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# Lower Order Perturbations of Critical Fractional Laplacian Equations

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"Lower Order Perturbations of Critical Fractional Laplacian Equations"
a dissertation by Khalid Fanoukh Al Oweidi

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# ABSTRACT

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by

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We give sufficient conditions for the existence of nontrivial solutions to a class of critical nonlocal problems of the Brezis-Nirenberg type. Our result extends some results in the literature for the local case to the nonlocal setting. It also complements the known results for the nonlocal case.

# Contents

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# Dedication

This dissertation is dedicated to my 2005 graduating class in Babylon, Iraq, my Florida Institute of Technology classmates, and family and friends.

# 1 Introduction

Nonlinear elliptic equations involving critical Sobolev exponents have been extensively studied in the literature, beginning with the following celebrated result of Brezis and Nirenberg [1].

**Theorem 1.1.** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$ and consider the problem*

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ \quad u > 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

*where $\lambda > 0$ is a parameter and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of $-\Delta$ in $\Omega$.*

(1) *If $n \geq 4$, then problem (1.1) has a solution for all $\lambda \in (0, \lambda_1)$.*

(2) *If $n = 3$, then there exists $\lambda_* \in [0, \lambda_1]$ such that problem (1.1) has a solution for all $\lambda \in (\lambda_*, \lambda_1)$.*

(3) *If $n = 3$ and $\Omega = B_1(0)$ is the unit ball, then $\lambda_* = \lambda_1/4$ and problem (1.1) has no solution for $\lambda \leq \lambda_1/4$.*

Following [1], Gazzola and Ruf [5] considered the more general problem

$$\begin{cases} -\Delta u = g(x,u) + |u|^{2^*-2}\, u & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $g$ is a Caratheodory function on $\Omega \times \mathbb{R}$ with subcritical growth:

$$\lim_{|t| \to +\infty} \frac{g(x,t)}{|t|^{2^*-1}} = 0 \quad \text{uniformly a.e. on } \Omega.$$

Let $0 < \lambda_1 < \lambda_2 \le \cdots \longrightarrow +\infty$ be the sequence of Dirichet eigenvalues of $-\Delta$ in $\Omega$, repeated according to multiplicity. The following extensions of Theorem 1.1 were obtained in [5].

**Theorem 1.2.** *Assume the following conditions on $g$:*

(1) *for all $\varepsilon > 0$, there exists $a_\varepsilon \in L^{2n/(n+2)}(\Omega)$ such that $|g(x,t)| \le a_\varepsilon(x) + \varepsilon\, |t|^{2^*-1}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;*

(2) $G(x,t) := \displaystyle\int_0^t g(x,\tau)\, d\tau \ge 0$ *for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;*

(3) *there exist $k \in \mathbb{N}$, $\delta, \sigma > 0$, and $\mu \in (\lambda_k, \lambda_{k+1})$ such that $\frac{1}{2}\,(\lambda_k + \sigma)\, t^2 \le G(x,t) \le \frac{1}{2}\,\mu\, t^2$ for a.a. $x \in \Omega$ and $|t| \le \delta$;*

(4) $G(x,t) \ge \frac{1}{2}\,(\lambda_k + \sigma)\, t^2 - \frac{1}{2^*}\, t^{2^*}$ *for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;*

(5) *if* $n = 3$*, there exists a nonempty open subset* $\Omega_0$ *of* $\Omega$ *such that*

$$\lim_{t \to +\infty} \frac{G(x,t)}{t^4} = +\infty \quad \textit{uniformly a.e. on } \Omega_0.$$

*Then problem* (1.2) *has a nontrivial solution.*

**Theorem 1.3.** *Assume conditions* (1), (2), *and*

(5) *there exists* $\delta > 0$, $k \in \mathbb{N}$, *and* $\mu \in (\lambda_k, \lambda_{k+1})$ *such that* $\frac{1}{2}\, \lambda_k\, t^2 \le G(x,t) \le \frac{1}{2}\, \mu\, t^2$ *for a.a.* $x \in \Omega$ *and* $|t| \le \delta$*;*

(6) *there exists* $\sigma \in (0, 1/2^*)$ *such that* $G(x,t) \ge \frac{1}{2}\, \mu\, t^2 - \left(\frac{1}{2^*} - \sigma\right) |t|^{2^*}$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*

(7) *there exists a nonempty open subset* $\Omega_0$ *of* $\Omega$ *such that*

$$\lim_{t \to +\infty} \frac{G(x,t)}{t^{8n/(n^2-4)}} = +\infty \quad \textit{uniformly a.e. on } \Omega_0.$$

*Then problem* (1.2) *has a nontrivial solution.*

Other extensions and generalizations can be found, e.g., in [2, 3, 13]. More recently, Servadei and Valdinoci [9, 11] considered the nonlocal critical problem

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}\, u & \text{in } \Omega \\ \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3}$$

where $s \in (0,1)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n > 2s$ with Lipschitz boundary, $\lambda > 0$ is a parameter, and $2_s^* = 2n/(n-2s)$ is the fractional critical Sobolev exponent. Here $(-\Delta)^s$ is the fractional Laplacian operator, defined, up to a normalization factor, on smooth functions by

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \quad x \in \mathbb{R}^n.$$

Let us recall the definition of a weak solution of problem (1.3). Let

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty \right\}$$

be the usual fractional Sobolev space endowed with the Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left( \|u\|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2},$$

and let

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

Then $H_0^s(\Omega)$ is a closed linear subspace of $H^s(\mathbb{R}^n)$, equivalently renormed by the Gagliardo seminorm

$$[u]_s := \left( \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2},$$

and the imbedding $H_0^s(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r \in [1, 2_s^*]$ and compact for $r \in [1, 2_s^*)$ (see [4]). A weak solution of problem (1.3) is a function $u \in H_0^s(\Omega)$ satisfying

$$\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_\Omega \big( \lambda u(x) + |u(x)|^{2_s^*-2} \, u(x) \big) \, v(x) \, dx$$

for all $v \in H_0^s(\Omega)$.

Let $0 < \lambda_1 < \lambda_2 \leq \cdots \longrightarrow +\infty$ denote the sequence of eigenvalues of the nonlocal eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.4}$$

repeated according to multiplicity (see [10, Proposition 9]). Servadei and Valdinoci obtained the following result in [9, 11, 14, 15].

**Theorem 1.4.** *Problem* (1.3) *has a nontrivial weak solution in the following cases:*

(*a*) $2s < n < 4s$ *and* $\lambda > 0$ *is sufficiently large;*

(*b*) $n = 4s$ *and* $\lambda > 0$ *is not an eigenvalue of* (1.4)*;*

(*c*) $n > 4s$ *and* $\lambda > 0$.

In [12], they also considered the more general problem

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}\, u + f(x,u) & \text{in } \Omega \\ \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.5}$$

where $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$, and obtained the following result.

**Theorem 1.5.** *Assume the following conditions:*

(1) *for all* $M > 0$, $\sup\{|f(x,t)| : x \in \Omega,\ |t| \le M\} < +\infty$*;*

(2) $\displaystyle\lim_{t \to 0} \frac{f(x,t)}{t} = 0$ *uniformly a.e. on* $\Omega$*;*

(3) $\displaystyle\lim_{|t| \to +\infty} \frac{f(x,t)}{|t|^{2_s^*-1}} = 0$ *uniformly a.e. on* $\Omega$.

*If* $n \ge 4s$, *then problem* (1.5) *has a nontrivial weak solution for all* $\lambda \in (0, \lambda_1)$.

In this dissertation we consider the problem

$$\begin{cases} (-\Delta)^s u = g(x,u) + |u|^{2_s^*-2}\, u & \text{in } \Omega \\ \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.6}$$

where $s \in (0,1)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n > 2s$ with Lipschitz boundary, and $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$. Our main result is the following theorem.

**Theorem 1.6.** *Assume the following conditions:*

$(H_1)$ *there exist* $p \in [1, 2_s^*)$ *and* $C > 0$ *such that* $|g(x,t)| \leq C\,(|t|^{p-1} + 1)$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*

$(H_2)$ $G(x,t) := \int_0^t g(x,\tau)\, d\tau \geq 0$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*

$(H_3)$ *there exist* $k \in \mathbb{N}$, $\delta, \sigma > 0$, *and* $\mu \in (\lambda_k, \lambda_{k+1})$ *such that* $\frac{1}{2}\,(\lambda_k + \sigma)\, t^2 \leq G(x,t) \leq \frac{1}{2}\,\mu\, t^2$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$*;*

$(H_4)$ $G(x,t) \geq \frac{1}{2}\,(\lambda_k + \sigma)\, t^2 - \frac{1}{2_s^*}\,|t|^{2_s^*}$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}$*;*

$(H_5)$ *there exists a nonempty open subset* $\Omega_0$ *of* $\Omega$ *such that*

$$\lim_{t \to +\infty} \frac{G(x,t)}{t^{(n+2s)/(n-2s)}} = +\infty \quad \textit{uniformly a.e. on } \Omega_0.$$

*Then problem* (1.6) *has a nontrivial weak solution.*

Theorem 1.6 extends the results of Gazzola and Ruf [5] to the nonlocal case and complements the results of Servadei and Valdinoci [9, 11, 12]. This theorem will be proved after some preliminaries in the next section.

## 2 Preliminaries

A function $u \in H^s_0(\Omega)$ is a weak solution of problem (1.6) if

$$\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\, dx\, dy = \int_\Omega \big(g(x, u) + |u(x)|^{2^*_s - 2}\, u(x)\big)\, v(x)\, dx$$

for all $v \in H^s_0(\Omega)$. Weak solutions coincide with critical points of the $C^1$-functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\, dx\, dy - \int_\Omega \left(G(x, u) + \frac{1}{2^*_s}\, |u|^{2^*_s}\right) dx, \quad u \in H^s_0(\Omega).$$

Recall that $E$ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or the $(\mathrm{PS})_c$ condition for short, if every sequence $(u_j) \subset H^s_0(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. Let

$$S = \inf_{u \in H^s_0(\Omega) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}}\, dx\, dy}{\left(\displaystyle\int_\Omega |u|^{2^*_s}\, dx\right)^{2/2^*_s}} \tag{2.1}$$

be the best constant for the fractional Sobolev imbedding $H^s_0(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$. Proof of Theorem 1.6 will be based on the following proposition.

**Proposition 2.1.** *If* $0 < c < \dfrac{s}{n}\, S^{n/2s}$*, then every* $(\mathrm{PS})_c$ *sequence has a subsequence that converges weakly to a nontrivial critical point of* $E$*.*

*Proof.* Let $(u_j)$ be a $(\mathrm{PS})_c$ sequence. Then

$$E(u_j) = \frac{1}{2} \int_{\mathbb{R}^{2n}} \frac{(u_j(x) - u_j(y))^2}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega \left(G(x, u_j) + \frac{1}{2^*_s}\, |u_j|^{2^*_s}\right) dx = c + \mathrm{o}(1) \tag{2.2}$$

and

$$E'(u_j)\, u_j = \int_{\mathbb{R}^{2n}} \frac{(u_j(x) - u_j(y))^2}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega \left(u_j\, g(x, u_j) + |u_j|^{2^*_s}\right) dx = \mathrm{o}(1)\, \|u_j\|\,. \tag{2.3}$$

Dividing (2.3) by 2 and subtracting from (2.2) gives

$$\int_\Omega \left[\frac{1}{2}\, u_j\, g(x, u_j) - G(x, u_j) + \frac{s}{n}\, |u_j|^{2^*_s}\right] dx = \mathrm{o}(1)\, \|u_j\| + \mathrm{O}(1),$$

which together with $(H_1)$ and the Hölder and Young's inequalities gives

$$\int_\Omega |u_j|^{2^*_s}\, dx \le \mathrm{o}(1)\, \|u_j\| + \mathrm{O}(1).$$

This together with $(H_1)$ and (2.2) implies that $(u_j)$ is bounded in $H^s_0(\Omega)$. So a renamed subsequence converges to some $u$ weakly in $H^s_0(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, 2^*_s)$, and a.e. in $\Omega$. Then $u$ is a critical point of $E$ by the weak continuity of $E'$.

Suppose $u = 0$. Since $(u_j)$ is bounded in $H^s_0(\Omega)$ and converges to 0 in $L^p(\Omega)$, (2.3), $(H_1)$, and (2.1) give

$$\mathrm{o}(1) = \int_{\mathbb{R}^{2n}} \frac{(u_j(x) - u_j(y))^2}{|x - y|^{n+2s}}\, dx\, dy - \int_\Omega |u_j|^{2^*_s}\, dx \ge \|u_j\|^2 \left(1 - \frac{\|u_j\|^{2^*_s - 2}}{S^{2^*_s/2}}\right).$$

If $\|u_j\| \to 0$, then $E(u_j) \to 0$, contradicting $c > 0$, so this implies

$$\|u_j\|^2 \ge S^{n/2s} + \mathrm{o}(1)$$

for a renamed subsequence. Dividing (2.3) by $2^*_s$ and subtracting from (2.2) then gives

$$c = \frac{s}{n} \int_{\mathbb{R}^{2n}} \frac{(u_j(x) - u_j(y))^2}{|x - y|^{n+2s}}\, dx\, dy + \mathrm{o}(1) \ge \frac{s}{n}\, S^{n/2s} + \mathrm{o}(1),$$

contradicting $c < \frac{s}{n}\, S^{n/2s}$. □

To produce $(\mathrm{PS})_c$ sequences with $0 < c < \frac{s}{n}\, S^{n/2s}$, we will use the following linking theorem of Rabinowitz [7, 8].

**Theorem 2.2.** *Let $E$ be a $C^1$-functional on a Banach space $V$, and let $V = V^- \oplus V^+$ be a direct sum decomposition with $\dim V^- < \infty$. Assume that there exist $R > \rho > 0$ and $w_0 \in V^+$ with $\|w_0\| = 1$ such that*

$$\max_{u \in \partial Q} E(u) < \inf_{u \in \partial B_\rho \cap V^+} E(u),$$

*where*

$$Q = \left\{ v + tw_0 : v \in V^-,\ \|v\| \leq R,\ t \in [0, R] \right\}.$$

*Let $\Gamma = \left\{ h \in C(Q, V) : h|_{\partial Q} = id \right\}$ and set*

$$c := \inf_{h \in \Gamma} \max_{u \in h(Q)} E(u).$$

*Then*

$$\inf_{u \in \partial B_\rho \cap V^+} E(u) \leq c \leq \max_{u \in Q} E(u)$$

*and $E$ has a $(\mathrm{PS})_c$ sequence.*

## 3 Proof of Theorem 1.6

In this section we prove Theorem 1.6. Let $e_1, \ldots, e_k$ be $L^2$-orthonormal eigenfunctions for $\lambda_1, \ldots, \lambda_k$, let $H^- = \operatorname{span}\{e_1, \ldots, e_k\}$, and let $H^+ = (H^-)^\perp$. Without loss of generality we may assume that $0 \in \Omega_0$. For $m \in \mathbb{N}$ so large that $B_{4/m} := \{x \in \mathbb{R}^n : |x| < 4/m\} \subset \Omega_0$, let

$$\zeta_m(x) = \begin{cases} 0, & x \in B_{1/m} \\ m|x| - 1, & x \in A_m = B_{2/m} \setminus B_{1/m} \\ 1, & x \in \Omega \setminus B_{2/m}. \end{cases}$$

It is easily seen that

$$|\zeta_m(x) - \zeta_m(y)| \le m|x - y| \quad \forall x, y \in \Omega. \tag{3.1}$$

Let $e_j^m = \zeta_m \, e_j$, $j = 1, \ldots, k$ and let $H_m^- = \operatorname{span}\{e_1^m, \ldots, e_k^m\}$.

**Lemma 3.1.** *Let $f \in L^\infty(\Omega)$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in $\Omega$. Then*

$$\|\zeta_m u\|^2 \le \|u\|^2 + \frac{C\,|f|_\infty^2}{m^{n-2s}},$$

*where $C = C(n, \Omega, s) > 0$.*

To prove this lemma we will need the following estimates from [6].

**Lemma 3.2** ( [6], Lemma 2.3)**.** *Let* $f \in L^q(\Omega)$, $1 < q \leq \infty$ *and let* $u \in H_0^s(\Omega)$ *be a weak solution of* $(-\Delta)^s u = f$ *in* $\Omega$*. Then*

$$|u|_r \leq C\,|f|_q\,,$$

*where*

$$r = \begin{cases} nq/(n-2sq), & 1 < q < n/2s \\ \infty, & n/2s < q \leq \infty \end{cases}$$

*and* $C = C(n, \Omega, s, q) > 0$*. In particular, if* $f \in L^\infty(\Omega)$*, then* $|u|_\infty \leq C\,|f|_\infty$*.*

**Lemma 3.3** ( [6], Lemma 2.5)**.** *Let* $f \in L^q(\Omega)$, $n/2s < q \leq \infty$ *and let* $u \in H_0^s(\Omega)$ *be a weak solution of* $(-\Delta)^s u = f$ *in* $\Omega$*. Then*

$$\|\varphi u\|^2 \leq C\,|f|_q^2 \left(|\varphi|_{2q'}^2 + \|\varphi\|^2\right) \quad \forall \varphi \in L^{2q'}(\Omega) \cap H_0^s(\Omega),$$

*where* $C = C(n, \Omega, s, q) > 0$ *and* $q' = q/(q-1)$*.*

*Proof of Lemma 3.1.* We have

$$\|\zeta_m u\|^2 \le \int_{A_1} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{A_2} \frac{|\zeta_m(x)\,u(x)-\zeta_m(y)\,u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$$
$$+\,2\int_{A_3} \frac{|\zeta_m(x)\,u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy =: I_1 + I_2 + I_3,$$

where $A_1 = B^c_{2/m} \times B^c_{2/m}$, $A_2 = B_{3/m} \times B_{3/m}$, and $A_3 = B_{2/m} \times B^c_{3/m}$. We have $I_1 \le \|u\|^2$. To estimate $I_2$, let

$$\varphi_m(x) = \begin{cases} \zeta_m(x), & x \in B_{3/m} \\ 4 - m|x|, & x \in B_{4/m} \setminus B_{3/m} \\ 0, & x \in B^c_{4/m}. \end{cases}$$

Applying Lemma 3.3 to $\varphi_m$ with $q = \infty$,

$$I_2 \le \|\varphi_m u\|^2 \le C\,|f|^2_\infty \left(|\varphi_m|^2_2 + \|\varphi_m\|^2\right),$$

where $C = C(n, \Omega, s) > 0$. Since $\varphi_m(x) = \varphi_1(mx)$,

$$|\varphi_m|^2_2 = \int_{\mathbb{R}^n} \varphi_m(x)^2\,dx = \int_{\mathbb{R}^n} \varphi_1(mx)^2\,dx = \frac{|\varphi_1|^2_2}{m^n}$$

and

$$\|\varphi_m\|^2 = \int_{\mathbb{R}^{2n}} \frac{|\varphi_m(x) - \varphi_m(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^{2n}} \frac{|\varphi_1(mx) - \varphi_1(my)|^2}{|x-y|^{n+2s}}\, dx\, dy = \frac{\|\varphi_1\|^2}{m^{n-2s}},$$

so

$$I_2 \leq \frac{C\,|f|_\infty^2}{m^{n-2s}}.$$

For $(x, y) \in A_3$, $|x - y| \geq |y| - |x| > |y| - 2/m \geq |y| - (2/3)|y| = |y|/3$, so

$$I_3 \leq C\,|u|_\infty^2 \int_{A_3} \frac{1}{|y|^{n+2s}}\, dx\, dy \leq \frac{C\,|f|_\infty^2}{m^{n-2s}}$$

by Lemma 3.2. The desired conclusion follows. □

**Lemma 3.4.** *We have $e_j^m \to e_j$ in $H_0^s(\Omega)$ as $m \to \infty$, and*

$$\max_{\left\{u \in H_m^-:\, \int_\Omega u^2\, dx = 1\right\}} \|u\|^2 \leq \lambda_k + \frac{C}{m^{n-2s}} \tag{3.2}$$

*for some constant $C > 0$.*

*Proof.* We have

$$\begin{aligned}
\left\|e_j^m - e_j\right\|^2 &= \int_{\mathbb{R}^{2n}} \frac{[(\zeta_m(x)\, e_j(x) - e_j(x)) - (\zeta_m(y)\, e_j(y) - e_j(y))]^2}{|x-y|^{n+2s}}\, dx\, dy \\
&= \int_{\mathbb{R}^{2n}} \frac{|e_j(x)\, [\zeta_m(x) - \zeta_m(y)] + [\zeta_m(y) - 1][e_j(x) - e_j(y)]|^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\leq 2 \int_{\mathbb{R}^{2n}} \frac{e_j(x)^2\, [\zeta_m(x) - \zeta_m(y)]^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\quad + 2 \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y) - 1]^2\, [e_j(x) - e_j(y)]^2}{|x-y|^{n+2s}}\, dx\, dy \\
&\leq 2 \left(|e_j|_\infty^2\, I_1 + I_2\right), \qquad (3.3)
\end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(x) - \zeta_m(y)]^2}{|x-y|^{n+2s}}\, dx\, dy, \qquad I_2 = \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y) - 1]^2\, [e_j(x) - e_j(y)]^2}{|x-y|^{n+2s}}\, dx\, dy.$$

We will show that $I_1$ and $I_2$ go to 0 as $m \to \infty$.

Since $\zeta_m = 1$ in $B_{2/m}^c$,

$$I_1 = \int_{B_{2/m} \times B_{2/m}} \frac{[\zeta_m(x) - \zeta_m(y)]^2}{|x-y|^{n+2s}}\, dx\, dy + 2 \int_{B_{2/m} \times B_{2/m}^c} \frac{[1 - \zeta_m(x)]^2}{|x-y|^{n+2s}}\, dx\, dy =: I_3 + 2I_4.$$

Write

$$I_4 = \int_{B_{2/m} \times B_{3/m}^c} \frac{[1 - \zeta_m(x)]^2}{|x-y|^{n+2s}}\, dx\, dy + \int_{B_{2/m} \times (B_{3/m} \setminus B_{2/m})} \frac{[1 - \zeta_m(x)]^2}{|x-y|^{n+2s}}\, dx\, dy =: I_5 + I_6.$$

Clearly, $I_3$ and $I_6$ are less than or equal to

$$\int_{B_{2/m}\times B_{3/m}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}}\,dx\,dy =: I_7,$$

so $I_1 \le 2I_5 + 3I_7$. To estimate $I_5$ and $I_7$, we change variables from $(x,y)$ to $(x,\xi)$, where $\xi = x-y$. For $(x,y)\in B_{2/m}\times B^c_{3/m}$, $|\xi| \ge |y|-|x| > 1/m$ and hence

$$I_5 \le \int_{B_{2/m}\times B^c_{3/m}} \frac{dx\,dy}{|x-y|^{n+2s}} \le \int_{B_{2/m}\times B^c_{1/m}} \frac{dx\,d\xi}{|\xi|^{n+2s}} \le \frac{C}{m^{n-2s}}. \tag{3.4}$$

For $(x,y)\in B_{2/m}\times B_{3/m}$, $|\xi| \le |x|+|y| < 5/m$ and hence (3.1) gives

$$I_7 \le m^2 \int_{B_{2/m}\times B_{3/m}} \frac{dx\,dy}{|x-y|^{n-2(1-s)}} \le m^2 \int_{B_{2/m}\times B_{5/m}} \frac{dx\,d\xi}{|\xi|^{n-2(1-s)}} \le \frac{C}{m^{n-2s}}.$$

Thus, $I_1 \le C/m^{n-2s}$.

Now we estimate $I_2$. We have

$$I_2 = \int_{\mathbb{R}^n\times B_{2/m}} \frac{[1-\zeta_m(y)]^2\,[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}}\,dx\,dy \le I_8 + 4\,|e_j|_\infty^2\,I_9,$$

where

$$I_8 = \int_{B_{3/m}\times B_{2/m}} \frac{[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}}\,dx\,dy, \qquad I_9 = \int_{B^c_{3/m}\times B_{2/m}} \frac{dx\,dy}{|x-y|^{n+2s}}.$$

Since $e_j \in H^s_0(\Omega)$ and $|B_{3/m} \times B_{2/m}| \to 0$, $I_8 \to 0$. As in (3.4), $I_9 \leq C/m^{n-2s}$. Thus, $I_2 \leq C/m^{n-2s} + \mathrm{o}(1)$.

To prove (3.2), let $v = \sum_{j=1}^{k} \alpha_j\, e_j \in H^-$. By Lemma 3.1,

$$\|\zeta_m v\|^2 \leq \|v\|^2 + \frac{C\, |f|_\infty^2}{m^{n-2s}}, \tag{3.5}$$

where

$$f = (-\Delta)^s v = \sum_{j=1}^{k} \lambda_j\, \alpha_j\, e_j \in H^-.$$

Since $\dim H^- < \infty$,

$$|f|_\infty^2 \leq c_1\, |f|_2^2 = c_1 \sum_{j=1}^{k} \lambda_j^2\, \alpha_j^2 \leq c_1\, \lambda_k^2 \sum_{j=1}^{k} \alpha_j^2 = c_2\, |v|_2^2$$

for some constants $c_1, c_2 > 0$. Since $\|v\|^2 \leq \lambda_k\, |v|_2^2$, this together with (3.5) gives

$$\|\zeta_m v\|^2 \leq \left(\lambda_k + \frac{C}{m^{n-2s}}\right) |v|_2^2\,. \tag{3.6}$$

On the other hand,

$$|\zeta_m v|_2^2 = \int_{\Omega \setminus B_{2/m}} v^2\, dx + \int_{B_{2/m}} (\zeta_m v)^2\, dx \geq \int_\Omega v^2\, dx - \int_{B_{2/m}} v^2\, dx$$

and

$$\int_{B_{2/m}} v^2\, dx \le c_3\, \frac{|v|_\infty^2}{m^n} \le c_4\, \frac{|v|_2^2}{m^n}$$

for some constants $c_3, c_4 > 0$, so

$$|\zeta_m v|_2^2 \ge \left(1 - \frac{c_4}{m^n}\right) |v|_2^2 . \tag{3.7}$$

Combining (3.6) and (3.7) gives

$$\|\zeta_m v\|^2 \le \left(\lambda_k + \frac{C}{m^{n-2s}}\right) |\zeta_m v|_2^2 .$$

Since $H_m^- = \{\zeta_m v : v \in H^-\}$, (3.2) follows from this. □

**Lemma 3.5.** *For all sufficiently large $m$, $H_0^s(\Omega) = H_m^- \oplus H^+$.*

*Proof.* Let $P : H_0^s(\Omega) \to H^-$ be the orthogonal projection. First we show that $PH_m^- = H^-$ for all sufficiently large $m$. Since $PH_m^- \subset H^-$ and $\dim H^- = k$, it suffices to show that $Pe_1^m, \dots, Pe_k^m$ are linearly independent. Suppose not. Then there exists $\alpha^m = (\alpha_1^m, \dots, \alpha_k^m) \in S^{n-1}$ such that

$$\sum_{j=1}^k \alpha_j^m\, Pe_j^m = 0, \tag{3.8}$$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Passing to a subsequence, we may assume that $\alpha^m \to \alpha = (\alpha_1, \ldots, \alpha_n) \in S^{n-1}$. Since $Pe_j^m \to Pe_j = e_j$ by Lemma 3.4, then passing to the limit is (3.8) gives

$$\sum_{j=1}^{k} \alpha_j \, e_j = 0.$$

Since $e_1, \ldots, e_k$ are linearly independent, then $\alpha_1 = \cdots = \alpha_k = 0$, contradicting $\alpha \in S^{n-1}$.

Given $u \in H_0^s(\Omega)$, write $u = v+w$ with $v \in H^-$, $w \in H^+$. Since $PH_m^- = H^-$, there exists $z \in H_m^-$ such that $Pz = v$. Then $u = z + (v - z + w)$ and $v - z + w \in H^+$ since $P(v - z + w) = 0$. Finally, suppose $u \in H_m^- \cap H^+$. Since $u \in H_m^-$,

$$u = \sum_{j=1}^{k} \alpha_j \, e_j^m$$

for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Since $u \in H^+$,

$$Pu = \sum_{j=1}^{k} \alpha_j \, Pe_j^m = 0.$$

Since $Pe_1^m, \ldots, Pe_k^m$ are linearly independent for sufficiently large $m$, then $\alpha_1 = \cdots = \alpha_k = 0$ and hence $u = 0$. □

As in [11], set

$$U_\varepsilon(x) = \frac{c(n,s)\,\varepsilon^{(n-2s)/2}}{(\varepsilon^2 + |x|^2)^{(n-2s)/2}}, \quad \varepsilon > 0,$$

where $c(n,s) > 0$ is such that

$$\|U_\varepsilon\|^2 = |U_\varepsilon|_{2_s^*}^{2_s^*} = S^{n/2s}.$$

Then take a smooth function $\eta_m : \mathbb{R}^n \to [0,1]$ such that $\eta_m = 1$ in $B_{1/4m}$ and $\eta = 0$ outside $B_{1/2m}$, and set $u_\varepsilon^m = \eta_m\, U_\varepsilon$. The following estimates were obtained in [11]:

$$\|u_\varepsilon^m\|^2 = S^{n/2s} + \mathrm{O}(\varepsilon^{n-2s}), \qquad |u_\varepsilon^m|_{2_s^*}^{2_s^*} = S^{n/2s} + \mathrm{O}(\varepsilon^n) \tag{3.9}$$

as $\varepsilon \to 0$. We prove Theorem 1.6 by applying Theorem 2.2 using the direct sum decomposition $H_0^s(\Omega) = H_m^- \oplus H^+$ and taking $w_0 = u_\varepsilon^m$. We will show that

$$\max_{u \in \partial Q_\varepsilon^m} E(u) \le 0 < \inf_{u \in \partial B_\rho \cap H^+} E(u)$$

if $\rho, \varepsilon > 0$ are sufficiently small and $m$, $R > \rho$ are sufficiently large, where

$$Q_\varepsilon^m = \left\{ v + t u_\varepsilon^m : v \in H_m^-,\ \|v\| \le R,\ t \in [0,R] \right\}.$$

Let $\Gamma = \left\{ h \in C(Q_\varepsilon^m, H_0^s(\Omega)) : h|_{\partial Q_\varepsilon^m} = id \right\}$ and set

$$c := \inf_{h \in \Gamma} \max_{u \in h(Q_\varepsilon^m)} E(u).$$

Then Theorem 2.2 gives a $(\mathrm{PS})_c$ sequence with

$$\inf_{u \in \partial B_\rho \cap H^+} E(u) \leq c \leq \max_{u \in Q_\varepsilon^m} E(u).$$

We will show that

$$\max_{u \in Q_\varepsilon^m} E(u) < \frac{s}{n} S^{n/2s} \tag{3.10}$$

if $\varepsilon$ is sufficiently small and apply Proposition 2.1 to obtain a nontrivial critical point of $E$.

**Lemma 3.6.** *If $\rho > 0$ is sufficiently small, then*

$$\inf_{u \in \partial B_\rho \cap H^+} E(u) > 0.$$

*Proof.* By $(H_1)$ and $(H_3)$,

$$G(x,t) \leq \frac{1}{2} \mu t^2 + c_5 |t|^p \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}$$

for some constant $c_5 > 0$. For $u \in H^+$, this together with the fact that $\dfrac{\|u\|^2}{|u|_2^2} \geq \lambda_{k+1}$ and the fractional Sobolev embedding theorem gives

$$\begin{aligned} E(u) &\geq \frac{1}{2}\, \|u\|^2 - \int_\Omega \left(\frac{1}{2}\, \mu\, u^2 + c_5 |u|^p + \frac{1}{2_s^*}\, |u|^{2_s^*}\right) dx \\ &\geq \frac{1}{2} \left(1 - \frac{\mu}{\lambda_{k+1}}\right) \|u\|^2 - c_6 \left( \|u\|^p + \|u\|^{2_s^*} \right) \end{aligned}$$

for some constant $c_6 > 0$. Since $\mu < \lambda_{k+1}$ and $2 < p < 2_s^*$, the desired conclusion follows from this for sufficiently small $\rho$. □

**Lemma 3.7.** *If $m$ and $R > \rho$ are sufficiently large and $\varepsilon > 0$ is sufficiently small, then*

$$\max_{u \in \partial Q_\varepsilon^m} E(u) \leq 0. \tag{3.11}$$

*Proof.* For $v \in H_m^-$ with $\|v\| \leq R$ and $t \in [0, R]$,

$$E(v + t u_\varepsilon^m) = E(v) + E(t u_\varepsilon^m) - 4t \int_{B_{1/m}^c \times B_{1/2m}} \frac{v(x)\, u_\varepsilon^m(y)}{|x - y|^{n+2s}}\, dx\, dy \tag{3.12}$$

since $v = 0$ in $B_{1/m}$ and $u_\varepsilon^m = 0$ outside $B_{1/2m}$.

By Lemma 3.4 and $(H_4)$,

$$\begin{aligned} E(v) &\leq \frac{1}{2}\left(\lambda_k + \frac{C}{m^{n-2s}}\right)\int_\Omega v^2\,dx - \frac{1}{2}\left(\lambda_k + \sigma\right)\int_\Omega v^2\,dx \\ &= -\frac{1}{2}\left(\sigma - \frac{C}{m^{n-2s}}\right)\int_\Omega v^2\,dx \\ &\leq -\frac{\sigma}{4}\int_\Omega v^2\,dx \end{aligned}$$

for sufficiently large $m$. Since $H_m^-$ is finite dimensional, it follows from this that

$$E(v) \leq -c_7\,\|v\|^2 \tag{3.13}$$

for some constant $c_7 > 0$, in particular, $E(v) \leq 0$.

By $(H_2)$ and (3.9),

$$\begin{aligned} E(tu_\varepsilon^m) &\leq \frac{t^2}{2}\,\|u_\varepsilon^m\|^2 - \frac{t^{2_s^*}}{2_s^*}\,|u_\varepsilon^m|_{2_s^*}^{2_s^*} \\ &\leq \left(\frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*}\right) S^{n/2s} + c_8\,R^{2_s^*}\,\varepsilon^{n-2s} \end{aligned} \tag{3.14}$$

for some constant $c_8 > 0$.

The last integral in (3.12) is bounded by

$$c(n,s)\,|v|_\infty\,\varepsilon^{(n-2s)/2}\int_{B_{1/m}^c \times B_{1/2m}} \frac{dx\,dy}{|x-y|^{n+2s}\left(\varepsilon^2 + |y|^2\right)^{(n-2s)/2}}.$$

Changing variables from $(x,y)$ to $(\xi,y)$, where $\xi = x - y$, $|\xi| \geq |x| - |y| > 1/2m$

and hence the integral on the right is bounded by

$$\int_{B^c_{1/2m} \times B_{1/2m}} \frac{d\xi\, dy}{|\xi|^{n+2s} |y|^{n-2s}},$$

and the scaling $(\xi, y) \mapsto (m\xi, my)$ shows that this integral is independent of $m$. Since $\|v\| \le R$, it now follows that

$$\left| \int_{B^c_{1/m} \times B_{1/2m}} \frac{v(x)\, u^m_\varepsilon(y)}{|x-y|^{n+2s}}\, dx\, dy \right| \le c_9\, R\, \varepsilon^{(n-2s)/2} \tag{3.15}$$

for some constant $c_9 > 0$.

Combining (3.12)–(3.15) gives

$$E(v + tu^m_\varepsilon) \le -c_7 \|v\|^2 + \left( \frac{t^2}{2} - \frac{t^{2^*_s}}{2^*_s} \right) S^{n/2s} + c_8\, R^{2^*_s}\, \varepsilon^{n-2s} + c_{10}\, R^2\, \varepsilon^{(n-2s)/2},$$

where $c_{10} = 4c_9$. For $v + tu^m_\varepsilon \in \partial Q^m_\varepsilon \setminus H^-_m$, either $\|v\| = R$ or $t = R$, so it follows from this that there exists $R > \rho$ such that (3.11) holds for all sufficiently small $\varepsilon$. □

Turning to (3.10), by contradiction, suppose

$$\max_{u \in Q^m_{\varepsilon_j}} E(u) \ge \frac{s}{n}\, S^{n/2s}$$

for some sequence $\varepsilon_j \searrow 0$. Since $H^-_m$ is finite dimensional, $Q^m_{\varepsilon_j}$ is compact and hence the above maximum is attained at some point $u_j = v_j + t_j u^m_{\varepsilon_j} \in Q^m_{\varepsilon_j}$. Then

$$\begin{aligned}
\frac{s}{n}\, S^{n/2s} &\le E(u_j)\\
&= E(v_j) + E(t_j u^m_{\varepsilon_j}) - 4t_j \int_{B^c_{1/m} \times B_{1/2m}} \frac{v_j(x)\, u^m_{\varepsilon_j}(y)}{|x-y|^{n+2s}}\, dx\, dy\\
&\le \frac{t_j^2}{2}\, \|u^m_{\varepsilon_j}\|^2 - \frac{t_j^{2^*_s}}{2^*_s}\, |u^m_{\varepsilon_j}|^{2^*_s}_{2^*_s} - \int_\Omega G(x, t_j u^m_{\varepsilon_j})\, dx + c_{11}\, \varepsilon_j^{(n-2s)/2} \qquad (3.16)
\end{aligned}$$

for some constant $c_{11} > 0$ as in the proof of Lemma 3.7. The estimates in (3.9) give

$$\begin{aligned}
\frac{t_j^2}{2}\, \|u^m_{\varepsilon_j}\|^2 - \frac{t_j^{2^*_s}}{2^*_s}\, |u^m_{\varepsilon_j}|^{2^*_s}_{2^*_s} &\le \left(\frac{t_j^2}{2} - \frac{t_j^{2^*_s}}{2^*_s}\right) S^{n/2s} + c_{12}\, \varepsilon_j^{n-2s} \qquad (3.17)\\
&\le \max_{t \in [0,\infty)} \left(\frac{t^2}{2} - \frac{t^{2^*_s}}{2^*_s}\right) S^{n/2s} + c_{12}\, \varepsilon_j^{n-2s}\\
&= \frac{s}{n}\, S^{n/2s} + c_{12}\, \varepsilon_j^{n-2s} \qquad (3.18)
\end{aligned}$$

for some constant $c_{12} > 0$, so (3.16) gives

$$\int_\Omega G(x, t_j u^m_{\varepsilon_j})\, dx \le c_{13}\, \varepsilon_j^{(n-2s)/2} \qquad (3.19)$$

for some constant $c_{13} > 0$.

Since $t_j \in [0, R]$, $t_j$ converges to some $t_0 \in [0, R]$ for a renamed subsequence.

By (3.16), (3.17), and $(H_2)$,

$$\frac{s}{n}\, S^{n/2s} \le \left(\frac{t_j^2}{2} - \frac{t_j^{2_s^*}}{2_s^*}\right) S^{n/2s} + c_{14}\, \varepsilon_j^{(n-2s)/2}$$

for some constant $c_{14} > 0$, and passing to the limit gives

$$\frac{t_0^2}{2} - \frac{t_0^{2_s^*}}{2_s^*} \ge \frac{s}{n}.$$

Since the function $[0, \infty) \to \mathbb{R}$, $t \mapsto \dfrac{t^2}{2} - \dfrac{t^{2_s^*}}{2_s^*}$ attains its maximum value of $\dfrac{s}{n}$ only at $t = 1$, it follows that $t_0 = 1$.

We now show that (3.19) together with $(H_2)$ and $(H_5)$ leads to a contradiction. For $j$ so large that $B_{\varepsilon_j} \subset B_{4/m}$, $(H_2)$ gives

$$\int_\Omega G(x, t_j u_{\varepsilon_j}^m)\, dx \ge \int_{B_{\varepsilon_j}} G(x, t_j U_{\varepsilon_j})\, dx \tag{3.20}$$

since $\eta_m = 1$ in $B_{1/4m}$. Set

$$\varphi(t) = \inf_{x \in \Omega_0,\, \tau \ge t} \frac{G(x, \tau)}{\tau^{(n+2s)/(n-2s)}}, \quad t \ge 0.$$

Then $\varphi$ is nondecreasing,

$$\lim_{t \to +\infty} \varphi(t) = +\infty \tag{3.21}$$

by $(H_5)$, and

$$G(x,t) \geq \varphi(t)\, t^{(n+2s)/(n-2s)} \quad \text{for a.a. } x \in \Omega_0 \text{ and } t \geq 0.$$

Since $B_{\varepsilon_j} \subset B_{4/m} \subset \Omega_0$, this together with (3.20) gives

$$\int_\Omega G(x, t_j u^m_{\varepsilon_j})\, dx \geq \int_{B_{\varepsilon_j}} \varphi(t_j U_{\varepsilon_j})\, (t_j U_{\varepsilon_j})^{(n+2s)/(n-2s)}\, dx. \tag{3.22}$$

For $x \in B_{\varepsilon_j}$,

$$U_{\varepsilon_j}(x) = U_{\varepsilon_j}(|x|) \geq U_{\varepsilon_j}(\varepsilon_j) = c_{15}\, \varepsilon_j^{-(n-2s)/2}$$

for some constant $c_{15} > 0$. Since $t_j \to 1$ and $\varphi$ is nondecreasing, this together with (3.22) gives

$$\begin{aligned} \int_\Omega G(x, t_j u^m_{\varepsilon_j})\, dx &\geq c_{16} \int_{B_{\varepsilon_j}} \varphi(c_{17}\, \varepsilon_j^{-(n-2s)/2})\, \varepsilon_j^{-(n+2s)/2}\, dx \\ &= c_{18}\, \varphi(c_{17}\, \varepsilon_j^{-(n-2s)/2})\, \varepsilon_j^{(n-2s)/2} \end{aligned}$$

for some constants $c_{16}, c_{17}, c_{18} > 0$ and all sufficiently large $j$. This together with (3.19) implies that $\varphi(c_{17}\, \varepsilon_j^{-(n-2s)/2})$ is bounded, contradicting (3.21). This completes the proof of Theorem 1.6.

# 4 Conclusion

In this dissertation we considered the problem

$$\begin{cases} (-\Delta)^s u = g(x,u) + |u|^{2_s^*-2}\, u & \text{in } \Omega \\ \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n > 2s$ with Lipschitz boundary, and $g$ is a Carathéodory function on $\Omega \times \mathbb{R}$. Our main conclusion is that the following conditions are sufficient to guarantee the existence of a nontrivial weak solution to this problem:

$(H_1)$ there exist $p \in [1, 2_s^*)$ and $C > 0$ such that $|g(x,t)| \leq C\,(|t|^{p-1} + 1)$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;

$(H_2)$ $G(x,t) := \displaystyle\int_0^t g(x,\tau)\, d\tau \geq 0$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;

$(H_3)$ there exist $k \in \mathbb{N}$, $\delta, \sigma > 0$, and $\mu \in (\lambda_k, \lambda_{k+1})$ such that $\dfrac{1}{2}\,(\lambda_k + \sigma)\, t^2 \leq G(x,t) \leq \dfrac{1}{2}\,\mu\, t^2$ for a.a. $x \in \Omega$ and $|t| \leq \delta$;

$(H_4)$ $G(x,t) \geq \dfrac{1}{2}\,(\lambda_k + \sigma)\, t^2 - \dfrac{1}{2_s^*}\, |t|^{2_s^*}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$;

$(H_5)$ there exists a nonempty open subset $\Omega_0$ of $\Omega$ such that

$$\lim_{t \to +\infty} \frac{G(x,t)}{t^{(n+2s)/(n-2s)}} = +\infty \quad \text{uniformly a.e. on } \Omega_0.$$

This result extends the results of Gazzola and Ruf in [5] to the nonlocal case and complements those of Servadei and Valdinoci in [9, 11, 12].

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