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Nonlocal Boundary Value Problems for Linear Hyperbolic Systems with Two Independent Variables

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By

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"Nonlocal Boundary Value Problems for Linear Hyperbolic Systems with Two Independent Variables"
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ABSTRACT

Title: *Nonlocal Boundary Value Problems for Linear Hyperbolic Systems with Two Independent Variables*

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Nonlocal boundary value problems in a characteristic rectangle for second order linear hyperbolic systems are considered. There are established:

(i) Unimprovable sufficient conditions for general boundary value problems to possess the Fredholm property;

(ii) Optimal sufficient conditions of unique solvability of general boundary value problems;

(iii) Effective sufficient conditions for doubly periodic problems to possess the Fredholm property;

(iv) Unimprovable sufficient conditions of unique solvability of doubly periodic problems;

(v) Effective sufficient conditions for boundary value problems of periodic type to possess the Fredholm property;

(vi) Unimprovable sufficient conditions of unique solvability of boundary value problems of periodic type;

(vii) Necessary and sufficient conditions of unique solvability of periodic and periodic type problems for linear hyperbolic systems with constant coefficients.

CONTENTS

List of Notations

$\mathbb{Z}$ is the set of integers.

$\mathbb{R}^{m\times n}$ is the space of $m\times n$ matrices $\mathbf{X}=(x_{kl})$ with real components x_{kl} $(k=1,....,m, \quad l=1,...,n)$ and the norm

$$\|x\|=\sum_{k=1}^{m}\sum_{l=1}^{n}|x_{kl}|.$$

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$; I is the unit matrix.

By the absolute value of the matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$ we understand the matrix $|X|=(|x_{ij}|)\in\mathbb{R}^{m\times n}$ with components $|x_{ij}|$ $(i=1,\dots,m; j=1,\dots,n)$.

A matrix $X=(x_{ij})\in\mathbb{R}^{m\times n}$ is called non-negative if $x_{ij}\geq 0$ $(i=1,\dots,m; j=1,\dots,n)$.

The inequalities between the matrices $X=(x_{ij})$ and $Y=(y_{ij})\in\mathbb{R}^{m\times n}$ are understood componentwise, i.e.,

$$X\leq Y\Leftrightarrow x_{ij}\leq y_{ij}\quad (i=1,\dots,m; j=1,\dots,n).$$

If $X_k=(x_{ijk})\in\mathbb{R}^{m\times n}$ $(k=1,\dots,k_0)$, then

$$\max_{1\leq k\leq k_0}X_k=\big(\max_{1\leq k\leq k_0}x_{ijk}\big).$$

$\Omega=[0,\omega_1]\times[0,\omega_2]$.

$$u^{(j,k)}=\frac{\partial^{j+k}u}{\partial x^j\partial y^k}.$$

$\operatorname{supp} f$ is the support of the function f.

$C^k([0,\omega];\mathbb{R}^{m\times n})$ is the Banach space of functions $u:[0,\omega]\to\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i)}$ $(i=0,\dots,k)$, endowed with the norm

$$\|u\|_{C^k([0,\omega])}=\sum_{i=0}^{k}\|u^{(i)}\|_{C([0,\omega])}.$$

$C^k(\Omega;\mathbb{R}^{m\times n})$ is the Banach space of functions $u:\Omega\to;\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i,j)}$ $(i+j=0,\dots,k)$, endowed with the norm

$$\|u\|_{C^k(\Omega)}=\sum_{i+j=0}^{k}\|u^{(i,j)}\|_{C(\Omega)}.$$

$C^{k,l}(\Omega;\mathbb{R}^{m\times n})$ is the Banach space of functions $u:\Omega\to;\mathbb{R}^{m\times n}$, having continuous partial derivatives $u^{(i,j)}$ $(i=0,\dots,k;\ j=0,\dots,l)$, endowed with the norm

$$\|u\|_{C^{k,l}(\Omega)}=\sum_{i=0}^{k}\sum_{j=0}^{l}\|u^{(i,j)}\|_{C(\Omega)}.$$

$C^k_\omega(\mathbb{R};\mathbb{R}^{m\times n})$ is the space of k-times continuously differentiable ω-periodic functions, endowed with the norm

$$\|u\|_{C^k_\omega}=\sum_{i=0}^{k}\|u^{(i)}\|_{C([0,\omega])}.$$

$C^{k,l}_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{m\times n})$ is the Banach space of functions that are ω_1-periodic with respect to the first variable and ω_2-periodic with respect to the second variable, having continuous partial derivatives $u^{(i,j)}$ $(i=0,\dots,k;\ j=0,\dots,l)$, endowed with the norm

$$\|u\|_{C^{k,l}_{\omega_1\,\omega_2}}=\sum_{i=0}^{k}\sum_{j=0}^{l}\|u^{(i,j)}\|_{C(\Omega)}.$$

$L(\Omega)$ is the Banach space of Lebesgue integrable functions $u:\Omega\to\mathbb{R}$ endowed with the norm

$$\|u\|_L=\iint_\Omega|u(\mathbf{x})|\,d\mathbf{x}.$$

$L^\infty(\Omega)$ is the spaces of essentially bounded measurable functions $u:\Omega\to\mathbb{R}$ endowed with the norm

$$\|u\|_{L^\infty}=\operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}|u(\mathbf{x})|.$$

$AC([0,\omega])$ is the Banach space of absolutely continuous functions $u:[0,\omega]\to\mathbb{R}$ endowed with the norm

$$\|u\|_{AC}=|u(0)|+\int_0^\omega\left|u'(x)\right|dx.$$

$AC(\Omega)$ is the Banach space of absolutely continuous functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC(\Omega)} = |u(0,0)| + \sum_{j=0}^{1}\sum_{k=0}^{1}\int_0^{\omega_1}\int_0^{\omega_2} |u^{(j,k)}(x,y)|\, dx\, dy.$$

Introduction

0.1. **Statement of the problem.** Beginning from the 60ies, problems on periodic solutions in a strip or in the large as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see, e.g., [1-13, 18-29]). These problems naturally lead us to boundary value problems in a rectangle with general functional boundary conditions. In [16] the initial–boundary value problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{0.1}$$

$$\begin{aligned} u(0,y) &= \varphi(y) \quad \text{for} \quad y \in [0,\omega_2], \\ h(u(x,\cdot)) &= \psi(x) \quad \text{for} \quad x \in [0,\omega_1] \end{aligned} \tag{0.2}$$

was studied and a complete theory of problem (0.1), (0.2) was constructed.

In [17] for system (0.1) the problem on doubly periodic solutions

$$u(x+\omega_1,y) = u(x,y), \quad u(x,y+\omega_2) = u(x,y) \tag{0.3}$$

was studied.

In the present thesis for system (0.1) we consider the general nonlocal boundary value conditions

$$\begin{aligned} \ell(u(\cdot,y)) &= \varphi(y) \quad \text{for} \quad y \in [0,\omega_2], \\ h(u(x,\cdot)) &= \psi(x) \quad \text{for} \quad x \in [0,\omega_1], \end{aligned} \tag{0.4}$$

where $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ bounded linear (vector) functionals, and $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ are vector functions satisfying the *consistency condition*

$$h(\varphi) = l(\psi).$$

0.2. **Boundary value problems for systems of linear ordinary differential equations with a parameter.** For the system of linear differential equations

$$\frac{dz}{dx} = P(x,y)z + q(x,y) \tag{0.5}$$

depending on a parameter $y \in [0, \omega_2]$, consider the boundary conditions

$$h(z(\cdot, y)) = \varphi(y), \tag{0.6}$$

where $P \in C(\Omega; \mathbb{R}^{n\times n})$, $q \in C(\Omega_1; \mathbb{R}^n)$ and $h : AC([0,\omega]; \mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear (vector) functional.

Along with $(0.5), (0.6)$ (2.1),(2.2) consider the corresponding homogeneous boundary value problem

$$\frac{dz}{dx} = P(x,y)z, \tag{0.7}$$

$$h(z(\cdot, y)) = 0. \tag{0.8}$$

By $Z(x,y)$ denote the fundamental matrix of system (0.7), satisfying the initial condition

$$Z(0,y) = I. \tag{0.9}$$

Lemma 0.1. *h is a bounded linear operator acting from $AC([0,\omega_1])$ to $\mathbb{R}^n$ if and only if h admits the representation*

$$h(z) = H_0 z(0) + \int_0^{\omega_1} H(s) z'(s)\, ds \tag{0.10}$$

where $H_0 \in \mathbb{R}^{n\times n}$ and the matrix function $H \in L^\infty(\Omega; \mathbb{R}^{n\times n})$. Moreover, the matrix H_0 and the matrix function H are uniquely determined by the operator h.

Lemma 0.1 is a particular case of Lemma 2.1_1 from [16], which itself is the result of Dunford-Pettis' theorem ([14],Ch.XI, §1, Theorem 6).

Set:

$$M_0(y) = h\big(Z(\cdot,y)\big) = H_0 + \int_0^{\omega_1} H(s)\frac{\partial}{\partial s}Z(s,y)\,ds$$
$$= H_0 + \int_0^{\omega_1} H(s)P(s,y)Z(s,y)\,ds \tag{0.11}$$

and

$$M(x,y) = H(x)Z(x,y) + \int_x^{\omega_1} H(s)\frac{\partial}{\partial s}Z(s,y)dt$$
$$= H(x)Z(x,y) + \int_x^{\omega_1} H(s)P(s,y)Z(s,y)\,ds. \tag{0.12}$$

Lemma 0.2. *If $P \in C(\Omega;\mathbb{R}^{n\times n})$, then then*

$$Z \in C^{1,0}(\Omega;\mathbb{R}^{n\times n}). \tag{0.13}$$

Lemma 0.2 is a particular case of Lemma 2.2_2 from [16].

Lemma 0.3. *If $P \in C(\Omega;\mathbb{R}^{n\times n})$ and $h : AC([0,\omega];\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear functional, then*

$$M_0 \in C([0,\omega_2];\mathbb{R}^{n\times n}), \quad M \in L^\infty(\Omega;\mathbb{R}^{n\times n})$$

and

$$\int_0^x M(s,y)\,ds \in C(\Omega;\mathbb{R}^{n\times n}).$$

Lemma 0.3 is a particular case of Lemma 2.3_2 from [16].

Lemma 0.4. *Let*

$$\det M_0(y) \neq 0 \quad \textit{for} \quad y \in [0,\omega_2].$$

Then problem (0.5), (0.6) *has a unique solution z admitting the representation*

$$z(x,y) = \int_0^{\omega_1} G(x,s;y)q(s,y)\,ds,$$

where

$$G(x,s;y) = \begin{cases} -Z(x,y)M_0^{-1}(y)M(s,y)Z^{-1}(s,y) \\ \qquad\qquad\qquad +Z(x,y)Z^{-1}(s,y) \quad \text{for} \quad 0 \le s < x \\ -Z(x,y)M_0^{-1}(y)M(s,y)Z^{-1}(s,y) \quad \text{for} \quad x < s \le \omega_1 \end{cases}$$

is Green's matrix of problem (0.7), (0.8).

Lemma 0.4 is a particular case of Theorem 1.1 from [15].

1. **General boundary Value Problems**

Consider the linear hyperbolic system

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y) \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} \ell(u(\cdot,y)) &= \varphi(y) \quad \text{for} \quad y \in [0,\omega_2], \\ h(u(x,\cdot)) &= \psi(x) \quad \text{for} \quad x \in [0,\omega_1], \end{aligned} \tag{1.2}$$

where $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j = 0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $\ell : C([0,\omega_1];\mathbb{R}^n) \to \mathbb{R}^n$ and $h : C([0,\omega_2];\mathbb{R}^n) \to \mathbb{R}^n$ bounded linear (vector) functionals, and $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ are vector functions satisfying the *consistency condition*

$$h(\varphi) = l(\psi). \tag{1.3}$$

Along with problem (1.1), (1.2) consider its corresponding homogeneous problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1.1_0}$$

$$\begin{aligned} \ell(u(\cdot,y)) &= 0 \quad \text{for} \quad y \in [0,\omega_2], \\ h(u(x,\cdot)) &= 0 \quad \text{for} \quad x \in [0,\omega_1]. \end{aligned} \tag{1.2_0}$$

Let $Z_1(x,y)$ be the solution of the initial value problem

$$Z_y = P_1(x,y)Z, \qquad Z(x,0) = I, \tag{1.4}$$

and let $Z_2(x,y)$ be the solution of the initial value problems

$$Z_x = P_2(x,y)Z, \qquad Z(0,y) = I. \tag{1.5}$$

According to Lemma 0.1, the operators ℓ and h uniquely define the matrix functions $L_0 : [0,\omega_1] \to \mathbb{R}^{n\times n}$, $L : [0,\omega_1] \to \mathbb{R}^{n\times n}$ and $H_0 : [0,\omega_2] \to \mathbb{R}^{n\times n}$, $H : [0,\omega_2] \to \mathbb{R}^{n\times n}$ such that

$$\ell(v) = L_0\, v(0) + \int_0^{\omega_1} L(s)\, v'(s)\, ds \quad \text{for} \quad v \in AC([0,\omega_1];\mathbb{R}^n)$$

and

$$h(z) = H_0\, z(0) + \int_0^{\omega_2} H(t)\, z'(t)\, dt \quad \text{for} \quad z \in AC([0,\omega_2];\mathbb{R}^n).$$

Set:

$$M_0(x) = h\big(Z_1(x,\cdot)\big) = H_0 + \int_0^{\omega_2} H(t)\frac{\partial}{\partial t}Z_1(x,t)dt$$
$$= H_0 + \int_0^{\omega_2} H(t)P_1(x,t)Z_1(x,t)\,dt; \tag{1.6}$$

$$N_0(y) = \ell\big(Z_2(\cdot,y)\big) = L_0 + \int_0^{\omega_1} L(s)\frac{\partial}{\partial s}Z_2(s,y)\,ds$$
$$= L_0 + \int_0^{\omega_1} L(s)P_2(s,y)Z_2(s,y)\,ds; \tag{1.7}$$

$$M(x,y) = H(y)Z_1(x,y) + \int_y^{\omega_2} H(t)\frac{\partial}{\partial t}Z_1(x,t)dt$$
$$= H(y)Z_1(x,y) + \int_y^{\omega_2} H(t)P_1(x,t)Z_1(x,t)\,dt$$

and

$$N(x,y) = L(x)Z_2(x,y) + \int_x^{\omega_1} L(s)\frac{\partial}{\partial s}Z_2(s,y)\,ds$$
$$= L(x)Z_2(x,y) + \int_x^{\omega_1} L(s)P_2(s,y)Z_2(s,y)\,ds.$$

If $\det N_0(y) \neq 0$ for some $y \in [0,\omega_2]$, then the problem

$$\frac{dz}{dx} = P_2(x,y)z; \qquad \ell(z(\cdot)) = 0, \tag{1.8}$$

depending on the parameter y, has only the trivial solution, and thus possesses Green's matrix $G_1(x,s;y)$. Notice that

$$G_1(x,s;y) = \begin{cases} -Z_2(x,y)N_0^{-1}(y)N(s,y)Z_2^{-1}(s,y) & \\ \qquad\qquad +Z_2(x,y)Z_2^{-1}(s,y) & \text{for} \quad 0 \le s < x \\ -Z_2(x,y)N_0^{-1}(y)N(s,y)Z_2^{-1}(s,y) & \text{for} \quad x < s \le \omega_1 \end{cases} \tag{1.9}$$

and

$$G_1(x+,x;y) - G_1(x-,x;y) = I.$$

Similarly, if $\det M_0(x) \neq 0$ for some $x \in [0,\omega_1]$, then the problem

$$\frac{dz}{dy} = P_1(x,y)z; \qquad h(z(\cdot)) = 0, \tag{1.10}$$

depending on the parameter x, has only the trivial solution, and thus possesses Green's matrix $G_2(y,t;x)$ such that

$$G_2(y,t;x) = \begin{cases} -Z_1(x,y)M_0^{-1}(x)M(x,t)Z_1^{-1}(x,t) \\ \qquad\qquad\qquad +Z_1(x,y)Z_1^{-1}(x,t) & \text{for} \quad 0 \le t < y \\ -Z_1(x,y)M_0^{-1}(x)M(x,t)Z_1^{-1}(x,t) & \text{for} \quad y < t \le \omega_2 \end{cases} \tag{1.11}$$

and

$$G_2(y+,y;x) - G_2(y,y;x) = I.$$

1.1. **Formulation of the Main Results.**

Theorem 1.1. *Let $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, and let*

$$det\, M_0(x) \neq 0 \quad for \quad x \in [0,\omega_1] \tag{1.12}$$

and

$$det\, N_0(y) \neq 0 \quad for \quad y \in [0,\omega_2]. \tag{1.13}$$

Then problem $(1.1),(1.2)$ has the Fredholm property:

(i) the space of solutions of problem $(1.1_0),(1.2_0)$ is finite dimensional;

(ii) problem $(1.1),(1.2)$ is uniquely solvable for arbitrary $q \in C(\Omega;\mathbb{R}^n)$, and $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$, satisfying condition (1.3), if and only if problem $(1.1_0),(1.2_0)$ has only the trivial solution.

Definition 1.1. A matrix function $Z : \Omega \to \mathbb{R}^{m\times n}$ is called M_0-continuous if it admits in Ω the representation

$$Z(x,y) = M_0(x,y)Z_0(x,y),$$

where $Z_0 : \Omega \to \mathbb{R}^{m\times n}$ is continuous.

Theorem 1.2. *Let the following conditions hold:*
(i) the set $J_{M_0} = \{x \in [0,\omega_1] : det\, M_0(x) = 0\}$ is nowhere dense in $[0,\omega_1]$;
(ii) the set $J_{N_0} = \{y \in [0,\omega_2] : det\, N_0(y) = 0\}$ is nowhere dense in $[0,\omega_2]$;
(iii) $M(x,y)Z_1^{-1}(x,y)P_i(x,y)$ $(i = 0,1,2)$, $M(x,y)Z_1^{-1}(x,y)q(x,y)$, $\psi(x)$ and

$\psi'(x)$ are $M_0(x)$–continuous;

(iii) $N(x,y)Z_2^{-1}(x,y)P_i(x,y)$ $(i=0,1,2)$, $N(x,y)Z_2^{-1}(x,y)q(x,y)$, $\varphi(y)$ and $\varphi'(y)$ are $N_0(y)$–continuous.

Then problem (1.1), (1.2) is uniquely solvable if and only if problem $(1.1_0), (1.2_0)$ has only the trivial solution.

Theorem 1.3. *Let conditions (1.12) and (1.13) hold, let $\Gamma \in \mathbb{R}_+^{n\times n}$ be a nonnegative matrix with the* ***spectral radius less than 1****, and let either $P_1 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$,*

$$\ell(P_1(\cdot,y)v(\cdot)) = 0, \quad \textit{if} \;\; \ell(v(\cdot)) = 0 \quad \textit{for} \;\; v \in C([0,\omega_1]), \quad y \in [0,\omega_2] \tag{1.14}$$

and

$$\int_0^{\omega_2}\int_0^{\omega_1} \Big| G_2(y,t;x)G_1(x,s;t) \times \Big(P_0(s,t) + P_2(s,t)P_1(s,t) - \frac{\partial}{\partial s}P_1(s,t)\Big)\Big| \, ds\, dt \le \Gamma, \tag{1.15}$$

or $P_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n})$,

$$h(P_2(x,\cdot)z(\cdot)) = 0, \quad \textit{if} \;\; h(z(\cdot)) = 0 \quad \textit{for} \;\; z \in C([0,\omega_2]), \quad x \in [0,\omega_1] \tag{1.16}$$

and

$$\int_0^{\omega_1}\int_0^{\omega_2} \Big| G_1(x,s;y)G_2(y,t;s) \times \Big(P_0(s,t) + P_1(s,t)P_2(s,t) - \frac{\partial}{\partial t}P_2(s,t)\Big)\Big| \, dt\, ds \le \Gamma. \tag{1.17}$$

Then problem (1.1) (1.2) is uniquely solvable.

Lastly, consider the case $n = 1$, i.e. the equation

$$u_{xy} = p_0(x,y)u + p_1(y)u_x + p_2(x)u_y + q(x,y). \tag{1.18}$$

By $g_1(x,s)$ and $g_2(y,t)$, respectively, denote Green's functions of the problems

$$\frac{dz}{dx} = p_2(x)z; \qquad \ell(z(\cdot)) = 0 \tag{1.19}$$

and

$$\frac{dz}{dy} = p_1(y)z; \qquad h(z(\cdot)) = 0. \tag{1.20}$$

Corollary 1.1. *Let the following inequalities hold*

$$\ell\Big(\exp\Big(\int_0^y p_1(t)\,dt\Big)\Big) \neq 0, \tag{1.21}$$

$$h\Big(\exp\Big(\int_0^x p_2(s)\,ds\Big)\Big) \neq 0. \tag{1.22}$$

Then problem (1.18) (1.2) *has the Fredholm property. Furthermore, if*

$$\max_{(x,y)\in\Omega} \int_0^{\omega_2} \int_0^{\omega_1} \Big| g_2(y,t) g_1(x,s) \big(p_0(s,t) + p_1(t)p_2(s)\big)\Big|\, ds\, dt < 1, \tag{1.23}$$

then problem (1.18), (1.2) *is uniquely solvable.*

Remark 1.1. In Theorem 1.1 conditions (1.12) and (1.13), and in Theorem 1.2 conditions (*i*) and (*ii*) are essential and they cannot be weakened. Indeed, consider the problem

$$u_{xy} = p^2(y)u + p(y)u_x - p(y)u_y + q(x,y), \tag{1.24}$$

$$u(0,y) - u(2\pi,y) = 0, \quad u(x,0) - u(x,2\pi) = 0, \tag{1.25}$$

where $p \in C^\infty_{2\pi}(\mathbb{R})$ such that

$$\int_0^{2\pi} p(t)\,dt \neq 0. \tag{1.26}$$

For problem (1.24), (1.25),

$$M(x) = 1 - \exp\Big(\int_0^{2\pi} p(t)\,dt\Big)$$

and

$$N(y) = 1 - \exp\big(2\pi\, p(y)\big).$$

In view of (1.26), condition (1.12) holds, since

$$M(x) = 1 - \exp\Big(\int_0^{2\pi} p(t)\,dt\Big) \neq 0 \quad \text{for} \quad x \in [0,\omega_1]. \tag{1.27}$$

As for condition (1.13),

$$N(y) = 1 - \exp\big(2\pi\, p(y)\big) = 0 \quad \text{if and only if } \ p(y) = 0. \tag{1.28}$$

Set:

$$J_p = \{y \in [0, \omega_2] : p(y) = 0\}.$$

Along with (1.24) consider its corresponding homogeneous equation

$$u_{xy} = p^2(y)u + p(y)u_x - p(y)u_y. \tag{1.24$_0$}$$

Then problem $(1.24_0), (1.25)$ has:

(i) unique classical solution (which is the trivial solution) if and only if

$$\operatorname{supp} p = [0, \omega_2], \tag{1.29}$$

i.e. J_p is a nowhere dense set in $[0, \omega_2]$;

(ii) infinite dimensional space of classical solutions if

$$\operatorname{supp} p \neq [0, \omega_2]; \tag{1.30}$$

(iii) unique absolutely continuous solution (which is the trivial solution) if and only if

$$\operatorname{mes} J_p = 0; \tag{1.31}$$

(iv) infinite dimensional space of *nonclassical* absolutely continuous solutions if

$$\operatorname{mes} J_p > 0; \tag{1.32}$$

(v) unique classical solution (which is the trivial solution) and infinite dimensional space of *nonclassical* absolutely continuous solutions if

$$\operatorname{supp} p = [0, \omega_2] \quad \text{and} \quad \operatorname{mes} J_p > 0. \tag{1.33}$$

If p is a function satisfying (1.29) such that

$$|p| \in C^\infty_{2\pi}(\mathbb{R}), \tag{1.34}$$

$$\begin{aligned} J_p^- &= \{y \in [0, \omega_2] : p(y) < 0\} \neq \emptyset, \\ J_p^+ &= \{y \in [0, \omega_2] : p(y) > 0\} \neq \emptyset, \end{aligned} \tag{1.35}$$

and $q(x,y) = |p(y)|$, then problem (1.24), (1.25) has the unique absolutely continuous solution

$$u(y) = \int_y^{y+2\pi} \frac{\exp\Big(\int_t^y p(t)\,dt\Big)}{\exp\Big(-\int_0^{2\pi} p(t)\,dt\Big) - 1}\,\mathrm{sign}(p(t))\,dt$$

which is *nonclassical*, since the derivative

$$u'(y) = p(y)u(y) + \mathrm{sign}(p(y))$$
$$= p(y) \int_y^{y+2\pi} \frac{\exp\Big(\int_t^y p(t)\,dt\Big)}{\exp\Big(-\int_0^{2\pi} p(t)\,dt\Big) - 1}\,\mathrm{sign}(p(t))\,dt + \mathrm{sign}(p(y))$$

is discontinuous at along the set $\overline{J_p^-} \cap \overline{J_p^+}$. Consequently, problem (1.24), (1.25) has no classical solution despite the fact that the coefficients of equation (1.24) are infinitely smooth.

If p is a function satisfying (1.29), (1.34) and (1.35), and $q(x,y) \equiv 1$, then problem (1.24), (1.25) does not have even an absolutely continuous solution, despite the fact that the homogeneous problem (1.24_0), (1.25) has only the trivial solution.

1.2. **Proofs of the Main Results.** *Proof of Theorem* 1.1. Let $\gamma : [0,\omega_1] \to \mathbb{R}$ be a continuous function such that the problem

$$\frac{dz}{dx} = \gamma(x)z, \tag{1.36}$$
$$\ell(z(\cdot)) = 0 \tag{1.37}$$

has only the trivial solution. By $G_\gamma(x,s)$ denote Green's function of problem (1.36), (1.37).

Let $u(x,y)$ be a solution of problem (1.1), (1.2), and let

$$v(x,y) = u_x(x,y) - \gamma(x)u(x,y). \tag{1.38}$$

Then v is a solution of the problem

$$v_y = P_1(x,y)v + \big(P_0(x,y)+\gamma(x)P_1(x,y)\big)u(x,y) \\ +\big(P_2(x,y)-\gamma(x)I\big)\,u_y(x,y)+q(x,y), \tag{1.39}$$

$$h(v(x,\cdot)) = \psi'(x) - \gamma(x)\psi(x) \quad \text{for} \quad x\in[0,\omega_1]. \tag{1.40}$$

Consequently, v admits the representations

$$\begin{aligned} M_0(x)v(x,0) =&\psi'(x)-\gamma(x)\psi(x) \\ &+\int_0^{\omega_2} M(x,t)Z_1^{-1}(x,t)\Big(\big(P_0(x,t)+\gamma(x)P_1(x,t)\big)\,u(x,t) \\ &+\big(P_2(x,t)-\gamma(x)I\big)\,u_t(x,t)+q(x,t)\Big)\,dt, \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} v(x,y) =&Z_1(x,y)M_0^{-1}(x)\big(\psi'(x)-\gamma(x)\psi(x)\big) \\ &+\int_0^{\omega_2} G_2(y,t;x)\Big(\big(P_0(x,t)+\gamma(x)P_1(x,t)\big)\,u(x,t) \\ &+\big(P_2(x,t)-\gamma(x)I\big)\,u_t(x,t)+q(x,t)\Big)\,dt, \end{aligned} \tag{1.42}$$

where

$$G_2(y,t;x) = \begin{cases} -Z_1(x,y)M_0^{-1}(x)M(x,t)Z_1^{-1}(x,t) & \\ \qquad\qquad +Z_1(x,y)Z_1^{-1}(x,t) & \text{for} \quad 0\le t<y \\ -Z_1(x,y)M_0^{-1}(x)M(x,t)Z_1^{-1}(x,t) & \text{for} \quad y<t\le\omega_2 \end{cases}$$

is Green's matrix of the problem

$$\frac{dz}{dy} = P_1(x,y)z; \qquad h(z(\cdot)) = 0.$$

$(1.38), (1.42)$ imply that u is a solution of the problem

$$\begin{aligned} u_x(x,y) = \gamma(x)u(x,y)+&Z_1(x,y)M_0^{-1}(x)\big(\psi'(x)-\gamma(x)\psi(x)\big) \\ &+\int_0^{\omega_2} G_2(y,t;x)\Big(\big(P_0(x,t)+\gamma(x)P_1(x,t)\big)\,u(x,t) \\ &+\big(P_2(x,t)-\gamma(x)I\big)\,u_t(x,t)+q(x,t)\Big)\,dt, \end{aligned} \tag{1.43}$$

$$\ell(u(\cdot,y)) =\varphi(y). \tag{1.44}$$

Consequently, u admits the representation

$$
\begin{aligned}
u(x,y) =& Z_2(x,y)N_\gamma^{-1}(y)\varphi(y) \\
&+ \int_0^{\omega_1}\int_0^{\omega_2} G_\gamma(x,s)\,G_2(y,t;s)\Big(\big(P_0(s,t)+\gamma(s)P_1(s,t)\big)\,u(s,t) \\
&+\big(P_2(s,t)-\gamma(s)I\big)\,u_t(s,t)+q(s,t)\Big)\,dt\,ds, \qquad (1.45)
\end{aligned}
$$

where Z_γ is the solution of the initial value problem

$$
Z_x = \gamma(x)\,Z, \qquad Z(0,y) = I \qquad (1.46)
$$

and

$$
N_\gamma(y) = \ell\big(Z_\gamma(\cdot,y)\big).
$$

Similarly to (1.41) and (1.42), we get the representations:

$$
\begin{aligned}
N_0(y)u_y(0,y) = \varphi'(y) + \int_0^{\omega_1} N(y,s)Z_2^{-1}(s,y)\Big(P_0(s,y)u(s,y) \\
+P_1(s,y)\,u_s(s,y)+q(s,y)\Big)\,ds \qquad (1.47)
\end{aligned}
$$

and

$$
\begin{aligned}
u_y(x,y) = Z_2(x,y)N_0^{-1}(y)\varphi'(y) + \int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y)u(s,y) \\
+P_1(s,y)\,u_s(s,y)+q(s,y)\Big)\,ds, \qquad (1.48)
\end{aligned}
$$

where

$$
G_1(x,s;y) = \begin{cases} -Z_2(x,y)N_0^{-1}(y)N(s,y)Z_2^{-1}(s,y) \\ \qquad\qquad +Z_2(x,y)Z_2^{-1}(s,y) & \text{for} \quad 0 \le s < x \\ -Z_2(x,y)N_0^{-1}(y)N(s,y)Z_2^{-1}(s,y) & \text{for} \quad x < s \le \omega_1 \end{cases}
$$

is Green's matrix of the problem

$$
\frac{dz}{dx} = P_2(x,y)z; \qquad \ell(z(\cdot)) = 0.
$$

By substituting (1.43) and (1.45) into (1.48) we get:

$$
\begin{aligned}
u_y(x,y) =& Q_2(x,y) \\
&+ \int_0^{\omega_1}\int_0^{\omega_2} \Big(K_{21}(x,y,s,t)\,u(s,t) + K_{22}(x,y,s,t)\,u_t(s,t)\Big)\,dt\,ds, \qquad (1.49)
\end{aligned}
$$

where

$$\begin{aligned}K_{21}(x,y,s,t) =& G_1(x,s;y)P_1(s,y)G_2(y,t;s)P_0(s,t)\\&+\int_0^{\omega_1} G_1(x,\xi;y)P_0(\xi,y)G_\gamma(\xi,s)\,d\xi\\&\times G_2(y,t;s)\big(P_0(s,t)+\gamma(s)P_1(s,t)\big),\end{aligned}$$

$$\begin{aligned}K_{22}(x,y,s,t) =& G_1(x,s;y)P_1(s,y)G_2(y,t;s)P_2(s,t)\\&+\int_0^{\omega_1} G_1(x,\xi;y)P_0(\xi,y)G_\gamma(\xi,s)\,d\xi\\&\times G_2(y,t;s)\big(P_2(s,t)-\gamma(s)I\big),\end{aligned}$$

$$\begin{aligned}Q_2(x,y) =& Z_2(x,y)N_0^{-1}(y)\varphi'(y)\\&+\int_0^{\omega_1}\int_0^{\omega_2} G_1(x,s;y)P_1(s,y)G_2(y,t;s)q(s,t)\,ds\,dt\\&+\int_0^{\omega_1} G_1(x,s;y)P_1(s,y)Z_1(s,y)M_0^{-1}(s)\psi'(s)ds\\&+\int_0^{\omega_1} G_1(x,s;y)P_0(s,y)\Big(\int_0^{\omega_1} G_\gamma(s,\xi)Z_1(\xi,t)M_0^{-1}(\xi)\big(\psi'(\xi)-\gamma(\xi)\psi(\xi)\big)d\xi\\&+Z_\gamma(s)N_\gamma^{-1}\psi(t)\Big)\,ds+\int_0^{\omega_1} G_1(x,s;y)q(s,y)\,ds.\end{aligned}$$

Set:

$$K_{11}(x,y,s,t) = G_\gamma(x,s)\,G_2(y,t;s)\big(P_0(s,t)+\gamma(s)P_1(s,t)\big),$$
$$K_{12}(x,y,s,t) = G_\gamma(x,s)\,G_2(y,t;s)\big(P_2(s,t)-\gamma(s)I\big),$$

$$\begin{aligned}Q_1(x,y) =& Z_2(x,y)N_\gamma^{-1}(y)\varphi(y)\\&+\int_0^{\omega_1}\int_0^{\omega_2} G_\gamma(x,s)\,G_2(y,t;s)q(s,t)\,dt\,ds.\end{aligned}$$

In the space $C(\Omega;\mathbb{R}^{2n})$ consider the Fredholm integral equation

$$z(x,y) = \int_0^{\omega_1}\int_0^{\omega_2} K(x,y,s,t)z(s,t)dsdt + Q(x,y), \tag{1.50}$$

where

$$K(x,y,s,t) = \begin{pmatrix} K_{11}(x,y,s,t) & K_{12}(x,y,s,t) \\ \\ K_{21}(x,y,s,t) & K_{22}(x,y,s,t) \end{pmatrix}$$

and

$$Q(x,y) = \begin{pmatrix} Q_1(x,y) \\ \\ Q_2(x,y) \end{pmatrix}.$$

It is not difficult to show that if

$$z = \begin{pmatrix} z_1(x,y) \\ z_2(x,y) \end{pmatrix}$$

is a solution of equation (1.50), then $z_1 \in C^{1,1}(\Omega; \mathbb{R}^n)$, $z_2(x,y) = z_{1\,y}(x,y)$, and $u(x,y) = z_1(x,y)$ is a solution of problem (1.1), (1.2).

Thus problem (1.1), (1.2) is equivalent to the Fredholm integral equation (1.50). Consequently, problem (1.1), (1.2) has the Fredholm property. $\square$

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1.

Proof of Theorem 1.3. By Theorem 1.1, in view of conditions (1.12) and (1.13), problem (1.1), (1.2) has the Fredholm property. In order to prove the theorem it remains to show that problem $(1.1_0), (1.2_0)$ has only the trivial solution.

Let $P_1 \in C^{1,0}(\Omega; \mathbb{R}^{n\times n})$, and let $u(x,y)$ be a solution of problem $(1.1_0), (1.2_0)$. Set:

$$v(x,y) = u_x(x,y) - P_1(x,y)u(x,y). \tag{1.51}$$

Then v is a solution of the system

$$v_x = P_2(x,y)v + \Big(P_0(x,y) + P_2(x,y)P_1(x,y) - \frac{\partial}{\partial x}P_1(x,y)\Big)u(x,y). \tag{1.52}$$

Furthermore, in view of (1.14) and (1.2_0), v satisfies the boundary condition

$$h\big(v(x,\cdot)\big) = h\big(u_x(x,\cdot)\big) - h\big(P_1(x,\cdot)u(x,\cdot)\big) = 0. \tag{1.53}$$

(1.52) and (1.53) imply the representation

$$v(x,y) = \int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y) + P_2(s,y)P_1(s,y) - \frac{\partial}{\partial s}P_1(s,y)\Big)u(s,y)\,ds. \tag{1.54}$$

(1.51) and (1.54) imply that u is a solution of the problem

$$u_x(x,y) = P_1(x,y)u(x,y) + \int_0^{\omega_1} G_1(x,s;y)\Big(P_0(s,y) + P_2(s,y)P_1(s,y) - \frac{\partial}{\partial s}P_1(s,y)\Big)u(s,y)\,ds. \tag{1.55}$$

$$h(u(x,\cdot)) = 0. \tag{1.56}$$

Consequently, u admits the representation

$$u(x,y) = \int_0^{\omega_2}\int_0^{\omega_1} G_2(y,t;x)\,G_1(x,s;t) \times\Big(P_0(s,t) + P_2(s,t)P_1(s,t) - \frac{\partial}{\partial s}P_1(s,t)\Big)\,u(s,t)\,ds\,dt, \tag{1.57}$$

whence, in view of (1.15), we get

$$\begin{aligned} |u(x,y)| &\le \int_0^{\omega_2}\int_0^{\omega_1} \Big|G_2(y,t;x)\,G_1(x,s;t)\Big(P_0(s,t) + P_2(s,t)P_1(s,t) - \frac{\partial}{\partial s}P_1(s,t)\Big)\Big|\,ds\,dt \max_{(s,t)\in\Omega}|u(s,t)| \\ &\le \Gamma \max_{(s,t)\in\Omega}|u(s,t)|, \end{aligned} \tag{1.58}$$

and

$$\max_{(x,y)\in\Omega}|u(x,y)| \le \Gamma \max_{(x,y)\in\Omega}|u(x,y)|. \tag{1.59}$$

Taking into account inequality (1.59) and the fact that the spectral radius of Γ is less than 1, we get

$$\max_{(x,y)\in\Omega}|u(x,y)| = 0.$$

The other case can be considered similarly. $\Box$

Corollary 1.1 is a particular case of Theorem 1.3

2. **Doubly Periodic Solutions**

In this section we study the problem on doubly periodic solutions

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{2.1}$$

$$u(x+\omega_1, y) = u(x,y), \quad u(x, y+\omega_2) = u(x,y). \tag{2.2}$$

Throughout the section it will be assumed that

$$P_i \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n}) \quad (i=0,1,2), \qquad q \in C_{\omega_1\,\omega_2}(\mathbb{R}^2;\mathbb{R}^n). \tag{2.3}$$

It is rather obvious that if (2.3) holds, then conditions (2.3) are equivalent to the conditions

$$u(0,y) = u(\omega_1, y) \quad \text{for} \quad y \in [0,\omega_2], \quad u(x,0) = u(x,\omega_2) \quad \text{for} \quad x \in [0,\omega_1]. \tag{2.4}$$

More precisely, the restriction of an arbitrary solution u of problem (2.1), (2.2) on the domain Ω is a solution of problem (2.1), (2.4) and vice versa, every solution of problem (2.1), (2.4) admits a doubly periodic continuation, which is a solution of problem (2.1), (2.2).

Throughout this section by $G_1(x,s;y)$ we denote Green's matrix of the problem

$$\frac{dz}{dx} = P_2(x,y)z; \qquad z(0) - z(\omega_1) = 0,$$

and by $G_2(y,t;x)$ we denote Green's matrix of the problem

$$\frac{dz}{dy} = P_1(x,y)z; \qquad z(0) - z(\omega_2) = 0.$$

2.1. **Formulation of the Main Results.**

Theorem 2.1. *Let the following inequalities hold:*

$$\det\big(I - Z_1(x,\omega_2)\big) \neq 0 \quad \textit{for} \quad x \in [0,\omega_1], \tag{2.5}$$

$$\det\big(I - Z_2(\omega_1,y)\big) \neq 0 \quad \textit{for} \quad y \in [0,\omega_2]. \tag{2.6}$$

Then problem (2.1) (2.2) *has the Fredholm property.*

Set:

$$M_0(x) = I - Z_1(x, \omega_2), \quad P_{01}(x) = \int_0^{\omega_2} P_0(x, t)\, dt$$
$$N_0(y) = I - Z_2(\omega_1, y), \quad P_{02}(y) = \int_0^{\omega_1} P_0(s, y)\, ds.$$

By $H_1 : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ and $H_2 : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ denote the solutions of the matrix differential equations

$$\frac{\partial H_1(x, y)}{\partial y} = Z_2(x, y) N_0^{-1}(y) Z_2^{-1}(x, y) P_{02}(y)\, H_2(x, y)$$

and

$$\frac{\partial H_2(x, y)}{\partial x} = Z_1(x, y) M_0^{-1}(x) Z_1^{-1}(x, y) P_{01}(x)\, H_1(x, y),$$

satisfying the initial conditions

$$H_1(x, 0) = I \quad \text{for } x \in \mathbb{R}$$

and

$$H_2(0, y) = I \quad \text{for } y \in \mathbb{R}.$$

Remark 2.1. Theorem 2.1 is an improvement of Theorem 1.1 from [17], since Theorem 1.1, along with conditions (2.5) and (2.6), requires the fulfillment of one of the inequalities

$$\det\big(H_1(x, \omega_2) - I\big) \neq 0 \quad \text{for} \quad x \in [0, \omega_1],$$

or

$$\det\big(H_2(\omega_1, y) - I\big) \neq 0 \quad \text{for} \quad y \in [0, \omega_2].$$

holds

Corollary 2.1. *Let $P_1(x, y) \equiv P_1(x)$, $P_2(x, y) \equiv P_2(y)$, and let*

$$\det\big(I - \exp(\omega_2 P_1(x))\big) \neq 0 \quad \textit{for} \quad x \in [0, \omega_1], \tag{2.7}$$
$$\det\big(I - \exp(\omega_1 P_2(y))\big) \neq 0 \quad \textit{for} \quad y \in [0, \omega_2]. \tag{2.8}$$

Then problem (2.1) (2.2) *has the Fredholm property.*

Corollary 2.2. *Let*

$$\alpha_i(x) = \int_0^{\omega_2} p_{1ii}(x,t)\,dt \neq 0 \quad \textit{for} \quad x \in [0,\omega_1] \quad (i=1,\dots,n). \tag{2.9}$$

and

$$\beta_i(y) = \int_0^{\omega_1} p_{2ii}(s,y)\,ds \neq 0 \quad \textit{for} \quad y \in [0,\omega_2] \quad (i=1,\dots,n), \tag{2.10}$$

*and let there exist **nonnegative** continuous functions* $a_{ik} \in C([0,\omega_1])$ $\big(b_{ik} \in C([0,\omega_2])\big)$ $(i \neq k;\ i,k=1,\dots,n)$ *such that the spectral radius the matrix* $A(x) = (a_{ik}(x))_{i,k=1}^n$ $\big(B(y) = (b_{ik}(y))_{i,k=1}^n\big)$, *where* $a_{ii}(x) \equiv 0$ $\big(b_{ii}(y) \equiv 0\big)$ $(i=1,\dots,n)$, *is less than* 1 *for every* $x \in [0,\omega_1]$ $\big(y \in [0,\omega_2]\big)$. *Furthermore, let the inequalities*

$$\int_y^{y+\omega_2} e^{\int_t^y p_{1ii}(x,\tau)d\tau}|p_{1ik}(x,t)|dt \\ \leq a_{ik}(x)\big(1-e^{-\alpha_i(x)}\big) \quad (i \neq k;\ i,k=1,\dots,n) \tag{2.11}$$

and

$$\int_x^{x+\omega_1} e^{\int_s^x p_{2ii}(\xi,y)d\xi}|p_{2ik}(s,y)|ds \\ \leq b_{ik}(y)\big(1-e^{-\beta_i(y)}\big) \quad (i \neq k;\ i,k=1,\dots,n) \tag{2.12}$$

hold on Ω. *Then problem* (2.1) (2.2) *has the Fredholm property.*

Corollary 2.3. *Let the inequalities*

$$\sigma_{1i}p_{1ii}(x,y) \leq \alpha_{ii}, \quad \sigma_{2i}p_{2ii}(x,y) \leq \beta_{ii} \quad \textit{for} \quad (x,y) \in \Omega \quad (i=1,\dots,n), \tag{2.13}$$

$$|p_{1ik}(x,y)| \leq \alpha_{ik},\ |p_{2ik}(x,y)| \leq \beta_{ik} \ \textit{for} \ (x,y) \in \Omega \ (i \neq k;\ i,k=1,\dots,n), \tag{2.14}$$

hold, where $\sigma_{ij} \in \{-1,1\}$ $(i=1,2;\ j=1,\dots,n)$, α_{ik} *and* β_{ik} $(i,k=1,\dots)$ *are constants, and the real parts of eigenvalues of the matrices* $(\alpha_{ik})_{i,k=1}^n$ *and* $(\beta_{ik})_{i,k=1}^n$ *are negative. Then problem* (2.1) (2.2) *has the Fredholm property.*

For an arbitrary matrix $P \in \mathbb{R}^{n\times n}$ set:

$$\widehat{P} = \frac{1}{2}\big(P + P^T\big),$$

where P^T is the transpose of the matrix P.

Corollary 2.4. *Let there exist* $\sigma_i \in \{-1, 1\}$ $(i = 1, 2)$ *such that*

$$\sigma_1 \widehat{P}_1(x, y) \text{ is positive definite for } (x, y) \in \Omega, \tag{2.15}$$

and

$$\sigma_2 \widehat{P}_2(x, y) \text{ is positive definite for } (x, y) \in \Omega. \tag{2.16}$$

Then problem (2.1) (2.2) *has the Fredholm property.*

Theorem 2.2. *Let conditions* (2.5) *and* (2.6) *hold, let* $\Gamma \in \mathbb{R}_+^{n\times n}$ *be a non-negative matrix with the* ***spectral radius less than 1****, and let either* $P_1 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R}^{n\times n})$ *and*

$$\int_0^{\omega_2}\int_0^{\omega_1} \Big| G_2(y, t; x) G_1(x, s; t) \times \Big(P_0(s, t) + P_2(s, t) P_1(s, t) - \frac{\partial}{\partial s} P_1(s, t) \Big) \Big|\, ds\, dt \le \Gamma, \tag{2.17}$$

or $P_2 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R}^{n\times n})$ *and*

$$\int_0^{\omega_1}\int_0^{\omega_2} \Big| G_1(x, s; y) G_2(y, t; s) \times \Big(P_0(s, t) + P_1(s, t) P_2(s, t) - \frac{\partial}{\partial t} P_2(s, t) \Big) \Big|\, dt\, ds \le \Gamma. \tag{2.18}$$

Then problem (2.1) (2.2) *is uniquely solvable.*

Consider the system

$$u_{xy} = P_0(x, y) u + u_x + u_y + q(x, y). \tag{2.19}$$

Theorem 2.3. *Let*

$$P_0(x, y) = P_0^T(x, y) \text{ for } (x, y) \in \Omega, \tag{2.20}$$

and let one of the following three conditions hold:

(i) $P_0 \in C^{1,0}_{\omega_1\omega_2}(\mathbb{R}^2; \mathbb{R}^{n\times n})$,

$$P_0(x, y) + \frac{1}{2}\frac{\partial P_0(x, y)}{\partial x} \text{ is negative semi–definite for } (x, y) \in \Omega, \tag{2.21}$$

$$\int_0^{\omega_1} P_0(s, y)\, ds \text{ is negative definite for } y \in [0, \omega_2]; \tag{2.22}$$

(ii) $P_0 \in C^{0,1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$,

$$P_0(x,y) + \frac{1}{2}\frac{\partial P_0(x,y)}{\partial y} \ \ is\ \ negative\ \ semi\text{–}definite\ \ \ for\ \ \ (x,y)\in\Omega, \tag{2.23}$$

$$\int_0^{\omega_2} P_0(x,t)\,dt\ \ is\ \ negative\ \ definite\ \ \ for\ \ \ x\in[0,\omega_1]; \tag{2.24}$$

(ii) $P_0 \in C^{1}_{\omega_1\omega_2}(\mathbb{R}^2;\mathbb{R}^{n\times n})$,

$$P_0(x,y) + \frac{1}{4}\Big(\frac{\partial P_0(x,y)}{\partial x} + \frac{\partial P_0(x,y)}{\partial y}\Big)$$
$$is\ \ negative\ \ semi\text{–}definite\ \ \ for\ \ \ (x,y)\in\Omega, \tag{2.25}$$

$$\int_0^{\omega_1}\int_0^{\omega_2} P_0(s,t)\,dt\,ds\ \ is\ \ negative\ \ definite. \tag{2.26}$$

Then the problem (2.19), (2.2) *is uniquely solvable.*

Now consider the case, where $P_i(x,y) \equiv P_i$ $(i = 0,1,2)$, i.e. consider the equation

$$u_{xy} = P_0u + P_1u_x + P_2u_y + q(x,y). \tag{2.27}$$

Theorem 2.4. *Let*

$$\det\big(I - \exp(\omega_2P_1)\big) \neq 0, \tag{2.28}$$

$$\det\big(I - \exp(\omega_1P_2)\big) \neq 0. \tag{2.29}$$

Then problem (2.27), (2.2) *has the Fredholm property. Moreover, problem* (2.27), (2.2) *is uniquely solvable* ***if and only if***

$$\det\Big(P_0 + i\frac{2\pi}{\omega_2}mP_1 + i\frac{2\pi}{\omega_1}kP_2 + mk\,I\Big) \neq 0\ \ \ for\ \ \ m,\,k\in\mathbb{Z}. \tag{2.30}$$

Lastly, consider the equation

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y + q(x,y). \tag{2.31}$$

Theorem 2.5. *Let*

$$p_0(y)\,p_1(y)\,p_2(y) < 0\ \ \ for\ \ \ y\in[0,\omega_1]. \tag{2.32}$$

Then problem (2.31), (2.2) *is uniquely solvable.*

2.2. **Proofs of the Main Results.** *Proof of Theorem* 2.1. Theorem 2.1 is a particular case of Theorem 1.1 and its proof is similar to the proof of Theorem 1.1.

Proof of Corollary 2.1. Corollary 2.1 is a direct consequence of Theorem 2.1, since

$$Z_1(x,y) = \exp\big(yP_1(x)\big),$$
$$Z_2(x,y) = \exp\big(xP_2(y)\big).$$

□

Proof of Corollary 2.2. By Lemma 2.9 from [16], the conditions of Corollary 2.2 guarantee the fulfillment of inequalities (2.5) and 2.6. Then, by Theorem 2.1, problem $(2.1),(2.2)$ has the Fredholm property. □

Proof of Corollary 2.3. By Lemma 2.10 from [16], the conditions of Corollary 2.3 guarantee the fulfillment of all of the conditions of Corollary 2.2. Then, by Corollary 2.2, problem $(2.1),(2.2)$ has the Fredholm property. □

Proof of Corollary 2.4. Consider the problem

$$\frac{dz}{dy} = P_1(x,y)z, \tag{2.33}$$
$$z(0) - z(\omega_2) = 0. \tag{2.34}$$

Let z be an arbitrary solution for some $x \in [0,\omega_1]$. Then we have

$$\frac{dz(y)}{dy} \cdot z(y) = P_1(x,y)z(y) \cdot z(y) \tag{2.35}$$

and

$$\int_0^{\omega_2} \frac{dz(y)}{dy} \cdot z(y)\,dy = \int_0^{\omega_2} \widehat{P}_1(x,y)z(y) \cdot z(y)\,dy. \tag{2.36}$$

In view of (2.34), we have

$$\int_0^{\omega_2} \frac{dz(y)}{dy} \cdot z(y)\,dy = \frac{1}{2}\int_0^{\omega_1} \frac{d}{dy}\big(z(y)\cdot z(y)\big)\,dy = 0. \tag{2.37}$$

(2.36), (2.37) and (2.15) immediately imply

$$z(y) \equiv 0.$$

Thus, problem (2.33), (2.34) has only the trivial solution for every $x \in [0, \omega_1]$.

Similarly one can show that the problem

$$\frac{dz}{dx} = P_2(x, y)z, \tag{2.38}$$

$$z(0) - z(\omega_1) = 0 \tag{2.39}$$

has only the trivial solution for every $y \in [0, \omega_2]$.

Consequently, inequalities (2.5) and (2.6) hold. By Theorem 2.1, problem (2.1), (2.2) has the Fredholm property. □

Proof of Theorem 2.2. Theorem 2.2 is a particular case of Theorem 1.3, since

$$P_1(0, y)u(0, y) - P_1(\omega_1, y)u(\omega_1, y) = 0 \quad \text{if} \quad u(0, y) - u(\omega_1, y) = 0$$

and

$$P_2(x, 0)u(x, 0) - P_2(x, \omega_2)u(x, \omega_2) = 0 \quad \text{if} \quad u(x, 0) - u(x, \omega_2) = 0.$$

□

Proof of Theorem 2.3. By Corollary 2.1, problem (2.19), (2.2) has the Fredholm property. In order to prove the theorem, it remains to show that the homogeneous system

$$u_{xy} = P_0(x, y)u + u_x + u_y \tag{2.19$_0$}$$

has only the trivial solution satisfying conditions (2.2).

Indeed, let u be an arbitrary solution of problem $(2.19_0), (2.2)$, and let conditions (i) of Theorem 2.3 hold. Then we have

$$\begin{aligned} u_{xy}(x,y)\cdot u(x,y) =& P_0(x,y)u(x,y)\cdot u(x,y) \\ &+ u_x(x,y)\cdot u(x,y) + u_y(x,y)\cdot u(x,y), \\ u_{xy}(x,y)\cdot u_x(x,y) =& P_0(x,y)u(x,y)\cdot u_x(x,y) \\ &+ u_x(x,y)\cdot u_x(x,y) + u_y(x,y)\cdot u_x(x,y), \end{aligned}$$

$$\begin{aligned} &\int_0^{\omega_1}\int_0^{\omega_2} u_{xy}(x,y)\cdot u(x,y)\,dy\,dx \\ &\quad = \int_0^{\omega_1}\int_0^{\omega_2}\Big(P_0(x,y)u(x,y)\cdot u(x,y) + u_x(x,y)\cdot u(x,y) \\ &\qquad + u_y(x,y)\cdot u(x,y)\Big)\,dy\,dx, \end{aligned} \tag{2.40}$$

$$\begin{aligned} &\int_0^{\omega_1}\int_0^{\omega_2} u_{xy}(x,y)\cdot u_x(x,y)\,dy\,dx \\ &\quad = \int_0^{\omega_1}\int_0^{\omega_2}\Big(P_0(x,y)u(x,y)\cdot u_x(x,y) + u_x(x,y)\cdot u_x(x,y) \\ &\qquad + u_y(x,y)\cdot u_x(x,y)\Big)\,dy\,dx. \end{aligned} \tag{2.41}$$

After integrating by parts and taking into account conditions (2.2), from (2.40) and (2.41), respectively, we get

$$\begin{aligned} &-\int_0^{\omega_1}\int_0^{\omega_2} u_x(x,y)\cdot u_y(x,y)\,dy\,dx \\ &\qquad = \int_0^{\omega_1}\int_0^{\omega_2} P_0(x,y)u(x,y)\cdot u(x,y)\,dy\,dx, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} &\int_0^{\omega_1}\int_0^{\omega_2}\Big(-\frac{1}{2}P_{0\,x}(x,y)u(x,y)\cdot u(x,y) + u_x(x,y)\cdot u_x(x,y) \\ &\qquad + u_y(x,y)\cdot u_x(x,y)\Big)\,dy\,dx = 0. \end{aligned} \tag{2.43}$$

(2.42) and (2.43) yield

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(-\Big[P_0(x,y)+\frac{1}{2}P_{0\,x}(x,y)\Big]u(x,y)\cdot u(x,y)$$
$$+u_x(x,y)\cdot u_x(x,y)\Big)\,dy\,dx=0. \tag{2.44}$$

(2.21) and (2.44) imply that

$$u_x(x,y)\equiv 0,\quad \text{and}\quad u(x,y)\equiv u(y),$$

while (2.22) and (2.44) imply

$$u(y)\equiv 0.$$

Case (ii) can be proved similarly by establishing the equality

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(-\Big[P_0(x,y)+\frac{1}{2}P_{0\,y}(x,y)\Big]u(x,y)\cdot u(x,y)$$
$$+u_y(x,y)\cdot u_y(x,y)\Big)\,dy\,dx=0. \tag{2.45}$$

Case (iii) can be proved by establishing the equality

$$\int_0^{\omega_1}\int_0^{\omega_2}\Big(-2\Big[P_0(x,y)+\frac{1}{4}P_{0\,x}(x,y)+\frac{1}{4}P_{0\,y}(x,y)\Big]u(x,y)\cdot u(x,y)$$
$$+u_x(x,y)\cdot u_x(x,y)+u_y(x,y)\cdot u_y(x,y)\Big)\,dy\,dx=0, \tag{2.46}$$

which is the sum of (2.44) (2.45). □

Proof of Theorem 2.4. By Corollary 2.1, problem (2.27), (2.2) has the Fredholm property. In order to prove the theorem, it remains to show that the homogeneous system

$$u_{xy}=P_0u+P_1u_x+P_2u_y \tag{2.27_0}$$

has only the trivial solution satisfying conditions (2.2).

Indeed, let u be an arbitrary solution of problem (2.27_0), (2.2). Then u admits the representation

$$u(x,y)=\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}u_{mk}e^{i\left(m\frac{2\pi}{\omega_1}x+k\frac{2\pi}{\omega_2}y\right)}. \tag{2.47}$$

After substituting (2.47) in system (2.27_0), we get

$$\Big(P_0 + i\frac{2\pi}{\omega_2}mP_1 + i\frac{2\pi}{\omega_1}kP_2 + mk\,I\Big)u_{mk} = 0 \quad \text{for} \quad m,\, k \in \mathbb{Z}. \tag{2.48}$$

(2.30) implies

$$u_{mk} = 0 \quad \text{for} \quad m,\, k \in \mathbb{Z}.$$

Consequently, $u(x, y) \equiv 0$. $\square$

Proof of Theorem 2.5. By (2.32), we have

$$p_1(y) \neq 0 \quad \text{for} \quad y \in [0, \omega_2] \tag{2.49}$$

and

$$p_2(y) \neq 0 \quad \text{for} \quad y \in [0, \omega_2]. \tag{2.50}$$

In view of (2.49) and (2.50), by Corollary 2.1, problem (2.31), (2.2) has the Fredholm property. In order to prove the theorem, it remains to show that the homogeneous system

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y \tag{2.31_0}$$

has only the trivial solution satisfying conditions (2.2).

Indeed, let u be an arbitrary solution of problem (2.31_0), (2.2). Then u admits the representation

$$u(x, y) = \sum_{m\in\mathbb{Z}} u_m(y)e^{im\frac{2\pi}{\omega_1}x}. \tag{2.51}$$

After substituting (2.51) in equation (2.31_0), we get

$$u'_m(y) = \rho_m(y)u_m(y), \quad u_m(0) = u_m(\omega_2), \quad m \in \mathbb{Z}, \tag{2.52}$$

where

$$\rho_m(y) = \frac{p_0(y) + im\frac{2\pi}{\omega_1}p_1(y)}{im\frac{2\pi}{\omega_1} - p_2(y)} = -\frac{p_0(y)p_2(y) - m^2\frac{4\pi^2}{\omega_1^2}p_1(y)}{m^2\frac{4\pi^2}{\omega_1^2} + p_2^2(y)} - im\frac{2\pi}{\omega_1}\,\frac{p_0(y) + p_1(y)p_2(y)}{m^2\frac{4\pi^2}{\omega_1^2} + p_2^2(y)}, \qquad m \in \mathbb{Z}. \tag{2.53}$$

(2.32) yields

$$\operatorname{Re}\rho_m(y) \neq 0 \quad \text{for} \quad y \in [0, \omega_2], \quad m \in \mathbb{Z} \tag{2.54}$$

and, consequently,

$$1 - \exp\Big(\int_0^{\omega_2} \rho_m(y)\,dy\Big) \neq 0, \quad m \in \mathbb{Z}. \tag{2.55}$$

(2.55) immediately implies

$$u_m(y) \equiv 0, \quad m \in \mathbb{Z},$$

and, consequently, $u(x, y) \equiv 0$. $\square$

3. **Problems of Periodic Type**

In this section we study the problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{3.1}$$

$$u(0,y) = Au(\omega_1,y) + \varphi(y), \quad u(x,0) = Bu(x,\omega_2) + \psi(x), \tag{3.2}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, and $\varphi \in C^1([0,\omega_2];\, \mathbb{R}^n)$ and $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ are vector functions satisfying the *consistency condition*

$$\varphi(0) - B\varphi(\omega_2) = \psi(0) - A\psi(\omega_1).$$

Throughout this section by $G_1(x,s;y)$ we denote Green's matrix of the problem

$$\frac{dz}{dx} = P_2(x,y)z; \qquad z(0) - Az(\omega_1) = 0,$$

and by $G_2(y,t;x)$ we denote Green's matrix of the problem

$$\frac{dz}{dy} = P_1(x,y)z; \qquad z(0) - Bz(\omega_2) = 0.$$

3.1. **Formulation of the Main Results.**

Theorem 3.1. *Let the following inequalities hold:*

$$\det\big(I - Z_1(x,\omega_2)\,A\big) \neq 0 \quad \textit{for} \quad x \in [0,\omega_1], \tag{3.3}$$

$$\det\big(I - Z_2(\omega_1,y)\,B\big) \neq 0 \quad \textit{for} \quad y \in [0,\omega_2]. \tag{3.4}$$

Then problem (3.1) (3.2) *has the Fredholm property.*

Corollary 3.1. *Let $P_1(x,y) \equiv P_1(x)$, $P_2(x,y) \equiv P_2(y)$, and let*

$$\det\big(I - \exp(\omega_2 P_1(x))\,B\big) \neq 0 \quad \textit{for} \quad x \in [0,\omega_1], \tag{3.5}$$

$$\det\big(I - \exp(\omega_1 P_2(y))\,A\big) \neq 0 \quad \textit{for} \quad y \in [0,\omega_2]. \tag{3.6}$$

Then problem (3.1) (3.2) *has the Fredholm property.*

Corollary 3.2. *Let there exist* $\sigma_i \in \{-1, 1\}$ $(i = 1, 2)$ *such that*

$$\sigma_1\big(A^T A - I\big) \ \text{ is positive semi–definite}, \tag{3.7}$$

$$\sigma_1 \widehat{P}_1(x, y) \ \text{ is positive definite for } \ (x, y) \in \Omega \tag{3.8}$$

and

$$\sigma_2\big(B^T B - I\big) \ \text{ is positive semi–definite}, \tag{3.9}$$

$$\sigma_2 \widehat{P}_2(x, y) \ \text{ is positive definite for } \ (x, y) \in \Omega. \tag{3.10}$$

Then problem (3.1) (3.2) *has the Fredholm property.*

Theorem 3.2. *Let conditions* (3.3) *and* (3.4) *hold, let* $\Gamma \in \mathbb{R}_+^{n\times n}$ *be a nonnegative matrix with the* ***spectral radius less than 1****, and let either*

$$P_1 \in C^{1,0}(\Omega; \mathbb{R}^{n\times n}), \ \ P_1(0, y) = P_1(\omega_1, y), \ \ P_1(\omega_1, y)\, A = A\, P_1(\omega_1, y), \tag{3.11}$$

and

$$\int_0^{\omega_2}\int_0^{\omega_1} \Big| G_2(y, t; x) G_1(x, s; t) \times \Big(P_0(s, t) + P_2(s, t)P_1(s, t) - \frac{\partial}{\partial s}P_1(s, t)\Big)\Big|\, ds\, dt \le \Gamma, \tag{3.12}$$

or

$$P_2 \in C^{0,1}(\Omega; \mathbb{R}^{n\times n}), \ \ P_2(x, 0) = P_2(x, \omega_2), \ \ P_2(x, \omega_2)\, B = B\, P_2(x, \omega_2), \tag{3.13}$$

and

$$\int_0^{\omega_1}\int_0^{\omega_2} \Big| G_1(x, s; y) G_2(y, t; s) \times \Big(P_0(s, t) + P_1(s, t)P_2(s, t) - \frac{\partial}{\partial t}P_2(s, t)\Big)\Big|\, dt\, ds \le \Gamma. \tag{3.14}$$

Then problem (3.1) (3.2) *is uniquely solvable.*

Consider the system

$$u_{xy} = P_0(x, y)u + u_x + u_y + q(x, y). \tag{3.15}$$

Theorem 3.3. *Let*

$$P_0(x,y) = P_0^T(x,y) \quad for \quad (x,y) \in \Omega, \tag{3.16}$$

$$A^T A - I \ \ be \ \ positive \ \ semi\text{–}definite, \tag{3.17}$$

$$B^T B - I \ \ be \ \ positive \ \ semi\text{–}definite, \tag{3.18}$$

$$I - A^T A - B^T + B^T A^T AB \ \ be \ \ positive \ \ semi\text{–}definite, \tag{3.19}$$

and let one of the following three conditions hold:
(i) $P_0 \in C(\Omega; \mathbb{R}^{n\times n})$ *and*

$$P_0(\omega_1, y) - A^t P_0(0,y)A \ \ is \ \ positive \ \ semi\text{–}definite \quad for \quad y \in [0,\omega_2], \tag{3.20}$$

$$P_0(x,y) + \frac{1}{2}\frac{\partial P_0(x,y)}{\partial x} \ \ is \ \ negative \ \ semi\text{–}definite \quad for \quad (x,y) \in \Omega, \tag{3.21}$$

$$\int_0^{\omega_1} P_0(s,y)\,ds \ \ is \ \ negative \ \ definite \quad for \quad y \in [0,\omega_2]; \tag{3.22}$$

(ii) $P_0 \in C^{0,1}(\Omega; \mathbb{R}^{n\times n})$ *and*

$$P_0(x,\omega_2) - B^t P_0(x,\omega_2)B \ \ is \ \ positive \ \ semi\text{–}definite \quad for \quad x \in [0,\omega_1], \tag{3.23}$$

$$P_0(x,y) + \frac{1}{2}\frac{\partial P_0(x,y)}{\partial y} \ \ is \ \ negative \ \ semi\text{–}definite \quad for \quad (x,y) \in \Omega, \tag{3.24}$$

$$\int_0^{\omega_2} P_0(x,t)\,dt \ \ is \ \ negative \ \ definite \quad for \quad x \in [0,\omega_1]; \tag{3.25}$$

(ii) $P_0 \in C^1(\Omega; \mathbb{R}^{n\times n})$ *and*

$$P_0(\omega_1, y) - A^t P_0(0,y)A \ \ is \ \ positive \ \ semi\text{–}definite \quad for \quad y \in [0,\omega_2], \tag{3.26}$$

$$P_0(x,\omega_2) - B^t P_0(x,\omega_2)B \ \ is \ \ positive \ \ semi\text{–}definite \quad for \quad x \in [0,\omega_1], \tag{3.27}$$

$$P_0(x,y) + \frac{1}{4}\Big(\frac{\partial P_0(x,y)}{\partial x} + \frac{\partial P_0(x,y)}{\partial y}\Big) \\ is \ \ negative \ \ semi\text{–}definite \quad for \quad (x,y) \in \Omega, \tag{3.28}$$

$$\int_0^{\omega_1}\int_0^{\omega_2} P_0(s,t)\,dt\,ds \ \ is \ \ negative \ \ definite. \tag{3.29}$$

Then the problem (3.15), (3.2) *is uniquely solvable.*

Now consider the case, where $P_i(x,y) \equiv P_i$ $(i=0,1,2)$ and $A=I$, i.e. consider the problem

$$u_{xy} = P_0 u + P_1 u_x + P_2 u_y + q(x,y), \tag{3.30}$$

$$u(0,y) = u(\omega,y) + \varphi(y), \quad u(x,0) = Bu(x,\omega_2) + \psi(x). \tag{3.31}$$

Theorem 3.4. *Let*

$$\det\big(I - \exp(\omega_2 P_1)B\,\big) \neq 0, \tag{3.32}$$

$$\det\big(I - \exp(\omega_1 P_2)\big) \neq 0. \tag{3.33}$$

Then problem (3.30), (3.31) *has the Fredholm property. Moreover, problem* (3.30), (3.31) *is uniquely solvable* ***if and only if***

$$\det\big(I - \exp(\omega_1 \Lambda_k\big)\,B\big) \neq 0 \quad \text{for} \quad k \in \mathbb{Z}, \tag{3.34}$$

where

$$\Lambda_k = \Big(i\frac{2\pi}{\omega_1}kI - P_2\Big)\Big(P_0 + i\frac{2\pi}{\omega_1}kP_1\Big).$$

Lastly, consider the case $n=1$. For the equation

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y + q(x,y) \tag{3.35}$$

consider the boundary conditions

$$u(0,y) = u(\omega,y) + \varphi(y), \quad u(x,0) = bu(x,\omega_2) + \psi(x). \tag{3.36}$$

Theorem 3.5. *Let the following inequalities hold:*

$$p_0(y)\,p_1(y)\,p_2(y) < 0 \quad \text{for} \quad y \in [0,\omega] \tag{3.37}$$

and

$$(1-b)\,p_1(y) \geq 0 \quad \text{for} \quad y \in [0,\omega]. \tag{3.38}$$

Then problem (3.35), (3.36) *is uniquely solvable.*

3.2. **Proofs of the Main Results.** *Proof of Theorem* 3.1. Theorem 3.1 is a particular case of Theorem 1.1 and its proof is similar to the proof of Theorem 1.1.

The proof of Corollary 3.1 is similar to the proof of Corollary 2.1.

The proof of Corollary 3.2 is similar to the proof of Corollary 2.2.

The proof of Corollary 3.1 is similar to the proof of Corollary 2.1.

Proof of Theorem 3.2. Theorem 3.2 is a particular case of Theorem 1.3, since conditions (3.13) and (3.15) imply (1.14) and (1.16). □

The proof of Theorem 3.3 is similar to the proof of Theorem 2.3.

Proof of Theorem 3.4. In view of (3.32) and (3.33), by Theorem 3.1, problem (3.30), (3.31) has the Fredholm property. Consider the homogeneous problem

$$u_{xy} = P_0 u + P_1 u_x + P_2 u_y, \tag{3.30_0}$$

$$u(0, y) = u(\omega, y), \quad u(x, 0) = Bu(x, \omega_2). \tag{3.31_0}$$

In order to prove the theorem, it remains to show that the homogeneous problem $(3.30_0), (3.31_0)$ has only the trivial solution.

Indeed, let u be an arbitrary solution of problem $(3.30_0), (3.31_0)$. Then u admits the representation

$$u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y) e^{ik \frac{2\pi}{\omega_1} x}. \tag{3.39}$$

After substituting (3.39) in system (3.30_0), we get

$$u_k'(y) = \Lambda_k\, u_k(y), \quad u_k(0) = B\, u_k(\omega_2), \qquad k \in \mathbb{Z}. \tag{3.40}$$

By (3.34), problem (3.40) has only the trivial solution for every $k \in \mathbb{Z}$. Consequently, $u(x, y) \equiv 0$. □

Proof of Theorem 3.5. In view of (3.37) and (3.38), by Theorem 3.1, problem (3.35), (3.36) has the Fredholm property. In order to prove the theorem, it remains to show that the homogeneous problem

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y, \tag{3.35$_0$}$$

$$u(0,y) = u(\omega,y), \quad u(x,0) = bu(x,\omega_2) \tag{3.36$_0$}$$

has only the trivial solution.

Indeed, let u be an arbitrary solution of problem (3.35_0), (3.36_0). Then u admits the representation

$$u(x,y) = \sum_{m\in\mathbb{Z}} u_m(y)e^{im\frac{2\pi}{\omega_1}x}. \tag{3.41}$$

After substituting (3.41) in equation (3.35_0), we get

$$u_m'(y) = \rho_m(y)u_m(y), \quad u_m(0) = u_m(\omega_2), \ \ m\in\mathbb{Z}, \tag{3.42}$$

where

$$\rho_m(y) = \frac{p_0(y) + im\frac{2\pi}{\omega_1}p_1(y)}{im\frac{2\pi}{\omega_1} - p_2(y)} = -\frac{p_0(y)p_2(y) - m^2\frac{4\pi^2}{\omega_1^2}p_1(y)}{m^2\frac{4\pi^2}{\omega_1^2} + p_2^2(y)} - im\frac{2\pi}{\omega_1}\,\frac{p_0(y) + p_1(y)p_2(y)}{m^2\frac{4\pi^2}{\omega_1^2} + p_2^2(y)}, \qquad m\in\mathbb{Z}. \tag{3.43}$$

(3.37) and (3.38) yield

$$(1-b)\mathrm{Re}\,\rho_m(y) > 0 \quad \text{for} \quad y\in[0,\omega_2], \quad m\in\mathbb{Z} \tag{3.44}$$

and, consequently,

$$1 - b\,\exp\Big(\int_0^{\omega_2} \rho_m(y)\,dy\Big) \neq 0, \quad m\in\mathbb{Z}. \tag{3.45}$$

(3.45) immediately implies

$$u_m(y) \equiv 0, \quad m\in\mathbb{Z},$$

and, consequently, $u(x,y)\equiv 0$. □

Theorem 3.2 is a particular case of Theorem 1.3, since conditions (3.13) and (3.15) imply (1.14) and (1.16). □

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