^2}{4}}\]

3. \(\alpha^2 \int e^{-\frac{u^2}{4}} \, du\)

Let \(v = \frac{u}{2}\)  \(\frac{dv}{du} = \frac{1}{2}\)  \(du = 2 \, dv\)

\[\sqrt{\pi} \alpha^2 \int \frac{2}{\sqrt{\pi}} e^{-v^2} \, dv\]

\[\sqrt{\pi} \alpha^2 \ \text{erf}\left(\frac{u}{2}\right) = \sqrt{\pi} \alpha^2 \ \text{erf}\left(t - \frac{\alpha}{2}\right)\]

The integral III is:
\[
\frac{1}{8} E e^{\frac{\alpha^2}{4}} \left[ \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \right) + \left( -4 \alpha e^{-\frac{u^2}{4}} \right) + \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) \right]
\]

IV \quad E \int e^{-t^2} e^{\frac{1}{2}(B-A)t} dt

Let \( \alpha = \frac{1}{2} (B - A) \)

\[
E \int e^{\alpha t - t^2} dt
\]

Complete the square \( \alpha t - t^2 = \frac{\alpha^2}{4} - \left( t - \frac{\alpha}{2} \right)^2 \)

\[
E \int e^{\frac{\alpha^2}{4} \left( t - \frac{\alpha}{2} \right)^2} dt
\]

Let \( u = t - \frac{\alpha}{2} \) \quad \frac{du}{dt} = 1 \quad dt = du

\[
E \int e^{\frac{\alpha^2}{4} \cdot u^2} du
\]

\[
\frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \int \frac{2}{\sqrt{\pi}} e^{-u^2} du
\]

\[
\frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( u \right) \quad \frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B-A)}{4} \right)
\]

Then the integral IV is \( \frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B-A)}{4} \right) \)
\[ v_2(t) = -\frac{1}{B}[I - II + III - IV] \]

\[ v_2(t) = -\frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2}(B - A)t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{-\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2}(B - A)t-t^2} \right) \right] \]

\[ \quad + \frac{1}{8} E e^{\frac{\alpha^2}{4}} \left[ \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf}\left( t - \frac{\alpha}{2} \right) \right) + \left( -4 \alpha e^{-\frac{u^2}{4}} \right) \right] \]

\[ \quad + \sqrt{\pi} \alpha^2 \text{erf}\left( t - \frac{\alpha}{2} \right) - \left( \frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B - A)}{4} \right) \right) \]

where \( \alpha = \frac{1}{2} (B - A) \)
Then the particular solution \( y_p(t) = v_1(t)y_{p1}(t) + v_2(t)y_{p2}(t) \)

\[
y_p(t) = -\frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2}(B - A)t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2}(B - A)t - t^2} \right) \right]
\]

\[
+ \frac{1}{4} E \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) + \left( -4 \alpha e^{-\frac{u^2}{4}} \right) \right)
\]

\[
+ \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) - \left( \frac{E \sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \]
\]

\[
* e^{-\frac{1}{2} \left( \frac{K_s}{\bar{K}_s} \sqrt{\left( \frac{K_s}{\bar{K}_s} \right)^2 - \frac{4K_s}{\bar{K}_s}} \right)^2 t}
\]

\[
- \frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2}(B - A)t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2}(B - A)t - t^2} \right) \right]
\]

\[
+ \frac{1}{8} E \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) + \left( -4 \alpha e^{-\frac{u^2}{4}} \right) \right)
\]

\[
+ \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) - \left( \frac{E \sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \]
\]

\[
* e^{-\frac{1}{2} \left( \frac{K_s}{\bar{K}_s} \sqrt{\left( \frac{K_s}{\bar{K}_s} \right)^2 - \frac{4K_s}{\bar{K}_s}} \right)^2 t}
\]

The final solution

\[
y(t) = i_o(t) = y_g(t) + y_p(t)
\]
\[ i_0(t) = C_1 e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} + C_2 e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} \]

\[ - \frac{1}{B} \left[ 2C e^{2(B-A)t} \right] \]

\[ - \left( D \propto \frac{\sqrt{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B-A)}{4} \right) - \frac{D}{2} e^{2(B-A)t-t^2} \right) \]

\[ + \frac{1}{8} E e^{\frac{\alpha^2}{4}} \left( -2ue^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf}\left( t - \frac{\alpha}{2} \right) + \left( -4 \propto e^{-\frac{u^2}{4}} \right) \right) \]

\[ + \sqrt{\pi} \propto \text{erf}\left( t - \frac{\alpha}{2} \right) \right) - \left( \frac{E\sqrt{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf}\left( t - \frac{(B-A)}{4} \right) \right) \]

\[ \frac{1}{2} \alpha \propto \text{erf}\left( t - \frac{\alpha}{2} \right) \]

\[ * e^{\left( \frac{K_4}{K_3} \sqrt{K_4^2 - 4K_3K_5} \right) \frac{t}{2K_3}} \]

\[ 3.58 \]
3.4.1. Dark count rate modeling

To calculate dark count rate, the formula below is used (Tosi 2009):

\[
DCR = -\frac{1}{T_{ON}} \ln \left( 1 - \frac{\text{Actual counts}}{f_{\text{Gate}}} \right)
\]

Where Actual counts is given by: \( \text{Actual counts} = \frac{i_0(t) \cdot t}{q} \)

\[
DCR = -\frac{1}{T_{ON}} \ln \left( 1 - \frac{t}{q \cdot f_{\text{Gate}}} \left( C_1 e^{\left(-K_3 - \sqrt{K_3^2 - 4K_4K_5}\right) \frac{t}{2K_4}} + C_2 e^{\left(-K_3 + \sqrt{K_3^2 - 4K_4K_5}\right) \frac{t}{2K_4}} \right) \right)
\]

\[
-\frac{1}{B} \left[ \frac{2C}{B - A} e^{\left(B - A\right) t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{x^2}{4}} \right)^{\text{erf} \left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2} \left(B - A\right) t}} \right]
\]

\[
+ \frac{1}{4} B e^{\frac{x^2}{4}} \left( -2u e^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \right)
+ \left( -4 \alpha e^{-\frac{\alpha^2}{4}} + \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) \right)
\]

\[
- \left( \frac{E \sqrt{\pi}}{2} e^{\frac{x^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \right) \right)
\]

\[
- \frac{1}{B} \left[ \frac{2C}{B - A} e^{\left(B - A\right) t} - \left( \frac{D \alpha \sqrt{\pi}}{4} e^{\frac{x^2}{4}} \right)^{\text{erf} \left( t - \frac{(B - A)}{4} \right) - \frac{D}{2} e^{\frac{1}{2} \left(B - A\right) t}} \right]
\]

\[
+ \frac{1}{6} B e^{\frac{x^2}{4}} \left( -2u e^{-\frac{u^2}{4}} + 2\sqrt{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \right)
+ \left( -4 \alpha e^{-\frac{\alpha^2}{4}} + \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) \right)
\]

\[
- \left( \frac{E \sqrt{\pi}}{2} e^{\frac{x^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \right) \right)
\]
3.4.2. Photon detection efficiency modeling

Photon detection efficiency is calculated based on the actual counts detected and dark count rates:

\[
PDE = \frac{\text{Actual counts} - \text{DCR}}{\text{Actual counts}}
\]

\[
PDE = 1 - \frac{DCR}{q} \left( C_1 e^{\left( -K_4 - \sqrt{K_4^2 - 4K_3 R_0} \right) t_{2K_5}} + C_2 e^{\left( -K_4 + \sqrt{K_4^2 - 4K_3 R_0} \right) t_{2K_5}} \right)
\]

\[
- \frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2}(B - A)t} \left( \frac{D \alpha \sqrt[4]{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) - \frac{D}{2} e^{\frac{1}{2}(B - A)t - t^2} \right]
\]

\[
+ \frac{1}{4} E e^{\frac{\alpha^2}{4}} \left[ \left( -2ue^{-\frac{\alpha^2}{4}} - 2\sqrt[2]{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \right) - \left( -4 \alpha e^{-\frac{\alpha^2}{4}} \right) \right]
\]

\[
+ \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) - \left( E \left( -\frac{\sqrt[2]{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \right)
\]

\[
* e^{\left( -\frac{K_4}{K_3} + \sqrt{\left( \frac{K_4^2}{K_3^2} - 4K_3 R_0 \right)} t_{2K_5} \right)}
\]

\[
- \frac{1}{B} \left[ \frac{2C}{B - A} e^{\frac{1}{2}(B - A)t} \left( \frac{D \alpha \sqrt[4]{\pi}}{4} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) - \frac{D}{2} e^{\frac{1}{2}(B - A)t - t^2} \right]
\]

\[
+ \frac{1}{4} E e^{\frac{\alpha^2}{4}} \left[ \left( -2ue^{-\frac{\alpha^2}{4}} - 2\sqrt[2]{\pi} \text{erf} \left( t - \frac{\alpha}{2} \right) \right) - \left( -4 \alpha e^{-\frac{\alpha^2}{4}} \right) \right]
\]

\[
+ \sqrt{\pi} \alpha^2 \text{erf} \left( t - \frac{\alpha}{2} \right) - \left( E \left( -\frac{\sqrt[2]{\pi}}{2} e^{\frac{\alpha^2}{4}} \text{erf} \left( t - \frac{(B - A)}{4} \right) \right) \right)
\]

\[
* e^{\left( -\frac{K_4}{K_3} + \sqrt{\left( \frac{K_4^2}{K_3^2} - 4K_3 R_0 \right)} t_{2K_5} \right)}
\]
3.5. Comparing mathematical results with experimental

The coefficients of NHODE and the coefficient of the circuit and device components are shown in Table 1 with the range of values chosen to optimize SPAD performance.

<table>
<thead>
<tr>
<th>Circuit and Device Components</th>
<th>Range Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$2\pi$GHz</td>
</tr>
<tr>
<td>$C_i$</td>
<td>$1 - 300$nF</td>
</tr>
<tr>
<td>$V_{dc}$</td>
<td>$10 - 30$V</td>
</tr>
<tr>
<td>$R_{dc}$</td>
<td>$1 - 3k\Omega$</td>
</tr>
<tr>
<td>$R_s$</td>
<td>$1 - 100k\Omega$</td>
</tr>
<tr>
<td>$R_o$</td>
<td>$1 - 10k\Omega$</td>
</tr>
<tr>
<td>$C_{as}$</td>
<td>$1 - 100pF$</td>
</tr>
<tr>
<td>$C_{ks}$</td>
<td>$1 - 100pF$</td>
</tr>
<tr>
<td>$R_d$</td>
<td>$100 - 500k\Omega$</td>
</tr>
</tbody>
</table>
Fig. 3.3. DCR as a function of excess voltage when $R_s$ is varied, and compared to experimental results (Tosi 2014)

The input resistor, $R_s$ controls the gating voltage and has a slight effect on DCR when changing within the defined range. As Fig. 3.3 demonstrates, by increasing $R_s$, DCR will marginally increase. At excess voltage of 7 V, DCR is 5400 cps with input resistance of 1 kΩ. However, increasing $R_s$ to 30 kΩ raises DCR to around 6100 cps. Thus, 1 kΩ is assigned as a default value for $R_s$ to produce the lowest DCR while providing protection for the SPAD.

As shown in Fig. 3.4, increasing $C_i$ from 1 nF to 200 nF increases DCR from 800 cps to more than 6000 cps at an excess voltage of 7 V. The series capacitor blocks low
frequencies and allows high frequencies to pass with low distortion or attenuation. Although increasing $C_i$ increases DCR, a high capacitance value is needed for reducing cut off frequency, $f_c = \frac{1}{2\pi RC}$, to assure more events are detected by the SPAD. For this, the input capacitance is set at 200 nF.

Fig. 3.4 DCR as a function of excess voltage when $C_i$ is varied, and compared to experimental results (Tosi 2014)

$R_{dc}$ is used to control the current flow to the SPAD and protect the SPAD from high currents. However, as shown in Fig. 3.5, DCR is affected by $R_{dc}$ where increasing $R_{dc}$ decreases DCR. If $R_{dc}$ is raised from 1 kΩ to 1.5 kΩ, DCR reduces from 12200 cps to 6700 cps at excess voltage of 7 V. Choosing a lower resistance for $R_{dc}$ results in higher DCR due to the higher current in the SPAD. Choosing a higher $R_{dc}$ reduces the
current, but also reduces the probability of having an output reading on the
detector. Thus, a mid-range value of 2 kΩ is chosen for $R_{dc}$.

Fig. 3.5 DCR as a function of excess voltage when $R_{dc}$ is varied, and compared to
experimental results (Tosi 2014)

Using the optimal values from the model with $R_s$, $C_i$, and $R_{dc}$ set at 1 kΩ, 200 nF,
and 2 kΩ, respectively, Fig. 3.6 is produced from the model and presented with
experimental data from Tosi et al. (Tosi 2014). As shown in experiments and the
mathematical model, increasing excess voltage increases DCR. The standard
deviation is calculated to show how the mathematical model for calculating DCR
deviates from experimental data. To calculate the standard deviation, $s$, the
formula used is:
\[ s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \]

where, \( n \) is the number of studies, and \( x_i \) is the value of element \( i \), and \( \bar{x} \) is the mean given by:

\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \]

The mathematical model varies from the experimental data by 1.17 kcps while the excess voltage is varied from 3 to 9 volts.

![Graph of DCR vs Excess Voltage](image-url)

Fig. 3.6. DCR VS excess voltage, mathematical model and experimental data (Tosi 2014)
DCR is also determined when $T_{\text{ON}}$ swapped from 1 – 20 nm as shown in Fig. 3.7 with CIP off and as shown in Fig. 3.8 with CIP on. It is observable that DCR increases with the increase of gating time. DCR is higher at the case of CIP on because there are more factors that contribute on DCR such as increasing the number of avalanche probability and the increase of temperature due to carriers’ movement.

Fig. 3.7. DCR versus time when CIP off
In Fig. 3.9, dark count rate is simulated when CIP is on and compared with experiments of Lui et al (Liu 2007).
Trade off between DCR and PDE

The trade off between dark count rate and photon detection efficiency is shown in Fig. 3.10. DCR is always desired to be as low as possible while PDE increased. However, in these photodetectors, PDE is decreased with the decrease of DCR for the fact that higher incoming photons, even they increase PDE, they raise the probability of false counts and DCR.

Fig. 3.10 Trade off between DCR and PDE with different excess voltages
4. Conclusions

The output current of a gated single-photon avalanche detector (SPAD) is mathematically modeled based on the circuit shown in Fig. 3.1 with consideration of the material parameters of the SPAD. The SPAD is gated with sinusoidal gating signal first and output current of the SPAD is modeled. After calculating the output current, dark count rate (DCR) and photon detection efficiency (PDE) are modeled based on the model of the SPAD output current. The circuit has a controlled incident photon source (CIP) which is a 1550 nm laser source allowing incident photons to hit the device at certain times. The SPAD and its parameters are modeled on two different occasions. Once when CIP is turned off, and once when CIP is turned back ON.

Using the proposed model, reducing input resistor $R_s$ by 20 kΩ, when $V_{EX}$ is set to 6 V, results in decreasing dark count rate by 20% which considered a huge improvement. At the same $V_{EX}$, when $R_{dc}$ decreased by 500 Ω, dark count rate is improved by 26%. Dark count rate is also improved from 6.7 Kcps to 4.8 Kcps when input capacitance increased from 200 nF to 300 nF at the same $V_{EX}$ of 6 V.
5. References


Jiang 2007  

Kadhim 2017  

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Korzh 2014  

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Lacaita 1996  

Lee 2014  

Liang 2011  

Liu 2007  

Lu 2013  


<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Year</th>
<th>Title</th>
<th>Journal/Conference</th>
</tr>
</thead>
</table>


**Zhang 2013**  

**Zheng 2015**  