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On the Inverse Multiphase Stefan Problem

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Abstract

Title: On the Inverse Multiphase Stefan Problem

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We consider inverse multiphase Stefan problem, where information on the heat flux on the fixed boundary is missing and must be found along with the temperature and free boundaries. Optimal control framework is pursued, where boundary heat flux is the control, and optimality criteria consists of the minimization of the L_2-norm declination of the trace of the solution to the Stefan problem from the temperature measurement on the fixed boundary. State vector solves multiphase Stefan problem in a weak formulation, which is equivalent to Neumann problem for the quasilinear parabolic PDE with discontinuous coefficient. Full discretization through finite differences is implemented and discrete optimal control problem is introduced. We prove well-posedness in Sobolev spaces framework and convergence of discrete optimal control problems to the original problem both with respect to cost functional and control. Along the way the convergence of the method of finite differences for the weak solution of the multiphase Stefan problem is proved. The proof is based on the proof of uniform L_∞ bound, and $W_2^{1,1}$-energy estimate for the discrete multiphase Stefan problem.

Contents

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Notation of Spaces Used

$L_2[0,T]$ - Space of Lebesgue square-integrable functions. It is a Hilbert space with inner product

$$(u,v) = \int_0^T uv\,dt$$

$L_\infty[0,T]$ - Space of essentially bounded functions. It is a Banach space with norm

$$\|u\|_{L_\infty[0,T]} = \operatorname*{esssup}_{0\le t\le T} |u(t)|$$

$W_2^k[0,T], k = 1,2,...$ - Hilbert space of all elements of $L_2[0,T]$ whose weak derivatives up to order k exist and belong to $L_2[0,T]$. The inner product is defined as

$$(u,v) = \int_0^T \sum_{s=0}^{k} \frac{d^s u}{dt^s}\frac{d^s v}{dt^s}\,dt$$

$L_2(D)$ - Hilbert space with inner product

$$(u,v) = \int_D uv\,dx\,dt$$

$W_2^{1,0}(D)$ - Hilbert space of all elements of $L_2(D)$ that have a weak derivative in the x direction, $\frac{\partial u}{\partial x}$, and such that it belongs to $L_2(D)$. The inner product is defined as

$$(u,v) = \int_D \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) dx\,dt$$

$W_2^{1,1}(D)$ - Hilbert space of all elements of $L_2(\Omega)$ with weak derivatives of first order, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$. Also its weak derivatives must belong to $L_2(D)$. The inner product is defined as

$$(u,v) = \int_D \left(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\right) dx\,dt$$

Chapter 1

Introduction

1.1 The Multiphase Stefan Problem

Consider the following one dimensional multiphase Stefan problem: Find $u(x,t),\ \xi_j(t), j = \overline{1,J}$ in $D = \{0 < x < \ell, \quad 0 < t \leq T\}$ satisfying (1.1)-(1.6) below:

$$\alpha(u)u_t - (k(u)u_x)_x = f(x,t), \qquad (x,t) \in D, \quad u(x,t) \neq u^j, j = \overline{1,J} \tag{1.1}$$

$$u|_{x=\xi_j(t)} = u^j, \qquad 0 < t \leq T, \quad j = \overline{1,J} \tag{1.2}$$

$$[u]|_{x=\xi_j(t)} = 0, \qquad 0 < t \leq T, \quad j = \overline{1,J} \tag{1.3}$$

$$[k(u)u_x]|_{x=\xi_j(t)} = \gamma_j \frac{d\xi_j}{dt}, \qquad 0 < t \leq T, \quad j = \overline{1,J} \tag{1.4}$$

$$u(x,0) = \phi(x), \qquad 0 \leq x \leq \ell \tag{1.5}$$

$$k(u)u_x|_{x=0} = g(t), \qquad k(u)u_x|_{x=\ell} = p(t), \qquad 0 < t \leq T \tag{1.6}$$

Where α, k are positive C^1 functions on each segment $[u^j, u^{j+1}]$ with 1st type discontinuity at $u = u^j,\ j = \overline{1,J}$, with $u^1 < u^2 < \cdots < u^J$, each $\gamma_j, j = \overline{1,J}$ is a positive number, and $[u]|_{x=\xi_j}$ is the saltus of u at ξ_j, defined as

$$[u]|_{x=\xi_j} = u|^+_{x=\xi_j} - u|^-_{x=\xi_j}$$

In the physical context, f characterizes the density of the sources, ϕ is the initial temperature, g and p are the heat fluxes on the left and right fixed boundary respectively, each u^j represents a phase transition temperature, and (1.4) is the Stefan condition, which relates the free boundary heat fluxes to a push in the corresponding boundary.

Weak formulation of the multiphase Stefan problem, as well as existence and uniqueness of the weak solution to the multiphase Stefan problem was first proved in [23, 30]. We refer to monographies [25, 27, 15] for the details of this approach.

1.2 The Inverse Multiphase Stefan Problem

Suppose now that a certain piece of data is not available. For example, suppose that the heat flux, g, at the fixed boundary $x = 0$ is not known and must be found along with the temperature u and the phase transition boundaries ξ_j. As compensation for not knowing this function, we must have access to additional information, which for instance may come as a measurement of the temperature at the fixed boundary $x = \ell$:

$$u(\ell, t) = \nu(t), \qquad 0 < t \leq T \tag{1.7}$$

Inverse Multiphase Stefan Problem (IMSP). Find the functions $u(x, t),\ \xi_j(t),\ j = \overline{1, J}$, and the boundary heat flux $g(t)$ satisfying (1.1)-(1.7).

Inverse Stefan problem is not well posed in the sense of Hadamard. That is, if the data is not sufficiently coordinated, there may be no solution. Or even if a solution exists, it may not do so uniquely. Or worst of all, the solution may not depend continuously on the data functions.

We refer to a recent paper [1] for broad review of the literature on Inverse Stefan Problems. The one-phase inverse Stefan problem (ISP) was first mentioned in [10], where unknown heat flux is to be determined under the given free boundary. The variational approach for solving this ill-posed inverse Stefan problem was used in [7, 8]. The first result on the optimal control of the Stefan problem appeared in [35], where an optimal temperature along the fixed boundary must be determined to guarantee that the solutions of the Stefan problem stay close to the measurements taken at the final time. In [35], the existence result was proved. In [36], the Frechet derivative was found and the convergence of the finite difference scheme was proved, and Tikhonov regularization was suggested. Later development of the inverse Stefan problem proceeded in these two directions: Inverse

Stefan problems with given phase boundaries were considered in [3, 5, 6, 9, 11, 12, 13, 18, 33, 16]; optimal control of Stefan problems, or equivalently inverse problems with unknown phase boundaries were investigated in [4, 14, 19, 20, 21, 22, 24, 26, 29, 28, 31, 32, 34, 16]. We refer to the monography [16] for a complete list of references of both types of inverse Stefan problems, both for linear and quasilinear parabolic equations.

In two recent papers [1, 2] a new variational formulation of the the one-phase ISP was developed. Optimal control framework was implemented where boundary heat flux and the free boundary are components of the control vector and optimality criteria consist of the minimization of the sum of L_2-norm declinations from the available measurement of the temperature flux on the fixed boundary and available information on the phase transition temperature on the free boundary. This approach allows one to tackle situations when the phase transition temperature is not known explicitly, and is available through measurement with possible error. It also allows for the development of iterative numerical methods of least computational cost due to the fact that for every given control vector, the parabolic PDE is solved in a fixed region instead of full free boundary problem. In [1] the well-posedness in Sobolev spaces framework and convergence of time-discretized optimal control problems is proved. In [2] full discretization was implemented and the convergence of the discrete optimal control problems to the original problem both with respect to cost functional and control is proved. The main advantage of this method is that numerically at each step, the problem to be solved is only a Neumann problem, and not a full free boundary problem.

For the IMSP, some results are seen in [17]. The main goal of this research project is to apply to the IMSP the optimal control framework often used as an approach to one-dimensional Inverse Stefan problems. As in [2], the optimal control problem will be solved by performing a full discretization through finite differences and proving the convergence of discrete optimal control problems to the original problem both with respect to the cost functional and the control.

1.3 Reformulation of the Inverse Multiphase Stefan Problem and Weak Solution

Following the usual reformulation of the IMSP (see [25, 30]), we consider the transformation

$$v(x,t) := \int_{u_1}^{u(x,t)} k(y)\,dy \tag{1.8}$$

Then $v^j = \int_{u^1}^{u^j} k(y)\,dy$, $v^1 = 0 < \cdots < v^J$, and our conditions become:

$$\beta(v)v_t - v_{xx} = f(x,t), \qquad (x,t) \in D, v(x,t) \neq v^j \tag{1.9}$$

$$v|_{x=\xi_j(t)} = v^j, \qquad 0 < t \leq T \tag{1.10}$$

$$[v]|_{x=\xi_j(t)} = 0, \qquad 0 < t \leq T, \quad j = \overline{1,J} \tag{1.11}$$

$$[v_x]|_{x=\xi_j(t)} = \gamma_j \frac{d\xi_j}{dt}, \qquad 0 < t \leq T, \quad j = \overline{1,J} \tag{1.12}$$

$$v(x,0) = \Phi(x) = \int_{u^1}^{\phi(x)} k(y)\,dy, \qquad 0 \leq x \leq \ell \tag{1.13}$$

$$v_x|_{x=0} = g(t), \qquad 0 < t \leq T \tag{1.14}$$

$$v_x|_{x=\ell} = p(t), \qquad 0 < t \leq T \tag{1.15}$$

$$v(\ell,t) = \Gamma(t) = \int_{u^1}^{\nu(t)} k(y)\,dy, \qquad 0 < t \leq T \tag{1.16}$$

Now, we can invoke a function $b(v)$ such that $b'(v) = \beta(v)$. Naturally then, $\frac{\partial b(v)}{\partial t} = \beta(v)v_t$. Our partial differential equation becomes

$$\frac{\partial b(v)}{\partial t} - v_{xx} = f(x,t), \qquad (x,t) \in D, v(x,t) \neq v^j \tag{1.17}$$

Moreover, we're free to choose the jump of b at the values $v = v^j$. Call $\xi_0(t) = 0$, $\xi_{J+1}(t) = \ell$, and define the surfaces

$$\mathscr{C}_j := \{(x,t) \in D \mid v(x,t) = v^j\}, \qquad j = \overline{1,J} \tag{1.18}$$

Our IMSP is thus redefined as: Find $\{u(x,t), \quad \xi_j(t), j=\overline{1,J}, \quad g(t)\}$ such that

$$\frac{\partial b(v)}{\partial t} - v_{xx} = f(x,t), \qquad (x,t)\in D, v(x,t)\neq v^j \tag{1.19}$$

$$v(x,0) = \Phi(x) \qquad 0\le x\le \ell \tag{1.20}$$

$$v_x|_{x=0} = g(t), \qquad v_x|_{x=\ell} = p(t), \qquad 0<t\le T \tag{1.21}$$

$$v|_{x=\xi_j(t)} = v^j, \quad [v]|_{x=\xi_j(t)} = 0, \quad [v_x]|_{x=\xi_j(t)} = \gamma_j \frac{d\xi_j}{dt}, \qquad 0<t\le T, \quad j=\overline{1,J} \tag{1.22}$$

$$v(\ell,t) = \Gamma(t) \qquad 0<t\le T \tag{1.23}$$

We derive the weak solution to the Stefan problem through the following formal argument: Assume that the surfaces $\mathscr{C}_j$ are piecewise smooth. Multiply (1.17) by a function $\psi\in C^1(D)$ such that $\psi(x,T)=0$, and integrate over D to get:

$$\int_0^T \sum_{j=1}^{J+1} \int_{\xi_{j-1}(t)}^{\xi_j(t)} \Big[\frac{\partial b(v(x,t))}{\partial t}\psi(x,t) - v_{xx}(x,t)\psi(x,t) - f(x,t)\psi(x,t)\Big]\,dx\,dt = 0$$

Now perform integration by parts on the first term with respect to t and the second with respect to x. We have

$$\int_0^T \sum_{j=1}^{J+1} \int_{\xi_{j-1}(t)}^{\xi_j(t)} \Big[-b(v)\psi_t + v_x\psi_x - f\psi\Big]\,dx\,dt + \int_0^\ell b(v)\psi\,dx\Big|_0^T - \int_0^T v_x\psi\,dt\Big|_0^\ell -$$
$$-\sum_{j=1}^{J}\int_{\mathscr{C}_j}\Big([b(v)]|_{\mathscr{C}_j}\cos(n,t) - [v_x]|_{\mathscr{C}_j}\cos(n,x)\Big)\psi\,dS = 0 \tag{1.24}$$

The difference in the last integrand of (1.24) can be made 0 if we select the saltus of b at v^j so that

$$[b(v)]|_{v=v^j} = \gamma_j \tag{1.25}$$

Indeed, in this case, applying (1.12) vanishes the last term. Furthermore, from (1.13), (1.14), (1.15) and our selection of ψ, (1.24) becomes

$$\int_0^T\int_0^\ell \Big[-b(v(x,t))\psi_t(x,t)+v_x(x,t)\psi_x(x,t)-f(x,t)\psi(x,t)\Big]\,dx\,dt-\int_0^\ell b(\Phi(x))\psi(x,0)\,dx- \\ -\int_0^T p(t)\psi(\ell,t)\,dt+\int_0^T g(t)\psi(0,t)\,dt=0 \tag{1.26}$$

It is important to see that b is non-decreasing and C^1 in each of the segments $[v^j, v^{j+1}]$ and has 1st type discontinuities at $v = v^j$ according to (1.25). Integral identity (1.26) is satisfied by a classical solution v of the Stefan problem for every function $\psi \in C^1(D)$ such that $\psi(x,T) = 0$. We will use (1.26) in our definition for the weak solution, but at the moment, the value of $b(v)$ on $\mathscr{C}_j$ is not defined. Hence we introduce the following

Definition. We say that a measurable function $B(x,t,v)$ is *of type* $\mathscr{B}$ if

(a) $B(x,t,v) = b(v), \qquad v \neq v^j, \quad \forall j = \overline{1,J}$

(b) $B(x,t,v) \in [b(v^j)^-, b(v^j)^+], \qquad v = v^j$ for some j.

Given g, a solution to the Stefan problem (1.19)-(1.22) is understood in the following sense:

Definition. $v \in W_2^{1,1}(D) \cap L_\infty(D)$ is called a *weak solution of the Stefan problem* (1.19)-(1.22) if for any two functions B, B_0 of type $\mathscr{B}$, the following integral identity is satisfied:

$$\int_0^T\int_0^\ell \Big[-B(x,t,v(x,t))\psi_t+v_x\psi_x-f\psi\Big]\,dxdt-\int_0^\ell B_0(x,0,\Phi(x))\psi(x,0)\,dx- \\ -\int_0^T p(t)\psi(\ell,t)\,dt+\int_0^T g(t)\psi(0,t)\,dt=0, \qquad \forall\psi\in W_2^{1,1}(D), \quad \psi(x,T)=0 \tag{1.27}$$

1.4 Optimal Control Problem.

Consider the control set

$$\mathscr{G}_R = \left\{ g : g \in W_2^1[0,T],\ \|g\|_{W_2^1[0,T]} \le R \right\}$$

We wish to minimize the cost functional $\mathscr{J}$ given by

$$\mathscr{J}(g) = \|v(\ell,t;g) - \Gamma(t)\|^2_{L_2[0,T]} \tag{1.28}$$

on $\mathscr{G}_R$, where $v = v(x,t;g) \in W_2^{1,1}(D) \cap L_\infty(D)$ is a weak solution of the Stefan problem in the sense of (1.27). This optimization problem will be called *Problem* $\mathcal{I}$.

1.5 Discretization.

Let

$$\omega_\tau = \{t_k, k = \overline{1,n}\}, \tau = \frac{T}{n}, t_k = k\tau, \qquad \omega_h = \{x_i, i = \overline{1,m}\}, h = \frac{\ell}{m}, x_i = ih$$

be grids in the time and space domains, respectively, and we'll assume from here on that

$$m \to \infty \quad \text{as } n \to \infty$$

Define the Steklov averages

$$a_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} a(t)\,dt \qquad \Phi_i = \frac{1}{h}\int_{x_i}^{x_{i+1}} \Phi(x)\,dx, \quad \Phi_m = \Phi(\ell) \tag{1.29}$$

$$f_{ik} = \frac{1}{\tau h}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}} f(x,t)\,dxdt, \qquad k = \overline{1,n}, \quad i = \overline{0,m-1}$$

where a stands for any of the functions p, Γ, g, or g^n. Introduce the discretized control set

$$\mathscr{G}_R^n = \{[g]_n \in \mathbb{R}^{n+1} : \|[g]_n\|_{w_2^1} \le R\}$$

where $[g]_n = (g_0, g_1, \ldots, g_n)$, and

$$\|[g]_n\|^2_{w_2^1} = \sum_{k=1}^{n} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{k\bar{t}}^2$$

with $g_{k\bar{t}} = \frac{g_k - g_{k-1}}{\tau}$. Consider now the mappings between the discrete and continuous control sets, $\mathscr{Q}_n : W_2^1[0,T] \to \mathbb{R}^{n+1}, \quad \mathscr{P}_n : \mathbb{R}^{n+1} \to W_2^1[0,T]$ as

$$\mathscr{Q}_n(g) = [g]_n, \quad \text{for } g \in \mathscr{G}_R, \text{ where } g_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} g(t)\,dt, \quad k = \overline{1,n},\; g_0 = g(0) \tag{1.30}$$

$$\mathscr{P}_n([g]_n) = g^n,\; \text{for } [g]_n \in \mathscr{G}_R^n;\;\; g^n(t) = g_{k-1} + \frac{g_k - g_{k-1}}{\tau}(t - t_{k-1}), t \in [t_{k-1}, t_k),\; k = \overline{1,n} \tag{1.31}$$

Approximate the function $b(v)$ by averagings $b_n(v)$ using an infinitely differentiable nonnegative kernel $\omega_\rho(|v|)$ of radius $\rho = \frac{1}{n}$, defined as

$$\omega_\rho(|v|) = \begin{cases} \mathcal{C}\rho^{-1}e^{-\frac{\rho^2}{\rho^2 - v^2}}, & |v| \leq \rho \\ 0, & |v| > \rho \end{cases} \tag{1.32}$$

where $\mathcal{C}$ is a constant chosen so that $\int\limits_{\mathbb{R}} \omega_1(|u|)\,du = 1$. With b_n under our belt, we seek now to define a matrix of values, which we'll call the "discrete state vector", where this matrix will represent the discrete analogue of the function $u(x,t)$.

Discrete State Vector. Given $[g]_n$, the vector function $[v([g]_n)]_n = \big(v(0), v(1), \ldots, v(n)\big)$; $v(k) \in \mathbb{R}^{m+1}, \quad k = 0, \ldots, n$ is called a *discrete state vector* if

(a) $v_i(0) = \Phi_i, \quad i = \overline{0,m}$

(b) For arbitrary $k = 1, \ldots, n$, the vector $v(k) \in \mathbb{R}^{m+1}$ satisfies

$$\sum_{i=0}^{m-1} h\Big[\big(b_n(v_i(k))\big)_{\bar{t}}\eta_i + v_{ix}(k)\eta_{ix} - f_{ik}\eta_i\Big] - p_k\eta_m + g_k^n\eta_0 = 0, \qquad \forall\eta = (\eta_i) \in \mathbb{R}^{m+1} \tag{1.33}$$

Discrete Optimal Control Problem. Given $[g]_n \in \mathscr{G}_R^n$, the discrete cost functional $\mathscr{I}_n$ is defined as

$$\mathscr{I}_n([g]_n) = \sum_{k=1}^{n} \tau \Big(v_m(k) - \Gamma_k \Big)^2 \tag{1.34}$$

Where $v_m(k)$ are components of the discrete state vector $[v([g]_n)]_n$. We define

$$\mathscr{I}_{n_*} := \inf_{[g]_n \in \mathscr{G}^n} \mathscr{I}_n([g]_n)$$

The discrete optimal control problem will be labeled *Problem* $\mathcal{I}_n$.

Chapter 2

Main Results

2.1 Assumptions

Suppose that

$$f \in L_\infty(D), \quad p \in W_2^1[0,T], \quad \Phi \in W_2^1[0,\ell] \tag{2.1}$$

and in addition that $\Phi(x)$ is such that the critical values $v^j, j = 1, \ldots, m$ are taken by Φ on a set of measure 0 in the $x-$space. This set of assumptions will be called $\mathscr{A}$.

2.2 Existence of a Solution to the Optimal Control Problem

Theorem 2.1 *Take the set of assumptions $\mathscr{A}$. The Problem $\mathcal{I}$ has a solution, i.e. the set*

$$\mathscr{G}_* = \left\{ g \in \mathscr{G}_R \middle| \mathscr{J}(g) = \mathscr{J}_* := \inf_{g \in \mathscr{G}_R} \mathscr{J}(g) \right\}$$

is not empty.

2.2.1 Method of Proof to Theorem 2.1

The proof of Theorem 2.1 hinges upon showing the weak continuity of the cost functional $\mathscr{J}$. To this end, existence and uniqueness of the weak solution to the Stefan problem for fixed $g \in \mathscr{G}_R$ is essential, as are the $L_\infty(D)$ and $W_2^{1,1}(D)$ uniform estimations that the solution satisfies, since then weak convergence of $\{g_n\}$ to g in $W_2^1[0,T]$ will imply weak convergence of $\{u(x,t;g_n)\}$ to $u(x,t;g)$ in $W_2^{1,1}(D)$, and this in turn will guarantee the weak continuity of $\mathscr{J}$. The requisite estimations are deduced from corresponding estimations for the discrete state vector, which are derived in detail in Chapter 3.

2.3 Convergence of discrete problems $\mathcal{I}_n$ to problem $\mathcal{I}$

Theorem 2.2 *Take the set of assumptions $\mathscr{A}$. The sequence of discrete optimal control problems $\mathcal{I}_n$ approximates the optimal control problem $\mathcal{I}$ with respect to functional, i.e.*

$$\lim_{n\to+\infty} \mathscr{I}_{n_*} = \mathscr{J}_*, \tag{2.2}$$

where

$$\mathscr{I}_{n_*} = \inf_{\mathscr{G}_R^n} \mathscr{I}_n([g]_n),\ n = 1, 2, ...$$

If $[g]_{n_\varepsilon} \in \mathscr{G}_R^n$ is chosen such that

$$\mathscr{I}_{n_*} \le \mathscr{I}_n([g]_{n_\varepsilon}) \le \mathscr{I}_{n_*} + \varepsilon_n,\ \varepsilon_n \downarrow 0,$$

then the sequence $g^n = \mathscr{P}_n([g]_{n_\varepsilon})$ converges to some element $g_* \in \mathscr{G}_*$ weakly in $W_2^1[0,T]$ and strongly in $L_2[0,T]$. Moreover, piecewise linear interpolation $\hat{v}^\tau$ of the discrete state vector $[v([g]_{n_\varepsilon})]_n$ converges to the weak solution $v(x,t;g_*) \in W_2^{1,1}(D)$ of the Stefan Problem weakly in $W_2^{1,1}(D)$.

2.3.1 Method of Proof to Theorem 2.2

We make use of a result in [35], which provides necessary and sufficient conditions under which the main portion of Theorem 2.2 will hold. Therefore, the proof of this theorem is dependent on proving that the conditions of the general criteria will hold. We present the lemma in Chapter 3, and split the proof that our situation satisfies the conditions of the lemma into several portions.

Chapter 3

Preliminary Results

3.1 Existence and Uniqueness of the Discrete State Vector

Proposition 3.1 *Given any $[g]_n \in \mathscr{G}^n$, and any h, τ, a discrete state vector exists uniquely.*

Proof. First we prove uniqueness. Suppose v and $\tilde{v}$ both are discrete state vectors for a given $[g]_n$. Due to (a) from the d.s.v. definition, we have that $v(0) = \tilde{v}(0)$. For a fixed $k \geq 1$, suppose that $v(k-1) = \tilde{v}(k-1)$. (1.33) is satisfied for both v and $\tilde{v}$. Substract the identities for $\eta = v(k) - \tilde{v}(k)$ to get:

$$\sum_{i=0}^{m-1} \Big[\big(b_n(v_i(k))_{\bar{t}} - b_n(\tilde{v}_i(k))_{\bar{t}}\big)\big(v_i(k) - \tilde{v}_i(k)\big) + \big(v_{ix}(k) - \tilde{v}_{ix}(k)\big)^2 \Big] = 0$$

However,

$$b_n(v_i(k))_{\bar{t}} - b_n(\tilde{v}_i(k))_{\bar{t}} = \frac{b_n(v_i(k)) - b_n(v_i(k-1))}{\tau} - \frac{b_n(\tilde{v}_i(k)) - b_n(\tilde{v}_i(k-1))}{\tau}$$

$$= \frac{b_n(v_i(k)) - b_n(\tilde{v}_i(k))}{\tau}$$

Thus that the previous summation identity becomes:

$$\sum_{i=0}^{m-1}\left[\frac{1}{\tau}\big(b_n(v_i(k)) - b_n(\tilde{v}_i(k))\big)\big(v_i(k) - \tilde{v}_i(k)\big) + \big(v_{ix}(k) - \tilde{v}_{ix}(k)\big)^2\right] = 0$$

If $v_i(k) \neq \tilde{v}_i(k)$, then owing to the strictly increasing property of $b_n(v)$, the whole summand is non-negative. If $v_i(k) = \tilde{v}_i(k)$, then the whole summand is non-negative as well since the first term vanishes. It follows that the whole summation is non-negative. Therefore, that it is equal to 0 implies that $v_i(k) = \tilde{v}_i(k), \quad \forall i = \overline{0, m}$. Hence, by induction, $v = \tilde{v}$. □

Now we seek to prove existence. Again we'll rely on induction. Construct $v(0)$ as given in (a) of the Discrete State Vector Definition. Note that $\|v(0)\| := \max_i |v_i(0)| = \max_i |\Phi_i| \leq \|\Phi\|_{L_\infty[0,\ell]}$. Now fix $k \geq 1$, and assume that $v(k-1)$ has been constructed successfully so that (1.33) is satisfied for all $K < k$. Moreover, assume that $\|v(k-1)\| < +\infty$. Notice that the summation identity (1.33) is equivalent to solving the following system of non-linear equations:

$$\begin{cases} \Big(v_0(k) + \frac{h^2}{\tau} b_n(v_0(k))\Big) - v_1(k) & = \quad \frac{h^2}{\tau} b_n(v_0(k-1)) + h^2 f_{0k} - h g_k^n \\ -v_{i-1}(k) + \Big(2v_i(k) + \frac{h^2}{\tau} b_n(v_i(k))\Big) - v_{i+1}(k) & = \quad \frac{h^2}{\tau} b_n(v_i(k-1)) + h^2 f_{ik}, \quad i = \overline{1, m-1} \\ -v_{m-1}(k) + v_m(k) & = \quad h p_k \end{cases} \tag{3.1}$$

We will construct $v(k)$ by the method of successive approximations. It is critical to remember that h, τ will be fixed here. Choose $v^0 = v(k-1)$. Having obtained v^N,we search v^{N+1} as a solution of the following system:

$$\begin{cases} \Big(v_0^{N+1}(k) + \frac{h^2}{\tau} b_n(v_0^{N+1}(k))\Big) - v_1^N(k) & = \quad \frac{h^2}{\tau} b_n(v_0(k-1)) + h^2 f_{0k} - h g_k^n \\ -v_{i-1}^N(k) + 2v_i^{N+1}(k) + \frac{h^2}{\tau} b_n(v_i^{N+1}(k)) - v_{i+1}^N(k) & = \quad \frac{h^2}{\tau} b_n(v_i(k-1)) + h^2 f_{ik}, \quad i = \overline{1, m-1} \\ -v_{m-1}^{N+1}(k) + v_m^{N+1}(k) & = \quad h p_k \end{cases} \tag{3.2}$$

which we transform as

$$\begin{cases} \left[1+\frac{h^2}{\tau}\zeta_n^0\right]v_0^{N+1} = v_1^N + h^2 f_{0k} - hg_k^n + \frac{h^2}{\tau}\zeta_n^0 v_0(k-1) \\ \left[2+\frac{h^2}{\tau}\zeta_n^i\right]v_i^{N+1} = v_{i-1}^N + v_{i+1}^N + h^2 f_{ik} + \frac{h^2}{\tau}\zeta_n^i v_i(k-1), \qquad i=1,\dots,m-1 \\ -v_{m-1}^{N+1} + v_m^{N+1} = hp_k \end{cases} \tag{3.3}$$

where

$$\zeta_n^i = \int_0^1 b_n'(\theta v_i^{N+1} + (1-\theta)v_i(k-1))d\theta, \quad i=\overline{0,m-1} \tag{3.4}$$

Due to $b_n'(w) \geq \bar{b} > 0 \; \forall w$, we have

$$\bar{b} \leq \zeta_n^i \leq M_n, \quad i=\overline{0,m-1} \tag{3.5}$$

independent of N. Obviously the system (3.3) has a solution. We now proceed to show the solution of (3.3) is uniformly bounded; that is, that $\exists M > 0$ such that $\|v^N\| := \max_i |v_i^N| \leq M, \quad \forall N$. From (3.3), (3.4), (3.5) it follows

$$|v_0^{N+1}| \leq \frac{1}{1+\frac{h^2}{\tau}\zeta_n^0}\left(h^2|f_{0k}| + h|g_k^n|\right) + \frac{h^2}{\tau}\frac{\zeta_n^0}{1+\frac{h^2}{\tau}\zeta_n^0}|v_0(k-1)| + \frac{1}{1+\frac{h^2}{\tau}\zeta_n^0}\|v^N\|$$

$$|v_i^{N+1}| \leq \frac{1}{2+\frac{h^2}{\tau}\zeta_n^i}h^2|f_{ik}| + \frac{h^2}{\tau}\frac{\zeta_n^i}{2+\frac{h^2}{\tau}\zeta_n^i}|v_i(k-1)| + \frac{2}{2+\frac{h^2}{\tau}\zeta_n^i}\|v^N\|$$

$$|v_m^{N+1}| \leq h|p_k| + |v_{m-1}^{N+1}| \tag{3.6}$$

We do have that $\frac{1}{1+\frac{h^2}{\tau}\zeta_n^i} \leq \frac{1}{1+\frac{h^2}{\tau}\bar{b}} < 1$ and $\frac{\zeta_n^i}{1+\frac{h^2}{\tau}\zeta_n^i} < \frac{\tau}{h^2}$, $\forall i \; \forall N$. From these observation and (3.6), we obtain the estimation:

$$\|v^{N+1}\| \leq \left[h^2 \max_i |f_{ik}| + h|g_k^n| + h|p_k| + \|v(k-1)\|\right] + \frac{1}{1+\frac{h^2}{2\tau}\bar{b}}\|v^N\| \tag{3.7}$$

Through an inductive use of (3.7), we obtain:

$$\|v^{N+1}\| \leq \left(1+\frac{h^2}{2\tau}\bar{b}\right)^{-N}\|v^0\|+\sum_{l=0}^{N}\left(1+\frac{h^2}{2\tau}\bar{b}\right)^{-l}\left[h^2\max_i|f_{ik}|+h|g_k^n|+h|p_k|+\|v(k-1)\|\right] \quad (3.8)$$

The first term on the right hand side of (3.8) goes to 0 as $N \to \infty$ since $v^0 = v(k-1)$. The summation term in (3.8) is bounded, since

$$\sum_{l=0}^{N}\left(1+\frac{h^2}{2\tau}\bar{b}\right)^{-l} \leq \sum_{l=0}^{\infty}\left(1+\frac{h^2}{2\tau}\bar{b}\right)^{-l} = \frac{1}{1-\left(1+\frac{h^2}{2\tau}\bar{b}\right)^{-1}} = 1+\frac{2\tau}{h^2\bar{b}}$$

By our assumption on $\|v(k-1)\|$, it follows that $\exists M$ such that $\|v^N\| \leq M$, $\forall N$. By the Bolzano-Weierstrass Theorem, we can take a subsequence $v^{N_l} \in \mathbb{R}^m$ that converges to $v \in \mathbb{R}^m$. Passing to limit along this subsequence, from (3.3), (3.1) follows. Since (3.1) has a unique solution we deduce that the sequence v^N converges to the unique solution of (3.1). The proposition is proved. □

Furthemore, the following interpolations will be considered:

$$\begin{aligned}
\tilde{v}(x,t) &= v_i(k), \qquad x \in [x_i, x_{i+1}], \quad t \in [t_{k-1}, t_k], \qquad i = \overline{0, m-1}, \quad k = \overline{0, n} \\
\hat{v}(x;k) &= v_i(k) + v_{ix}(k)(x - x_i), \qquad x \in [x_i, x_{i+1}], \quad i = \overline{0, m-1} \\
v^\tau(x,t) &= \hat{v}(x;k), \qquad t \in [t_{k-1}, t_k] \\
\hat{v}^\tau(x,t) &= \hat{v}(x;k-1) + \hat{v}_{\bar{t}}(x;k)(t - t_{k-1}), \qquad t \in [t_{k-1}, t_k], \quad k = \overline{1, n}
\end{aligned} \quad (3.9)$$

Given the existence and uniqueness of the discrete state vector for fixed n, we can uniquely define for each $k = 1, \ldots, n$ the vector ζ_k whose m components ζ_k^i are given by

$$\zeta_k^i = \int_0^1 b_n'(\theta v_i(k) + (1-\theta)v_i(k-1))d\theta, \quad i = \overline{0, m-1} \quad (3.10)$$

3.2 Necessary and Sufficient Convergence Conditions

Lemma 3.1 *[35] Sequence of discrete optimal control problems $\mathcal{I}_n$ approximates the continuous optimal control problem I if and only if the following conditions are satisfied:*

(1) *for arbitrary sufficiently small $\varepsilon > 0$ there exists number $M_1 = M_1(\varepsilon)$ such that $\mathscr{Q}_M(g) \in \mathscr{G}_R^M$ for all $g \in \mathscr{G}_{R-\varepsilon}$ and $M \geq M_1$; and for any fixed $\varepsilon > 0$ and for all $g \in \mathscr{G}_{R-\varepsilon}$ the following inequality is satisfied:*

$$\limsup_{M\to\infty} \Big(\mathscr{I}_M(\mathscr{Q}_M(g)) - \mathscr{J}(g) \Big) \leq 0. \tag{3.11}$$

(2) *for arbitrary sufficiently small $\varepsilon > 0$ there exists number $M_2 = M_2(\varepsilon)$ such that $\mathscr{P}_M([g]_M) \in \mathscr{G}_{R+\varepsilon}$ for all $[g]_M \in \mathscr{G}_R^M$ and $M \geq M_2$; and for all $[g]_M \in \mathscr{G}_R^M$, $M \geq 1$ the following inequality is satisfied:*

$$\limsup_{M\to\infty} \Big(\mathscr{J}(\mathscr{P}_M([g]_M)) - \mathscr{I}_M([g]_M) \Big) \leq 0. \tag{3.12}$$

(3) *the following inequalities are satisfied:*

$$\limsup_{\varepsilon\to 0} \mathscr{J}_*(\varepsilon) \geq \mathscr{J}_*, \quad \liminf_{\varepsilon\to 0} \mathscr{J}_*(-\varepsilon) \leq \mathscr{J}_*, \tag{3.13}$$

where $\mathscr{J}_*(\pm\varepsilon) = \inf_{\mathscr{G}_{R\pm\varepsilon}} \mathscr{J}(g)$.

3.3 On the Mappings $\mathscr{P}_n$ and $\mathscr{Q}_n$

Lemma 3.2 *The mappings $\mathscr{P}_n, \mathscr{Q}_n$ satisfy the conditions of Lemma 3.1.*

Proof. Fix $\varepsilon > 0$, and let $g \in \mathscr{G}_{R-\varepsilon}$, $[g]_n = \mathscr{Q}_n(g)$. We observe that

$$\sum_{k=1}^{n} \tau g_k^2 = \sum_{k=1}^{n} \tau \left(\frac{1}{\tau} \int\limits_{t_{k-1}}^{t_k} g(t)\,dt \right)^2 \leq \int\limits_0^T g^2(t)\,dt = \|g\|^2_{L_2[0,T]} \tag{3.14}$$

$$\sum_{k=1}^{n} \tau g_{k\bar{t}}^2 = \frac{1}{\tau} \left(\frac{1}{\tau} \int\limits_0^{\tau} g(t)\,dt - g(0) \right)^2 + \sum_{k=2}^{n} \frac{1}{\tau} \left(\frac{1}{\tau} \int\limits_{t_{k-1}}^{t_k} g(t)\,dt - \frac{1}{\tau} \int\limits_{t_{k-2}}^{t_{k-1}} g(t)\,dt \right)^2 \leq$$

$$= \frac{1}{\tau^3} \left(\int\limits_0^{\tau} \int\limits_0^{t} g'(\xi)\,d\xi \right)^2 + \sum_{k=2}^{n} \frac{1}{\tau^3} \left(\int\limits_{t_{k-1}}^{t_k} \int\limits_{t-\tau}^{t} g'(\xi)\,d\xi \right)^2 \overset{\text{CBS}}{\leq}$$

$$\leq \frac{1}{\tau} \int\limits_0^{\tau} \int\limits_0^{t} |g'(\xi)|^2\,d\xi + \sum_{k=2}^{n} \frac{1}{\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{t-\tau}^{t} |g'(\xi)|^2\,d\xi \quad \overset{\text{Fubini's}}{\leq}$$

$$\leq \|g'\|^2_{L_2[0,\tau]} + \|g'\|^2_{L_2[0,T]} \tag{3.15}$$

From (3.14),(3.15), we get

$$\|[g]_n\|^2_{w_2^1} \leq \|g\|^2_{W_2^1[0,T]} + \|g'\|^2_{L_2[0,\tau]} \leq (R-\varepsilon)^2 + \|g'\|^2_{L_2[0,\tau]} \tag{3.16}$$

Since $g \in W_2^1[0,T]$ and $\tau \to 0$ as $n \to \infty$, we know

$$\lim_{n\to\infty} \|g'\|^2_{L_2[0,\tau]} = 0$$

Consequently, we can choose τ so small that the whole right-hand side of the above equation is bounded by R^2. By definition then, $[g]_n \in \mathscr{G}_R^n$.

Now let $[g]_n \in \mathscr{G}_R^n$ be given and write $g^n = \mathscr{P}_n([g]_n)$. We see that

$$\int_0^T \left|\frac{dg^n(t)}{dt}\right|^2 dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} g_{k\bar{t}}^2\, dt = \sum_{k=1}^n \tau g_{k\bar{t}}^2 \tag{3.17}$$

$$\int_0^T |g^n(t)|^2\, dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \Big(g_{k-1} + g_{k\bar{t}}(t - t_{k-1})\Big)^2 dt =$$
$$= \sum_{k=1}^n \tau g_{k-1}^2 + \sum_{k=1}^n \tau^2 g_{k-1} g_{k\bar{t}} + \frac{1}{3}\sum_{k=1}^n \tau^3 g_{k\bar{t}}^2 = \sum_{k=1}^n \tau g_k g_{k-1} + \frac{1}{3}\sum_{k=1}^n \tau^3 g_{k\bar{t}}^2 \tag{3.18}$$

Actually, since $[g]_n \in \mathscr{G}_R^n$, it is the case that $\sum\limits_{k=1}^n \tau g_{k\bar{t}}^2 \le C^2$ where C is a constant independent of n. This of course implies $\tau g_{k\bar{t}}^2 \le C^2$ for any k, or equivalently,

$$|g_k - g_{k-1}| \le C\sqrt{\tau}, \qquad k = 1, \ldots, n \tag{3.19}$$

Using (3.19) on (3.18), we can write

$$\int_0^T |g^n(t)|^2\, dt \le \sum_{k=1}^n \tau g_k^2 + C\sqrt{\tau}\sum_{k=1}^n \tau g_k + \frac{1}{3}\tau^2 \sum_{k=1}^n \tau g_{k\bar{t}}^2 \le$$
$$\le (1 + CT\sqrt{\tau})\sum_{k=1}^n \tau g_k^2 + \frac{1}{3}\tau^2 \sum_{k=1}^n \tau g_{k\bar{t}}^2 \tag{3.20}$$

Combining (3.17) and (3.20) we have that

$$\|g^n\|_{W_2^1[0,T]}^2 \le \|[g]_n\|_{w_2^1}^2 + CT\sqrt{\tau}\sum_{k=1}^n \tau g_k^2 + \frac{1}{3}\tau^2 \sum_{k=1}^n \tau g_{k\bar{t}}^2 \le R^2 + O(\sqrt{\tau}) \tag{3.21}$$

Owing to (3.21), we can choose n so large that τ will be small enough to guarantee that the right-hand side will be bounded by $(R+\varepsilon)^2$. Hence $g^n \in \mathscr{G}_{R+\varepsilon}$ for all n large enough. □

Remark. Actually, (3.21) tells us more. By Morrey's Inequality, g^n is in particular a uniformly continuous function on $[0,T]$, and for n large enough it is satisfied by Lemma 3.2 that

$$\|g^n\|_{C[0,T]} \le C\|g^n\|_{W_2^1[0,T]} \le C(R+1)$$

where C is independent of n. Since

$$|g_k| = |g^n(t_k)| \leq \|g^n\|_{C[0,T]}, \qquad k = 0, \ldots, n$$

it is therefore the case that

$$\sup_{n\in\mathbb{N}} \left(\max_{0\leq k\leq n} |g_k| \right) \leq C(R+1)$$

In other words, the vectors $[g]_n$ are uniformly bounded.

3.4 Uniqueness of the Weak Solution to the Stefan Problem

Lemma 3.3 *Suppose that $\Phi(x)$ is such that the critical values $v^j, j = 1, \ldots, m$ are taken by Φ on a set of measure 0 in the $x-$space. Then there is at most one solution to the Stefan problem in the sense of (1.27).*

That a solution to the Stefan problem in the sense of (1.27) is unique follows by an argument analogous to that presented in Section 9 of Chapter V of [25]. Indeed, we will prove uniqueness in a wider class of solutions than that given in (1.27). Suppose that $v \in L_\infty(D)$ only, not necessarily in the Sobolev space $W_2^{1,1}(D)$, and that for any two functions B, B_0 of type $\mathscr{B}$ it satisfies the identity

$$\int_0^T \int_0^\ell \Big[B(x,t,v)\psi_t + v\psi_{xx} + f\psi \Big] \, dxdt + \int_0^\ell B_0(x,0,\Phi(x))\psi(x,0)\, dx +$$
$$\int_0^T p(t)\psi(\ell,t)\, dt - \int_0^T g(t)\psi(0,t)\, dt = 0, \qquad \forall \psi \in W_2^{2,1}(D), \psi(x,T) = 0, \psi_x(0,t) = \psi_x(\ell,t) = 0 \tag{3.22}$$

The class of functions satisfying the above definition contains the class of solutions given in (1.27). Suppose v and $\tilde{v}$ are two solutions in the sense of (3.22). Due to our assumption on Φ, substracting (3.22) in $\tilde{v}$ from that in v guarantees that the second integral in (3.22) vanishes, and we obtain:

$$\int_0^T \int_0^\ell \left(B(x,t,v) - \tilde{B}(x,t,\tilde{v})\right) \left(\psi_t + \frac{v - \tilde{v}}{B(x,t,v) - \tilde{B}(x,t,\tilde{v})}\psi_{xx}\right) dx\, dt = 0$$

Write $a(x,t) = \frac{v-\tilde{v}}{B(x,t,v)-\tilde{B}(x,t,\tilde{v})}$. For $(x,t) \in D$ such that $v(x,t) = \tilde{v}(x,t)$, it is the case that $a(x,t) = 0$. Otherwise, since B and $\tilde{B}$ are strictly increasing on v a.e. $(x,t) \in D$, it follows that a is non-negative for a.e. (x,t). Moreover, the a.e. positiveness of $b'(v(x,t))$ implies that $\bar{b} = \text{essinf } b' > 0$ and that b is strictly increasing, and so for almost every (x,t) (assume that $\tilde{v}(x,t) < v(x,t)$ for the sake of notational simplicity),

$$|a(x,t)| = \left|\frac{v - \tilde{v}}{B(x,t,v) - \tilde{B}(x,t,\tilde{v})}\right| = \left|\frac{v - \tilde{v}}{\int\limits_{\tilde{v}(x,t)}^{v(x,t)} b'(w)\, dw + \sum\limits_{i: v^i \in (\tilde{v}(x,t), v(x,t))} (b(v^i)^+ - b(v^i)^-)}\right|$$

$$\leq \left|\frac{v - \tilde{v}}{\int\limits_{\tilde{v}}^{v} \bar{b}\, dv}\right| = \frac{1}{\bar{b}}$$

Thus a is essentially bounded, and $\text{esssup } a(x,t) = a_0 < +\infty$. Fix $\varepsilon > 0$, and take as $\psi(x,t)$ the solution of the Neumann problem

$$\psi_t + (a(x,t) + \varepsilon)\psi_{xx} = F(x,t), \qquad \psi_x(0,t) = \psi_x(\ell,t) = 0, \quad \psi(x,T) = 0 \tag{3.23}$$

Where the ε is added to ensure the uniform parabolicity of the conjugate diffusion coefficient, and F is an arbitrary smooth bounded function in D. Note that (3.23) is the conjugate heat equation, and call its unique solution ψ^ε (that this solution exists uniquely is well known; in particular, it follows from [25]). Our goal here is to use the arbitrariness of F to obtain that $B - \tilde{B} = 0$ a.e.; to this end, notice that through the use of (3.23), we can write

$$\int_0^T \int_0^\ell \left(B(x,t,v) - \tilde{B}(x,t,\tilde{v})\right) \left(F - \varepsilon\psi^\varepsilon_{xx}\right) dx\, dt = 0 \tag{3.24}$$

Thus our goal will be attained if we have an energy estimate on ψ_{xx} for solutions of (3.23). In the following, we prove a sufficient estimation for the analogous Heat Equation (the result follows immediately for the conjugate one by a simple change of variables). Let $a^\varepsilon(x,t) = a(x,t) + \varepsilon$, and for simplicity we don't write the superscript. Multiply the non-conjugate version of (3.23) by ψ_{xx} and integrate it over the rectangle $D_t := (0,\ell) \times (0,t)$ to get

$$-\int_0^t \int_0^\ell (\psi_\tau - a\psi_{xx})\psi_{xx}\,dx\,d\tau = -\int_0^t \int_0^\ell F\psi_{xx}\,dx\,d\tau = \int_0^t \int_0^\ell F_x\psi_x\,dx\,d\tau - \int_0^t F\psi_x\Big|_0^\ell d\tau$$

$$\int_0^t \int_0^\ell \left((\psi_\tau)_x\psi_x + a\psi_{xx}^2\right)dx\,d\tau - \int_0^t \psi_\tau\psi_x\Big|_0^\ell d\tau = \int_0^t \int_0^\ell \left((\psi_x)_\tau\psi_x + a\psi_{xx}^2\right)dx\,d\tau = \int_0^t \int_0^\ell F_x\psi_x\,dx\,d\tau$$

$$\frac{1}{2}\int_0^\ell \psi_x^2(x,t)\,dx - \frac{1}{2}\int_0^\ell \psi_x^2(x,0)\,dx + \int_0^t \int_0^\ell a\psi_{xx}^2\,dx\,d\tau \le \frac{1}{2}\int_0^t \int_0^\ell (\psi_x^2 + F_x^2)\,dx\,d\tau$$

$$\int_0^\ell \psi_x^2(x,t)\,dx + 2\int_0^t \int_0^\ell a\psi_{xx}^2\,dx\,d\tau \le \int_0^t \int_0^\ell \psi_x^2\,dx\,d\tau + \int_0^t \int_0^\ell F_x^2\,dx\,d\tau \qquad (3.25)$$

Letting now $y(t) = \int_0^t \int_0^\ell \psi_x^2\,dx\,d\tau$, it is clear that $y'(t) = \int_0^\ell \psi_x^2(x,t)\,dx$, thus that (3.25) implies

$$y'(t) \le y(t) + \int_0^t \int_0^\ell F_x^2\,dx\,d\tau$$

By Gronwall's Inequality now (more precisely Lemma 5.5 from [25]), we deduce from the above differential inequality that $y(t) \le \left[e^t - 1\right] \int_0^t \int_0^\ell F_x^2\,dx\,d\tau$, or in other words

$$\int_0^t \int_0^\ell \psi_x^2(x,t)\,dx \le \left[e^t - 1\right] \int_0^t \int_0^\ell F_x^2\,dx\,d\tau, \qquad \forall t \in (0,T]$$

$$(3.25) \qquad \Longrightarrow \qquad \int_0^\ell \psi_x^2(x,t)\,dx + 2\int_0^t \int_0^\ell a\psi_{xx}^2\,dx\,d\tau \le e^t \int_0^t \int_0^\ell F_x^2\,dx\,d\tau, \qquad \forall t \in (0,T]$$

The first of the above inequalities implies that $\operatorname*{ess\,sup}_{0\le t\le T}\int_0^\ell \psi_x^2(x,t)\,dx \le e^T \int_0^T\int_0^\ell F_x^2\,dx\,d\tau$. Now, since $\psi_t = a\psi_{xx} + F$, we have

$$\|\psi_t\|^2_{L_2(D_t)} = \|a\psi_{xx}+F\|^2_{L_2(D_t)} \le (\|a\psi_{xx}\|_{L_2(D_t)} + \|F\|_{L_2(D_t)})^2 \le 2\|a\psi_{xx}\|^2_{L_2(D_t)} + 2\|F\|^2_{L_2(D_t)}$$
$$\le 2\|F\|^2_{L_2(D_t)} + 2a_0\int_0^t\int_0^\ell a\psi_{xx}^2\,dx\,d\tau \le 2\Big(\|F\|^2_{L_2(D_t)} + a_0e^t\|F_x\|^2_{L_2(D_t)}\Big)$$

These results combined provide the energy estimate we need:

$$\int_0^T\int_0^\ell \Big(\psi_t^2 + a(x,t)\psi_{xx}^2\Big)\,dx\,dt + \operatorname*{ess\,sup}_{0\le t\le T}\|\psi_x\|^2_{L_2[0,\ell]} \le 2\Big(\|F\|^2_{L_2(D)} + a_0e^T\|F_x\|^2_{L_2(D)}\Big) \tag{3.26}$$

Having (3.26), we can now observe that

$$\left|\int_0^T\int_0^\ell (B-\tilde{B})\varepsilon\psi^\varepsilon_{xx}\,dx\,dt\right| = \left|\int_0^T\int_0^\ell (B-\tilde{B})\frac{\varepsilon}{(a+\varepsilon)^{\frac12}}(a+\varepsilon)^{\frac12}\psi^\varepsilon_{xx}\,dx\,dt\right| \overset{\text{Holder's}}{\le}$$
$$\le 2\text{esssup}\ b(v)\left(\int_0^T\int_0^\ell \frac{\varepsilon^2}{(a+\varepsilon)}\,dx\,dt\right)^{\frac12}\left(\int_0^T\int_0^\ell (a+\varepsilon)(\psi^\varepsilon_{xx})^2\,dx\,dt\right)^{\frac12} \le$$
$$\le C\sqrt{\varepsilon}\text{esssup}\ b(v)\left(\int_0^T\int_0^\ell \frac{\varepsilon}{(a+\varepsilon)}\,dx\,dt\right)^{\frac12}\|F\|_{W_2^{1,0}(D)} \to 0 \ \text{ as } \ \varepsilon\to 0$$

Where C is a constant depending only on T and a_0. Recall that $\varepsilon \le a+\varepsilon$, and so the integral on the right-hand side of the above inequality is bounded above by the area of the rectangle D. Therefore, (3.24) now implies

$$\int_0^T\int_0^\ell \big(B(x,t,v(x,t)) - \tilde{B}(x,t,\tilde{v}(x,t))\big)F\,dx\,dt = 0$$

Owing to the arbitrariness of F, the above equality implies that $B(x,t,v(x,t)) = \tilde{B}(x,t,\tilde{v}(x,t))$ a.e.$(x,t)\in D$, meaning $b(v(x,t)) = b(\tilde{v}(x,t))$, a.e. (x,t) s.t. $v(x,t)\ne v^j, j=1,\ldots,m$. Since b

is strictly increasing, we therefore have $v(x,t) = \tilde{v}(x,t)$ a.e. (x,t), so v and $\tilde{v}$ coincide as solutions in the sense of (3.22), and thus we have proven uniqueness in this large class of solutions. □

Corollary 3.1 *Under the conditions of Lemma 3.3 and the assumption that a weak solution exists, all of the sets* $\mathscr{C}_j, j = \overline{1,J}$ *have 2-dimensional measure 0.*

Proof. The proof of uniqueness gives us that $B_1(x,t,v(x,t)) = B_2(x,t,v(x,t))$ a.e. on D, for *any* two functions B_1, B_2 of type $\mathscr{B}$. The functions of type $\mathscr{B}$ generally differ on the sets $\mathscr{C}_j$, so if one of them has positive measure, we arrive at a contradiction to Lemma 3.3. The proof of existence of a weak solution for a given $g \in \mathscr{G}_R$ is found in Theorem 4.3, or alternatively, in the Appendix. □

Chapter 4

Energy Estimates and Compactness Theorem

4.1 Uniform Boundedness of the Discrete State Vector

Theorem 4.1 *Suppose that* $p \in L_\infty[0,T], \Phi \in L_\infty[0,\ell], f \in L_\infty(D)$. *For* $[g]_n \in \mathscr{G}_R^n$ *and* n, m *large enough, the discrete state vector* $[v([g]_n)]_n$ *satisfies the following estimate:*

$$\|[v]_n\|_{\ell_\infty} := \max_{0\le k\le n}\Big(\max_{0\le i\le m}|v_i(k)|\Big) \le C_\infty\Big(\|f\|_{L_\infty(D)} + \|p\|_{L_\infty[0,T]} + \|g^n\|_{W_2^1[0,T]} + \|\Phi\|_{L_\infty[0,\ell]}\Big) \tag{4.1}$$

where C_∞ *is a constant independent of* n *and* m.

Proof. Fix n arbitrarily large. Note $\max|v_i(0)| \le \|\Phi\|_{L_\infty[0,\ell]}$. Consider a positive function $\gamma(x) \in C^2[0,\ell]$ satisfying

$$\gamma(0) = \frac{1}{2}, \qquad \gamma(\ell) = \frac{1}{2}, \qquad \gamma'(0) = 1, \quad \gamma'(\ell) = -1, \qquad \frac{1}{4} \le \gamma(x) \le 1,\ x \in [0,\ell] \tag{4.2}$$

Define $\gamma_i = \gamma(x_i),\ i = \overline{0,m}$, and denote as x^i the value in $[x_i, x_{i+1}]$ that satisfies (by MVT) $\gamma(x_{i+1}) - \gamma(x_i) = \gamma'(x^i)h$. Transform the discrete state vector as

$$w_i(k) = v_i(k)\gamma_i, \qquad i = \overline{0,m}, \qquad k = \overline{0,n}$$

System (3.1) can be rewritten as:

$$\begin{cases} h\zeta_k^0 v_{0\bar{t}}(k) - v_{0x}(k) & = & hf_{0k} - g_k^n \\ \zeta_k^i v_{i\bar{t}}(k) - v_{ix\bar{x}}(k) & = & f_{ik}, \qquad i = 1, \ldots, m-1 \\ v_{m-1,x}(k) & = & p_k \end{cases} \tag{4.3}$$

Since

$$v_i(k) = \frac{1}{\gamma_i} w_i(k), \qquad v_{i\bar{t}}(k) = \frac{1}{\gamma_i} w_{i\bar{t}}(k)$$

$$v_{ix}(k) = \frac{1}{\gamma_{i+1}} w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_x w_i(k) = \frac{1}{\gamma_i} w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_x w_{i+1}(k)$$

$$v_{ix\bar{x}}(k) = \frac{1}{\gamma_{i-1}} w_{ix\bar{x}}(k) + \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right] w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} w_i(k) =$$

$$= \frac{1}{\gamma_{i+1}} w_{ix\bar{x}}(k) + \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right] w_{i\bar{x}}(k) + \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} w_i(k)$$

$$\left(\frac{1}{\gamma_i}\right)_x = -\frac{1}{\gamma_i\gamma_{i+1}}\gamma_{ix}, \qquad \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} = -\frac{1}{\gamma_i\gamma_{i+1}}\gamma_{ix\bar{x}} + \frac{\gamma_{ix}+\gamma_{i\bar{x}}}{\gamma_{i-1}\gamma_i\gamma_{i+1}}\gamma_{i\bar{x}}$$

Thus $w_i(0) = \gamma_i \Phi_i, \quad i = \overline{0,m}$, and for $k = \overline{1,n}$,

$$\begin{gathered} \tfrac{h}{\gamma_0}\zeta_k^0 w_{0\bar{t}}(k) - \tfrac{1}{\gamma_1} w_{0x}(k) - \left(\tfrac{1}{\gamma_0}\right)_x w_0(k) = hf_{0k} - g_k^n \\ \tfrac{1}{\gamma_i}\zeta_k^i w_{i\bar{t}}(k) - \tfrac{1}{\gamma_{i-1}} w_{ix\bar{x}}(k) - \left[\left(\tfrac{1}{\gamma_i}\right)_{\bar{x}} + \left(\tfrac{1}{\gamma_i}\right)_x\right] w_{ix}(k) - \left(\tfrac{1}{\gamma_i}\right)_{x\bar{x}} w_i(k) = f_{ik}, \qquad i = \overline{1,m-1} \\ \tfrac{1}{\gamma_{m-1}} w_{m-1,x}(k) + \left(\tfrac{1}{\gamma_{m-1}}\right)_x w_m(k) = p_k \end{gathered} \tag{4.4}$$

Furthermore, transform $w_i(k)$ as:

$$u_i(k) = w_i(k)e^{-\lambda t_k}, \qquad i = \overline{0,m}, \quad k = \overline{0,n} \tag{4.5}$$

where

$$\lambda = \frac{65}{\bar{b}}\left(\|\gamma''\|_{C[0,\ell]} + \|\gamma'\|^2_{C[0,\ell]}\right) \tag{4.6}$$

and if $t^k \in [t_{k-1}, t_k]$ satisfies through the MVT that $e^{\lambda t_k} - e^{\lambda t_{k-1}} = \lambda e^{\lambda t^k}(t_k - t_{k-1}) = \lambda e^{\lambda t^k}\tau$, then

$$w_{i\bar{t}}(k) = e^{\lambda t_{k-1}}u_{i\bar{t}}(k) + \lambda e^{\lambda t^k}u_i(k)$$

So $u_i(0) = w_i(0) = \gamma_i\Phi_i, \quad i = \overline{0,m}$, and for $k = \overline{1,n}$, the vector $u(k)$ satisfies the system

$$\frac{h}{\gamma_0}\zeta_k^0 e^{-\lambda\tau}u_{0\bar{t}}(k) - \frac{1}{\gamma_1}u_{0x}(k) + \left[\frac{\lambda h}{\gamma_0}\zeta_k^0 e^{-\lambda(t_k - t^k)} - \left(\frac{1}{\gamma_0}\right)_x\right]u_0(k) = e^{-\lambda t_k}\left(hf_{0k} - g_k^n\right)$$

$$\begin{aligned}\frac{1}{\gamma_i}\zeta_k^i e^{-\lambda\tau}u_{i\bar{t}}(k) - \frac{1}{\gamma_{i-1}}u_{ix\bar{x}}(k) - \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right]u_{ix}(k) + \\ + \left[\frac{\lambda}{\gamma_i}\zeta_k^i e^{-\lambda(t_k - t^k)} - \left(\frac{1}{\gamma_i}\right)_{x\bar{x}}\right]u_i(k) = f_{ik}e^{-\lambda t_k}, \qquad i = \overline{1, m-1}\end{aligned} \tag{4.7}$$

$$\frac{1}{\gamma_{m-1}}u_{m-1,x}(k) + \left(\frac{1}{\gamma_{m-1}}\right)_x u_m(k) = e^{-\lambda t_k}p_k$$

Now fix $k_1 \leq n$, and define the following sets of indexes for convenience:

$$\begin{aligned}\mathscr{M}_{k_1} &= \{(i,k) | i = 0, \ldots, m, \quad k = 0, \ldots, k_1\},\\ \mathscr{N} &= \{(i,k) | i = 1, \ldots, m-1, \quad k = 1, \ldots, k_1\},\\ \mathscr{T}_0 &= \{(i,k) | i = 0, k = 1, \ldots, k_1\},\\ \mathscr{T}_m &= \{(i,k) | i = m, k = 1, \ldots, k_1\},\\ \mathscr{X}_0 &= \{(i,k) | i = 0, \ldots, m, \quad k = 0\}\end{aligned}$$

Unless confusion may arise, we omit the subscript to $\mathscr{M}_{k_1}$. It is clear that

$$\mathscr{M} = \mathscr{N} \cup \mathscr{T}_0 \cup \mathscr{T}_m \cup \mathscr{X}_0.$$

If $u_i(k) \leq 0$ in $\mathscr{M}$, then $\max\limits_{\mathscr{M}} u_i(k) \leq 0$. Suppose that $\exists(i,k)$ such that $u_i(k) > 0$. Then $\max\limits_{\mathscr{M}} u_i(k) > 0$. Let $(i^*, k^*) \in \mathscr{M}$ be such that $u_{i^*}(k^*) = \max\limits_{\mathscr{M}} u_i(k)$.

If $(i^*,k^*) \in \mathscr{X}_0$, then $u_{i^*}(k^*) = \max_i \gamma_i \Phi_i \le \max_i \Phi_i \le \max_{[0,\ell]} \Phi(x)$.

If $(i^*,k^*) \in \mathscr{T}_m$, then $i^* = m$, $u_{m-1,x}(k^*) \ge 0$ and we can choose h small enough that $\gamma_{m-1,x} = \gamma'(x^{m-1}) \in (-\frac{3}{2}, -\frac{1}{2})$ so that

$$-\frac{\gamma_{m-1,x}}{\gamma_m\gamma_{m-1}} u_m(k^*) \le e^{-\lambda t_{k^*}} p_{k^*} \quad \Longrightarrow \quad u_m(k^*) \le \frac{\gamma_m\gamma_{m-1}}{-\gamma'(x^{m-1})} e^{-\lambda t_{k^*}} p_{k^*} \le e^{-\lambda t_{k^*}} p_{k^*}$$

If $(i^*,k^*) \in \mathscr{T}_0$, then $i^* = 0, u_{0\bar{t}}(k^*) \ge 0$, $u_{0x}(k^*) \le 0$. Notice that $\left(\frac{1}{\gamma_0}\right)_x = -\frac{1}{\gamma_0\gamma_1}\gamma_{0x}$. Note $\gamma_{0x} = \gamma'(x^0)$, so for h small enough, we can ascertain $\gamma_{0x} = \gamma'(x^0) \in (\frac{1}{2}, \frac{3}{2})$. It follows

$$\frac{\gamma'(x^0)}{\gamma_0\gamma_1} u_0(k^*) \le e^{-\lambda t_{k^*}} \left(hf_{0k^*} - g^n_{k^*}\right) \quad \Longrightarrow \quad u_0(k^*) \le e^{-\lambda t_{k^*}} \left(hf_{0k^*} - g^n_{k^*}\right)$$

If $(i^*,k^*) \in \mathscr{N}$, then $u_{i^*\bar{t}}(k^*) \ge 0$, $u_{i^*x\bar{x}}(k^*) = \frac{1}{h^2}\big(u_{i^*+1}(k^*) - 2u_{i^*}(k^*) + u_{i^*-1}(k^*)\big) \le 0$. For $(i,k) \in \mathscr{N}$, the corresponding equation in (4.7) is equivalent to

$$\frac{1}{\gamma_i}\zeta^i_k e^{-\lambda\tau} u_{i\bar{t}}(k) - \frac{1}{\gamma_{i+1}} u_{ix\bar{x}}(k) - \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right] u_{i\bar{x}}(k) + \\ + \left[\frac{\lambda}{\gamma_i}\zeta^i_k e^{-\lambda(t_k - t^k)} - \left(\frac{1}{\gamma_i}\right)_{x\bar{x}}\right] u_i(k) = f_{ik} e^{-\lambda t_k} \tag{4.8}$$

Define the sets

$$\mathscr{N}_+ = \left\{(i,k) \in \mathscr{N} \,\middle|\, \left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x \ge 0\right\}, \quad \mathscr{N}_- = \left\{(i,k) \in \mathscr{N} \,\middle|\, \left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x < 0\right\}$$

And it's clear $\mathscr{N} = \mathscr{N}_+ \cup \mathscr{N}_-$. Suppose $(i^*,k^*) \in \mathscr{N}_+$. Then owing to (4.7) and $u_{i^*x}(k^*) \le 0$, we can write

$$\left[\frac{\lambda}{\gamma_{i^*}}\zeta^{i^*}_{k^*} e^{-\lambda(t_{k^*} - t^{k^*})} - \left(\frac{1}{\gamma_{i^*}}\right)_{x\bar{x}}\right] u_{i^*}(k^*) \le f_{i^*k^*} e^{-\lambda t_{k^*}} \tag{4.9}$$

If instead $(i^*,k^*) \in \mathscr{N}_-$, then we can use (4.8) and the fact that $u_{i^*\bar{x}}(k^*) \geq 0$ to achieve again (4.9). Therefore, (4.9) is achieved in any case. We can choose τ so small that $e^{-\lambda(t_k - t^k)} > \frac{1}{2}, \quad \forall k$. Observe that

$$\frac{1}{\zeta_{k^*}^{i^*}} e^{\lambda(t_{k^*} - t^{k^*})} \left[-\frac{1}{\gamma_{i^*+1}} \gamma_{i^* x\bar{x}} + \frac{\gamma_{i^* x} + \gamma_{i^* \bar{x}}}{\gamma_{i^*-1}\gamma_{i^*+1}} \gamma_{i^* \bar{x}} \right] \leq \frac{2}{\bar{b}} \left[4(2\|\gamma''\|_{C[0,\ell]}) + 16(2\|\gamma'\|^2_{C[0,\ell]}) \right] \leq$$
$$\leq \frac{64}{\bar{b}} \left(\|\gamma''\|_{C[0,\ell]} + \|\gamma'\|^2_{C[0,\ell]} \right)$$

Then by (4.6), it is the case that the coefficient of $u_{i^*}(k^*)$ is positive independently of i^*, k^*. Therefore,

$$u_{i^*}(k^*) \leq C_\gamma f_{i^* k^*} e^{-\lambda t_{k^*}}$$

where C_γ is a constant depending only on γ and $\bar{b}$.

We can put together the obtained estimations to deduce that for $(i,k) \in \mathscr{M}_{k_1}$,

$$u_i(k) \leq \max_{\mathscr{M}} u_i(k) \leq \max \left\{ 0, \quad \|\Phi\|_{C[0,\ell]}, \quad \|p\|_{C[0,T]}, \quad \|g^n\|_{C[0,T]} + \|f\|_{L_\infty(D)}, \quad C_\gamma \|f\|_{L_\infty(D)} \right\}$$

But because $u_i(k) = \gamma_i e^{-\lambda t_k} v_i(k)$, we have the following uniform upper bound for the discrete state vector:

$$v_i(k) \leq 4e^{\lambda T} \max \left\{ 0, \ \|\Phi\|_{C[0,\ell]}, \ \|p\|_{C[0,T]}, \ \|g^n\|_{C[0,T]} + \|f\|_{L_\infty(D)}, \ C_\gamma \|f\|_{L_\infty(D)} \right\}, \quad (i,k) \in \mathscr{M}_{k_1}$$

In a fully analogous manner, we arrive at a uniform lower bound for the discrete state vector: For $(i,k) \in \mathscr{M}_{k_1}$,

$$v_i(k) \geq 4e^{\lambda T} \min \left\{ 0, \quad -\|\Phi\|_{C[0,\ell]}, \quad -\|p\|_{C[0,T]}, \quad -\|g^n\|_{C[0,T]} - \|f\|_{L_\infty(D)}, \quad -C_\gamma \|f\|_{L_\infty(D)} \right\}$$

Combining the uniform upper and lower bounds imply (4.1) up to k_1. But k_1 was arbitrary in $1, \dots, n$. Theorem is proved. □

Remark. Theorem 4.1 directly implies the estimate

$$\max_{0 \le k \le n} \left(\sum_{i=0}^{m-1} h u_i^2(k) \right) \le \ell \|[v]_n\|_{\ell_\infty} \le C_2 \Big(\|f\|_{L_\infty(D)} + \|p\|_{L_\infty[0,T]} + \|g^n\|_{W_2^1[0,T]} + \|\Phi\|_{L_\infty[0,\ell]} \Big) \tag{4.10}$$

where C_2 is independent of n and m.

4.2 Energy Estimate for the Discrete State Vector

Theorem 4.2 *Suppose that $p \in W_2^1[0,T], \Phi \in W_2^1[0,\ell], f \in L_\infty(D)$. For $[g]_n \in \mathscr{G}_R^n$ and n, m large enough, the discrete state vector $[v([g]_n)]_n$ satisfies the following estimate:*

$$\begin{aligned} \|[v]_n\|_{\mathscr{E}}^2 := \sum_{k=1}^{n} \tau \sum_{i=0}^{m-1} h v_{i\bar{t}}^2(k) + \max_{1 \le k \le n} \left(\sum_{i=0}^{m-1} h v_{ix}^2(k) \right) + \sum_{k=1}^{n} \tau^2 \sum_{i=0}^{m-1} h v_{ix\bar{t}}^2(k) \quad \le \\ \le \quad \tilde{C}_\infty \Big(\|\Phi\|_{W_2^1[0,\ell]}^2 + \|f\|_{L_\infty(D)}^2 + \|p\|_{W_2^1[0,T]}^2 + \|g^n\|_{W_2^1[0,T]}^2 \Big) \end{aligned} \tag{4.11}$$

where $\tilde{C}_\infty$ is a constant independent of n and m.

Proof. Consider n and m large enough that Theorem 4.1 is satisfied. In (1.33), choose $\eta = 2\tau v_{\bar{t}}(k)$. Using (3.10), write $(b_n(v_i(k)))_{\bar{t}} = \zeta_k^i v_{i\bar{t}}(k)$. Also, use the fact that

$$2\tau v_{ix}(k)(v_{i\bar{t}}(k))_x = 2\tau v_{ix}(k)(v_{ix}(k))_{\bar{t}} =$$

$$= v_{ix}^2(k) + v_{ix}^2(k) - 2v_{ix}(k)v_{ix}(k-1) + v_{ix}^2(k-1) - v_{ix}^2(k-1) =$$

$$= v_{ix}^2(k) + \Big(v_{ix}(k) - v_{ix}(k-1)\Big)^2 - v_{ix}^2(k-1) =$$

$$= v_{ix}^2(k) - v_{ix}^2(k-1) + \tau^2 v_{ix\bar{t}}(k)$$

We thus have

$$2\tau \sum_{i=0}^{m-1} h\zeta_k^i v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} hv_{ix}^2(k) - \sum_{i=0}^{m-1} hv_{ix}^2(k-1) + \tau^2 \sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) =$$
$$= 2\tau \sum_{i=0}^{m-1} hf_{ik}v_{i\bar{t}}(k) + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k) \tag{4.12}$$

Estimate the right hand side of (4.12) by applying Cauchy Inequality with $\varepsilon > 0$ the first term. Recall that $b_n'(v) \geq \bar{b}$, $\forall v$. We will have:

$$2\tau \sum_{i=0}^{m-1} h\zeta_k^i v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} hv_{ix}^2(k) - \sum_{i=0}^{m-1} hv_{ix}^2(k-1) + \tau^2 \sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \leq$$
$$\leq \bar{b}\tau \sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{1}{\bar{b}}\tau \sum_{i=0}^{m-1} hf_{ik}^2 + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k) \tag{4.13}$$

We can absorb the first term on the right hand side of (4.13) to the first term on the left hand side. Hence:

$$\tau \sum_{i=0}^{m-1} h\bar{b}v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} hv_{ix}^2(k) - \sum_{i=0}^{m-1} hv_{ix}^2(k-1) + \tau^2 \sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \leq$$
$$\leq \frac{1}{\bar{b}}\tau \sum_{i=0}^{m-1} hf_{ik}^2 + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k), \qquad \forall k = 1, \ldots, n \tag{4.14}$$

Perform summation of (4.14) for k from 1 to q, $2 \le q \le n$. The second and third term on the left hand side telescope, and we gather:

$$\bar{b}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)+\sum_{i=0}^{m-1}hv_{ix}^2(q)+\sum_{k=1}^{q}\tau^2\sum_{i=0}^{m-1}hv_{ix\bar{t}}^2(k)\le$$
$$\le\sum_{i=0}^{m-1}hv_{ix}^2(0)+\frac{1}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hf_{ik}^2+2\sum_{k=1}^{q}\tau p_k v_{m\bar{t}}(k)-2\sum_{k=1}^{q}\tau g_k^n v_{0\bar{t}}(k) \tag{4.15}$$

Use the summation by parts technique on the p and g sums:

$$\sum_{k=1}^{q}\tau p_k v_{m\bar{t}}(k)=\sum_{k=1}^{q}p_k v_m(k)-\sum_{k=1}^{q}p_k v_m(k-1)=\sum_{k=1}^{q}p_k v_m(k)-\sum_{k=0}^{q-1}p_{k+1}v_m(k)=$$
$$=-\sum_{k=1}^{q-1}\tau p_{kt}v_m(k)+p_q v_m(q)-p_1 v_m(0)$$
$$\sum_{k=1}^{q}\tau g_k^n v_{0\bar{t}}(k)=-\sum_{k=1}^{q-1}\tau g_{kt}^n v_0(k)+g_q^n v_0(q)-g_1^n v_0(0) \tag{4.16}$$

In view of (4.16) and borrowing (4.1) from Theorem 4.1, (4.15) yields (through Cauchy Inequality):

$$\bar{b}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)+\sum_{i=0}^{m-1}hv_{ix}^2(q)+\sum_{k=1}^{q}\tau^2\sum_{i=0}^{m-1}hv_{ix\bar{t}}^2(k)\le$$
$$\le\sum_{i=0}^{m-1}h\Phi_{ix}^2+\frac{1}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hf_{ik}^2+\sum_{k=1}^{q-1}\tau p_{kt}^2+\sum_{k=1}^{q-1}\tau\big(g_{kt}^n\big)^2 \;+$$
$$+\;2t_{q-1}\|[v]_n\|_{\ell_\infty}^2+2\big(\|p\|_{L_\infty[0,T]}+\|g^n\|_{L_\infty[0,T]}\big)\|[v]_n\|_{\ell_\infty} \tag{4.17}$$

Through the definition of the Steklov average, the Cauchy-Schwarz inequality and Fubini's Theorem, for h small enough we have the following results:

$$\sum_{i=0}^{m-1} h\Phi_{ix}^2 \leq \|\Phi'\|^2_{L_2[0,\ell]} + \|\Phi'\|^2_{L_2[\ell-h,\ell]} \leq \|\Phi\|^2_{W_2^1[0,\ell]}$$
$$\|g^n\|_{L_\infty[0,T]} \leq \|g^n\|_{W_2^1[0,T]} \leq R$$
$$\sum_{k=1}^{q-1} \tau\big(g^n_{kt}\big)^2 = \sum_{k=2}^{q} \tau(g^n_{k\bar{t}})^2 \leq \|(g^n)'\|^2_{L_2[0,T]} \leq \|g^n\|^2_{W_2^1[0,T]} \leq R^2$$
$$\sum_{k=1}^{q-1} \tau p_{kt}^2 = \sum_{k=2}^{q} \tau p_{k\bar{t}}^2 \leq \|p'\|^2_{L_2[0,T]} \leq \|p\|^2_{W_2^1[0,T]}$$
$$\sum_{k=1}^{q} \tau \sum_{i=0}^{m-1} h f_{ik}^2 \leq \|f\|^2_{L_2(D)} \tag{4.18}$$

Applying the results in (4.18) to (4.17),

$$\sum_{k=1}^{q} \tau \sum_{i=0}^{m-1} h v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} h v_{ix}^2(q) + \sum_{k=1}^{q} \tau^2 \sum_{i=0}^{m-1} h v_{ix\bar{t}}^2(k) \leq$$
$$\leq \tilde{C}_\infty\Big(\|\Phi\|^2_{W_2^1[0,\ell]} + \frac{1}{\bar{b}}\|f\|^2_{L_2(D)} + \|p\|^2_{W_2^1[0,T]} + \|g^n\|^2_{W_2^1[0,T]}\Big) \tag{4.19}$$

Where $\tilde{C}_\infty$ is a constant dependent on the L_∞ norms of the functions f, p, Φ, and on $\bar{b}, T, R$, but independent of n, m and q. (4.19) is true for $q = \overline{1, n}$. As such, it's true for $q = n$ and for q such that the left hand side of (4.19) is maximum. In this way, (4.11) is concluded from (4.19).□

4.3 Compactness of the Sequence of Interpolations of the Discrete State Vector

Theorem 4.3 *Take set of assumptions $\mathscr{A}$. Let $\{[g]_n\}$ be a sequence in $\mathscr{G}_R^n$ such that the sequence of interpolations $\{\mathscr{P}_n([g]_n)\}$ converges weakly to $g \in W_2^1[0,T]$. Then the whole sequence of interpolations $\{\hat{v}^\tau\}$ of the associated discrete state vectors converge weakly to $v = v(x,t;g) \in W_2^{1,1}(D)$, with v the unique weak solution to the Stefan Problem in the sense of (1.27).*

Proof. By the definitions of the interpolations given in (3.9), we observe that

$$\|\hat{v}^\tau\|_{L_\infty(D)} = \operatorname*{esssup}_{(x,t)\in D} |\hat{v}^\tau(x,t)| = \max_{0\le k\le n}\Big(\max_{0\le i\le m}|v_i(k)|\Big) = \|[v]_n\|_{\ell_\infty} \tag{4.20}$$

$$\int_0^T\int_0^\ell (\hat{v}^\tau)^2\,dx\,dt \le T\ell\|\hat{v}^\tau\|_{L_\infty(D)} = T\ell\|[v]_n\|_{\ell_\infty}$$

$$\int_0^T\int_0^\ell (\hat{v}_x^\tau)^2\,dx\,dt \le 2\sum_{k=0}^{n-1}\sum_{i=0}^{m-1}\tau h v_{ix}^2(k) + 2\sum_{k=1}^{n}\sum_{i=0}^{m-1}\frac{1}{3}\tau^3 h v_{ix\bar{t}}^2(k) \le 2(\|[v]_n\|_{\mathscr{E}}^2 + \|\Phi\|_{W_2^1[0,\ell]}^2)$$

$$\int_0^T\int_0^\ell (\hat{v}_t^\tau)^2\,dx\,dt \le 2\sum_{k=1}^{n}\sum_{i=0}^{m-1}\tau h\big(v_{i\bar{t}}^2(k) + \frac{1}{3}h^2 v_{ix\bar{t}}^2(k)\big) \le \frac{2}{3}\sum_{k=1}^{n}\left[\sum_{i=0}^{m-1}\Big(7\tau h v_{i\bar{t}}^2(k)\Big) + 2\tau h v_{m\bar{t}}^2(k)\right]$$

$$\sum_{k=1}^{n}\tau h v_{m\bar{t}}^2(k) \le 2\sum_{k=1}^{n}\tau h\Big(v_{m-1,\bar{t}}^2(k) + h^2 p_{k\bar{t}}^2\Big) \le 2\|[v]_n\|_{\mathscr{E}}^2 + 2h^3\|p\|_{W_2^1[0,T]}^2 \tag{4.21}$$

Since $[g]_n \in \mathscr{G}^n$, then $\|g^n\|_{W_2^1[0,T]} \le R+1$ for large enough n. From the energy estimates (4.1), (4.11) and calculations (4.20), (4.21) it is therefore the case that $\{\hat{v}^\tau\}$ is uniformly bounded in the spaces $W_2^{1,1}(D)$ and $L_\infty(D)$. As such, we may choose a subsequence of $\{\hat{v}^\tau\}$ that converges weakly in $W_2^{1,1}(D)$ to some function $v \in W_2^{1,1}(D)\cap L_\infty(D)$, and thus strongly in $L_2(D)$, by virtue of which we can choose a further subsequence that converges to v pointwise almost everywhere. It is our intent to show now that v satisfies the definition of a weak solution to the Stefan Problem. To do this, first realize that sequences $\{v^\tau\}, \{\hat{v}^\tau\}$ are equivalent in $W_2^{1,0}(D)$, and sequences $\{v^\tau\}, \{\tilde{v}\}$

are equivalent in $L_2(D)$, as shown by the following calculations:

$$\int_0^T\int_0^\ell |v^\tau-\hat{v}^\tau|^2\,dx\,dt=\sum_{k=1}^n\sum_{i=0}^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\big(\hat{v}(x;k)-\hat{v}(x;k-1)-\hat{v}_{\bar{t}}(x;k)(t-t_{k-1})\big)^2\,dx\,dt=$$

$$=\sum_{k=1}^n\sum_{i=0}^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\hat{v}_{\bar{t}}^2(x;k)(t_k-t)^2\,dx\,dt=\frac{1}{3}\tau^2\int_0^T\int_0^\ell(\hat{v}_t^\tau)^2\,dx\,dt\longrightarrow 0 \tag{4.22}$$

$$\int_0^T\int_0^\ell |v_x^\tau-\hat{v}_x^\tau|^2\,dx\,dt=\sum_{k=1}^n\sum_{i=0}^{m-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\big(v_{ix}(k)-v_{ix}(k-1)-v_{ix\bar{t}}(k)(t-t_{k-1})\big)^2\,dx\,dt=$$

$$=\sum_{k=1}^n\sum_{i=0}^{m-1}\int_{t_{k-1}}^{t_k}hv_{ix\bar{t}}^2(k)(t_k-t)^2\,dx\,dt=\frac{1}{3}\tau\left(\sum_{k=1}^n\tau^2\sum_{i=0}^{m-1}hv_{ix\bar{t}}^2(k)\right)\longrightarrow 0 \tag{4.23}$$

$$\int_0^T\int_0^\ell |v^\tau-\tilde{v}|^2\,dx\,dt=\sum_{k=1}^n\sum_{i=0}^{m-1}\tau\int_{x_i}^{x_{i+1}}\Big|v_i(k)+v_{ix}(k)(x-x_i)-v_i(k)\Big|^2\,dx\,dt=$$

$$=\sum_{k=1}^n\sum_{i=0}^{m-1}\frac{1}{3}\tau h^3v_{ix}^2(k)\,dx\,dt\longrightarrow 0 \tag{4.24}$$

As n,m go to $+\infty$. Accordingly, $v^\tau\to v$ weakly in $W_2^{1,0}(D)$ and $\tilde{v}\to v$ strongly in $L_2(D)$ and pointwise a.e. on D along a subsequence. Fix arbitrary $\psi\in W_2^{1,1}(D)$ with $\psi|_{t=T}=0$. Actually, without loss of generality we can consider $\psi\in C^1(\overline{D})$ and $\psi|_{t=T}=0$. Define $\psi_i(k)=\psi(x_i,t_k)$, $\forall i\ \forall k$, and consider the interpolations:

$$\psi^\tau(x,t):=\psi_i(k),\qquad \psi_x^\tau(x,t):=\psi_{ix}(k)\qquad \psi_t^\tau(x,t):=\psi_{it}(k)$$
$$,\qquad x_i\le x<x_{i+1},\quad t_{k-1}<t\le t_k,\qquad i=\overline{0,m},\ k=\overline{0,n} \tag{4.25}$$

It is readily checked that $\psi^\tau,\psi_x^\tau,\psi_t^\tau$ converge uniformly on $\overline{D}$ as $n,m\to\infty$ to the functions ψ,ψ_x,ψ_t respectively. Fix n. For each k in (1.33) as satisfied by the discrete state vector $[v([g]_n)]_n$, choose $\eta_i=\tau\psi_i(k)$, $\forall i$ and sum all equalities (1.33) over $k=1,\dots,n$. The resulting expression is as follows:

$$\sum_{k=1}^n\tau\sum_{i=0}^{m-1}h\Big[\big(b_n(v_i(k))\big)_{\bar{t}}\psi_i(k)+v_{ix}(k)\psi_{ix}(k)-f_{ik}\psi_i(k)\Big]-\sum_{k=1}^n\tau p_k\psi_m(k)+\sum_{k=1}^n\tau g_k^n\psi_0(k) \tag{4.26}$$

We transform the first term through summation by parts:

$$\sum_{k=1}^{n}\tau\sum_{i=0}^{m-1}h\big(b_n(v_i(k))\big)_{\bar{t}}\psi_i(k)=\sum_{k=1}^{n}\sum_{i=0}^{m-1}hb_n(v_i(k))\psi_i(k)-\sum_{k=1}^{n}\sum_{i=0}^{m-1}hb_n(v_i(k-1))\psi_i(k)=$$

$$=\sum_{k=1}^{n}\sum_{i=0}^{m-1}hb_n(v_i(k))\psi_i(k)-\sum_{k=0}^{n-1}\sum_{i=0}^{m-1}hb_n(v_i(k))\psi_i(k+1)=$$

$$=-\sum_{k=1}^{n-1}\tau\sum_{i=0}^{m-1}hb_n(v_i(k))\psi_{it}(k)+\sum_{i=0}^{m-1}hb_n(v_i(n))\psi_i(n)-\sum_{i=0}^{m-1}hb_n(v_i(0))\psi_i(1)=$$

$$=-\int_0^{T-\tau}\int_0^{\ell}b_n(\tilde{v}(x,t))\psi_t^\tau\,dx\,dt-\int_0^{\ell}b_n(\tilde{\Phi}(x))\psi^\tau(x,\tau)\,dx$$

Thus, (4.26) can be rewritten as:

$$\int_0^T\int_0^{\ell}\Big[-b_n(\tilde{v})\psi_t^\tau+v_x^\tau\psi_x^\tau-f\psi^\tau\Big]\,dx\,dt-\int_0^{\ell}b_n(\tilde{\Phi})\psi^\tau(x,\tau)\,dx-$$

$$-\int_0^T p(t)\psi^\tau(\ell,t)\,dt+\int_0^T g^n(t)\psi^\tau(0,t)\,dt+\int_{T-\tau}^{T}\int_0^{\ell}b_n(\tilde{v})\psi_t^\tau\,dx\,dt=0 \tag{4.27}$$

Theorem 4.1 implies that if $\mathscr{V}_n := \big\{y\in\mathbb{R}\mid\exists(x,t)\in D\text{ s.t. }\tilde{v}(x)=y\big\}$ (i.e. $\mathscr{V}_n$ is the range of $\tilde{v}$), then the set $\mathscr{V}=\bigcup_{n=1}^{\infty}\mathscr{V}_n$ is bounded in $\mathbb{R}$, hence its closure $\overline{\mathscr{V}}$ is compact in $\mathbb{R}$. Because of the piecewise continuity of b, it follows that $b(\tilde{v}(x,t))\in L_\infty(D)$, and therefore $\|b_n(\tilde{v})\|_{L_\infty(D)}\le C$. Since D is a set of finite measure, $\|b_n(\tilde{v})\|_{L_2(D)}\le C$, so that a subsequence $\{b_{n_l}(\tilde{v}(x,t))\}$ can be constructed so that it converges weakly in $L_2(D)$ to a function $\tilde{b}(x,t)\in L_2(D)$. Through a similar argument, a subsequence can be constructed so that $b_{n_l}(\tilde{\Phi}(x))$ converges weakly in $L_2[0,\ell]$ to a function $\tilde{b}_0(x)\in L_2[0,\ell]$. Take a diagonal of these subsequences as the whole sequence. We see that

$$\int_{T-\tau}^{T}\int_0^{\ell}b_n(\tilde{v})\psi_t^\tau\,dx\,dt\le\left(\int_{T-\tau}^{T}\int_0^{\ell}b_n^2(\tilde{v})\,dx\,dt\right)^{\frac{1}{2}}\left(\int_{T-\tau}^{T}\int_0^{\ell}(\psi_t^\tau)^2\,dx\,dt\right)^{\frac{1}{2}}\le$$

$$\le\|b_n^2(\tilde{v})\|_{L_2(D)}\|\psi_t^\tau\|_{L_2\left([0,\ell]\times[T-\tau,T]\right)}\longrightarrow 0\text{ as }n\to\infty \tag{4.28}$$

Now, due to (4.28), the uniform convergence of $\psi^\tau,\ \psi^\tau_x,\ \psi^\tau_t$ respectively to $\psi,\ \psi_x,\ \psi_t$ and weak convergence of $b_n(\tilde{v}),\ v^\tau_x,\ \ b_n(\tilde{\Phi}),\ g^n$ to $\tilde{b},\ v_x,\ \tilde{b}_0,\ g$ in the respective L_2 spaces, then as $n \to \infty$, (4.27) implies

$$\int_0^T \int_0^\ell \Big[-\tilde{b}(x,t)\psi_t + v_x \psi_x - f\psi \Big]\, dx\, dt - \int_0^\ell \tilde{b}_0(x)\psi(x,0)\, dx - \\ - \int_0^T p(t)\psi(\ell,t)\, dt + \int_0^T g(t)\psi(0,t)\, dt = 0 \tag{4.29}$$

It can be checked that both $\tilde{b}$ and $\tilde{b}_0$ are functions of type $\mathscr{B}$. If at the point (x,t)

$$\tilde{v}(x,t) \to v(x,t) \neq v^j$$

then we have

$$b_n(\tilde{v}(x,t)) = \int_{\tilde{v}(x,t)-\frac{1}{n}}^{\tilde{v}(x,t)+\frac{1}{n}} \omega_{1/n}(|\tilde{v}(x,t)-u|)b(u)\, du \ \longrightarrow\ b(v(x,t)).$$

On the contrary, if at the point (x,t) we have

$$\tilde{v}(x,t) \to v(x,t) = v^j$$

then we have

$$b(v^j)^- \le \liminf_{n\to\infty} b_n(\tilde{v}(x,t)) \le \limsup_{n\to\infty} b_n(\tilde{v}(x,t)) \le b(v^j)^+$$

Since the sequence $\{b_n(\tilde{v})\}$ converges to $\tilde{b}(x,t)$ weakly in $L_2(D)$, by Mazur's lemma there is a sequence of convex combinations of elements of $\{b_n(\tilde{v})\}$ which converges to $\tilde{b}(x,t)$ strongly in $L_2(D)$. Therefore, there is a subsequence of convex combinations which converges to $\tilde{b}(x,t)$ a.e. in D. It easily follows that $\tilde{b} = B(x,t,v(x,t))$ is a function of type $\mathscr{B}$. In a very similar way, it is seen that $\tilde{b}_0 = B(x,0,\Phi(x))$ is of type $\mathscr{B}$. Hence, by definition, v is a weak solution to the Stefan Problem in the sense of (1.27). From Lemma 3.3 then, v is the unique solution, which implies that v is the only weak limit point of the sequence $\{\hat{v}^\tau\}$. Therefore, the whole sequence $\{\hat{v}^\tau\}$ converges to v weakly in $W_2^{1,1}(D)$. $\square$

Chapter 5

Proofs of the Main Results

5.1 Proof of Theorem 2.1

Consider a sequence $\{g_l\} \in \mathscr{G}_R$ such that $\mathscr{J}(g_l) \searrow \mathscr{J}_*$ (this sequence can be constructed by the definition of the infimum). Since $\{g_l\}$ is uniformly bounded in $W_2^1[0,T]$, it is weakly precompact in $\mathscr{G}_R$. Therefore, there exists a subsequence $\{g_{l_k}\}$ which converges weakly in $W_2^1[0,T]$, say, to $g \in W_2^1[0,T] \in \mathscr{G}_R$. For ease of notation, take this subsequence as the sequence $\{g_l\}$.

Let $v_l = v(x,t;g_l)$ and $v = v(x,t;g)$ be solutions to the Stefan problem in the sense of (1.27) with g_l and g respectively. Then for fixed l, the sequence of vectors $\{[g_l]_n\}$ given by $[g_l]_n = \mathscr{Q}_n(g_l)$ is such that the interpolations $g_l^n = \mathscr{P}_n([g_l]_n)$ converge weakly in $W_2^1[0,T]$ to $g_l \in W_2^1[0,T]$ as $n \to \infty$.

Therefore, Theorem 4.3 applies, and so associated to $[g_l]_n$ the interpolations $\hat{v}_l^\tau$ of the discrete state vectors $[v([g_l]_n)]_n$ converge weakly in $W_2^{1,1}(D)$ to v_l. As such,

$$\|v_l\|_{W_2^{1,1}(D)} \leq \liminf_{n\to\infty} \|\hat{v}_l^\tau\|_{W_2^{1,1}(D)} \leq C \liminf_{n\to\infty} \Big(\|[v_l]_n\|_{\ell_\infty} + \|[v_l]_n\|_{\mathscr{E}} \Big) \tag{5.1}$$

Where C is independent of n, m and l. Thanks to $\{g_l\} \subset \mathscr{G}_R$, it it is clear from (4.1) and (4.11) that the right hand side of (5.1) is uniformly bounded. Similarly, one can conclude that

$$\|v_l\|_{L_\infty(D)} \leq \liminf_{n\to\infty} \|[v_l]_n\|_{\ell_\infty}$$

Accordingly, $\{v_l\} \in W_2^{1,1}(D) \cap L_\infty(D)$ is a weakly precompact sequence in $W_2^{1,1}(D)$, so that it contains a subsequence $\{v_{l_k}\}$ which converges weakly to a function $\tilde{v} \in W_2^{1,1}(D)$, and thus strongly in $L_2(D)$. Due to this strong convergence in $L_2(D)$, a further subsequence of $\{v_{l_k}\}$ can be extracted which converges almost everywhere to $\tilde{v}$ on D. Then the uniform boundedness of this subsequence in $L_\infty(D)$ implies that $\tilde{v} \in L_\infty(D)$. Now, take this subsequence as the whole sequence. Each of the v_l satisfies (1.27) with g_l and with an arbitrarily fixed function B of type $\mathscr{B}$. Going to infinity along the sequence, we have that we can replace g_l with g and v_l with $\tilde{v}$ in (1.27). Indeed, $B(x,t,v_l(x,t)) \to B(x,t,v(x,t))$ a.e. on D because of Corollary 3.1 and the fact that $v_l \to v$ a.e. on D. Consequently, $\tilde{v}$ is a solution to the Stefan problem with g. But, due to uniqueness of such a solution, it follows that $v = \tilde{v}$ in $W_2^{1,1}(D) \cap L_\infty(D)$. Next, note:

$$\begin{aligned}
\lim_{l\to\infty} \Big| \mathscr{J}(g) - \mathscr{J}(g_l) \Big| &= \lim_{l\to\infty} \Big| \|v(\ell,t) - \Gamma(t)\|^2_{L_2[0,T]} - \|v_l(\ell,t) - \Gamma(t)\|^2_{L_2[0,T]} \Big| \\
&= \lim_{l\to\infty} \Big| < v - \Gamma, v - \Gamma >_{L_2[0,T]} - < v_l - \Gamma, v_l - \Gamma >_{L_2[0,T]} \Big| \\
&= \lim_{l\to\infty} \Big| \|v(\ell,t) - v_l(\ell,t)\|^2_{L_2[0,T]} + 2 < v - v_l, v_l - \Gamma >_{L_2[0,T]} \Big| \\
&= \lim_{l\to\infty} \left| \int_0^T |v(\ell,t) - v_l(\ell,t)|^2\, dt + 2 \int_0^T \big(v(\ell,t) - v_l(\ell,t)\big)\big(v_l(\ell,t) - \Gamma(t)\big)\, dt \right|
\end{aligned} \tag{5.2}$$

By the weak convergence of the sequence $\{v_l\}$ to v in $W_2^{1,1}(D)$, it follows that we have strong convergence in the space of traces. In particular, the integrals in (5.2) vanish as $l \to \infty$. Hence $\lim\limits_{l\to\infty} \mathscr{J}(g_l) = \mathscr{J}(g)$. This limit is unique though, therefore it is the case that $\mathscr{J}(g) = \mathscr{J}_*$, so that $g \in \mathscr{G}_*$. $\square$

5.2 Proof of Theorem 2.2

The proof of Theorem 2.2 is split into three separate lemmas, as shown below.

5.2.1 Lemma A

Lemma A. *Let* $\mathscr{J}_*(\pm\varepsilon) = \inf_{\mathscr{G}_{R\pm\varepsilon}} \mathscr{J}(g),\ \varepsilon > 0.$ *Then* $\lim_{\varepsilon\to 0} \mathscr{J}_*(\varepsilon) = \mathscr{J}_* = \lim_{\varepsilon\to 0} \mathscr{J}_*(-\varepsilon)$.

Proof. The proof of this lemma is very similar to the analogous lemma from [1]. If $0 < \varepsilon_1 < \varepsilon_2$, then

$$\mathscr{J}_*(\varepsilon_2) \leq \mathscr{J}_*(\varepsilon_1) \leq \mathscr{J}_* \leq \mathscr{J}_*(-\varepsilon_1) \leq \mathscr{J}_*(-\varepsilon_2)$$

Hence $\lim_{\varepsilon\to 0} \mathscr{J}_*(\varepsilon) \leq \mathscr{J}_*$ and $\lim_{\varepsilon\to 0} \mathscr{J}_*(-\varepsilon) \geq \mathscr{J}_*$. Choose $g_\varepsilon \in \mathscr{G}_{R+\varepsilon}$ such that

$$\lim_{\varepsilon\to 0} \Big(\mathscr{J}(g_\varepsilon) - \mathscr{J}_*(\varepsilon) \Big) = 0.$$

Since $\{g_\varepsilon\}$ is weakly pre-compact in $W_2^1[0,T]$, there exists a subsequence ε' such that $g_{\varepsilon'} \to g_*$ weakly in $W_2^1[0,T]$ as $\varepsilon' \to 0$. Since $\mathscr{J}$ is weakly continuous, $\mathscr{J}(g_{\varepsilon'}) \to \mathscr{J}(g_*)$ as $\varepsilon \to 0$. Hence $\mathscr{J}_*(\varepsilon') \to \mathscr{J}(g_*) \geq \mathscr{J}_*$ as $\varepsilon' \to 0$. Thus $\lim_{\varepsilon\to 0} \mathscr{J}_*(\varepsilon) = \mathscr{J}_*$.

From the other side, by Theorem 2.1 we know $\exists g_* \in \mathscr{G}_R$ s.t. $\mathscr{J}(g_*) = \mathscr{J}_*$. If $g_* \in \mathscr{G}_R \backslash \partial\mathscr{G}_R$, then $\exists \varepsilon_* > 0$ such that $g_* \in \mathscr{G}_{R-\varepsilon}$, $\forall \varepsilon < \varepsilon_*$, and in this case $\mathscr{J}_*(-\varepsilon) = \mathscr{J}_*$, $\forall \varepsilon < \varepsilon_*$. If $g_* \in \partial\mathscr{G}_R$, then $\exists \{g_\varepsilon\}$ with $g_\varepsilon \in \mathscr{G}_{R-\varepsilon}$ such that $g_\varepsilon \to g_*$ in $W_2^1[0,T]$ as $\varepsilon \to 0$. The continuity of $\mathscr{J}$ gives us that $\lim_{\varepsilon\to 0} \mathscr{J}(g_\varepsilon) = \mathscr{J}(g_*) = \mathscr{J}_*$. Since on the other hand, $\mathscr{J}(g_\varepsilon) \geq \mathscr{J}_*(-\varepsilon)$, it follows that $\lim_{\varepsilon\to 0} \mathscr{J}_*(-\varepsilon) = \mathscr{J}_*$. □

5.2.2 Lemma B

Lemma B. $\forall g \in \mathscr{G}_R$, $\lim_{n\to\infty} \mathscr{I}_n(\mathscr{Q}_n(g)) = \mathscr{J}(g)$.

Proof. Take $g \in \mathscr{G}_R$ arbitrarily. If $\mathscr{Q}(g) = [g]_n$, and $g^n = \mathscr{P}_n([g]_n)$, then $g^n \to g$ strongly in $W_2^1[0,T]$ as $n \to \infty$. Applying Theorem 4.3, we have that the interpolations $\hat{v}^\tau$ of the discrete state vectors $[v([g]_n)]_n$ converge to $v = v(x,t;g)$ weakly in $W_2^{1,1}(D)$ as $n \to \infty$, and thus the traces $\hat{v}^\tau(\ell,\cdot)$ converge strongly in $L_2[0,T]$ to trace $v(\ell,\cdot)$. By calculations (4.22) and (4.23), the sequences $\{v^\tau\}$, $\{\hat{v}^\tau\}$ are equivalent in $W_2^{1,0}(D)$, so that the $v^\tau(\ell,\cdot)$ traces too converge to $v(\ell,\cdot)$

strongly in $L_2[0,T]$. If we define

$$\tilde{\Gamma}(t) = \Gamma_k = \frac{1}{\tau}\int\limits_{t_{k-1}}^{t_k} \Gamma(t)\,dt, \qquad t_{k-1} < t \le t_k, \quad k = \overline{1,n} \tag{5.3}$$

Then $\tilde{\Gamma} \to \Gamma$ in $L_2[0,T]$ as $n \to \infty$. Therefore,

$$\begin{aligned} |\mathscr{I}_n(\mathscr{Q}_n(g)) - \mathscr{J}(g)| &= \left| \sum_{k=1}^{n} \tau(v_m(k) - \Gamma_k)^2 \; - \int\limits_0^T (v(\ell,t) - \Gamma(t))^2\,dt \right| \le \\ &\le \|v^\tau(\ell,\cdot) - v(\ell,\cdot)\|^2_{L_2[0,T]} + \|\tilde{\Gamma} - \Gamma\|^2_{L_2[0,T]} \quad + \\ &+ 2\int\limits_0^T \Big[|v^\tau(\ell,t) - v(\ell,t)||v(\ell,t) - \tilde{\Gamma}(t)| + |\Gamma(t) - \tilde{\Gamma}(t)||v(\ell,t) - \Gamma(t)| \Big]\,dt \\ &\longrightarrow 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{5.4}$$

(5.4) establishes the lemma. □

5.2.3 Lemma C

Lemma C. *For an arbitrary sequence $\{[g]_n\}$ such that $[g]_n \in \mathscr{G}_R^n$,*

$$\lim_{n\to\infty} \Big(\mathscr{J}(\mathscr{P}_n([g]_n)) - \mathscr{I}_n([g]_n) \Big) = 0$$

.

Proof. Let $g^n = \mathscr{P}_n([g]_n)$. This sequence is weakly precompact, so that a subsequence g^{n_l} converges weakly to a function g in $W_2^1[0,T]$. Take this subsequence as the whole sequence. Note that

$$|\mathscr{J}(g^n) - \mathscr{I}_n([g]_n)| \le |\mathscr{J}(g^n) - \mathscr{J}(g)| + |\mathscr{J}(g) - \mathscr{I}_n([g]_n)| \tag{5.5}$$

Since $\mathscr{J}$ is weakly continuous, $|\mathscr{J}(g^n) - \mathscr{J}(g)| \longrightarrow 0$ as $n \to \infty$. It remains to show that the second term on the right hand side of (5.5) goes to 0 as $n \to \infty$. Actually, the proof of this fact flows in a manner very similar to the proof of Lemma B. From (5.5) it follows that $\lim_{n_l \to \infty} \Big(\mathscr{J}(\mathscr{P}_{n_l}([g]_{n_l})) - \mathscr{I}_{n_l}([g]_{n_l}) \Big) = 0$. However, the subsequence chosen was arbitrary. Therefore, the same result can be achieved for any subsequence $\{g_{n_\alpha}\}$ of $\{g_n\}$. It is then the case that the whole sequence $\mathscr{J}(\mathscr{P}_n([g]_n)) - \mathscr{I}_n([g]_n)$ converges to 0 as $n \to \infty$. $\square$

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Appendix

Alternate Proof of Existence of a Weak Solution

Proposition. *There exists a weak solution to the Stefan problem in the sense of (1.27).*

Proof. Approximate the function $b(v)$ by averagings $b_n(v)$ using an infinitely differentiable nonnegative kernel $\omega_\rho(|v|)$ of radius $\rho = \frac{1}{n}$, the function $\Phi(x)$ by a family of functions Φ_n which are uniformly bounded and infinitely differentiable, and that converge to it in the norm of $W_2^1[0,\ell]$, and approximate (f,p,g) by the sequences of uniformly bounded functions $(\{f_n\},\{p_n\},\{g_n\})$ defined through averagings using an infinitely differentiable nonnegative kernel, living in $(C^\infty(D), C^\infty[0,T], C^\infty[0,T])$ that converge to the point (f,p,g) in the $L_2(D), W_2^1[0,T]$, $W_2^1[0,T]$ norms. Since there are finitely many phase transition values v^j, it follows that b' exists almost everywhere in $\mathbb{R}$, $b'(u) > 0$ a.e. $u \in \mathbb{R}$, and we have $\bar{b} := \operatorname*{essinf}_{u\in\mathbb{R}} b'(u) > 0$. In light of this, we have:

$$b_n'(v) = \left(\int_{v-\frac{1}{n}}^{v+\frac{1}{n}} \omega_{\frac{1}{n}}(|v-w|) b(w)\, dw \right)' = \left(\int_{\mathbb{R}} \omega_{\frac{1}{n}}(|u|) b(v-u)\, du \right)' =$$

$$= \int_{\mathbb{R}} \omega_{\frac{1}{n}}(|u|) b'(v-u)\, du \geq \operatorname*{essinf}_{u\in\mathbb{R}} b'(u) \int_{\mathbb{R}} \omega_{\frac{1}{n}}(|u|)\, du = \bar{b} > 0, \qquad \forall v \in \mathbb{R}, \quad \forall n \tag{A.1}$$

So that b_n is strictly increasing in $\mathbb{R}$ for every n, and b_n' is a finite distance away from 0 independently of n. The auxiliary problems

$$\frac{\partial b_n(v)}{\partial t} - v_{xx} = f_n(x,t), \qquad (x,t) \in D, \tag{A.2}$$

$$v(x,0) = \Phi_n(x) \qquad x \in [0,\ell] \tag{A.3}$$

$$v_x|_{x=0} = g_n(t), \qquad v_x|_{x=\ell} = p_n(t), \qquad t \in [0,T] \tag{A.4}$$

have solutions v_n in $H^{2,1}(\overline{D})$ from Theorem 7.4 in Chapter V of [25]. Indeed, observe we can write (A.2) equivalently as

$$v_t - \frac{1}{b_n'(v(x,t))} v_{xx} = \frac{1}{b_n'(v(x,t))} f_n(x,t), \qquad (x,t) \in D$$

From Theorem 2.3 of Chapter I from the same source, the following estimate is true for each of the v_n:

$$\|v_n\|_{L_\infty(D)} = \operatorname*{ess\,sup}_D |v_n| \le C \left(\|f_n\|_{L_\infty(D)} + \|g_n\|_{L_\infty[0,T]} + \|p_n\|_{L_\infty[0,T]} + \|\Phi_n\|_{L_\infty[0,\ell]} \right) \tag{A.5}$$

Where C is independent of v_n. Thus the solutions v_n are uniformly bounded in D. We now seek to obtain a uniform estimate on the $W_2^{1,1}(D)$ norm of these solutions.

Energy Estimate on Solutions to the Auxiliary Problems. Let v_n be a solution to (A.2)-(A.4) with the data functions having the aforementioned regularity. Extend v_n to the rectangle $\tilde{D} = \{-\ell \le x \le 2\ell, 0 \le t \le T\}$ as an element of $H^{2,1}(\tilde{D})$. Let $D_t = \big\{0 < x < \ell,\ 0 <$

$\tau < t\}$, $\forall t \in (0, T]$. A solution to (A.2)-(A.4) satisfies the integral identity

$$0 = \int_0^t \int_0^\ell \Big[b_n'(v_n) v_{n_t} \psi + v_{n_x} \psi_x - f_n \psi \Big] \, dx d\tau - \\ - \int_0^t p_n(\tau) \psi(\ell, \tau) \, d\tau + \int_0^t g_n(\tau) \psi(0, \tau) \, d\tau, \qquad \forall \psi \in W_2^{1,0}(D_t) \tag{A.6}$$

In particular, consider the family of functions $\psi_\sigma = v_{n_{\sigma_t}}$ as the space averaging of the function v_{n_t} with a sufficiently smooth nonnegative kernel. We can write:

$$\int_0^t \int_0^\ell \Big[b_n'(v_n) v_{n_t} v_{n_{\sigma_t}} + v_{n_x} (v_{n_{\sigma_t}})_x - f_n v_{n_{\sigma_t}} \Big] \, dx d\tau - \int_0^t \Big[p_n(\tau) v_{n_{\sigma_t}}(\ell, \tau) - g_n(\tau) v_{n_{\sigma_t}}(0, \tau) \Big] \, d\tau = 0$$

Send $\sigma \to 0$, to get:

$$\int_0^t \int_0^\ell \Big[b_n'(v_n) (v_{n_t})^2 + v_{n_x} (v_{n_x})_t - f_n v_{n_t} \Big] \, dx d\tau - \int_0^t p_n(\tau) v_{n_t}(\ell, t) \, d\tau + \int_0^t g_n(\tau) v_{n_t}(0, \tau) \, d\tau = 0$$

$$\int_0^t \int_0^\ell \big(b_n'(v_n) v_{n_t}^2 - f_n v_{n_t} \big) \, dx d\tau + \frac{1}{2} \int_0^\ell v_{n_x}^2(x, \tau) \Big|_0^t \, dx = \int_0^t p_n(\tau) v_{n_t}(\ell, \tau) \, d\tau - \int_0^t g_n(\tau) v_{n_t}(0, \tau) \, d\tau$$

$$\int_0^t \int_0^\ell b_n'(v_n) v_{n_t}^2 \, dx d\tau + \frac{1}{2} \int_0^\ell v_{n_x}^2(x, t) \, dx = \frac{1}{2} \int_0^\ell (\Phi_n')^2(x) \, dx + \int_0^t p_n(\tau) v_{n_t}(\ell, \tau) \, d\tau - \\ - \int_0^t g_n(\tau) v_{n_t}(0, \tau) \, d\tau + \int_0^t \int_0^\ell f_n v_{n_t} \, dx d\tau \tag{A.7}$$

At this point, we see that we will have to estimate the p_n and g_n integrals. We may produce an integration by parts on both integrals on the right-hand side of (A.7) to ascertain:

$$\int_0^t\int_0^\ell b_n'(v_n)v_{n_t}^2\,dxd\tau+\frac{1}{2}\int_0^\ell v_{n_x}^2(x,t)\,dx=\frac{1}{2}\int_0^\ell(\Phi_n')^2(x)\,dx+\int_0^t\int_0^\ell f_nv_{n_t}\,dxd\tau-$$
$$-\int_0^t p_n'(\tau)v_n(\ell,\tau)\,d\tau+p_n(t)v_n(\ell,t)-p_n(0)v_n(\ell,0)+$$
$$+\int_0^t g_n'(\tau)v_n(0,\tau)\,d\tau-g_n(t)v_n(0,t)+$$
$$+g_n(0)v_n(0,0) \tag{A.8}$$

Now, replace through inequality the RHS of (A.8) by its absolute value; then for all t, replace the appearance of $v_n(x,t)$ on the RHS of (A.8) with the essential supremum of v_n over D, which is bounded by the energy estimate (A.5). Moreover, apply a Cauchy Inequality with $\varepsilon=b_n'(v_n(x,t))>0$ on the inside of the f_n integral. The resulting inequality is:

$$\int_0^t\int_0^\ell b_n'(v_n)v_{n_t}^2\,dxd\tau+\frac{1}{2}\int_0^\ell v_{n_x}^2(x,t)\,dx\le\frac{1}{2}\int_0^t\int_0^\ell\frac{f_n^2(x,\tau)}{b_n'(v_n(x,\tau))}\,dxd\tau+\frac{1}{2}\int_0^t\int_0^\ell b_n'(v_n)v_{n_t}^2\,dxd\tau+$$
$$+\|v_n\|_{L_\infty(D)}\Big(2\|p_n\|_{C[0,T]}+2\|g_n\|_{C[0,T]}+$$
$$+\sqrt{t}\|p_n'\|_{L_2[0,T]}+\sqrt{t}\|g_n'\|_{L_2[0,T]}\Big)+\frac{1}{2}\int_0^\ell(\Phi_n')^2(x)\,dx \tag{A.9}$$

Now, from the definition of $\bar{b}_n$ and Morrey's Inequality, we finally obtain the energy estimate

$$\bar{b}_n\|(v_n)_t\|_{L_2(D)}^2+\operatorname*{ess\,sup}_{0\le t\le T}\int_0^\ell(v_n)_x^2(x,t)\,dx\le\frac{1}{\bar{b}_n}\|f_n\|_{L_2(D)}^2+\|\Phi_n'\|_{L_2[0,\ell]}^2+$$
$$+C\|v_n\|_{L_\infty(D)}\Big(\|p_n\|_{W_2^1[0,T]}+\|g_n\|_{W_2^1[0,T]}\Big) \tag{A.10}$$

Where C is a positive constant depending only on T. The energy estimates (A.5) and (A.10) indeed tell us that the sequence $\{v_n\}$ is bounded in the $W_2^{1,1}(D) \cap L_\infty(D)$ space. In particular, (A.10) implies that $\{v_n\} \in W_2^{1,1}(D) \cap L_\infty(D)$ is a weakly precompact sequence in $W_2^{1,1}(D)$, so that it contains a subsequence $\{v_{n_k}\}$ which converges weakly to a function $v \in W_2^{1,1}(D)$, and thus strongly to it in $L_2(D)$. Due to this strong convergence in $L_2(D)$, a further subsequence of $\{v_{n_k}\}$ can be extracted which converges almost everywhere to v on D. Then the uniform boundedness of this subsequence in $L_\infty(D)$ through (A.5) implies that $v \in L_\infty(D)$. Now, take this subsequence as the whole sequence. Each of the elements of this subsequence satisfies the integral identity (A.6) and hence the integral identity (1.27) for the corresponding arbitrary ψ (this is the case since the last integral identity can be arrived by a simple integration by parts on the t derivative of the first term). That is, presently we have:

$$\int_0^T \int_0^\ell \Big[-b_n(v_n(x,t))\psi_t + (v_n)_x \psi_x - f_n \psi \Big]\, dxdt - \int_0^\ell b_n(\Phi_n(x))\psi(x,0)\, dx -$$
$$- \int_0^T p_n(t)\psi(\ell,t)\, dt + \int_0^T g_n(t)\psi(0,t)\, dt = 0, \qquad \forall \psi \in W_2^{1,1}, \psi(x,T) = 0 \tag{A.11}$$

Due to the uniform boundedness of the subsequence, $b_n(v_n(x,t))$ can be chosen so that it converges weakly in $L_2(D)$ to some function $\tilde{b}(x,t)$, and so that $b_n(\Phi_n(x))$ converges weakly in $L_2[0,\ell]$ to a function $\tilde{b}_0(x)$. Moreover, by our assumption on the convergence of the functions f_n, g_n, p_n, Φ_n, and by the weak convergence of v_n to v in $W_2^{1,1}(D)$, it follows that we can send $n \to \infty$ in (A.11) to get

$$\int_0^T \int_0^\ell \Big[-\tilde{b}(x,t)\psi_t + v_x \psi_x - f\psi \Big]\, dxdt - \int_0^\ell \tilde{b}_0(x)\psi(x,0)\, dx -$$
$$- \int_0^T p(t)\psi(\ell,t)\, dt + \int_0^T g(t)\psi(0,t)\, dt = 0, \qquad \forall \psi \in W_2^{1,1}, \psi(x,T) = 0$$

Actually, $\tilde{b}, \tilde{b}_0$ are functions of type $\mathscr{B}$, so that v is a weak solution to our problem in the sense of (1.27). To prove the energy estimates that v satisfies, first note that (A.5) and $v_n \rightharpoonup v$

in $W_2^{1,1}(D)$ imply that

$$\|v\|_{L_\infty(D)} \leq \limsup_{n\to\infty} \|v_n(x,t)\|_{L_\infty(D)} \leq C\Big(\|f\|_{L_\infty(D)} + \|g\|_{C[0,T]} + \|p\|_{C[0,T]} + \|\Phi\|_{C[0,\ell]}\Big) \quad \text{(A.12)}$$

The above estimate and (A.10) provide us with the following $W_2^{1,1}(D)$ estimate for v:

$$\|v\|_{W_2^{1,1}(D)} \leq \|f\|^2_{L_2(D)} + \|\Phi'\|^2_{L_2[0,\ell]} + C\|v\|_{L_\infty(D)}\Big(\|p\|_{W_2^1[0,T]} + \|g\|_{W_2^1[0,T]}\Big) \quad \text{(A.13)}$$