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On a class of critical N-Laplacian problems

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by

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Abstract

Title:

On a class of critical N-Laplacian problems

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We establish some existence results for a class of critical N-Laplacian problems in a bounded domain in $\mathbb{R}^N$. In the absence of a suitable direct sum decomposition, we use an abstract linking theorem based on the $\mathbb{Z}_2$-cohomological index to obtain a nontrivial critical point.

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List of Notations

We say that $f(j) = O(g(j))$ if there exists a constant $M > 0$ such that for j large enough

$$f(j) \leq Mg(j)$$

We say that $f(j) = o(g(j))$ if for every $\epsilon > 0$, for j large enough

$$f(j) \leq \epsilon g(j)$$

Let Ω denote a smooth bounded domain in $\mathbb{R}^N$. The α-th derivative of a function u is denoted $D^\alpha u$. The gradient is denoted as ∇u, the divergence as $\nabla\cdot$, and the p-Laplacian operator Δ_p on a function u is $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$. We have the following notation for each function space.

$C^k(\Omega)$ - the space of k times continuously differentiable functions on Ω.

$C^k(\overline{\Omega})$ - the space of the space of k times continuously differentiable functions on Ω whose derivatives up to order k are uniformly continuous on subsets of Ω.

$C_0^k(\Omega)$ - the space of k times continuously differential functions with support in Ω.

There are analogous definitions for infinitely differentiable functions ($k = \infty$) and continuous functions ($k = 0$).

$L^p(\Omega)$ - the space of measurable functions u with finite L^p norm

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx\right)^{1/p}$$

When $p = \infty$, we have instead

$$\|u\|_\infty := \operatorname*{ess\,sup}_\Omega |u|$$

$L^p_{loc}(\Omega)$ - the space of measurable functions in $L^p(K)$ for all compact subsets $K \subset \Omega$.

$W^{k,p}(\Omega)$ - the space of k times weakly differentiable functions on Ω with the derivatives up to order k being in $L^p(\Omega)$. It has the norm

$$\|u\|_{k,p} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_p$$

$W_0^{k,p}(\Omega)$ - the closure of the space $C_0^\infty(\Omega)$ in the $W^{k,p}$ norm.

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I would like to thank my family for their support. Without them, I would not have been able to start this path to create a better future for my loved ones and myself.

Finally, I would like to thank my fiancée Mari for always being there for me, through the happy times and the rough times. I am able to complete my studies thanks to your love and encouragement.

Dedication

I dedicate this work to my mother Sandra, and to my future wife Mari.
Thank you for making my journey in this life the most fulfilling.

1 Main Problem

The aim of this dissertation is to establish some existence results for the class of critical N-Laplacian problems

$$\begin{cases} -\Delta_N u = h(u)\, e^{\alpha\, |u|^{N'}} & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $N' = N/(N-1)$ is the Hölder conjugate of N, and h is a continuous function such that

$$\lim_{|t|\to\infty} h(t) = 0 \tag{1.2}$$

and

$$0 < \beta := \liminf_{|t|\to\infty} t h(t) < \infty. \tag{1.3}$$

This problem is motivated by the Trudinger-Moser inequality

$$\sup_{\substack{u \in W_0^{1,N}(\Omega) \\ \|u\| \leq 1}} \int_\Omega e^{\alpha_N\, |u|^{N'}}\, dx < \infty, \tag{1.4}$$

where $W_0^{1,N}(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^N dx \right)^{1/N},$$

$$\alpha_N = N\, \omega_{N-1}^{1/(N-1)},$$

and

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

is the area of the unit sphere in $\mathbb{R}^N$ (see Trudinger [14] and Moser [10]). Problem (1.1) is critical with respect to this inequality and hence lacks compactness.

In dimension $N = 2$, problem (1.1) is semilinear and has been extensively studied in the literature (see, e.g., [2, 3, 4, 6]). In dimensions $N \geq 3$, this problem is quasilinear and has been studied mainly when

$$G(t) := \int_0^t h(s)\, e^{\alpha\, |s|^{N'}} ds \leq \lambda\, |t|^N \quad \text{for small } t \tag{1.5}$$

for some $\lambda \in (0, \lambda_1)$ (see, e.g., [1, 5, 9]). Here

$$\lambda_1 = \inf_{u \in W_0^{1,N}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^N dx}{\displaystyle\int_\Omega |u|^N dx} \tag{1.6}$$

is the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_N u = \lambda |u|^{N-2} u & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The case $h(t) = \lambda |t|^{N-2} t$ with $\lambda > 0$, for which $\beta = \infty$, was recently studied in Yang and Perera [15]. The remaining case, where $N \geq 3$, $\lambda \geq \lambda_1$, and $\beta < \infty$, does not seem to have been studied in the literature. This case is covered in our results here, which are for large $\beta < \infty$ and allow $\lambda \geq \lambda_1$ in (1.5).

We go over some background material before stating and proving the main results.

2 Sobolev Spaces

We first establish the setting for finding weak solutions of a Partial Differential Equation. It is assumed that the reader is familiar with the fundamental concepts in Real and Functional Analysis.

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.

Definition 2.1. For $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_p = \left(\int_\Omega |f|^p \, dx \right)^{1/p} < \infty$$

For $p = \infty$, $L^\infty(\Omega)$ consists of functions which are bounded almost everywhere in Ω. It has the norm

$$\|f\|_\infty = \inf\{C \geq 0 \mid |f| \leq C \text{ a.e } \Omega\}$$

We identify functions which are equal almost everywhere so that the function spaces defined before and after are normed vector spaces.

Theorem 2.1 (Holder's inequality)**.** *Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (we sometimes denote q as p' for such a pair). For $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have*

$$\int_\Omega |fg| \, dx < \left(\int_\Omega |f|^p \, dx \right)^{1/p} \left(\int_\Omega |g|^q \, dx \right)^{1/q} = \|f\|_p \|g\|_q$$

This inequality is a fundamental building block for other important inequalities, as well as being heavily used itself. One such application is an imbedding for L^p spaces:

$$\|f\|_p \leq |\Omega|^{1/p - 1/q} \|f\|_q, \ 1 \leq p \leq q, \text{ and } f \in L^q(\Omega).$$

Definition 2.2. Let $L^1_{loc}(\Omega)$ denote the space of locally integrable functions

$$L^1_{loc}(\Omega) = \{f : \, f|_{\Omega'} \in L^1(\Omega') \text{ for compact } \Omega' \subset \Omega\}$$

Definition 2.3 (Weak Derivatives)**.** Let α be a multi-index, and let D^α denote the α-th partial derivative. For $u, v \in L^1_{loc}(\Omega)$, we say that v is that α-th weak derivative of u if the following identity holds for all $\phi \in C_0^\infty(\Omega)$:

$$(-1)^{|\alpha|} \int_\Omega \phi v \, dx = \int_\Omega u D^\alpha \phi \, dx$$

Weak derivatives are equal almost everywhere and coincide with the usual derivative if it exists, so we also denote them as $v = D^\alpha u$. We will use ∇u to denote the gradient of u if its first order weak derivatives exist.

Definition 2.4 (Sobolev Spaces)**.** Denote the space of k times weakly differentiable functions as $W^k(\Omega)$. For $p \geq 1$, the Sobolev Space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in W^k(\Omega) \,|\, D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq k\}$$

It has the following equivalent norms

$$\|u\|_{k,p} = \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p \, dx \right)^{1/p}$$

and

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \left(\int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}$$

Definition 2.5. We also have the subspace $W_0^{k,p}(\Omega)$ of functions which vanish on the boundary in the trace sense.

Alternatively, it can also be defined as the closure of $C_0^\infty(\Omega)$ in the $W^{k,p}$ norm.

It is a more natural setting to search for solutions of problems with zero Dirichlet conditions on the boundary.

We now state the Sobolev embedding theorem and compactness result for $W_0^{1,p}(\Omega)$. These can be extended to $W^{1,p}(\Omega)$ for Lipschitz domains.

Theorem 2.2 (Sobolev Inequalities)**.** *For $1 \leq p < N$, define $p^* = np/(n-p)$. If $p < N$, we have*

$$W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega),$$

and there is a constant $C(N,p)$ depending only on N and p such that

$$\|u\|_{p^*} \leq C(n,p) \|\nabla u\|_p$$

If $p > N$, then we have

$$W_0^{1,p}(\Omega) \subset C(\overline{\Omega}),$$

and there is also a constant $C(N,p)$ depending only on N and p such that

$$\sup_\Omega |u| \leq C(n,p) |\Omega|^{1/N - 1/p} \|\nabla u\|_p$$

Theorem 2.3 (Rellich-Kondrachov)**.** *If $p < N$, then $W_0^{1,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$ for $1 \leq q < p^*$. If $p > N$, then it is compactly embedded in $C(\overline{\Omega})$.*

We will examine imbeddings for the case $p = N$ in the following section.

3 Trudinger-Moser inequality

In this section we give a brief overview of the Trudinger-Moser inequality:

$$\sup_{u \in W_0^{1,N}(\Omega),\ \|u\| \leq 1} \int_\Omega e^{\alpha(N)\,|u|^{N'}} dx < \infty, \tag{3.1}$$

where $W_0^{1,N}(\Omega)$ is the usual Sobolev space with the norm $\|u\| = \left(\int_\Omega |\nabla u|^N dx\right)^{1/N}$, and $\alpha(N)$ is a constant which depends only on N.

Trudinger and others have shown the existence of such a constant, but Moser showed that the sharpest constant is

$$\alpha_N = N\,\omega_{N-1}^{1/(N-1)}$$

The integral is actually finite for any constant α and $u \in W_0^{1,N}(\Omega)$, but for constants larger than the threshold it can be made arbitrarily large.

A treatment for this inequality can be found in Gilbarg and Trudinger by way of operators on $L^p(\Omega)$ defined with the Riesz potential.

3.1 Estimates based on the Riesz Potential

Let $s \in (0, 1]$ and define for $f \in L^p(\Omega)$

$$(V_s f)(x) = \int_\Omega \frac{f(y)}{|x-y|^{N(1-s)}} dy$$

The following lemma shows that this operator is well defined.

Lemma 3.1. *V_s maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ for $1 \leq q \leq \infty$ such that*

$$0 \leq \delta := \frac{1}{p} - \frac{1}{q} < s$$

Moreover, we have the inequality

$$\|V_s f\|_q \leq \left(\frac{1-\delta}{s-\delta}\right)^{1-\delta} \omega_N^{1-s}\, |\Omega|^{s-\delta} \|f\|_p \tag{3.2}$$

Proof. We first consider the case $1 \leq q < \infty$.

Let

$$h(x-y) = \frac{1}{|x-y|^{N(1-s)}}$$

so that

$$(V_s f)(x) = \int_\Omega h(x-y)f(y)dy$$

Denote by R the radius of the ball centered at $x \in \Omega$ such that $|B_R(x)| = |\Omega|$, and let $r := 1/(1-\delta)$.

First note that

$$\int_\Omega h^r(x-y)dy \leq \int_{B_R(x)} h^r(x-y)dy$$

This follows from the fact that $|B_R(x)| = |\Omega|$:

$$\int_{\Omega\backslash B_R(x)} h^r(x-y)dy \leq R^{rN(s-1)}\,|\Omega\backslash B_R(x)| = R^{rN(s-1)}\,|B_R(x)\backslash\Omega|$$
$$\leq \int_{B_R(x)\backslash\Omega} h^r(x-y)dy$$

Hence we obtain by direct calculation

$$\|h\|_r \leq \left(\frac{1-\delta}{s-\delta}\right)^{1-\delta} \omega_N^{1-s}\,|\Omega|^{s-\delta} \tag{3.3}$$

Noting that $\frac{1}{q} + (\frac{1}{p'}) + \delta = 1$, we apply the generalized Holder inequality to

$$h|f| = (h^r|f|^p)^{1/q}(h^r)^{1/p'}(|f|^p)^\delta$$

to obtain

$$\|V_s f(x)\|_q^q \leq \|h\|_r^{q/p'}\|f\|_p^{pq\delta}\int_\Omega\int_\Omega h^r(x-y)|f(y)|^p\,dy\,dx$$
$$\leq \|h\|_r^{1+q/p'}\|f\|_p^{p+pq\delta},$$

The last inequality comes from the use of Fubini's theorem.

Hence we obtain (3.2) from the above inequality and (3.3).

For the case $q=\infty$, $\frac{1}{p} < s$, we have using Holder's inequality

$$|V_s f(x)| \leq \left(\int_\Omega h^{p'}(x-y)dy\right)^{1/p'}\|f\|_p$$

for all $x \in \Omega$, so (3.2) follows. □

Lemma 3.2. *For $1 \le p < \infty$, $f \in L^p(\Omega)$, let $g = V_{1/p} f$.*

Then there exists constants c_1, c_2 depending only on n and p such that

$$\int_\Omega e^{(g/c_1 \|f\|_p)^{p'}} \le c_2 \, |\Omega| \tag{3.4}$$

Proof. For $q \ge p$, $q + 1 - q/p \le q$, hence we have from (3.2)

$$\|g\|_q \le q^{1-1/p+1/q} \omega_N^{1/p'} |\Omega|^{1/q} \|f\|_p,$$

which implies that

$$\int_\Omega |g|^q dx \le q^{1+q/p'} \omega_N^{q/p'} |\Omega| \|f\|_p^q$$

If $q \ge p - 1$, we have $p'q \ge p$, so applying the above inequality with $p'q$ gives

$$\int_\Omega |g|^{p'q} dx \le (p'q)^{1+q} \omega_N^q |\Omega| \|f\|_p^{p'q} = (p'q)(\omega_N p' q \|f\|_p^{p'})^q |\Omega|$$

Hence we have

$$\int_\Omega \sum_{k=\lfloor p \rfloor}^{K} \frac{1}{k!} \left(\frac{|g|}{c_1 \|f\|_p} \right)^{p'k} \le p' \, |\Omega| \sum_{k=\lfloor p \rfloor}^{K} \left(\frac{p' \omega_N}{c_1^{p'}} \right)^k \frac{k^k}{(k-1)!}, \tag{3.5}$$

where $\lfloor p \rfloor$ denotes the largest integer smaller than p. The right hand side converges if $\frac{p'\omega_N}{c_1^{p'}} < \frac{1}{e}$ by the ratio test.

Now for $q \le p - 1$, we have $p'q \le p$, so by the imbedding of L^p spaces and (3.2), we have

$$\|g\|_{p'q} \le |\Omega|^{1/p'q - 1/p} \|g\|_p \le p \omega_N^{1/p'} |\Omega|^{1/p} |\Omega|^{1/p'q-1/p} \|f\|_p,$$

so

$$\frac{\|g\|_{p'q}^{p'q}}{\|f\|_p^{p'q}} \le p^{p'q} \omega_N^q \, |\Omega|$$

Hence the summation on the left of (3.5) is bounded by a constant times $|\Omega|$ for all K. Since it is a sum of non-negative functions, by Lebesgue's Monotone Convergence theorem (see Theorem 1.27 in Rudin), we obtain (3.4). □

Lemma 3.3. *For $u \in W_0^{1,1}(\Omega)$, we have*

$$u(x) = \frac{1}{\omega_{N-1}} \int_\Omega \frac{\nabla u(y) \cdot (x-y)}{|x-y|^N} \, dy \tag{3.6}$$

almost everywhere in Ω.

Proof. We first assume that $u \in C_0^1(\Omega)$, and we define $u = 0$ outside of Ω.

For $|z| = 1$, we have by the fundamental theorem

$$u(x) = -\int_0^\infty u_r(x + rz) \, dr$$

Integrating both sides with respect to z, we have

$$\begin{aligned} \omega_{N-1} u(x) &= -\int_{|z|=1} \int_0^\infty u_r(x+rz) \, dr \, dz \\ &= -\int_0^\infty \int_{|z|=1} z \cdot \nabla u(x+rz) \, dz \, dr \end{aligned}$$

On the other hand, we have from the co-area formula

$$\begin{aligned} \int_\Omega \frac{\nabla u(y) \cdot (x-y)}{|x-y|^N} \, dy &= \int_0^\infty \int_{|z|=1} \frac{-rz \cdot \nabla u(x+rz)}{r^N} r^{N-1} \, dz \, dr \\ &= -\int_0^\infty \int_{|z|=1} z \cdot \nabla u(x+rz) \, dz \, dr \end{aligned}$$

So we have established (7.7) for $u \in C_0^1(\Omega)$.

Now let $\{u_j\}$ be a sequence in $C_0^1(\Omega)$ converging to $u \in W_0^{1,1}(\Omega)$.

We have the estimate by (3.2) for $s = 1/N$, $f = |\nabla u_j - \nabla u|$ and for some constant C

$$\left| u_j - \frac{1}{\omega_{N-1}} \int_\Omega \frac{\nabla u(y) \cdot (x-y)}{|x-y|^N} \, dy \right| \leq \frac{1}{\omega_{N-1}} \int_\Omega \frac{|\nabla u_j(y) - \nabla u(y)|}{|x-y|^{N-1}} \, dy$$

$$\leq C \|\nabla u_j - \nabla u\|_1 \to 0$$

Hence for almost every $x \in \Omega$, we have (7.7).

□

From the identity (7.7), we obtain the inequality for $u \in W_0^{1,1}(\Omega)$

$$|u| \leq \frac{1}{\omega_{N-1}} V_{1/N}|\nabla u| \tag{3.7}$$

This gives a nice proof of the Poincare inequality.

Theorem 3.4. *For $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$, we have*

$$\|u\|_p \leq (\omega_N^{-1}|\Omega|)^{1/N}\|Du\|_p \tag{3.8}$$

Proof. By the Sobolev imbedding we have

$$W_0^{1,p}(\Omega) \subset W_0^{1,1}(\Omega)$$

So for $u \in W_0^{1,p}(\Omega)$, we have by (3.2) and (3.7)

$$\begin{aligned} \|u\|_p \leq (N\omega_N)^{-1}\| \, V_{1/N}|\nabla u| \, \|_p &\leq \frac{N}{N\omega_N}\omega_N^{1-1/N}|\Omega|^{1/N}\|\nabla u\|_p \\ &= (\omega_N^{-1}|\Omega|)^{1/N}\|\nabla u\|_p \end{aligned}$$

□

Poincare's inequality allows us to interchange the equivalent norms $\|\nabla u\|_p$ and $\|u\|$ in $W_0^{1,p}(\Omega)$. We can also now obtain (3.1) directly.

Proof. For $u \in W_0^{1,N}(\Omega)$, let $g = V_{1/N}|\nabla u|$.
We have from (3.7)

$$|u| \leq \frac{g}{\omega_{N-1}},$$

hence for $\|u\| = 1$ in $W_0^{1,N}(\Omega)$, by (3.4)

$$\int_\Omega e^{(|u|/c_1)^{N'}}\, dx \leq \int_\Omega e^{(g/c_2)^{N'}}\, dx \leq c_3\,|\Omega|$$

for appropriately chosen constants. □

It follows from this inequality that $W_0^{1,N}(\Omega)$ can be imbedded into $L^q(\Omega)$ for $1 \leq q < \infty$.

3.2 The Moser sequence

We take the opportunity here to define the Moser sequence $\{\omega_j\}$. It will be needed later to obtain a critical value that falls within the compactness threshold from the critical point theorems.

For $j \ge 2$, let

$$\omega_j(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log j)^{(N-1)/N}, & |x| \le d/j \\ \dfrac{\log(d/|x|)}{(\log j)^{1/N}}, & d/j < |x| < d \\ 0, & |x| \ge d. \end{cases} \tag{3.9}$$

Proposition 3.5. *We have*

$$\int_\Omega \omega_j^m \, dx = \frac{m!\, \omega_{N-1}^{1-m/N} d^N}{N^{m+1} (\log j)^{m/N}} \left[1 - \frac{1}{j^N} \sum_{l=1}^m \frac{(N \log j)^{m-l}}{(m-l)!}\right], \quad m = 1, \dots, N \tag{3.10}$$

and

$$\int_\Omega |\nabla \omega_j|^m \, dx = \begin{cases} \dfrac{\omega_{N-1}^{1-m/N} d^{N-m}}{(N-m)(\log j)^{m/N}} \left(1 - \dfrac{1}{j^{N-m}}\right), & m = 1, \dots, N-1 \\ 1, & m = N. \end{cases} \tag{3.11}$$

Proof. We have

$$\int_\Omega \omega_j^m \, dx = \frac{\omega_{N-1}^{1-m/N} d^N}{(\log j)^{m/N}} \left[I_m + \frac{(\log j)^m}{N j^N}\right],$$

where

$$I_m = \int_{1/j}^1 (-\log s)^m \, s^{N-1} \, ds.$$

We have

$$I_1 = \frac{1}{N^2} \left[1 - \frac{1}{j^N} (N \log j + 1)\right],$$

and integrating by parts gives the recurrence relation

$$I_m = \frac{m}{N} I_{m-1} - \frac{(\log j)^m}{N j^N}, \quad m \ge 2.$$

So

$$I_m = \frac{m!}{N^{m+1}} \left[1 - \frac{1}{j^N} \sum_{l=0}^{m} \frac{(N \log j)^{m-l}}{(m-l)!} \right],$$

and (3.10) follows.

We have

$$|\nabla \omega_j(x)| = \begin{cases} \dfrac{1}{(\omega_{N-1} \log j)^{1/N}} \dfrac{1}{|x|}, & d/j < |x| < d \\ 0, & otherwise \end{cases}$$

So

$$\int_\Omega |\nabla \omega_j(x)|^m \, dx = \frac{\omega_{N-1}^{1-m/N}}{(\log j)^{m/N}} \int_{d/j}^{d} r^{N-m-1} \, dr$$

The integral on the right evaluates to

$$\int_{d/j}^{d} r^{N-m-1} \, dr = \begin{cases} \dfrac{d^{N-m}}{N-m} \left(1 - \dfrac{1}{j^{N-m}} \right), & m = 1, \ldots, N-1 \\ \log j, & m = N \end{cases}$$

from which (3.11) follows.

□

We note that this sequence can be used to show that $W_0^{1,N}(\Omega)$ cannot be imbedded into $L^\infty(\Omega)$ (There are examples of unbounded functions in $W_0^{1,N}(\Omega)$, but once again we are only using this section to define this sequence). Suppose that there is a constant C, depending only on N, such that for $u \in W_0^{1,N}(\Omega)$

$$\|u\|_\infty \leq C \|u\|$$

Then we have for ω_j

$$\|\omega_j\|_\infty \leq C \|\omega_j\| = C$$

since $\|\omega_j\| = 1$ for all $j \geq 2$. However, $\|\omega_j\|_\infty \to \infty$, so no such constant can exist.

4 Critical Point Theory

In this section we summarize the main ideas used to prove an abstract critical point theorem.

4.1 The Variational Setting

Let W be a Banach space over $\mathbb{R}$, and let W' denote its continuous dual.

Definition 4.1. A functional $E : W \to \mathbb{R}$ is Fréchet differentiable at a point $u \in W$ if there is a $D \in W'$ such that

$$E(u + v) - E(u) - Dv = o(\|v\|), \text{ for all } v \in W$$

This D is unique, and we denote it as $E'(u)$.

Additionally, we say that $E \in C^1$ if the map which takes u to $E'(u)$ is continuous.

If $E'(u) = 0$, then u is called a critical point of E, and $c = E(u)$ is called a critical value.

The Fréchet derivative is a generalization of the derivative to normed spaces. There is also a generalization of the directional derivative.

Definition 4.2. For $u, v \in W$, the Gateaux derivative of E at u in the direction of v is

$$\frac{d}{d\tau} E(u + \tau v)\Big|_{\tau=0} = E'(u)v$$

The class of variational Partial Differential Equations consists of problems where finding weak solutions corresponds to finding the critical points of functionals on some Banach space. The classical so-called direct methods of the Calculus of Variations involve adjusting the topology of the functional domain in order to strike a balance between continuity and compactness, so that minimizers can be obtained via methods from Functional Analysis.

In general, it is difficult to obtain sequential compactness for these functionals, but Palais and Smale introduced the following notion as a substitute.

Definition 4.3. $\{u_m\} \subset W$ is a Palais-Smale (PS) sequence of E if it satisfies the following:

1. $E(u_m)$ is bounded
2. $E'(u_m) \to 0$ in W'.

If the first condition is replaced with $E(u_m) \to c$ for some $c \in \mathbb{R}$, then $\{u_m\}$ is called a $(PS)_c$ sequence of E.

Definition 4.4. A functional E satisfies the (PS) (or $(PS)_c$) condition if all of its (PS) (or $(PS)_c$) sequences have a strongly convergent subsequence.

This slightly separate notion for a $(PS)_c$ sequence is needed later for critical problems, where we only have precompactness of the sequence if the value c lies in a threshold determined by the functional. Interestingly, these thresholds are related to the sharp embedding constants of certain Sobolev spaces.

4.2 Deformation Lemma and the Minimax Principle

With the above notion of compactness, we can formulate the Deformation Lemma (due to Palais), from which many critical point existence results can be derived via its corollary the Minimax Principle. We state the Deformation Lemma without proof for C^1 functionals satisfying the (PS) condition, but we note that the following will hold for the $(PS)_c$ condition as well.

Theorem 4.1 (Deformation Lemma). *Suppose that E is a C^1 functional which satisfies the (PS) condition. Also, let $c \in \mathbb{R}$, $E^c = \{u \in W : E(u) < c\}$, $\bar{\epsilon} > 0$, and N be any neighborhood of*

$$K_c = \{u \in W \,:\, E(u) = c, E'(u) = 0\}$$

Then there exists $\epsilon \in (0, \bar{\epsilon})$ and a continuous one parameter family of homeomorphisms of W into itself, $\eta(\cdot, t), t \in [0, \infty)$, such that

1. *$\eta(u, t) = u$ if either of the following hold:*
   i. $t = 0$
   ii. $E'(u) = 0$
   iii. $|E(u) - c| \geq \bar{\epsilon}$
2. *$E(\eta(u, t))$ is non-increasing in t for all $u \in W$*
3. *$\eta(E^{c+\epsilon} \backslash N, 1) \subset E^{c-\epsilon}$*

This family of homeomorphisms also satisfies the semi-group property. That is, for any $s, t \geq 0$, we have

$$\eta(\cdot, s) \circ \eta(\cdot, t) = \eta(\cdot, s + t)$$

Definition 4.5. Let $\mathcal{F}$ be a collection of subsets of W. For any η satisfying the properties of the one in the conclusion of the Deformation Lemma, we say that $\mathcal{F}$ is η-invariant if for all $F \in \mathcal{F}$, $t \geq 0$, we have $\eta(F, t) \in \mathcal{F}$.

Theorem 4.2 (Minimax Principle)**.** *Suppose that E is a C^1 functional which satisfies the (PS) condition, and $\mathcal{F}$ is an η-invariant collection of sets for any η as in the Deformation Lemma.*

Then if

$$c := \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) < \infty,$$

it is a critical value of E.

Proof. If c is not a critical value of E, then $K_c = \emptyset$, so choose $N = \emptyset$ and $\bar{\epsilon} = 1$, and let ϵ and η be as determined in the Deformation Lemma.

For this ϵ, there is an $F \in \mathcal{F}$ such that

$$\sup_{u \in F} E(u) < c + \epsilon \implies F \subset E^{c+\epsilon}$$

Let $G = \eta(F, 1)$. By assumption, G is also in $\mathcal{F}$. By the deformation lemma, we have $\eta(E^{c+\epsilon}, 1) \subset E^{c-\epsilon}$, which implies that $G \subset E^{c-\epsilon}$, so

$$\sup_{u \in G} E(u) < c - \epsilon,$$

which is a contradiction. □

We list some simple but remarkable corollaries of the Minimax Principle.

Corollary 4.3. *If E is C^1 and satisfies the (PS) condition, and if*

$$c = \sup_{u \in W} E(u) < \infty,$$

then c is a critical value of E and the maximum of E is achieved for some critical point. A similar result holds for the minimum if $\inf E(W) < \infty$.

Proof. Let $\mathcal{F} = \{W\}$. Then the result follows if

$$c = \inf_{F \in \mathcal{F}} \sup_{u \in F} E(u) = \sup_{u \in W} E(u) < \infty$$

The result for the minimum can be obtained by taking $\mathcal{F} = \{\{u\} : u \in W\}$. □

Another corollory is the famous Mountain Pass lemma due to Ambrosetti and Rabinowitz [13]. We state another formulation of the Minimax Principle in terms of path invariance first.

Definition 4.6. Let Γ be a collection of continuous maps γ from a topological space X to W. We say that Γ is η-invariant if for any $\gamma \in \Gamma$, we have $\eta(\cdot, 1) \circ \gamma \in \Gamma$.

This version of the Minimax Principle can now be stated in terms of

$$c \coloneqq \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u)$$

being bounded instead, by considering the collection $\mathcal{F} = \{\gamma(X) : \gamma \in \Gamma\}$ in the previous Minimax Principle.

Theorem 4.4 (Mountain Pass lemma)**.** *Suppose that E is a C^1 functional and satisfies the (PS) condition.*

Additionally, suppose that

1. *$E(0) = 0$*
2. *There are constants $\alpha, \rho > 0$ such that for $u \in W$, if $\|u\| = \rho$ then $E(u) \geq \alpha$*
3. *There exists $v \in W$ such that $\|v\| > \rho$ and $E(v) < \alpha$*

Let $\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0, \gamma(1) = v\}$, and define

$$c \coloneqq \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} E(u)$$

Then $c \geq \alpha$ is a critical value of E.

Proof. Let $\bar{\epsilon} = \min\{\alpha, \alpha - E(v)\}$. Then there exists by the Deformation Lemma an η such that Γ is η-invariant. The value c is finite since $[0,1]$ is compact.

Hence by the Minimax Principle, c is a critical value. □

We have another theorem due to Rabinowitz [13] which generalizes the previous one.

Theorem 4.5 (Linking theorem)**.** *Suppose that $W = V_1 \oplus V_2$, where V_1 is finite dimensional. Also, suppose that E is a C^1 functional on W which satisfies the (PS) condition and*

1. *There are constants $\alpha, \rho > 0$ such that $E(\partial B_\rho(0) \cap V_2) \geq \alpha$*
2. *There exists $v \in \partial B_1(0) \cap V_2$ and $R > \rho$ such that if*

   $$Q_{R,v} = (\overline{B_R(0)} \cap V_1) \oplus \{tv : t \in [0, R]\},$$

   then $E(\partial Q) \leq 0$.

Let

$$\Gamma = \{h \in C(\overline{Q}, W) : h|_{\partial Q} = id\}$$

and

$$c \coloneqq \inf_{h \in \Gamma} \max_{u \in Q} E(h(u))$$

Then $c \leq \alpha$ is a critical value of E.

4.3 An Abstract Critical Point Theorem

In practice, the constants and constructs in the above theorems depend on eigenvalues of problem (1.7). In the case of the linking theorem, when $N = 2$ we have a linear decomposition of $W_0^{1,2}(\Omega)$ into eigenspaces of the Laplacian which allows us to carry out the linking construction.

When $N > 2$ however, we do not have such a decomposition for the non-linear p-Laplacian operator. We get around this obstacle by using an abstract critical point theorem proved in Yang and Perera [15, Theorem 2.2]. This result generalizes the linking theorem of Rabinowitz [13]. First we state the definition of the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [8].

Definition 4.7. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\}, & A \neq \emptyset \\ 0, & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of the cohomological index (see Fadell and Rabinowitz [8]).

Proposition 4.6. *The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:*

(*i*) *Definiteness: $i(A) = 0$ if and only if $A = \emptyset$.*

(*ii*) *Monotonicity: If there is an odd continuous map from A to B (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism.*

(*iii*) *Dimension: $i(A) \leq \dim W$.*

(*iv*) *Continuity: If A is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of A such that $i(N) = i(A)$. When A is compact, N may be chosen to be a δ-neighborhood $N_\delta(A) = \{u \in W : dist(u, A) \leq \delta\}$.*

(*v*) *Subadditivity: If A and B are closed, then $i(A \cup B) \leq i(A) + i(B)$.*

(*vi*) *Stability: If SA is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$.*

(*vii*) *Piercing property: If* A*,* A_0 *and* A_1 *are closed, and* $\varphi : A \times [0,1] \to A_0 \cup A_1$ *is a continuous map such that* $\varphi(-u,t) = -\varphi(u,t)$ *for all* $(u,t) \in A \times [0,1]$*,* $\varphi(A \times [0,1])$ *is closed,* $\varphi(A \times \{0\}) \subset A_0$ *and* $\varphi(A \times \{1\}) \subset A_1$*, then* $i(\varphi(A \times [0,1]) \cap A_0 \cap A_1) \geq i(A)$*.*

(*viii*) *Neighborhood of zero: If* U *is a bounded closed symmetric neighborhood of* 0*, then* $i(\partial U) = \dim W$*.*

Theorem 4.7. *Let* E *be a* C^1*-functional defined on a Banach space* W *and let* A_0 *and* B_0 *be disjoint nonempty closed symmetric subsets of the unit sphere* $S = \{u \in W : \|u\| = 1\}$ *such that*

$$i(A_0) = i(S \setminus B_0) < \infty. \tag{4.1}$$

Assume that there exist $R > \rho > 0$ *and* $\omega \in S \setminus A_0$ *such that*

$$\sup E(A) \leq \inf E(B), \qquad \sup E(X) < \infty,$$

where

$$A = \{sv : v \in A_0,\, 0 \leq s \leq R\} \cup \{R\,\pi((1-t)\,v + t\omega) : v \in A_0,\, 0 \leq t \leq 1\},$$

$$B = \{\rho u : u \in B_0\},$$

$$X = \{sv + t\omega : v \in A_0,\, s,t \geq 0,\, \|sv + t\omega\| \leq R\},$$

and $\pi : W \setminus \{0\} \to S,\ u \mapsto u/\|u\|$ *is the radial projection onto* S*. Let*

$$\Gamma = \{\gamma \in C(X,W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\},$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).$$

Then $\inf E(B) \leq c \leq \sup E(X)$*, and* E *has a* $(\mathrm{PS})_c$ *sequence.*

Proof. We claim that for all $\gamma \in \Gamma$

$$\gamma(X) \cap B \neq \emptyset \tag{4.2}$$

Suppose there exists such a γ. Let

$$\tilde{A} = \{R\pi((1-|t|)v + t\omega) : v \in A_0, -1 \leq t \leq 1\}$$

This set is closed and symmetric since A_0 is also.

We have the following odd and continuous map from SA_0 to $\tilde{A}$

$$(v,t) \mapsto R\pi((1-|t|)v + t\omega),$$

hence by the stability and monotonicity properties of the index, we have

$$i(\tilde{A}) \geq i(SA_0) \geq i(A_0) + 1$$

Let $\varphi : \tilde{A} \times [0, 1] \to W \backslash B$ be the map

$$\varphi(u,t) = \begin{cases} \gamma(tv) & v \in \tilde{A} \cap A \\ \\ -\gamma(-tv) & v \in \tilde{A} \backslash A \end{cases}$$

This map is continuous by the pasting lemma, and the fact that γ is the identity on $\{sv : v \in A_0,\, 0 \leq s \leq R\}$

The following hold for this map:

- $\varphi(-v, t) = -\varphi(u, t)$
- $\varphi(\tilde{A} \times [0, 1]) = \gamma(X) \cup -\gamma(X)$ is closed
- $\varphi(\tilde{A} \times \{0\}) = \{0\}$ and $\varphi(\tilde{A} \times \{1\}) = \tilde{A}$ since γ is identity on A

Let S_ρ denote the sphere of radius ρ in W, and let B_ρ and $B_{\overline{\rho}}$ denote the open and closed balls of radius ρ respectively. Applying the piercing property of the index to $\tilde{A}_0 = B_\rho$ and $\tilde{A}_1 = W \backslash B_\rho$ as well as the monotonicity property gives us

$$i(\tilde{A}) \leq i(\varphi(\tilde{A} \times [0, 1]) \cap \tilde{A}_0 \cap \tilde{A}_1) \leq i((W \setminus B) \cap S_r) = i(S_r \setminus B) = i(S \setminus B_0)$$

This gives us $i(A_0) < i(\tilde{A}) \leq i(S \setminus B_0)$, which contradicts (4.1).

From this we have $c \geq \inf E(B)$ since for any γ

$$\inf_B E(u) \leq \inf_{\gamma(X)\cap B} E(u) \leq \sup_{\gamma(X)\cap B} E(u) \leq \sup_{\gamma(X)} E(u)$$

For our purposes we can assume $c > \inf E(B)$. Then taking $\overline{\epsilon} = c - \sup E(g(A))$ in the deformation lemma, we have Γ is η-invariant. So if E doesn't have any $(PS)_c$ sequences, it vacuously satisfies the $(PS)_c$ condition and hence has a critical point by the Minimax Principle, which is a contradiction. □

We note that (4.2) is a useful notion by itself.

Definition 4.8. Let X be a topological space, and let $A \subset X$ be closed. Let g be a continuous map from A to W such that $g(A)$ is closed and bounded. Finally, let $B \neq \emptyset$ be a closed subset of W, and let

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed}, \gamma|_A = g\}$$

We say that (A, g) homotopically links B with respect to X if for all $\gamma \in \Gamma$

$$\gamma(X) \cap B \neq \emptyset$$

As another example of homotopical linking, we have the Mountain Pass lemma if we set $A = \{0\} \cup \{1\}$, $X = [0, 1]$ and $B = B_\rho$.

5 Eigenvalues of the p-Laplacian

We give a brief overview in the literature concerning the Dirichlet eigenvalues of the p-Laplacian operator

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

More on this topic can be found in Lindqvist. Much is known about the eigenvalues of the Laplacian operator and other symmetric elliptic operators, such as the spectrum being countable and unbounded. For the p-Laplacian it is known that the first Dirichlet eigenvalue as defined in (1.6) is positive, simple, and isolated. An unbounded sequence of eigenvalues can also be constructed using the Krasnoselskii genus. Whether this sequence exhausts the spectrum is still unknown.

Definition 5.1. For a real Banach space W, consider the class of closed symmetric subsets

$$\mathcal{A} = \{A \subset W : A = -A, A \text{ is closed}\}$$

The Krasnoselskii genus $\gamma : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ is defined for non-empty A as

$$\gamma(A) = \begin{cases} \inf \{k : \exists f \in C(A; \mathbb{R}^k \backslash \{0\}), f \text{ is odd}\} \\ \infty, \text{ otherwise} \end{cases}$$

Finally, define $\gamma(\emptyset) = 0$

Denote $S_p = \{u \in L^p(\Omega) : \|u\|_p = 1\}$.
Let $\overline{\mathcal{A}}_k = \{A \subset W_0^{1,p}(\Omega) : A = -A, \gamma(A) \geq k, A \cap S_p \text{ is compact}\}$

Then the following minimax scheme generates an unbounded sequence of eigenvalues of (5.1)

$$\lambda_k = \inf_{A \in \overline{\mathcal{A}}_k} \max_{v \in A} \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx}$$

In the context of existence theory, many multiplicity results can be be proven with the genus. As an index theory it has similar properties as the cohomological index, but it lacks the crucial piercing property which is needed to prove Theorem (4.7). Hence for our problem, we use another sequence of eigenvalues constructed via the cohomological index.

First note that eigenvalues of problem (1.7) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^N dx}, \quad u \in S = \left\{ u \in W_0^{1,N}(\Omega) : \int_\Omega |\nabla u|^N dx = 1 \right\}.$$

We have the following proposition (see Perera [11] and Perera et al. [12, Proposition 3.52 and Proposition 3.53]).

Proposition 5.1. *Let $\mathcal{F}$ denote the class of symmetric subsets of S, let $i(\cdot)$ denote the cohomological index, and set*

$$\lambda_k := \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.7)*. Moreover, if $\lambda_{k-1} < \lambda_k$, then*

$$i(\Psi^{\lambda_{k-1}}) = i(S \setminus \Psi_{\lambda_k}) = k - 1,$$

where $\Psi^a = \{u \in S : \Psi(u) \leq a\}$ and $\Psi_a = \{u \in S : \Psi(u) \geq a\}$ for $a \in \mathbb{R}$.

The last property of these eigenvalues involving the index and sublevel sets is vital for finding linking sets to use in Theorem (6.2). As a last remark for this section, the above minimax schemes for the eigenvalues should be compared with a similar scheme for symmetric matrices via the Courant-Fisher principle. In this context, the notion of an index serves as a substitute for the dimension.

We end this section with the following result of Degiovanni and Lancelotti ([7, Theorem 2.3]).

Proposition 5.2. *If $\lambda_{k-1} < \lambda_k$, then $\Psi^{\lambda_{k-1}}$ contains a compact symmetric set C of index $k - 1$ that is bounded in $C^1(\overline{\Omega})$.*

6 A Compactness Result

We end this series of preliminaries with a key compactness result.

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$E(u) = \frac{1}{N} \int_\Omega |\nabla u|^N dx - \int_\Omega G(u)\, dx, \quad u \in W_0^{1,N}(\Omega).$$

We show that despite the lack of compactness on this functional, it is enough to show that it admits a $(PS)_c$ sequence within the compactness threshold to ensure the existence of a critical point. Proofs of Theorem 7.1 and Theorem 8.1 will be based on the following result.

Proposition 6.1. *Assume that* $\alpha > 0$ *and* h *satisfies* (1.2) *and* (1.3)*. Then for all* $c \neq 0$ *satisfying*

$$c < \frac{1}{N} \left(\frac{\alpha_N}{\alpha} \right)^{N-1},$$

every $(\mathrm{PS})_c$ *sequence of* E *has a subsequence that converges weakly to a nontrivial solution of problem* (1.1)*.*

Proof. Let $(u_j) \subset W_0^{1,N}(\Omega)$ be a $(\mathrm{PS})_c$ sequence of E. Then

$$E(u_j) = \frac{1}{N} \|u_j\|^N - \int_\Omega G(u_j)\, dx = c + \mathrm{o}(1) \tag{6.1}$$

and

$$E'(u_j)\, u_j = \|u_j\|^N - \int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} dx = \mathrm{o}(\|u_j\|). \tag{6.2}$$

First we show that (u_j) is bounded in $W_0^{1,N}(\Omega)$. Multiplying (6.1) by $2N$ and subtracting (6.2) gives

$$\|u_j\|^N + \int_\Omega \Big(u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} - 2NG(u_j) \Big)\, dx = 2Nc + \mathrm{o}(\|u_j\| + 1),$$

so it suffices to show that $th(t)\, e^{\alpha\, |t|^{N'}} - 2NG(t)$ is bounded from below. Let $0 < \varepsilon < \beta/(2N+1)$. By (1.2) and (1.3), for some constant $C_\varepsilon > 0$,

$$|G(t)| \le \varepsilon\, e^{\alpha\, |t|^{N'}} + C_\varepsilon \tag{6.3}$$

and

$$th(t)\, e^{\alpha\, |t|^{N'}} \ge (\beta - \varepsilon)\, e^{\alpha\, |t|^{N'}} - C_\varepsilon \tag{6.4}$$

for all t. So

$$th(t)\, e^{\alpha\,|t|^{N'}} - 2NG(t) \ge [\beta - (2N+1)\,\varepsilon]\, e^{\alpha\,|t|^{N'}} - (2N+1)\, C_\varepsilon,$$

which is bounded from below.

Since (u_j) is bounded in $W^{1,N}_0(\Omega)$, a renamed subsequence converges to some u weakly in $W^{1,N}_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in Ω. We have

$$E'(u_j)\, v = \int_\Omega |\nabla u_j|^{N-2}\, \nabla u_j \cdot \nabla v\, dx - \int_\Omega v\, h(u_j)\, e^{\alpha\,|u_j|^{N'}} dx \to 0 \tag{6.5}$$

for all $v \in W^{1,N}_0(\Omega)$. By (1.2), given any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|h(t)\, e^{\alpha\,|t|^{N'}}| \le \varepsilon\, e^{\alpha\,|t|^{N'}} + C_\varepsilon \quad \forall t. \tag{6.6}$$

By (6.2),

$$\sup_j \int_\Omega u_j\, h(u_j)\, e^{\alpha\,|u_j|^{N'}} dx < \infty,$$

which together with (6.4) gives

$$\sup_j \int_\Omega e^{\alpha\,|u_j|^{N'}} dx < \infty. \tag{6.7}$$

For $v \in C^\infty_0(\Omega)$, it follows from (6.6) and (6.7) that the sequence $(v\, h(u_j)\, e^{\alpha\,|u_j|^{N'}})$ is uniformly integrable and hence

$$\int_\Omega v\, h(u_j)\, e^{\alpha\,|u_j|^{N'}} dx \to \int_\Omega v\, h(u)\, e^{\alpha\,|u|^{N'}} dx$$

by Vitali's convergence theorem, so it follows from (6.5) that

$$\int_\Omega |\nabla u|^{N-2}\, \nabla u \cdot \nabla v\, dx - \int_\Omega v\, h(u)\, e^{\alpha\,|u|^{N'}} dx = 0.$$

Then this holds for all $v \in W^{1,N}_0(\Omega)$ by density, so the weak limit u is a solution of problem (1.1).

Suppose that $u = 0$. Then

$$\int_\Omega G(u_j)\, dx \to 0$$

since (6.3) and (6.7) imply that the sequence $(G(u_j))$ is uniformly integrable, so (6.1) gives $c \ge 0$ and

$$\|u_j\| \to (Nc)^{1/N}. \tag{6.8}$$

Let $Nc < \nu < (\alpha_N/\alpha)^{N-1}$. Then $\|u_j\| \le \nu^{1/N}$ for all $j \ge j_0$ for some j_0. Let $q = \alpha_N/\alpha\nu^{1/(N-1)} > 1$. By the Hölder inequality,

$$\left|\int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} dx\right| \le \left(\int_\Omega |u_j\, h(u_j)|^p\, dx\right)^{1/p} \left(\int_\Omega e^{q\alpha\, |u_j|^{N'}} dx\right)^{1/q},$$

where $1/p + 1/q = 1$. The first integral on the right-hand side converges to zero since h is bounded and $u_j \to 0$ in $L^p(\Omega)$, and the second integral is bounded by (1.4) since $q\alpha\, |u_j|^{N'} = \alpha_N\, |\widetilde{u}_j|^{N'}$, where $\widetilde{u}_j = u_j/\nu^{1/N}$ satisfies $\|\widetilde{u}_j\| \le 1$ for $j \ge j_0$, so

$$\int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} dx \to 0.$$

Then $u_j \to 0$ by (6.2) and hence $c = 0$ by (6.8), contrary to assumption. So u is a nontrivial solution. □

7 First Existence Result

7.1 Statement

Let d be the radius of the largest open ball contained in Ω. Our first result is the following theorem.

Theorem 7.1. *Assume that* $\alpha > 0$*,* h *satisfies* (1.2) *and* (1.3)*, and* G *satisfies*

$$G(t) \geq -\frac{1}{N}\,\sigma_0\,|t|^N \quad \text{for } t \geq 0, \tag{7.1}$$

$$G(t) \leq \frac{1}{N}\,(\lambda_1 - \sigma_1)\,|t|^N \quad \text{for } |t| \leq \delta \tag{7.2}$$

for some $\sigma_0 \geq 0$ *and* $\sigma_1, \delta > 0$*. If*

$$\beta > \frac{1}{\mathcal{M}\,\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{\sigma_0/(N-1)\,\kappa}, \tag{7.3}$$

where $\mathcal{M} = \lim\limits_{n \to \infty} \int_0^1 ne^{-n\,(t - t^{N'})}\,dt$ *and* $\kappa = \dfrac{1}{N!}\left(\dfrac{N}{d}\right)^N$*, then problem* (1.1) *has a nontrivial solution.*

In particular, we have the following corollary for $\sigma_0 = 0$.

Corollary 7.2. *Assume that* $\alpha > 0$*,* h *satisfies* (1.2) *and* (1.3)*, and* G *satisfies*

$$G(t) \geq 0 \quad \text{for } t \geq 0,$$

$$G(t) \leq \frac{1}{N}\,(\lambda_1 - \sigma_1)\,|t|^N \quad \text{for } |t| \leq \delta$$

for some $\sigma_1, \delta > 0$*. If*

$$\beta > \frac{1}{\mathcal{M}\,\alpha^{N-1}} \left(\frac{N}{d}\right)^N,$$

where $\mathcal{M} = \lim\limits_{n \to \infty} \int_0^1 ne^{-n\,(t - t^{N'})}\,dt$*, then problem* (1.1) *has a nontrivial solution.*

Corollary 7.2 should be compared with Theorem 1 of do Ó [9], where this result is proved under the stronger assumption $h(t) \geq 0$ for $t \geq 0$.

7.2 Proof of Theorem 7.1

We prove Theorem 7.1 by showing that the functional E has the mountain pass geometry with the mountain pass level $c \in (0, (1/N)(\alpha_N/\alpha)^{N-1})$ and applying Proposition 6.1.

Lemma 7.3. *There exists a $\rho > 0$ such that*

$$\inf_{\|u\|=\rho} E(u) > 0.$$

Proof. Since (1.2) implies that h is bounded, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \le C_\delta \, |t|^{N+1} \, e^{\alpha \, |t|^{N'}} \quad \text{for } |t| > \delta,$$

which together with (7.2) gives

$$\int_\Omega G(u) \, dx \le \frac{1}{N} \, (\lambda_1 - \sigma_1) \int_\Omega |u|^N dx + C_\delta \int_\Omega |u|^{N+1} \, e^{\alpha \, |u|^{N'}} dx. \tag{7.4}$$

By (1.6),

$$\int_\Omega |u|^N dx \le \frac{\rho^N}{\lambda_1}, \tag{7.5}$$

where $\rho = \|u\|$. By the Hölder inequality,

$$\int_\Omega |u|^{N+1} \, e^{\alpha \, |u|^{N'}} dx \le \left(\int_\Omega |u|^{2\,(N+1)} \, dx \right)^{1/2} \left(\int_\Omega e^{2\alpha \, |u|^{N'}} dx \right)^{1/2}. \tag{7.6}$$

The first integral on the right-hand side is bounded by $C\rho^{2\,(N+1)}$ for some constant $C > 0$ by the Sobolev embedding theorem. Since $2\alpha \, |u|^{N'} = 2\alpha \, \rho^{N'} |\widetilde{u}|^{N'}$, where $\widetilde{u} = u/\rho$ satisfies $\|\widetilde{u}\| = 1$, the second integral is bounded when $\rho^{N'} \le \alpha_N/2\alpha$ by (1.4). So combining (7.4)–(7.6) gives

$$\int_\Omega G(u) \, dx \le \frac{1}{N} \left(1 - \frac{\sigma_1}{\lambda_1} \right) \rho^N + \mathrm{O}(\rho^{N+1}) \quad \text{as } \rho \to 0.$$

Then

$$E(u) \ge \frac{1}{N} \, \frac{\sigma_1}{\lambda_1} \, \rho^N + \mathrm{O}(\rho^{N+1}),$$

and the desired conclusion follows from this for sufficiently small $\rho > 0$. □

We may assume without loss of generality that $B_d(0) \subset \Omega$. Let (ω_j) be the sequence of functions defined in (3.9).

Lemma 7.4. *We have*

(*i*) $E(t\omega_j) \to -\infty$ *as* $t \to \infty$ *for all* $j \geq 2$,

(*ii*) $\exists j_0 \geq 2$ *such that*

$$\sup_{t \geq 0} E(t\omega_{j_0}) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

Proof. (*i*) Fix $0 < \varepsilon < \beta$. By (1.3), $\exists M_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} > (\beta - \varepsilon)\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > M_\varepsilon. \tag{7.7}$$

Since $e^{\alpha\, |t|^{N'}} > \alpha^{2N-2}\, t^{2N}/(2N-2)!$ for all t, then there exists a constant $C_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} \geq \frac{1}{(2N-2)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{7.8}$$

and

$$G(t) \geq \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{7.9}$$

for all t. Since $\|\omega_j\| = 1$ and $\omega_j \geq 0$, then

$$E(t\omega_j) \leq \frac{t^N}{N} - \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} \int_\Omega \omega_j^{2N}\, dx + C_\varepsilon\, t \int_\Omega \omega_j\, dx,$$

and the conclusion follows.

(*ii*) Set

$$H_j(t) = E(t\omega_j) = \frac{t^N}{N} - \int_\Omega G(t\omega_j)\, dx, \quad t \geq 0.$$

If the conclusion is false, then it follows from (*i*) that for all $j \geq 2$, $\exists t_j > 0$ such that

$$H_j(t_j) = \frac{t_j^N}{N} - \int_\Omega G(t_j\omega_j)\, dx = \sup_{t \geq 0} H_j(t) \geq \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}, \tag{7.10}$$

$$H_j'(t_j) = t_j^{N-1} - \int_\Omega \omega_j\, h(t_j\omega_j)\, e^{\alpha\, t_j^{N'} \omega_j^{N'}}\, dx = 0. \tag{7.11}$$

Since $G(t) \geq -C_\varepsilon\, t$ for all $t \geq 0$ by (7.9), (7.10) gives

$$t_j^N \geq t_0^N - N\delta_j\, t_j, \tag{7.12}$$

where

$$t_0 = \Big(\frac{\alpha_N}{\alpha}\Big)^{(N-1)/N}$$

and

$$\delta_j = C_\varepsilon \int_\Omega \omega_j \, dx \to 0 \quad \text{as } j \to \infty \tag{7.13}$$

by Proposition 3.5. First we will show that $t_j \to t_0$.

By (7.12) and the Young's inequality,

$$(1 + \nu)\, t_j^N \ge t_0^N - \frac{N-1}{\nu^{1/(N-1)}}\, \delta_j^{N'} \quad \forall \nu > 0,$$

which together with (7.13) gives

$$\liminf_{j\to\infty} \, t_j \ge t_0. \tag{7.14}$$

Write (7.11) as

$$t_j^N = \int_{\{t_j\omega_j > M_\varepsilon\}} t_j\omega_j \, h(t_j\omega_j)\, e^{\alpha\, t_j^{N'} \omega_j^{N'}} \, dx + \int_{\{t_j\omega_j \le M_\varepsilon\}} t_j\omega_j \, h(t_j\omega_j)\, e^{\alpha\, t_j^{N'} \omega_j^{N'}} \, dx =: I_1 + I_2. \tag{7.15}$$

Set $r_j = d e^{-M_\varepsilon\, (\omega_{N-1} \log j)^{1/N}/t_j}$. Since $\liminf t_j > 0$, for all sufficiently large j, $d/j < r_j < d$ and $t_j\omega_j(x) > M_\varepsilon$ if and only if $|x| < r_j$. So (7.7) gives

$$I_1 \ge (\beta - \varepsilon) \int_{\{|x| < r_j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}} \, dx = (\beta - \varepsilon) \Bigg(\int_{\{|x| \le d/j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}} \, dx + \int_{\{d/j < |x| < r_j\}} e^{\alpha\, t_j^{N'} \omega_j^{N'}} \, dx \Bigg) =: (\beta - \varepsilon)\, (I_3 + I_4). \tag{7.16}$$

We have

$$I_3 = \frac{\omega_{N-1}}{N} \left(\frac{d}{j}\right)^N e^{\alpha\, t_j^{N'} \log j / \omega_{N-1}^{1/(N-1)}} = \frac{\omega_{N-1}}{N}\, d^N j^{\alpha\, (t_j^{N'} - t_0^{N'})/\omega_{N-1}^{1/(N-1)}}. \tag{7.17}$$

Since $th(t)\, e^{\alpha\, |t|^{N'}} \ge -C_\varepsilon\, t$ for all $t \ge 0$ by (7.8),

$$I_2 \ge -C_\varepsilon\, t_j \int_{\{t_j\omega_j \le M_\varepsilon\}} \omega_j \, dx \ge -\delta_j\, t_j. \tag{7.18}$$

Combining (7.15)–(7.18) and noting that $I_4 \geq 0$ gives

$$t_j^N \geq (\beta - \varepsilon)\, \frac{\omega_{N-1}}{N}\, d^N j^{\alpha\, (t_j^{N'} - t_0^{N'})/\omega_{N-1}^{1/(N-1)}} - \delta_j\, t_j.$$

It follows from this that

$$\limsup_{j \to \infty}\, t_j \leq t_0,$$

which together with (7.14) shows that $t_j \to t_0$.

Next we estimate I_4. We have

$$\begin{aligned}
I_4 &= \int_{\{d/j<|x|<r_j\}} e^{\alpha\, t_j^{N'} [\log\,(d/|x|)]^{N'}/(\omega_{N-1} \log j)^{1/(N-1)}}\, dx \\
&= \omega_{N-1} \left(\int_{d/j}^{d} e^{\alpha\, t_j^{N'} [\log\,(d/r)]^{N'}/(\omega_{N-1} \log j)^{1/(N-1)}}\, r^{N-1}\, dr \right. \\
&\quad \left. - \int_{r_j}^{d} e^{\alpha\, t_j^{N'} [\log\,(d/r)]^{N'}/(\omega_{N-1} \log j)^{1/(N-1)}}\, r^{N-1}\, dr \right) \\
&= \omega_{N-1}\, d^N \left(\log j \int_0^1 e^{-Nt\, [1 - (t_j/t_0)^{N'} t^{1/(N-1)}] \log j}\, dt \right. \\
&\quad \left. - \int_{s_j}^{1} s^{N-1}\, e^{\alpha\, t_j^{N'} (-\log s)^{N'}/(\omega_{N-1} \log j)^{1/(N-1)}}\, ds \right),
\end{aligned} \tag{7.19}$$

where $t = \log\,(d/r)/\log j$, $s = r/d$, and $s_j = r_j/d = e^{-M_\varepsilon\, (\omega_{N-1} \log j)^{1/N}/t_j} \to 0$. For $s_j < s < 1$, $\alpha\, t_j^{N'} (-\log s)^{N'}/(\omega_{N-1} \log j)^{1/(N-1)}$ is bounded by $\alpha M_\varepsilon^{N'}$ and goes to zero as $j \to \infty$, so the last integral converges to

$$\int_0^1 s^{N-1}\, ds = \frac{1}{N}.$$

So combining (7.15)–(7.19) and letting $j \to \infty$ gives

$$t_0^N \geq (\beta - \varepsilon)\, \frac{\omega_{N-1}}{N}\, d^N (L_1 + L_2 - 1),$$

where

$$L_1 = \liminf_{j \to \infty}\, e^{-n\, [1 - (t_j/t_0)^{N'}]},$$

$$L_2 = \liminf_{j \to \infty} \int_0^1 n e^{-n\, [t - (t_j/t_0)^{N'} t^{N'}]}\, dt,$$

and $n = N \log j \to \infty$. Letting $\varepsilon \to 0$ in this inequality gives

$$\beta \le \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N \frac{1}{L_1 + L_2 - 1}. \tag{7.20}$$

By (7.10), (7.1), and Proposition 3.5,

$$t_j^N - t_0^N \ge N \int_\Omega G(t_j \omega_j)\, dx \ge -\sigma_0\, t_j^N \int_\Omega \omega_j^N\, dx \ge -\frac{\sigma_0\, t_j^N}{\kappa n},$$

so

$$\left(\frac{t_j}{t_0}\right)^{N'} \ge \left(1 + \frac{\sigma_0}{\kappa n}\right)^{-1/(N-1)} \ge 1 - \frac{\sigma_0}{(N-1)\, \kappa n}.$$

This gives

$$L_1 \ge e^{-\sigma_0/(N-1)\, \kappa}$$

and

$$L_2 \ge \lim_{n \to \infty} \int_0^1 n e^{-n\, (t - t^{N'}) - \sigma_0\, t^{N'}/(N-1)\, \kappa}\, dt \ge \mathcal{M}\, e^{-\sigma_0/(N-1)\, \kappa}.$$

So (7.20) gives

$$\beta \le \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N \frac{1}{\mathcal{M}\, e^{-\sigma_0/(N-1)\, \kappa} - (1 - e^{-\sigma_0/(N-1)\, \kappa})} \le \frac{1}{\mathcal{M}\, \alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{\sigma_0/(N-1)\, \kappa},$$

contradicting (7.3). □

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. Let j_0 be as in Lemma 7.4 (*ii*). By Lemma 7.4 (*i*), $\exists R > \rho$ such that $E(R\omega_{j_0}) \le 0$, where ρ is as in Lemma 7.3. Let

$$\Gamma = \left\{\gamma \in C([0,1], W_0^{1,N}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = R\omega_{j_0}\right\}$$

be the class of paths joining the origin to $R\omega_{j_0}$, and set

$$c := \inf_{\gamma \in \Gamma}\ \max_{u \in \gamma([0,1])}\ E(u).$$

By Lemma 7.3, $c > 0$. Since the path $\gamma_0(t) = tR\omega_{j_0}$, $t \in [0,1]$ is in Γ,

$$c \le \max_{u \in \gamma_0([0,1])} E(u) \le \sup_{t \ge 0} E(t\omega_{j_0}) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

If there are no $(\mathrm{PS})_c$ sequences of E, then E satisfies the $(\mathrm{PS})_c$ condition vacuously and hence has a critical point u at the level c by the mountain pass theorem. Then u is a solution of problem (1.1) and u is nontrivial since $c > 0$. So we may assume that E has a $(\mathrm{PS})_c$ sequence. Then this sequence has a subsequence that converges weakly to a nontrivial solution of problem (1.1) by Proposition 6.1. □

8 Second Existence Result

8.1 Statement

To state our second result, let (λ_k) be the sequence of eigenvalues of problem (1.7) based on the $\mathbb{Z}_2$-cohomological index that was introduced in Perera [11] (see Section 5). We have the following theorem.

Theorem 8.1. *Assume that $\alpha > 0$, h satisfies* (1.2) *and* (1.3)*, and G satisfies*

$$G(t) \geq \frac{1}{N} (\lambda_{k-1} + \sigma_0) |t|^N \quad \forall t, \tag{8.1}$$

$$G(t) \leq \frac{1}{N} (\lambda_k - \sigma_1) |t|^N \quad \text{for } |t| \leq \delta \tag{8.2}$$

for some $k \geq 2$ and $\sigma_0, \sigma_1, \delta > 0$. Then there exists a constant $c > 0$ depending on Ω, α, and k, but not on σ_0, σ_1, or δ, such that if

$$\beta > \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{c/\sigma_0^{N-1}},$$

then problem (1.1) *has a nontrivial solution.*

Theorem 8.1 should be compared with Theorem 1.4 of de Figueiredo et al. [3, 4], where this result is proved in the case $N = 2$ under the additional assumption that $0 < 2G(t) \leq th(t)\, e^{\alpha t^2}$ for all $t \in \mathbb{R} \setminus \{0\}$. However, the linking argument used in [3, 4] is based on a splitting of $H^1_0(\Omega)$ that involves the eigenspaces of the Laplacian, and this argument does not extend to the case $N \geq 3$ where the N-Laplacian is a nonlinear operator and therefore has no linear eigenspaces. We will prove Theorem 8.1 using an abstract critical point theorem based on the $\mathbb{Z}_2$-cohomological index that was proved in Yang and Perera [15] (see Section 4.3).

8.2 Proof of Theorem 8.1

In the context of Theorem 4.7, we take A_0 to be the set C in Proposition 5.2 and $B_0 = \Psi_{\lambda_k}$. Since $i(S \setminus B_0) = k - 1$ by Proposition 5.1, (4.1) holds.

Lemma 8.2. *There exists a $\rho > 0$ such that* $\inf E(B) > 0$*, where $B = \{\rho u : u \in B_0\}$.*

Proof. As in the proof of Lemma 7.3, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \leq C_\delta\, |t|^{N+1}\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > \delta,$$

which together with (8.2) gives

$$G(t) \leq \frac{1}{N} (\lambda_k - \sigma_1) |t|^N + C_\delta\, |t|^{N+1}\, e^{\alpha\, |t|^{N'}} \quad \forall t. \tag{8.3}$$

For $u \in B_0$ and $\rho > 0$,

$$\int_\Omega |\rho u|^N dx \le \frac{\rho^N}{\lambda_k} \tag{8.4}$$

and

$$\int_\Omega |\rho u|^{N+1} e^{\alpha\, |\rho u|^{N'}} dx \le \rho^{N+1} \left(\int_\Omega |u|^{2\,(N+1)}\, dx\right)^{1/2} \left(\int_\Omega e^{2\alpha\, \rho^{N'} |u|^{N'}} dx\right)^{1/2}. \tag{8.5}$$

The first integral on the right-hand side of (8.5) is bounded by the Sobolev embedding theorem, and the second integral is bounded when $\rho^{N'} \le \alpha_N/2\alpha$ by (1.4). So combining (8.3)–(8.5) gives

$$\int_\Omega G(\rho u)\, dx \le \frac{1}{N} \left(1 - \frac{\sigma_1}{\lambda_k}\right) \rho^N + \mathrm{O}(\rho^{N+1}) \quad \text{as } \rho \to 0.$$

Then

$$E(\rho u) \ge \frac{1}{N} \frac{\sigma_1}{\lambda_k}\, \rho^N + \mathrm{O}(\rho^{N+1}),$$

and the desired conclusion follows from this for sufficiently small ρ. □

We may assume without loss of generality that $B_d(0) \subset \Omega$. Let (ω_j) be the sequence of functions defined in (3.9).

Lemma 8.3. *We have*

(i) $E(sv) \le 0 \;\; \forall v \in A_0,\ s \ge 0$*,*

(ii) for all $j \ge 2$*,*

$$\sup \{E(R\, \pi((1 - t)\, v + t\omega_j)) : v \in A_0,\ 0 \le t \le 1\} \to -\infty \text{ as } R \to \infty,$$

(iii) $\exists j_0 \ge 2$ *such that*

$$\sup \{E(sv + t\omega_{j_0}) : v \in A_0,\ s, t \ge 0\} < \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

Proof. (*i*) By (8.1),

$$E(u) \le \frac{1}{N} \left[\int_\Omega |\nabla u|^N dx - (\lambda_{k-1} + \sigma_0) \int_\Omega |u|^N dx\right]. \tag{8.6}$$

For $v \in A_0$ and $s \ge 0$,

$$\int_\Omega |sv|^N dx \ge \frac{s^N}{\lambda_{k-1}}$$

since $A_0 \subset \Psi^{\lambda_{k-1}}$, so (8.6) gives

$$E(sv) \le -\frac{1}{N}\, \frac{\sigma_0}{\lambda_{k-1}}\, s^N \le 0.$$

(*ii*) Fix $0 < \varepsilon < \beta$. As in the proof of Lemma 7.4 (*i*), $\exists M_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} > (\beta - \varepsilon)\, e^{\alpha\, |t|^{N'}} \quad \text{for } |t| > M_\varepsilon \tag{8.7}$$

and there exists a constant $C_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha\, |t|^{N'}} \ge \frac{1}{(2N-2)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{8.8}$$

and

$$G(t) \ge \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, t^{2N} - C_\varepsilon\, |t| \tag{8.9}$$

for all t. Let $A_1 = \{\pi((1-t)\, v + t\omega_j) : v \in A_0,\, 0 \le t \le 1\}$. For $u \in A_1$ and $R > 0$, (8.9) gives

$$E(Ru) \le \frac{R^N}{N} - \frac{2N-1}{(2N)!}\, (\beta - \varepsilon)\, \alpha^{2N-2}\, R^{2N} \int_\Omega |u|^{2N} dx + C_\varepsilon R \int_\Omega |u|\, dx.$$

The set A_1 is compact since A_0 is compact, so the first integral on the right-hand side is bounded away from zero on A_1. Since the second integral is bounded, the desired conclusion follows.

(*iii*) If the conclusion is false, then it follows from (*i*) and (*ii*) that for all $j \ge 2$, there exist $v_j \in A_0$, $s_j \ge 0$, $t_j > 0$ such that

$$E(s_j v_j + t_j \omega_j) = \sup \{E(sv + t\omega_j) : v \in A_0,\, s, t \ge 0\} \ge \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}.$$

Set $u_j = s_j v_j + t_j \omega_j$. Then

$$E(u_j) = \frac{1}{N}\, \|u_j\|^N - \int_\Omega G(u_j)\, dx \ge \frac{1}{N} \left(\frac{\alpha_N}{\alpha}\right)^{N-1}. \tag{8.10}$$

Moreover, $\tau u_j \in \{sv + t\omega_j : v \in A_0,\, s, t \ge 0\}$ for all $\tau \ge 0$ and $E(\tau u_j)$ attains its maximum at $\tau = 1$, so

$$\frac{\partial}{\partial \tau}\, E(\tau u_j)\bigg|_{\tau = 1} = E'(u_j)\, u_j = \|u_j\|^N - \int_\Omega u_j\, h(u_j)\, e^{\alpha\, |u_j|^{N'}} dx = 0. \tag{8.11}$$

Since $\|v_j\| = \|\omega_j\| = 1$ and $G(t) \ge 0$ for all t by (8.1), (8.10) gives

$$s_j + t_j \ge t_0,$$

where

$$t_0 = \left(\frac{\alpha_N}{\alpha}\right)^{(N-1)/N}.$$

First we show that $s_j \to 0$ and $t_j \to t_0$ as $j \to \infty$.

Combining (8.10) with (8.1) gives

$$\|s_j v_j + t_j \omega_j\|^N \geq (\lambda_{k-1} + \sigma_0) \int_\Omega |s_j v_j + t_j \omega_j|^N dx + t_0^N.$$

Set $\tau_j = s_j/t_j$. Then

$$\|\tau_j v_j + \omega_j\|^N \geq (\lambda_{k-1} + \sigma_0) \int_\Omega |\tau_j v_j + \omega_j|^N dx + \left(\frac{t_0}{t_j}\right)^N. \tag{8.12}$$

Since (v_j) is bounded in $C^1(\overline{\Omega})$, Proposition 3.5 gives

$$\|\tau_j v_j + \omega_j\|^N \leq \int_\Omega (\tau_j |\nabla v_j| + |\nabla \omega_j|)^N dx = \tau_j^N \int_\Omega |\nabla v_j|^N dx + \int_\Omega |\nabla \omega_j|^N dx$$

$$+ \sum_{m=1}^{N-1} \binom{N}{m} \tau_j^{N-m} \int_\Omega |\nabla v_j|^{N-m} |\nabla \omega_j|^m \, dx \leq \tau_j^N + 1 + c_1 \sum_{m=1}^{N-1} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

and

$$\int_\Omega |\tau_j v_j + \omega_j|^N dx \geq \int_\Omega (\tau_j |v_j| - \omega_j)^N dx = \tau_j^N \int_\Omega |v_j|^N dx$$

$$+ \sum_{m=1}^{N} (-1)^m \binom{N}{m} \tau_j^{N-m} \int_\Omega |v_j|^{N-m} \omega_j^m \, dx \geq \frac{\tau_j^N}{\lambda_{k-1}} - c_2 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

for some constants $c_1, c_2 > 0$. So (8.12) gives

$$\frac{\sigma_0}{\lambda_{k-1}} \tau_j^N + \left(\frac{t_0}{t_j}\right)^N \leq 1 + c_3 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}} \tag{8.13}$$

for some constant $c_3 > 0$, which implies that (τ_j) is bounded and

$$\liminf_{j \to \infty} t_j \geq t_0. \tag{8.14}$$

Next combining (8.11) with (8.7) and (8.8) gives

$$\|u_j\|^N = \int_{\{|u_j| > M_\varepsilon\}} u_j \, h(u_j) \, e^{\alpha |u_j|^{N'}} dx + \int_{\{|u_j| \leq M_\varepsilon\}} u_j \, h(u_j) \, e^{\alpha |u_j|^{N'}} dx$$

$$\geq (\beta - \varepsilon) \int_{\{|u_j| > M_\varepsilon\}} e^{\alpha |u_j|^{N'}} dx - C_\varepsilon \int_{\{|u_j| \leq M_\varepsilon\}} |u_j| \, dx. \tag{8.15}$$

For $|x| \le d/j$,

$$|u_j| \ge t_j \omega_j - s_j\, |v_j| \ge \frac{t_j}{\omega_{N-1}^{1/N}} \left[(\log j)^{(N-1)/N} - c_4 \tau_j\right]$$

for some constant $c_4 > 0$, and the last expression is greater than M_ε for all sufficiently large j since (τ_j) is bounded and $\liminf t_j > 0$. So

$$\int_{\{|u_j| > M_\varepsilon\}} e^{\alpha\, |u_j|^{N'}}\, dx \ge e^{\alpha\, t_j^{N'} [(\log j)^{(N-1)/N} - c_4 \tau_j]^{N'} / \omega_{N-1}^{1/(N-1)}} \int_{\{|x| \le d/j\}} dx$$

$$= \frac{\omega_{N-1}\, d^N}{N}\, j^{\alpha\, [t_j^{N'} (1 - c_4 \tau_j / (\log j)^{(N-1)/N})^{N'} - t_0^{N'}] / \omega_{N-1}^{1/(N-1)}}$$

for large j. On the other hand,

$$\int_{\{|u_j| \le M_\varepsilon\}} |u_j|\, dx \le \int_\Omega (s_j\, |v_j| + t_j \omega_j)\, dx \le c_5\, t_j \left[\tau_j + \frac{1}{(\log j)^{1/N}}\right]$$

for some constant $c_5 > 0$ by Proposition 3.5. So (8.15) gives

$$(\beta - \varepsilon)\, j^{\alpha\, [t_j^{N'} (1 - c_4 \tau_j / (\log j)^{(N-1)/N})^{N'} - t_0^{N'}] / \omega_{N-1}^{1/(N-1)}} \le \frac{N t_j^N (\tau_j + 1)^N}{\omega_{N-1}\, d^N} + c_6\, t_j \left[\tau_j + \frac{1}{(\log j)^{1/N}}\right] \qquad (8.16)$$

for some constant $c_6 > 0$. Since (τ_j) is bounded, it follows from this that

$$\limsup_{j \to \infty}\, t_j \le t_0,$$

which together with (8.14) shows that $t_j \to t_0$. Then (8.13) implies that $\tau_j \to 0$, so $s_j = \tau_j\, t_j \to 0$.

Now we show that there exists a constant $c > 0$ depending only on Ω, α, and k such that

$$\beta \le \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N e^{c/\sigma_0^{N-1}}. \qquad (8.17)$$

The right-hand side of (8.16) goes to $(N/d)^N / \alpha^{N-1}$ as $j \to \infty$. If $\beta \le (N/d)^N / \alpha^{N-1}$, then we may take any $c > 0$, so suppose $\beta > (N/d)^N / \alpha^{N-1}$. Then for $\varepsilon < \beta - (N/d)^N / \alpha^{N-1}$ and all sufficiently large j, (8.16) gives $j^{\alpha\, [t_j^{N'} (1 - c_4 \tau_j / (\log j)^{(N-1)/N})^{N'} - t_0^{N'}] / \omega_{N-1}^{1/(N-1)}} \le 1$, so

$$\frac{t_0}{t_j} \ge 1 - \frac{c_4 \tau_j}{(\log j)^{(N-1)/N}}.$$

Combining this with (8.13) gives

$$\frac{\sigma_0}{\lambda_{k-1}}\,\tau_j^N - \frac{N c_4 \tau_j}{(\log j)^{(N-1)/N}} \le c_3 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}},$$

so

$$\sigma_0 \tau_j^N \le c_7 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}$$

for some constant $c_7 > 0$. Set $\widetilde{\tau}_j = \tau_j\,(\log j)^{1/N}$. Then

$$\sigma_0 \widetilde{\tau}_j^N \le c_7 \sum_{m=1}^{N} \widetilde{\tau}_j^{N-m}. \tag{8.18}$$

We claim that

$$\widetilde{\tau}_j \le \frac{c_8}{\sigma_0} \tag{8.19}$$

for some constant $c_8 > 0$. Taking σ_0 smaller in (8.1) if necessary, we may assume that $\sigma_0 \le 1$. So if $\widetilde{\tau}_j < 1$, then (8.19) holds with $c_8 = 1$, so suppose $\widetilde{\tau}_j \ge 1$. Then (8.18) gives (8.19) with $c_8 = N c_7$. Now (8.13) gives

$$\left(\frac{t_0}{t_j}\right)^N \le 1 + \frac{c_3}{\log j} \sum_{m=1}^{N} \widetilde{\tau}_j^{N-m} \le 1 + \frac{c_9}{\sigma_0^{N-1} \log j}$$

for some constant $c_9 > 0$, so

$$\left(\frac{t_0}{t_j}\right)^{N'} \le \left(1 + \frac{c_9}{\sigma_0^{N-1} \log j}\right)^{1/(N-1)} \le 1 + \frac{c_9}{\sigma_0^{N-1} \log j}.$$

Then

$$\begin{aligned} t_j^{N'} \left[1 - \frac{c_4 \tau_j}{(\log j)^{(N-1)/N}}\right]^{N'} - t_0^{N'} &= t_j^{N'} \left[\left(1 - \frac{c_4 \widetilde{\tau}_j}{\log j}\right)^{N'} - \left(\frac{t_0}{t_j}\right)^{N'}\right] \\ \ge t_j^{N'} \left[\left(1 - \frac{c_{10}}{\sigma_0 \log j}\right)^{N'} - \left(1 + \frac{c_9}{\sigma_0^{N-1} \log j}\right)\right] &\ge -t_j^{N'} \left(\frac{N' c_{10}}{\sigma_0 \log j} + \frac{c_9}{\sigma_0^{N-1} \log j}\right) \\ &\ge -\frac{c_{11}}{\sigma_0^{N-1} \log j} \end{aligned}$$

for some constants $c_{10}, c_{11} > 0$, so

$$j^{\alpha\,[t_j^{N'}(1-c_4\tau_j/(\log j)^{(N-1)/N})^{N'}-t_0^{N'}]/\omega_{N-1}^{1/(N-1)}} \geq j^{-c/\sigma_0^{N-1}\log j} = e^{-c/\sigma_0^{N-1}}$$

for some constant $c > 0$. Combining this with (8.16) and passing to the limit gives

$$(\beta - \varepsilon)\, e^{-c/\sigma_0^{N-1}} \leq \frac{1}{\alpha^{N-1}} \left(\frac{N}{d}\right)^N,$$

and letting $\varepsilon \to 0$ gives (8.17). □

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Let $j_0 \geq 2$ be as in Lemma 8.3 (*iii*). By Lemma 8.3 (*ii*), $\exists R > \rho$ such that

$$\sup \{E(R\,\pi((1-t)\,v + t\omega_{j_0})) : v \in A_0,\, 0 \leq t \leq 1\} \leq 0, \tag{8.20}$$

where $\rho > 0$ is as in Lemma 8.2. Let

$$A = \{sv : v \in A_0,\, 0 \leq s \leq R\} \cup \{R\,\pi((1-t)\,v + t\omega_{j_0}) : v \in A_0,\, 0 \leq t \leq 1\},$$

$$X = \left\{sv + t\omega_{j_0} : v \in A_0,\, s, t \geq 0,\, \|sv + t\omega_{j_0}\| \leq R\right\}.$$

Combining Lemma 8.3 (*i*), (8.20), and Lemma 8.2 gives

$$\sup E(A) \leq 0 < \inf E(B), \tag{8.21}$$

while Lemma 8.3 (*iii*) gives

$$\sup E(X) \leq \sup \{E(sv + t\omega_{j_0}) : v \in A_0,\, s, t \geq 0\} < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1}. \tag{8.22}$$

Let

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\},$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).$$

By Theorem 4.7, $\inf E(B) \leq c \leq \sup E(X)$, and E has a $(\mathrm{PS})_c$ sequence. By (8.21) and (8.22),

$$0 < c < \frac{1}{N}\left(\frac{\alpha_N}{\alpha}\right)^{N-1},$$

so a subsequence of this $(\mathrm{PS})_c$ sequence converges weakly to a nontrivial solution of problem (1.1) by Proposition 6.1. □

9 Future Work

In this dissertation, we were able to show the existence of weak solutions of problem (1.1) where the growth of the associated energy functional depends on the eigenvalues of the p-Laplacian operator generated by the cohomological index. In the future we would like to investigate whether similar bifurcation and multiplicity results hold for this problem as in Yang and Perera [15]. Namely, we wish to know if there is a parameter in the problem that can be adjusted such that if it is within a range of values depending on λ_{k-1}, λ_k and the domain Ω, problem (1.1) will have at least m pairs of solutions, where m is the multiplicity of λ_k.

Another possible direction would be to establish existence results for the case where we have a singularity in the nonlinear term.

$$\begin{cases} -\Delta_N \, u = \dfrac{h(u)}{|x|^\gamma} \, e^{\alpha \, |u|^{N'}} & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

In this case, $0 \leq \gamma < N$. Another sequence of eigenvalues would be needed based on the corresponding singular version of the Dirichlet eigenvalue problem for the p-Laplacian.

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