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Erisa Hasani

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by
Erisa Hasani

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We the undersigned committee
hereby approve the attached thesis

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Erisa Hasani

Kanishka Perera, Ph.D.
Professor
Mathematical Sciences
Committee Chair

Ugur Abdulla, Ph.D., Dr.Sci., Dr.rer.nat.habil
Professor
Mathematical Sciences
Committee Member

Jean Perez, Ph.D.
Associate Professor
Aerospace, Physics and Space Sciences
Outside Committee Member

Munevver Subasi, Ph.D
Associate Professor and Department Head
Mathematical Sciences

ABSTRACT

Title:
The Brézis-Nirenberg Problem for the Generalized Kirchhoff Equation
Author:
Erisa Hasani
Major Advisor:
Kanishka Perera, Ph.D.

We study a class of critical Kirchhoff problems with a general nonlocal term. The main difficulty here is the absence of a closed-form formula for the compactness threshold. First we obtain a variational characterization of this threshold level. Then we prove a series of existence and multiplicity results based on this variational characterization.

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List of Notations

- For $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$, the space of Lebesgue-measurable functions with finite norm, $L^p(\Omega)$, is defined as follows

$$L^p(\Omega) := \{u : \Omega \to \mathbb{R}; u \text{ is measurable and } \int_\Omega |u|^p dx < \infty\}$$

with the norm

$$\|u\|_{L^p} := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$$

- Denote by $W^{m,p}(\Omega)$ the Sobolev space of $u \in L^p(\Omega)$ functions where $\left|D^k u\right|$ is also in $L^p(\Omega)$ for $|k| \leq m$, with the norm

$$\|u\|_{W^{m,p}} := \sum_{0 \leq |k| \leq m} \left\|D^k u\right\|_{L^p}$$

- Denote by $W_0^{m,p}(\Omega)$ the completion of C_0^∞ (space of real valued smooth functions with compact support in Ω) in the norm $\|\cdot\|_{W^{m,p}}$. If Ω is bounded then consider the equivalent norm given by

$$\|u\|_{W_0^{m,p}} := \sum_{|k|=m} \left\|D^k u\right\|_{L^p}$$

- For a bounded domain $\Omega \subset \mathbb{R}^N$ denote by $H_0^1(\Omega)$ the Sobolev space $W_0^{m,p}(\Omega)$ defined above, by taking $m = 1$ and $p = 2$, and denote the corresponding norm as follows

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2}$$

- Little o notation: the following two expressions are equivalent

$$f(x) = o(g(x)) \quad \text{as } x \to x_0 \quad \text{is the same as} \quad \frac{f(x)}{g(x)} \to 0 \quad \text{as } x \to x_0$$

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Chapter 1

Introduction

In this chapter we present an overview of problems from literature that led to the problem presented in this thesis. We start with results of nonlinear elliptic equations where the compactness of the Sobolev embedding $H_0^1(\Omega) \subset L^p(\Omega)$ is important. Such embedding is compact if $p \in [1, 2^*)$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. These problems can be solved using variational methods. Next, we move to the case where $p = 2^*$, for which we lose compactness of the Soboblev embedding. The first attempt to overcome such difficulty appeared in the 1983 paper by Brezis and Nirenberg [3]. The tools introduced in that paper can be used to solve more generalized types of problems involving critical exponents.

1.1 An Overview of Previous Results

Consider the following problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.1}$$

for any p, with $1 \leq p < 2^*$, and $\Omega \subset \mathbb{R}^N$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Weak solutions to this problem correspond to finding critical points of the following C^1 functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{p}\int_\Omega |u|^p dx$$

I satisfies Palais–Smale condition, and the mountain pass theorem (see [1]) can be used to get a nontrivial critical point.

In the Brézis-Nirenberg paper in 1983 [3], the case where $p = 2^*$ and $u > 0$ in problem (1.1.1) was considered. They introduced what came to be known as the Palais–Smale condition at level c, which is also denoted by $(\text{PS})_s$ (see Definition 2.1.1). They managed to overcome the difficulty that was due to lack of compactness of the embedding $H_0^1(\Omega) \subset L^p(\Omega)$, by looking at some type of compactness at certain levels c. In general, in problems where there is a critical Sobolev exponent involved, obtaining non-trivial solutions does not easily follow.

Next, consider the critical Kirchhoff equation

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1.2}$$

where a and b are nonenegative real numbers. Note that (1.1.2) is a generalization of problem (1.1.1) for the critical case by taking $a = 1$ and $b = 0$.

Problem (1.1.2) is related to the following equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = g(x,t)$$

proposed by Kirchhoff in 1883 [13], which is an extension of the wave equation for free vibrations of elastic strings introduced by D'Alembert. Problem (1.1.2) is the critical case of the stationary Kirchhoff equation. Changes in length of a string caused by transverse vibrations can be described by Kirchhoff's model. Here, we consider the critical case of such problem, for a subcritical result in dimensions $N \leq 3$, see Perera and Zhang [12].

As is usually the case with problems of critical growth, problem (1.1.2) lacks compactness. The standard approach to such problems is to determine a threshold level below which there is compactness, and construct minimax critical levels below this threshold. This approach has been used in this case in dimensions $N = 3$ and 4 to obtain nontrivial solutions in recent literature (see, e.g., Huang et al. [6], Liao et al. [8], Naimen [9, 10], Xie et al. [14], Yao and Mu [15], Zhang and Liu [16], and the references therein). For a further generalization we instead consider a positive power $\gamma > 1$ for the nonlocal term. We have the following model problem as the start of the current study

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2\,dx\right)^{\gamma-1}\right]\Delta u = \lambda u + |u|^{2^*-2}\,u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.3}$$

where $1 < \gamma < +\infty$, $a \geq 0$, $b > 0$, and $\lambda > 0$. Note that for $\gamma = 2$ we get problem (1.1.2). However, we notice that problem (1.1.3) can be further generalized (see next section). Our main results will be formulated and proved for a generalized case of which problems (1.1.1), (1.1.2) and (1.1.3) are special cases. Now we are ready to formulate our problem and state our main results.

1.2 Statement of the Problem and Main Results

Here is a generalized version of problem (1.1.3) with a general nonlocal term

$$\begin{cases} -h\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right) \Delta u = f(x,u) + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $h : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|f(x,t)| \leq a_1 |t|^{p-1} + a_2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{1.2.2}$$

for some constants $a_1, a_2 > 0$ and $1 < p < 2^*$. The class of nonlocal terms considered here includes sums of powers

$$h(t) = \sum_{i=1}^n a_i\, t^{\gamma_i - 1}, \quad t \geq 0,$$

where $a_1, \dots, a_n > 0$ and $1 \leq \gamma_1 < \cdots < \gamma_n < +\infty$. A model case is

$$h(t) = a + bt^{\gamma-1},$$

where $a, b \geq 0$ with $a + b > 0$ and $1 < \gamma < +\infty$. The classical case $h(t) = a + bt$ corresponds to $\gamma = 2$.

In the general case considered in the present work, such a threshold level cannot be found in closed form. Our first contribution here is a variational characterization of this threshold level (see Theorem 2.2.1). Then we give a series of existence and multiplicity results based on this variational characterization (see Section 3.1). This requires novel arguments due to the absence of a closed-form compactness threshold.

We will state and prove our compactness, existence, and multiplicity results for problem (1.2.1) with a general nonlocal term h in the next two sections. To illustrate our results while keeping the presentation simple, we state them here for the model problem

$$\begin{cases} -\left[a + b\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1}\right] \Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

where $1 < \gamma < +\infty$, $a \geq 0$, $b > 0$, and $\lambda > 0$.

Weak solutions of this problem coincide with critical points of the functional

$$J(u) = \frac{a}{2} \int_\Omega |\nabla u|^2\, dx + \frac{b}{2\gamma} \left(\int_\Omega |\nabla u|^2\, dx\right)^\gamma - \frac{\lambda}{2} \int_\Omega u^2\, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx, \quad u \in H^1_0(\Omega).$$

Recall that J satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or the $(\mathrm{PS})_c$ condition for short, if every sequence (u_j) in $H^1_0(\Omega)$ such that $J(u_j) \to c$ and $J'(u_j) \to 0$ has a strongly convergent subsequence. Let S be the best Sobolev constant (see (2.1.1)) and let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of the Laplacian in Ω. We have the following compactness results for the cases $\gamma < 2^*/2$, $\gamma = 2^*/2$, and $\gamma > 2^*/2$ (see Corollary 2.3.1, Corollary 2.3.5, and Corollary 2.3.7).

Let S denote the best Sobolev constant (see (2.1.1)) and let λ_1 denote the first Dirichlet eigenvalue of the Laplacian in Ω (see (2.1.5) for $\gamma = 1$).

Theorem 1.2.1. *Let $1 < \gamma < 2^*/2$, $a, b > 0$, and $0 < \lambda \le a\lambda_1$. Let t_0 be the unique positive solution of the equation $a + bt^{\gamma-1} = S^{-2^*/2}\, t^{2^*/2-1}$ and set*

$$c^* = \frac{1}{N}\, at_0 + \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) bt_0^\gamma.$$

Then J satisfies the $(\mathrm{PS})_c$ condition for all $c < c^*$.

Theorem 1.2.2. *Let $\gamma = 2^*/2$.*

(*i*) *Let $a > 0$, $0 < b < S^{-2^*/2}$, and $0 < \lambda < a\lambda_1$. Set*

$$c^* = \frac{1}{N} \left(\frac{a^{2^*/2}}{S^{-2^*/2} - b}\right)^{2/(2^*-2)}.$$

Then J satisfies the $(\mathrm{PS})_c$ condition for all $c < c^*$.

(*ii*) *If $a \ge 0$ and $b > S^{-2^*/2}$, then J satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$ for any $\lambda > 0$.*

Theorem 1.2.3. *If $\gamma > 2^*/2$ and*

$$a^{\gamma-2^*/2}\, b^{2^*/2-1} > \frac{(\gamma - 2^*/2)^{\gamma-2^*/2}\,(2^*/2 - 1)^{2^*/2-1}}{(\gamma-1)^{\gamma-1}}\, S^{-(2^*/2)(\gamma-1)},$$

then J satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$ for any $\lambda > 0$.

Theorems 1.2.1–1.2.3 have the following corollary for the classical case $\gamma = 2$, where $c^* = +\infty$ means that J satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$.

Corollary 1.2.4. *Let $\gamma = 2$.*

(*i*) *If $N = 3$, $a, b > 0$, and $0 < \lambda \le a\lambda_1$, then $c^* = \dfrac{1}{4}\, abS^3 + \dfrac{1}{24}\, b^3 S^6 + \dfrac{1}{24}\,(4aS + b^2 S^4)^{3/2}$.*

(*ii*) *If $N = 4$, $a > 0$, $0 < b < S^{-2}$, and $0 < \lambda < a\lambda_1$, then $c^* = \dfrac{a^2}{4\,(S^{-2} - b)}$.*

(*iii*) *If $N = 4$, $a \ge 0$, and $b > S^{-2}$, then $c^* = +\infty$ for any $\lambda > 0$.*

(*iv*) *If $N \ge 5$ and $a^{N-4}\, b^2 > \dfrac{4\,(N-4)^{N-4}}{(N-2)^{N-2}}\, S^{-N}$, then $c^* = +\infty$ for any $\lambda > 0$.*

Remark 1.2.5. The threshold levels in Corollary 1.2.4 (i)–(iii) were also obtained using different arguments in Naimen [10], Naimen [9], and Liao et al. [8], respectively.

We have the following existence and multiplicity results for problem (1.2.3) (see Corollary 3.2.2, Theorem 3.3.1, and Theorem 3.3.2).

Theorem 1.2.6. *If* $1 < \gamma < 2^*/2$, $a, b > 0$, *and* $N \geq 4$, *then problem* (1.2.3) *has a nontrivial solution for* $0 < \lambda < a\lambda_1$.

Theorem 1.2.7. *Let* $\gamma = 2^*/2$.

(i) *If* $a > 0$, $0 < b < S^{-2^*/2}$, *and* $N \geq 4$, *then problem* (1.2.3) *has a nontrivial solution for* $0 < \lambda < a\lambda_1$.

(ii) *If* $a = 0$ *and* $b > S^{-2^*/2}$, *then problem* (1.2.3) *has a nontrivial solution for all* $\lambda > 0$.

(iii) *If* $a > 0$ *and* $b > S^{-2^*/2}$, *then problem* (1.2.3) *has two nontrivial solutions for* $\lambda > a\lambda_1$.

Theorem 1.2.8. *If* $\gamma > 2^*/2$ *and*

$$a^{\gamma - 2^*/2}\, b^{2^*/2 - 1} > \frac{(\gamma - 2^*/2)^{\gamma - 2^*/2}\, (2^*/2 - 1)^{2^*/2 - 1}}{(\gamma - 1)^{\gamma - 1}}\, S^{-(2^*/2)(\gamma - 1)},$$

then problem (1.2.3) *has two nontrivial solutions for* $\lambda \geq a\lambda_1$.

Theorems 1.2.6–1.2.8 have the following corollaries for the classical case $\gamma = 2$.

Corollary 1.2.9. *Let* $\gamma = 2$ *and* $N = 4$.

(i) *If* $a > 0$ *and* $0 < b < S^{-2}$, *then problem* (1.2.3) *has a nontrivial solution for* $0 < \lambda < a\lambda_1$.

(ii) *If* $a = 0$ *and* $b > S^{-2}$, *then problem* (1.2.3) *has a nontrivial solution for all* $\lambda > 0$.

(iii) *If* $a > 0$ *and* $b > S^{-2}$, *then problem* (1.2.3) *has two nontrivial solutions for* $\lambda > a\lambda_1$.

Remark 1.2.10. The result in Corollary 1.2.9 (i) was also obtained in Naimen [9] using a different method. Liao et al. [8] obtained one nontrivial solution when $a \geq 0$, $b > S^{-2}$, and $\lambda > a\lambda_1$. See also Perera and Zhang [12] for a related result in the subcritical case in dimensions $N \leq 3$.

Corollary 1.2.11. *If* $\gamma = 2$, $N \geq 5$, *and*

$$a^{N-4}\, b^2 > \frac{4\,(N-4)^{N-4}}{(N-2)^{N-2}}\, S^{-N},$$

then problem (1.2.3) *has two nontrivial solutions for* $\lambda \geq a\lambda_1$.

Remark 1.2.12. Corollary 1.2.11 complements the results in Naimen and Shibata [11], where two positive solutions were obtained when $a = 1$, $b > 0$ is sufficiently small, and $0 < \lambda < \lambda_1$.

In the borderline case where $\gamma = 2^*/2$ and $b = S^{-2^*/2}$, lower-order terms come into play. Consider the problem

$$\begin{cases} -\left[a + S^{-2^*/2}\left(\int_\Omega |\nabla u|^2\,dx\right)^{2^*/2-1} + \eta\left(\int_\Omega |\nabla u|^2\,dx\right)^{\sigma-1}\right] = \lambda u + |u|^{2^*-2}\,u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.4}$$

where $a \geq 0$, $\eta > 0$, $1 < \sigma < 2^*/2$, and $\lambda > 0$. We have the following existence and multiplicity result (see Theorem 3.3.4).

Theorem 1.2.13. *Let $\eta > 0$ and $1 < \sigma < 2^*/2$.*

(*i*) *If $a = 0$, then problem* (1.2.4) *has a nontrivial solution for all $\lambda > 0$.*

(*ii*) *If $a > 0$, then problem* (1.2.4) *has two nontrivial solutions for $\lambda > a\lambda_1$.*

Chapter 2

Compactness Threshold

This chapter is devoted to showing compactness of the corresponding functional to our problem under a threshold which we determine. Depending on the problem and its assumptions, such threshold can be determined explicitly (see Remarks 2.3.2– 2.3.6) but for our generalized problem we do not have a closed form (see Theorem 2.2.1).

2.1 Preliminary Results

A weak solution of problem (1.2.1) is a function u that belongs to the Sobolev space $H_0^1(\Omega)$ and satisfies

$$h\left(\int_\Omega |\nabla u|^2\, dx\right) \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f(x,u)\, v\, dx + \int_\Omega |u|^{2^*-2}\, uv\, dx \quad \forall v \in H_0^1(\Omega).$$

Weak solutions coincide with critical points of the C^1-functional

$$J(u) = \frac{1}{2}\, H\left(\int_\Omega |\nabla u|^2\, dx\right) - \int_\Omega F(x,u)\, dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \quad u \in H_0^1(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)\, ds$ is the primitive of f.

Definition 2.1.1. The functional J satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or the $(\mathrm{PS})_c$ condition for short, if every sequence (u_j) in $H_0^1(\Omega)$ such that

$$J(u_j) \to c \qquad J'(u_j) \to 0,$$

called a $(\mathrm{PS})_c$ sequence, has a strongly convergent subsequence.

Let

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\left(\displaystyle\int_\Omega |u|^{2^*} dx\right)^{2/2^*}} \tag{2.1.1}$$

be the best Sobolev constant. The set

$$I = \left\{ t > 0 : h(t) \le S^{-2^*/2}\, t^{2^*/2 - 1} \right\}$$

will play an important role in our compactness results. We begin with a simple but useful proposition.

Proposition 2.1.2. *If (u_j) is a sequence in $H^1_0(\Omega)$ such that*

$$J'(u_j) \to 0, \qquad u_j \rightharpoonup u, \qquad \|u_j - u\|^2 \to t,$$

then either $t = 0$ or $t \in I$. In particular, if $I = \emptyset$, then every bounded sequence (u_j) in $H^1_0(\Omega)$ such that $J'(u_j) \to 0$ has a strongly convergent subsequence.

Proof. Since $J'(u_j) \to 0$,

$$h\left(\int_\Omega |\nabla u_j|^2\, dx \right) \int_\Omega \nabla u_j \cdot \nabla v\, dx - \int_\Omega f(x, u_j)\, v\, dx - \int_\Omega |u_j|^{2^* - 2}\, u_j\, v\, dx = \mathrm{o}(\|v\|) \quad (2.1.2)$$

for all $v \in H^1_0(\Omega)$, and since $u_j \rightharpoonup u$ and $\|u_j - u\|^2 \to t$,

$$\int_\Omega |\nabla u_j|^2\, dx \to \int_\Omega |\nabla u|^2\, dx + t =: s.$$

Passing to a renamed subsequence, we may assume that $u_j \to u$ strongly in $L^p(\Omega)$ and a.e. in Ω. So taking $v = u_j$ in (2.1.2) gives

$$h(s) \left(\int_\Omega |\nabla u|^2\, dx + t \right) - \int_\Omega u f(x, u)\, dx - \int_\Omega |u_j|^{2^*} dx = \mathrm{o}(1), \qquad (2.1.3)$$

while taking $v = u$ and passing to the limit gives

$$h(s) \int_\Omega |\nabla u|^2\, dx - \int_\Omega u f(x, u)\, dx - \int_\Omega |u|^{2^*} dx = 0. \qquad (2.1.4)$$

Since

$$\int_\Omega |u_j|^{2^*} dx - \int_\Omega |u|^{2^*} dx = \int_\Omega |u_j - u|^{2^*} dx + \mathrm{o}(1)$$

by the Brézis-Lieb lemma (see [2]), subtracting (2.1.4) from (2.1.3) and using (2.1.1) gives

$$t h(s) = \int_\Omega |u_j - u|^{2^*} dx + \mathrm{o}(1) \le S^{-2^*/2} \left(\int_\Omega |\nabla (u_j - u)|^2\, dx \right)^{2^*/2} + \mathrm{o}(1).$$

If $t > 0$, then passing to the limit and noting that $h(s) \ge h(t)$ since $s \ge t$ and h is nondecreasing gives $h(t) \le S^{-2^*/2}\, t^{2^*/2 - 1}$, so $t \in I$. $\square$

First we consider the case where I is nonempty. Let

$$H(t) = \int_0^t h(s)\, ds, \quad t \geq 0$$

be the primitive of h, and set

$$K(t) = \frac{1}{2}\, H(t) - \frac{1}{2^*}\, t h(t), \quad t \geq 0.$$

For $1 \leq \gamma \leq 2^*/2$, let

$$\lambda_1(\gamma) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right)^\gamma}{\displaystyle\int_\Omega |u|^{2\gamma}\, dx} \tag{2.1.5}$$

be the first eigenvalue of the nonlinear eigenvalue problem

$$\begin{cases} -\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1} \Delta u = \lambda\, |u|^{2\gamma-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is positive by the Sobolev embedding theorem. We note that $\lambda_1(1) = \lambda_1$, the first Dirichlet eigenvalue of the Laplacian in Ω, and $\lambda_1(2^*/2) = S^{2^*/2}$. We assume that

(A_1) for some constants $\alpha_1, \dots, \alpha_n > 0$, $1 \leq \gamma_1 < \cdots < \gamma_n < 2^*/2$, and $\mu_1 \leq \lambda_1(\gamma_1), \dots, \mu_n \leq \lambda_1(\gamma_n)$ with at least one of the inequalities strict,

$$K(t) \geq \sum_{i=1}^n \alpha_i\, t^{\gamma_i} \quad \forall t \geq 0$$

and

$$F(x,t) - \frac{1}{2^*}\, t f(x,t) \leq \sum_{i=1}^n \mu_i\, \alpha_i\, |t|^{2\gamma_i} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R};$$

(A_2) K is superadditive, i.e.,

$$K(t_1 + t_2) \geq K(t_1) + K(t_2) \quad \forall t_1, t_2 \geq 0;$$

(A_3) $h(t)/t^{2^*/2-1}$ is strictly decreasing for $t > 0$ and

$$0 \leq b := \lim_{t \to +\infty} \frac{h(t)}{t^{2^*/2-1}} < S^{-2^*/2} < \lim_{t \to 0} \frac{h(t)}{t^{2^*/2-1}} \leq +\infty.$$

We note that K is nonnegative by (A_1) and hence nondecreasing by (A_2). We have the following theorem.

2.2 Main Result

Theorem 2.2.1. *Assume that $I \neq \emptyset$ and (1.2.2), (A_1), and (A_2) hold. Set*

$$c^* = \inf_{t \in I} K(t). \tag{2.2.1}$$

Then J satisfies the $(\mathrm{PS})_c$ condition for all $c < c^*$. If, in addition, (A_3) holds, then the equation

$$h(t) = S^{-2^*/2}\, t^{2^*/2-1} \tag{2.2.2}$$

has a unique positive solution t_0 and

$$c^* = K(t_0), \tag{2.2.3}$$

in particular, $c^* > 0$.

Proof. First we note that for all $u \in H^1_0(\Omega)$,

$$K\bigg(\int_\Omega |\nabla u|^2\, dx\bigg) - \int_\Omega \bigg[F(x,u) - \frac{1}{2^*}\, u f(x,u)\bigg]\, dx \ge \sum_{i=1}^n \alpha_i \left(1 - \frac{\mu_i}{\lambda_1(\gamma_i)}\right) \bigg(\int_\Omega |\nabla u|^2\, dx\bigg)^{\gamma_i} \tag{2.2.4}$$

by (A_1) and (2.1.5).

Let $c < c^*$ and let (u_j) be a $(\mathrm{PS})_c$ sequence. Then

$$\frac{1}{2}\, H\bigg(\int_\Omega |\nabla u_j|^2\, dx\bigg) - \int_\Omega F(x,u_j)\, dx - \frac{1}{2^*} \int_\Omega |u_j|^{2^*} dx = c + \mathrm{o}(1) \tag{2.2.5}$$

and

$$h\bigg(\int_\Omega |\nabla u_j|^2\, dx\bigg) \int_\Omega \nabla u_j \cdot \nabla v\, dx - \int_\Omega f(x,u_j)\, v\, dx - \int_\Omega |u_j|^{2^*-2}\, u_j\, v\, dx = \mathrm{o}(\|v\|) \tag{2.2.6}$$

for all $v \in H^1_0(\Omega)$. Taking $v = u_j$ in (2.2.6), dividing by 2^*, and subtracting from (2.2.5) gives

$$K\bigg(\int_\Omega |\nabla u_j|^2\, dx\bigg) - \int_\Omega \bigg[F(x,u_j) - \frac{1}{2^*}\, u_j\, f(x,u_j)\bigg]\, dx = c + \mathrm{o}(1 + \|u_j\|),$$

which together with (2.2.4) implies that (u_j) is bounded in $H^1_0(\Omega)$. So a renamed subsequence of (u_j) converges to some u weakly in $H^1_0(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω. For a further subsequence, $\|u_j - u\|^2$ converges to some $t \ge 0$. We will show that $t = 0$.

Suppose $t \neq 0$. Then $t \in I$ by Proposition 2.1.2 and hence

$$K(t) \ge c^* \tag{2.2.7}$$

by (2.2.1). As in the proof of Proposition 2.1.2,

$$\int_\Omega |\nabla u_j|^2\, dx \to \int_\Omega |\nabla u|^2\, dx + t =: s$$

and

$$sh(s) - \int_\Omega uf(x,u)\,dx - \int_\Omega |u_j|^{2^*}\,dx = \mathrm{o}(1). \tag{2.2.8}$$

Moreover, passing to the limit in (2.2.5) gives

$$\frac{1}{2}\,H(s) - \int_\Omega F(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u_j|^{2^*}\,dx = c + \mathrm{o}(1),$$

and combining this with (2.2.8) gives

$$c = K(s) - \int_\Omega \left[F(x,u) - \frac{1}{2^*}\,uf(x,u)\right] dx.$$

Since

$$K(s) \ge K(t) + K\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)$$

by (A_2) and

$$K\bigg(\int_\Omega |\nabla u|^2\,dx\bigg) - \int_\Omega \left[F(x,u) - \frac{1}{2^*}\,uf(x,u)\right] dx \ge 0$$

by (2.2.4), then

$$c \ge K(t).$$

This together with (2.2.7) gives $c \ge c^*$, contrary to assumption. So $t = 0$.

If (A_3) holds, then the equation $h(t)/t^{2^*/2-1} = S^{-2^*/2}$ has a unique positive solution t_0 and $I = [t_0, \infty)$. Since K is nondecreasing, then $c^* = K(t_0)$, in particular, (A_1) implies that $c^* > 0$. □

Next we consider the case where I is empty. We assume that

(A_4) h satisfies one of the following conditions:

- (i) for some constants $\eta > 0$ and $p/2 < \gamma < 2^*/2$,

$$h(t) \ge S^{-2^*/2}\,t^{2^*/2-1} + \eta\,t^{\gamma-1} \quad \forall t \ge 0;$$

- (ii) for some constant $b > S^{-2^*/2}$,

$$h(t) \ge bt^{2^*/2-1} \quad \forall t \ge 0;$$

- (iii) for some constants $b > 0$ and $\gamma > 2^*/2$,

$$h(t) > \max\left\{S^{-2^*/2}\,t^{2^*/2-1}, bt^{\gamma-1}\right\} \quad \forall t > 0.$$

We have the following theorem.

Theorem 2.2.2. *Assume that* (1.2.2) *and* (A_4) *hold. Then* J *is bounded from below, coercive, and satisfies the* $(\mathrm{PS})_c$ *condition for all* $c \in \mathbb{R}$. *In particular,* J *has a global minimizer.*

Proof. By (1.2.2) and (2.1.1),

$$J(u) \geq \frac{1}{2}\, H\bigg(\int_\Omega |\nabla u|^2\, dx\bigg) - c_3 \int_\Omega |u|^p\, dx - c_4 - \frac{1}{2^*}\, S^{-2^*/2} \left(\int_\Omega |\nabla u|^2\, dx\right)^{2^*/2}$$

for some constants $c_3, c_4 > 0$. By (A_4), H satisfies one of the following:

(i) $H(t) \geq \dfrac{2}{2^*}\, S^{-2^*/2}\, t^{2^*/2} + \dfrac{\eta}{\gamma}\, t^\gamma$ for all $t \geq 0$, where $\eta > 0$ and $p/2 < \gamma < 2^*/2$;

(ii) $H(t) \geq \dfrac{2b}{2^*}\, t^{2^*/2}$ for all $t \geq 0$, where $b > S^{-2^*/2}$;

(iii) $H(t) \geq \dfrac{b}{\gamma}\, t^\gamma$ for all $t \geq 0$, where $b > 0$ and $\gamma > 2^*/2$.

It follows that J is bounded from below and coercive.

Let $c \in \mathbb{R}$ and let (u_j) be a $(\mathrm{PS})_c$ sequence. By coercivity, (u_j) is bounded. By (A_4), $h(t) > S^{-2^*/2}\, t^{2^*/2-1}$ for all $t > 0$, so $I = \emptyset$. So (u_j) has a strongly convergent subsequence by Proposition 2.1.2. □

2.3 Some Corollaries and Remarks

Finally we apply our results to the model problem

$$\begin{cases} -\left[a + b\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1}\right] \Delta u = f(x,u) + |u|^{2^*-2}\, u & \text{in } \Omega \\[2ex] u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.1}$$

where $a, b \geq 0$ with $a + b > 0$ and $1 < \gamma < +\infty$. Here

$$h(t) = a + bt^{\gamma-1}, \qquad H(t) = at + \frac{b}{\gamma}\, t^\gamma, \qquad K(t) = \frac{1}{N}\, at + \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) bt^\gamma$$

and

$$J(u) = \frac{a}{2} \int_\Omega |\nabla u|^2\, dx + \frac{b}{2\gamma} \left(\int_\Omega |\nabla u|^2\, dx\right)^\gamma - \int_\Omega F(x,u)\, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx, \quad u \in H^1_0(\Omega).$$

Theorem 2.2.1 has the following corollary for the case $\gamma < 2^*/2$.

Corollary 2.3.1. *Let* $1 < \gamma < 2^*/2$ *and* $a, b \geq 0$. *Assume that* f *satisfies* (1.2.2) *and*

$$F(x,t) - \frac{1}{2^*}\, tf(x,t) \leq \frac{1}{N}\, \lambda a t^2 + \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) \mu b\, |t|^{2\gamma} \quad \textit{for a.a. } x \in \Omega \textit{ and all } t \in \mathbb{R}$$

for some constants $\lambda \le \lambda_1$ *and* $\mu \le \lambda_1(\gamma)$ *with either* $a > 0$ *and* $\lambda < \lambda_1$*, or* $b > 0$ *and* $\mu < \lambda_1(\gamma)$*. Let* t_0 *be the unique positive solution of the equation*

$$a + bt^{\gamma-1} = S^{-2^*/2}\, t^{2^*/2-1}$$

and set

$$c^* = \frac{1}{N}\, at_0 + \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) bt_0^\gamma.$$

Then J *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c < c^*$*.*

Remark 2.3.2. For $a = 1$ and $b = 0$, Corollary 2.3.1 gives the well-known compactness threshold

$$c^* = \frac{1}{N}\, S^{N/2}$$

in the Brézis-Nirenberg problem (see [3]).

Remark 2.3.3. An interesting special case of problem (2.3.1) is

$$\begin{cases} -\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1} \Delta u = \mu\, |u|^{2\gamma-2}\, u + |u|^{2^*-2}\, u & \text{in } \Omega\\[10pt] u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\gamma > 1$ and $\mu > 0$. For $\gamma < 2^*/2$ and $\mu < \lambda_1(\gamma)$, Corollary 2.3.1 gives the compactness threshold

$$c^* = \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) S^{2^*\gamma/(2^*-2\gamma)}$$

for the associated variational functional

$$J(u) = \frac{1}{2\gamma}\left(\int_\Omega |\nabla u|^2\, dx\right)^\gamma - \frac{\mu}{2\gamma}\int_\Omega |u|^{2\gamma}\, dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \quad u \in H^1_0(\Omega).$$

Remark 2.3.4. The classical case $h(t) = a + bt$ when $N = 3$ is one of the few cases with both a and b positive and $\gamma < 2^*/2$ where c^* can be found in closed form. Here Corollary 2.3.1 gives

$$c^* = \frac{1}{4}\, abS^3 + \frac{1}{24}\, b^3 S^6 + \frac{1}{24}\left(4aS + b^2 S^4\right)^{3/2}.$$

This threshold level was also obtained in Naimen [10, Lemma 2.5] using concentration compactness arguments. Our approach here is simpler.

Theorem 2.2.1 also has the following corollary for the case $\gamma = 2^*/2$.

Corollary 2.3.5. *Let* $\gamma = 2^*/2$*,* $a > 0$*, and* $0 < b < S^{-2^*/2}$*. Assume that* f *satisfies* (1.2.2) *and*

$$F(x,t) - \frac{1}{2^*}\, tf(x,t) \le \frac{1}{N}\, \lambda a t^2 \quad \textit{for a.a. } x \in \Omega \textit{ and all } t \in \mathbb{R}$$

for some constant $\lambda < \lambda_1$. Set

$$c^* = \frac{1}{N} \left(\frac{a^{2^*/2}}{S^{-2^*/2} - b} \right)^{2/(2^*-2)}.$$

Then J satisfies the $(\mathrm{PS})_c$ *condition for all $c < c^*$.*

Remark 2.3.6. For the classical case $h(t) = a + bt$ with $N = 4$, $a > 0$, and $0 < b < S^{-2}$, Corollary 2.3.5 gives

$$c^* = \frac{a^2}{4\,(S^{-2} - b)}.$$

This threshold level was also obtained in Naimen [9, Lemma 2.1] by analyzing the behavior of Palais-Smale sequences. Our approach here is simpler again.

Theorem 2.2.2 has the following corollary for $\gamma \geq 2^*/2$.

Corollary 2.3.7. *Assume that f satisfies* (1.2.2). *Then J is bounded from below, satisfies the* $(\mathrm{PS})_c$ *condition for all $c \in \mathbb{R}$, and has a global minimizer in each of the following cases:*

(i) $\gamma = 2^*/2$, $a \geq 0$, *and* $b > S^{-2^*/2}$;

(ii) $\gamma > 2^*/2$ *and*

$$a^{\gamma - 2^*/2}\, b^{2^*/2-1} > \frac{(\gamma - 2^*/2)^{\gamma - 2^*/2}\, (2^*/2 - 1)^{2^*/2-1}}{(\gamma - 1)^{\gamma - 1}}\, S^{-(2^*/2)(\gamma-1)}.$$

Proof. The minimum of $a + bt^{\gamma-1} - S^{-2^*/2}\, t^{2^*/2-1}$, $t > 0$ is positive if and only if the last inequality holds (for details see [A.1]). □

Remark 2.3.8. For the classical case $h(t) = a + bt$ with $N = 4$, $a \geq 0$, and $b > S^{-2}$, Corollary 2.3.7 (i) implies that J satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. This was also observed in Liao et al. [8, Proposition 2.1].

Remark 2.3.9. For the classical case $h(t) = a + bt$ with $N \geq 5$, Corollary 2.3.7 (ii) implies that J satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$ if

$$a^{N-4}\, b^2 > \frac{4\,(N-4)^{N-4}}{(N-2)^{N-2}}\, S^{-N}.$$

Chapter 3

Existence and Multiplicity Results

In this chapter we state and prove our existence results in the general form of the nonlocal term h by considering separately the two cases when the set I is nonempty or empty. For the case where I is nonempty we prove existence of weak solutions to our problem by using the mountain pass theorem (see Ambrosetti and Rabinowitz [1]). In the case where I is empty we show existence of at least two nontrivial solutions by using the result of Brézis and Nirenberg [4, Theorem 4] (see Proposition 3.3.3).

3.1 Main Result for I nonempty

In the case where I is nonempty, our main existence result for problem (1.2.1) is the following theorem.

Theorem 3.1.1. *Assume* (1.2.2) *and* (A_1)–(A_3)*. Assume further that*

$$H(t) \geq b_0 t^{\gamma_0} \quad \text{for } 0 \leq t \leq \delta \tag{3.1.1}$$

for some constants $\delta, b_0 > 0$ *and* $1 \leq \gamma_0 < 2^*/2$,

$$F(x,t) \leq \frac{1}{2}\, \mu_0 b_0\, |t|^{2\gamma_0} \quad \text{for a.a. } x \in \Omega \text{ and } |t| \leq \delta \tag{3.1.2}$$

for some $\mu_0 < \lambda_1(\gamma_0)$*, and*

$$F(x,t) \geq \frac{1}{q}\, \nu t^q \quad \text{for a.a. } x \in B_r(x_0) \text{ and all } t \geq 0 \tag{3.1.3}$$

for some ball $B_r(x_0) \subset \Omega$, $\nu > 0$*, and* $2\gamma_0 \leq q \leq 2\gamma_n$*. Then problem* (1.2.1) *has a nontrivial solution in each of the following cases:*

(*i*) $N = 3$ *and* $q > 4$,

(*ii*) $N \geq 4$ *and* $q \geq N/(N-2)$.

We will show that the functional J has the mountain pass geometry and the mountain pass level is below the compactness threshold c^* in (2.2.3).

Lemma 3.1.2. *If* (1.2.2), (3.1.1), *and* (3.1.2) *hold, then* $\exists \rho > 0$ *such that*

$$\inf_{\|u\|=\rho} J(u) > 0. \tag{3.1.4}$$

Proof. By (1.2.2) and (3.1.2),

$$F(x,t) \leq \frac{1}{2}\,\mu_0 b_0\,|t|^{2\gamma_0} + c_5|t|^{2^*} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}$$

for some constant $c_5 > 0$. This together with (3.1.1) implies that for $\|u\| \leq \sqrt{\delta}$,

$$\begin{aligned} J(u) \geq \frac{1}{2}\,b_0\left[\left(\int_\Omega |\nabla u|^2\,dx\right)^{\gamma_0} - \mu_0 \int_\Omega |u|^{2\gamma_0}\,dx\right] - \left(c_5 + \frac{1}{2^*}\right)\int_\Omega |u|^{2^*} dx \\ \geq \frac{1}{2}\,b_0\left(1 - \frac{\mu_0}{\lambda_1(\gamma_0)} + \mathrm{o}(1)\right)\|u\|^{2\gamma_0} \quad \text{as } \|u\| \to 0 \end{aligned}$$

since $2^* > 2\gamma_0$. Since $\mu_0 < \lambda_1(\gamma_0)$, the desired conclusion follows from this. □

Next we show that for a suitably chosen $v \in H^1_0(\Omega) \setminus \{0\}$, $J(sv) \to -\infty$ as $s \to +\infty$ and the maximum of J on the ray sv, $s \geq 0$ is strictly less than c^*. Take a function $\psi \in C^\infty_0(B_r(x_0))$ such that $0 \leq \psi \leq 1$ on $B_r(x_0)$ and $\psi = 1$ on $B_{r/2}(x_0)$, and set

$$u_\varepsilon(x) = \frac{\psi(x)}{\left(\varepsilon + |x - x_0|^2\right)^{(N-2)/2}}$$

and

$$v_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|_{2^*}}$$

for $\varepsilon > 0$. Then

$$\int_\Omega |\nabla v_\varepsilon|^2\,dx = S + \mathrm{O}\big(\varepsilon^{(N-2)/2}\big) \tag{3.1.5}$$

and

$$\int_\Omega v_\varepsilon^q\,dx = \begin{cases} \kappa\varepsilon^{(2N-(N-2)q)/4} + \mathrm{O}\big(\varepsilon^{(N-2)q/4}\big) & \text{if } q > N/(N-2) \\ \kappa\varepsilon^{N/4}\,|\log\varepsilon| + \mathrm{O}\big(\varepsilon^{N/4}\big) & \text{if } q = N/(N-2) \end{cases} \tag{3.1.6}$$

for some constant $\kappa > 0$ (see, e.g., Drábek and Huang [5]).

Lemma 3.1.3. *For all sufficiently small* $\varepsilon > 0$,

$$J(sv_\varepsilon) \to -\infty \quad \textit{as } s \to +\infty. \tag{3.1.7}$$

Proof. Since $|v_\varepsilon|_{2^*} = 1$, $|v_\varepsilon|_p$ is bounded and (1.2.2) gives

$$J(sv_\varepsilon) \le \frac{1}{2}\, H\bigg(s^2 \int_\Omega |\nabla v_\varepsilon|^2\, dx\bigg) + c_6 s^p + c_7 - \frac{s^{2^*}}{2^*}, \quad s \ge 0 \tag{3.1.8}$$

for some constants $c_6, c_7 > 0$. Set

$$t = s^2 \int_\Omega |\nabla v_\varepsilon|^2\, dx.$$

Then $t \to +\infty$ as $s \to +\infty$ and (3.1.8) gives

$$J(sv_\varepsilon) \le \frac{1}{2}\, H(t) + c_6 t^{p/2} \left(\int_\Omega |\nabla v_\varepsilon|^2\, dx\right)^{-p/2} + c_7 - \frac{t^{2^*/2}}{2^*} \left(\int_\Omega |\nabla v_\varepsilon|^2\, dx\right)^{-2^*/2}. \tag{3.1.9}$$

By (A_3),

$$\lim_{t \to +\infty} \frac{H(t)}{t^{2^*/2}} = \frac{2b}{2^*},$$

so (3.1.9) gives

$$J(sv_\varepsilon) \le c_6 t^{p/2} \left(\int_\Omega |\nabla v_\varepsilon|^2\, dx\right)^{-p/2} + c_7 - \frac{t^{2^*/2}}{2^*} \left[\left(\int_\Omega |\nabla v_\varepsilon|^2\, dx\right)^{-2^*/2} - b + \mathrm{o}(1)\right]$$

as $t \to +\infty$. Since $\int_\Omega |\nabla v_\varepsilon|^2\, dx \to S$ as $\varepsilon \to 0$ by (3.1.5), $b < S^{-2^*/2}$, and $p < 2^*$, the desired conclusion follows. □

Lemma 3.1.4. *In each of the two cases in Theorem 3.1.1,*

$$\max_{s \ge 0} J(sv_\varepsilon) < c^* \tag{3.1.10}$$

for all sufficiently small $\varepsilon > 0$.

Proof. Since $v_\varepsilon = 0$ outside $B_r(x_0)$, (3.1.3) gives

$$J(sv_\varepsilon) \le \frac{1}{2}\, H\bigg(s^2 \int_\Omega |\nabla v_\varepsilon|^2\, dx\bigg) - \frac{1}{q}\, \nu s^q \int_\Omega v_\varepsilon^q\, dx - \frac{s^{2^*}}{2^*} =: z_\varepsilon(s),$$

so it suffices to show that

$$\max_{s \ge 0} z_\varepsilon(s) < c^*$$

for sufficiently small $\varepsilon > 0$. Suppose this is false. Then there are sequences (ε_j) and (s_j), with $\varepsilon_j, s_j > 0$ and $\varepsilon_j \to 0$, such that

$$z_{\varepsilon_j}(s_j) = \frac{1}{2}\, H\bigg(s_j^2 \int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx\bigg) - \frac{1}{q}\, \nu s_j^q \int_\Omega v_{\varepsilon_j}^q\, dx - \frac{s_j^{2^*}}{2^*} \ge c^* \tag{3.1.11}$$

and

$$s_j\, z_{\varepsilon_j}'(s_j) = h\bigg(s_j^2 \int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx\bigg)\, s_j^2 \int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx - \nu s_j^q \int_\Omega v_{\varepsilon_j}^q\, dx - s_j^{2^\ast} = 0. \qquad (3.1.12)$$

Set

$$t_j = s_j^2 \int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx.$$

Then (3.1.12) gives

$$\frac{h(t_j)}{t_j^{2^\ast/2-1}} = \frac{1}{\bigg(\displaystyle\int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx\bigg)^{2^\ast/2}} + \nu t_j^{-(2^\ast-q)/2}\, \frac{\displaystyle\int_\Omega v_{\varepsilon_j}^q\, dx}{\bigg(\displaystyle\int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx\bigg)^{q/2}}. \qquad (3.1.13)$$

If $t_j \to +\infty$ for a renamed subsequence, then the left-hand side goes to b by (A_3), while the right-hand side goes to $S^{-2^\ast/2}$ since $\int_\Omega |\nabla v_{\varepsilon_j}|^2\, dx \to S$ by (3.1.5) and $\int_\Omega v_{\varepsilon_j}^q\, dx \to 0$ by (3.1.6), contradicting our assumption that $b < S^{-2^\ast/2}$. So (t_j) is bounded, and hence converges to some $t \ge 0$ for a renamed subsequence. Then $s_j^2 \to S^{-1}\, t$ and hence passing to the limit in (3.1.11) gives

$$\frac{1}{2}\, H(t) - \frac{1}{2^\ast}\, S^{-2^\ast/2}\, t^{2^\ast/2} > 0$$

since $c^\ast > 0$, so $t > 0$. On the other hand, passing to the limit in (3.1.12) shows that t satisfies (2.2.2). Since t_0 is the unique positive solution of this equation, it follows that $t = t_0$.

Now combining (3.1.13) with (3.1.5) and (3.1.6) gives

$$\frac{h(t_j)}{t_j^{2^\ast/2-1}} = S^{-2^\ast/2} + \begin{cases} \sigma_j\, \varepsilon_j^{(2N-(N-2)q)/4} + \mathrm{O}\big(\varepsilon_j^{(N-2)/2}\big) & \text{if } q > N/(N-2) \\ \sigma_j\, \varepsilon_j^{N/4}\, |\log \varepsilon_j| + \mathrm{O}\big(\varepsilon_j^{\min\{(N-2)/2,N/4\}}\big) & \text{if } q = N/(N-2), \end{cases}$$

where $\sigma_j \to \kappa \nu S^{-q/2}\, t_0^{-(2^\ast-q)/2} > 0$. It follows from this that in each of the two cases in the lemma,

$$\frac{h(t_j)}{t_j^{2^\ast/2-1}} \ge S^{-2^\ast/2} = \frac{h(t_0)}{t_0^{2^\ast/2-1}}$$

for all sufficiently large j. Then $t_j \le t_0$ by (A_3). Since K is nondecreasing, then

$$K(t_j) \le K(t_0) = c^\ast.$$

However, dividing (3.1.12) by $2^\ast$ and subtracting from (3.1.11) gives

$$K(t_j) - \bigg(\frac{1}{q} - \frac{1}{2^\ast}\bigg)\, \nu s_j^q \int_\Omega v_{\varepsilon_j}^q\, dx \ge c^\ast,$$

so $K(t_j) > c^\ast$. This contradiction completes the proof. □

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Let ρ be as in Lemma 3.1.2 and fix $\varepsilon > 0$ such that (3.1.7) and (3.1.10) hold. Then $\exists R > \rho$ such that $J(Rv_\varepsilon) \leq 0$. Let

$$\Gamma = \left\{\varphi \in C([0,1], H_0^1(\Omega)) : \varphi(0) = 0,\, \varphi(1) = Rv_\varepsilon\right\}$$

be the class of paths in $H_0^1(\Omega)$ joining the origin to Rv_ε, and set

$$c := \inf_{\varphi \in \Gamma} \max_{u \in \varphi([0,1])} J(u).$$

By (3.1.4), $c > 0$. Since the path $\varphi_0(s) = sRv_\varepsilon$, $s \in [0,1]$ is in Γ,

$$c \leq \max_{u \in \varphi_0([0,1])} J(u) \leq \max_{s \geq 0} J(sv_\varepsilon) < c^*,$$

so J satisfies the $(\mathrm{PS})_c$ condition by Theorem 2.2.1. Hence J has a critical point u with $J(u) = c$ by the mountain pass theorem (see Ambrosetti and Rabinowitz [1]). Then u is a weak solution of problem (1.2.1) and u is nontrivial since $c > 0$. □

3.2 Some Corollaries

Theorem 3.1.1 has many interesting consequences, some of which we now present. First we consider the problem

$$\begin{cases} -h\left(\displaystyle\int_\Omega |\nabla u|^2\, dx\right) \Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.1}$$

where $\lambda > 0$. Assume that

$$K(t) \geq \alpha t \quad \forall t \geq 0 \tag{3.2.2}$$

for some constant $\alpha > 0$ and

$$H(t) \geq a_0\, t \quad \text{for } 0 \leq t \leq \delta \tag{3.2.3}$$

for some constants $\delta, a_0 > 0$. We have $f(x,t) = \lambda t$ and

$$F(x,t) = \frac{1}{2}\, \lambda t^2, \qquad F(x,t) - \frac{1}{2^*}\, t f(x,t) = \frac{1}{N}\, \lambda t^2,$$

so (A_1) holds with $\mu_1 = \lambda / N\alpha$ if $\lambda < N\alpha\lambda_1$, (3.1.2) holds with $\gamma_0 = 1$ and $\mu_0 = \lambda / a_0$ if $\lambda < a_0 \lambda_1$, and (3.1.3) holds with $q = 2$ if $\lambda > 0$. So Theorem 3.1.1 has the following corollary for problem (3.2.1).

Corollary 3.2.1. *Assume* (3.2.2)*,* (A_2)*,* (A_3)*, and* (3.2.3)*. If*

$$0 < \lambda < \min\{a_0, N\alpha\}\, \lambda_1$$

and $N \geq 4$*, then problem* (3.2.1) *has a nontrivial solution.*

In particular, we have the following corollary in the model case $h(t) = a + bt^{\gamma-1}$.

Corollary 3.2.2. *The problem*

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1}\right] \Delta u = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where either $1 < \gamma < 2^*/2$ *and* $b \geq 0$*, or* $\gamma = 2^*/2$ *and* $0 \leq b < S^{-2^*/2}$*, has a nontrivial solution if* $0 < \lambda < a\lambda_1$ *and* $N \geq 4$*.*

Next we consider the problem

$$\begin{cases} -h\left(\int_\Omega |\nabla u|^2\, dx\right) \Delta u = \mu\, |u|^{2\gamma-2}\, u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.4}$$

where $\mu > 0$ and $1 < \gamma < 2^*/2$. Assume that

$$K(t) \geq \beta t^\gamma \quad \forall t \geq 0 \tag{3.2.5}$$

for some constant $\beta > 0$ and

$$H(t) \geq b_0 t^\gamma \quad \text{for } 0 \leq t \leq \delta \tag{3.2.6}$$

for some constants $\delta, b_0 > 0$. We have $f(x,t) = \mu\, |t|^{2\gamma-2}\, t$ and

$$F(x,t) = \frac{1}{2\gamma}\, \mu\, |t|^{2\gamma}, \qquad F(x,t) - \frac{1}{2^*}\, tf(x,t) = \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right) \mu\, |t|^{2\gamma},$$

so (A_1) holds with $\mu_1 = (1/2\gamma - 1/2^*)\, \mu/\beta$ if

$$\mu < \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right)^{-1} \beta\lambda_1(\gamma),$$

(3.1.2) holds with $\gamma_0 = \gamma$ and $\mu_0 = \mu/\gamma b_0$ if $\mu < \gamma b_0 \lambda_1(\gamma)$, and (3.1.3) holds with $q = 2\gamma$ if $\mu > 0$. So Theorem 3.1.1 has the following corollary for problem (3.2.4).

Corollary 3.2.3. *Assume* (3.2.5)*,* (A_2)*,* (A_3)*, and* (3.2.6)*. If*

$$0 < \mu < \min\left\{\gamma b_0, \left(\frac{1}{2\gamma} - \frac{1}{2^*}\right)^{-1} \beta\right\} \lambda_1(\gamma)$$

and $N \geq 4$*, or* $N = 3$ *and* $\gamma > 2$*, then problem* (3.2.4) *has a nontrivial solution.*

In particular, we have the following corollary in the model case $h(t) = a + bt^{\gamma-1}$.

Corollary 3.2.4. *The problem*

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1}\right] \Delta u = \mu\, |u|^{2\gamma-2}\, u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \geq 0$ and $1 < \gamma < 2^*/2$, has a nontrivial solution if $0 < \mu < b\lambda_1(\gamma)$ and $N \geq 4$, or $N = 3$ and $\gamma > 2$.

Finally we consider the problem

$$\begin{cases} -h\left(\int_\Omega |\nabla u|^2\, dx\right) \Delta u = \nu\, |u|^{q-2}\, u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.7}$$

where $\nu > 0$ and $2 < q < 2^*$. Assume that for some constants $\alpha, \beta > 0$ and $q/2 < \gamma < 2^*/2$,

$$K(t) \geq \alpha t + \beta t^\gamma \quad \forall t \geq 0. \tag{3.2.8}$$

Since h is nonnegative, $H(t) \geq 2K(t) \geq 2\alpha t$, so (3.1.1) holds with $b_0 = 2\alpha$ and $\gamma_0 = 1$. We have $f(x,t) = \nu\, |t|^{q-2}\, t$ and

$$F(x,t) = \frac{1}{q}\, \nu\, |t|^q, \qquad F(x,t) - \frac{1}{2^*}\, t f(x,t) = \left(\frac{1}{q} - \frac{1}{2^*}\right) \nu\, |t|^q.$$

Since $q > 2$, (3.1.2) holds for any $\mu_0 > 0$ if $\delta > 0$ is sufficiently small. Theorem 3.1.1 has the following corollary for problem (3.2.7).

Corollary 3.2.5. *Assume* (3.2.8)*,* (A_2)*, and* (A_3)*. If*

$$0 < \nu < (2\gamma - 2) \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1} \left(\frac{\alpha\lambda_1}{2\gamma - q}\right)^{(2\gamma-q)/(2\gamma-2)} \left(\frac{\beta\lambda_1(\gamma)}{q-2}\right)^{(q-2)/(2\gamma-2)}$$

and $N \geq 4$, or $N = 3$ and $q > 4$, then problem (3.2.7) *has a nontrivial solution.*

Proof. To see that (A_1) holds, note that the minimum of

$$\lambda\alpha t^2 + \mu\beta|t|^{2\gamma} - \left(\frac{1}{q} - \frac{1}{2^*}\right) \nu\, |t|^q, \quad t \in \mathbb{R}$$

is nonnegative if and only if

$$\nu \leq (2\gamma - 2) \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1} \left(\frac{\alpha\lambda}{2\gamma - q}\right)^{(2\gamma-q)/(2\gamma-2)} \left(\frac{\beta\mu}{q-2}\right)^{(q-2)/(2\gamma-2)}. \qquad \square$$

(for details see Appendix A)

3.3 Main Result for I empty

In the case where I is empty, first we consider the model problem

$$\begin{cases} -\left[a + b\left(\int_\Omega |\nabla u|^2\, dx\right)^{\gamma-1}\right] \Delta u = \lambda u + g(x,u) + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3.1}$$

where $a, b \geq 0$ and $2^*/2 \leq \gamma < +\infty$ satisfy one of the two conditions in Corollary 2.3.7, $\lambda > 0$, and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$g(x,t) = \mathrm{o}(t) \quad \text{as } t \to 0, \text{uniformly a.e. in } \Omega \tag{3.3.2}$$

and

$$|g(x,t)| \leq c_8 |t|^{p-1} + c_9 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{3.3.3}$$

for some constants $c_8, c_9 > 0$ and $2 < p < 2^*$. The associated variational functional is

$$J(u) = \frac{a}{2} \int_\Omega |\nabla u|^2\, dx + \frac{b}{2\gamma} \left(\int_\Omega |\nabla u|^2\, dx\right)^{\gamma} - \frac{\lambda}{2} \int_\Omega u^2\, dx - \int_\Omega G(x,u)\, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} dx,$$

$$u \in H^1_0(\Omega),$$

where $G(x,t) = \int_0^t g(x,s)\, ds$ is the primitive of g. We note that

$$\int_\Omega G(x,u)\, dx = \mathrm{o}(\|u\|^2) \quad \text{as } \|u\| \to 0 \tag{3.3.4}$$

by (3.3.2) and (3.3.3).

When $a = 0$, we have the following existence result.

Theorem 3.3.1. *Assume that g satisfies* (3.3.2) *and* (3.3.3). *If $\gamma = 2^*/2$, $a = 0$, and $b > S^{-2^*/2}$, then problem* (3.3.1) *has a nontrivial solution for all $\lambda > 0$.*

Proof. By Corollary 2.3.7, J has a global minimizer u_0. For any $u \in H^1_0(\Omega) \setminus \{0\}$,

$$J(su) = -\frac{\lambda s^2}{2} \int_\Omega u^2\, dx + \mathrm{o}(s^2) \quad \text{as } s \to 0$$

by (3.3.4), so $J(su) < 0$ if $s > 0$ is sufficiently small. So $J(u_0) = \inf_{H^1_0(\Omega)} J < 0$ and hence u_0 is nontrivial. □

When $a > 0$, we prove a multiplicity result. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the Dirichlet eigenvalues of $-\Delta$ on Ω, repeated according to multiplicity.

Theorem 3.3.2. *Assume that g satisfies* (3.3.2) *and* (3.3.3).

(*i*) *If* $\gamma = 2^*/2$, $a > 0$, *and* $b > S^{-2^*/2}$, *then problem* (3.3.1) *has at least two nontrivial solutions in each of the following cases:*

(*a*) $a\lambda_k < \lambda < a\lambda_{k+1}$ *for some* $k \geq 1$;

(*b*) $a\lambda_k < \lambda = a\lambda_{k+1}$ *for some* $k \geq 1$ *and* $G(x,t) \leq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$ *for some* $\delta > 0$.

(*ii*) *If* $\gamma > 2^*/2$ *and*

$$a^{\gamma - 2^*/2}\, b^{2^*/2-1} > \frac{(\gamma - 2^*/2)^{\gamma-2^*/2}\, (2^*/2-1)^{2^*/2-1}}{(\gamma-1)^{\gamma-1}}\, S^{-(2^*/2)(\gamma-1)},$$

then problem (3.3.1) *has at least two nontrivial solutions in each of the following cases:*

(*a*) $a\lambda_k < \lambda < a\lambda_{k+1}$ *for some* $k \geq 1$;

(*b*) $a\lambda_k = \lambda < a\lambda_{k+1}$ *for some* $k \geq 1$ *and* $G(x,t) \geq 0$ *for a.a.* $x \in \Omega$ *and* $|t| \leq \delta$ *for some* $\delta > 0$.

We will prove this theorem using the following result of Brezis and Nirenberg [4, Theorem 4].

Proposition 3.3.3. *Let* J *be a* C^1*-functional on a Banach space* X. *Assume that* J *is bounded from below,* $\inf_X J < 0$, *and* J *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c \in \mathbb{R}$. *Assume further that* X *has a direct sum decomposition* $X = V \oplus W$, $u = v + w$ *with* $\dim V < +\infty$ *and*

$$\begin{cases} J(v) \leq 0 & \text{for } v \in V \cap B_r(0) \\ J(w) \geq 0 & \text{for } w \in W \cap B_r(0) \end{cases}$$

for some $r > 0$. *Then* J *has at least two nontrivial critical points.*

Proof of Theorem 3.3.2. By Corollary 2.3.7, J is bounded from below and satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. We have the direct sum decomposition $H^1_0(\Omega) = V \oplus W$, $u = v + w$, where V is the span of the eigenfunctions associated with $\lambda_1, \dots, \lambda_k$ and W is the orthogonal complement of V. For $v \in V$,

$$J(v) \leq -\frac{1}{2}\left(\frac{\lambda}{\lambda_k} - a\right)\int_\Omega |\nabla v|^2\,dx + \frac{b}{2\gamma}\left(\int_\Omega |\nabla v|^2\,dx\right)^\gamma - \int_\Omega G(x,v)\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,dx$$

$$= -\frac{1}{2}\left(\frac{\lambda}{\lambda_k} - a\right)\int_\Omega |\nabla v|^2\,dx + \mathrm{o}(\|v\|^2) \quad \text{as } \|v\| \to 0 \quad (3.3.5)$$

by (3.3.4), so $J(v) < 0$ if $\lambda > a\lambda_k$ and $\|v\| > 0$ is sufficiently small. For $w \in W$,

$$J(w) \ge \frac{1}{2}\left(a - \frac{\lambda}{\lambda_{k+1}}\right)\int_\Omega |\nabla w|^2\,dx + \frac{b}{2\gamma}\left(\int_\Omega |\nabla w|^2\,dx\right)^\gamma - \int_\Omega G(x,w)\,dx - \frac{1}{2^*}\int_\Omega |w|^{2^*}dx$$

$$= \frac{1}{2}\left(a - \frac{\lambda}{\lambda_{k+1}}\right)\int_\Omega |\nabla w|^2\,dx + \mathrm{o}(\|w\|^2) \quad \text{as } \|w\| \to 0$$

by (3.3.4), so $J(w) \ge 0$ if $\lambda < a\lambda_{k+1}$ and $\|w\|$ is sufficiently small. So J has at least two nontrivial critical points by Proposition 3.3.3 in the cases $(i)(a)$ and $(ii)(a)$.

In the case $(i)(b)$, (2.1.1) gives

$$J(w) \ge \int_\Omega \left[\frac{a}{2}\left(|\nabla w|^2 - \lambda_{k+1}\,w^2\right) - G(x,w)\right]dx + \frac{b - S^{-2^*/2}}{2^*}\left(\int_\Omega |\nabla w|^2\,dx\right)^{2^*/2} \qquad \forall w \in W.$$

The local sign condition on G in this case implies that the first integral on the right-hand side is nonnegative if $\|w\|$ is sufficiently small (see Li and Willem [7]). Since $b > S^{-2^*/2}$, then $J(w) \ge 0$ when $\|w\|$ is small. In the case $(ii)(b)$, (3.3.5) gives

$$J(v) \le \frac{b}{2\gamma}\left(\int_\Omega |\nabla v|^2\,dx\right)^\gamma - \int_{\{|v|>\delta\}} G(x,v)\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}dx \quad \forall v \in V.$$

Since V is a finite dimensional subspace of $H^1_0(\Omega)$ consisting of L^∞-functions and $\gamma > 2^*/2$, it follows from this that $J(v) < 0$ if $\|v\| > 0$ is sufficiently small. So J has two nontrivial critical points in these cases also. □

In the borderline case where $\gamma = 2^*/2$ and $b = S^{-2^*/2}$, lower-order terms come into play. We consider the problem

$$\begin{cases} -h\left(\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda u + g(x,u) + |u|^{2^*-2}\,u & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3.6}$$

where

$$h(t) = a + S^{-2^*/2}\,t^{2^*/2-1} + \eta\,t^{\sigma-1}, \quad t \ge 0,$$

$a \ge 0$, $\eta > 0$, $p/2 < \sigma < 2^*/2$, $\lambda > 0$, and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying (3.3.2) and (3.3.3). The associated functional is

$$J(u) = \frac{a}{2}\int_\Omega |\nabla u|^2\,dx + \frac{S^{-2^*/2}}{2^*}\left(\int_\Omega |\nabla u|^2\,dx\right)^{2^*/2} + \frac{\eta}{2\sigma}\left(\int_\Omega |\nabla u|^2\,dx\right)^\sigma$$

$$- \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega G(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx, \quad u \in H^1_0(\Omega),$$

where $G(x,t) = \int_0^t g(x,s)\,ds$ satisfies (3.3.4) as before. We have the following existence and multiplicity result.

Theorem 3.3.4. *Let $\eta > 0$ and $p/2 < \sigma < 2^*/2$, and assume that g satisfies* (3.3.2) *and* (3.3.3)*.*

(*i*) *If $a = 0$, then problem* (3.3.6) *has a nontrivial solution for all $\lambda > 0$.*

(*ii*) *If $a > 0$, then problem* (3.3.6) *has at least two nontrivial solutions in each of the following cases:*

(*a*) *$a\lambda_k < \lambda < a\lambda_{k+1}$ for some $k \geq 1$;*

(*b*) *$a\lambda_k < \lambda = a\lambda_{k+1}$ for some $k \geq 1$ and $G(x,t) \leq 0$ for a.a. $x \in \Omega$ and $|t| \leq \delta$ for some $\delta > 0$.*

Proof. (*i*) By Theorem 2.2.2, J has a global minimizer u_0. For any $u \in H^1_0(\Omega) \setminus \{0\}$,

$$J(su) = -\frac{\lambda s^2}{2} \int_\Omega u^2\, dx + \mathrm{o}(s^2) \quad \text{as } s \to 0$$

by (3.3.4), so $J(su) < 0$ if $s > 0$ is sufficiently small. So $J(u_0) = \inf_{H^1_0(\Omega)} J < 0$ and hence u_0 is nontrivial.

(*ii*) By Theorem 2.2.2, J is bounded from below and satisfies the $(\mathrm{PS})_c$ condition for all $c \in \mathbb{R}$. We have the direct sum decomposition $H^1_0(\Omega) = V \oplus W$, $u = v + w$, where V is the span of the eigenfunctions associated with $\lambda_1, \dots, \lambda_k$ and W is the orthogonal complement of V. For $v \in V$,

$$\begin{aligned} J(v) &\leq -\frac{1}{2}\left(\frac{\lambda}{\lambda_k} - a\right) \int_\Omega |\nabla v|^2\, dx + \frac{S^{-2^*/2}}{2^*} \left(\int_\Omega |\nabla v|^2\, dx\right)^{2^*/2} + \frac{\eta}{2\sigma} \left(\int_\Omega |\nabla v|^2\, dx\right)^\sigma \\ &- \int_\Omega G(x,v)\, dx - \frac{1}{2^*} \int_\Omega |v|^{2^*} dx = -\frac{1}{2}\left(\frac{\lambda}{\lambda_k} - a\right) \int_\Omega |\nabla v|^2\, dx + \mathrm{o}(\|v\|^2) \quad \text{as } \|v\| \to 0 \end{aligned}$$

by (3.3.4), so $J(v) < 0$ if $\lambda > a\lambda_k$ and $\|v\| > 0$ is sufficiently small. For $w \in W$,

$$\begin{aligned} J(w) &\geq \frac{1}{2}\left(a - \frac{\lambda}{\lambda_{k+1}}\right) \int_\Omega |\nabla w|^2\, dx + \frac{S^{-2^*/2}}{2^*} \left(\int_\Omega |\nabla w|^2\, dx\right)^{2^*/2} + \frac{\eta}{2\sigma} \left(\int_\Omega |\nabla w|^2\, dx\right)^\sigma \\ &- \int_\Omega G(x,w)\, dx - \frac{1}{2^*} \int_\Omega |w|^{2^*} dx = \frac{1}{2}\left(a - \frac{\lambda}{\lambda_{k+1}}\right) \int_\Omega |\nabla w|^2\, dx + \mathrm{o}(\|w\|^2) \quad \text{as } \|w\| \to 0 \end{aligned}$$

by (3.3.4), so $J(w) \geq 0$ if $\lambda < a\lambda_{k+1}$ and $\|w\|$ is sufficiently small. So J has at least two nontrivial critical points by Proposition 3.3.3 in the case (*a*). In the case (*b*), (2.1.1) gives

$$J(w) \geq \int_\Omega \left[\frac{a}{2}\left(|\nabla w|^2 - \lambda_{k+1}\, w^2\right) - G(x,w)\right] dx \quad \forall w \in W.$$

The local sign condition on G implies that the right-hand side is nonnegative when $\|w\|$ is small (see Li and Willem [7]). So J has two nontrivial critical points in this case also. □

Chapter 4

Future Problems

In this chapter we present a brief description of problems we wish to work on in the future. We start with the corresponding $p-$Laplacian problem for the generalized Kirchhoff equation with critical exponents. Next, we consider the corresponding fractional $p-$Laplacian problem.

4.1 $p-$Laplacian and fractioanl $p-$Laplacian

Define the p-Laplace operator as follows

$$\Delta_p u := \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right),$$

then consider the problem

$$\begin{cases} h\left(\displaystyle\int_\Omega |\nabla u|^p \, dx \right) (-\Delta)_p u = f(x,u) + |u|^{p^*-2}\, u & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.1}$$

where $1 < p < N$ Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $h : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function, $p^* = pN/(N-p)$ is the critical Sobolev exponent, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition

$$|f(x,t)| \leq a_1 |t|^{q-1} + a_2 \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R} \tag{4.1.2}$$

for some constants $a_1, a_2 > 0$ and $1 < q < p^*$.

The corresponding fractional case is given by

$$\begin{cases} h\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, dxdy \right) (-\Delta)_p^s u = f(x,u) + |u|^{p_s^*-2}\, u & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1.3}$$

where $s \in (0,1)$ and $p_s^* = Np/(N-sp)$ denotes the fractional critical Sobolev exponent and the fractional p-Laplace operator is defined as

$$(-\Delta)_p^s u(x) = 2\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N/B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

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Appendix A

Computations

A.1 Computations for Corollary 2.3.7 *(ii)*

We want to show that the minimum of $f(t) := a + bt^{\gamma-1} - S^{-2^*/2}\, t^{2^*/2-1}$, $t > 0$ is positive iff

$$a^{\gamma-2^*/2}\, b^{2^*/2-1} > \frac{(\gamma - 2^*/2)^{\gamma-2^*/2}\, (2^*/2-1)^{2^*/2-1}}{(\gamma-1)^{\gamma-1}}\, S^{-(2^*/2)(\gamma-1)}.$$

holds. This is a direct computation from explicitly finding a minimum for f and requiring that for such minimum value $f(t)$ stays positive at that point. We have

$$\begin{aligned} f'(t) &= 0 \\ b(\gamma-1)t^{\gamma-2} &= S^{-2^*/2}\,(2^*/2-1)\, t^{2^*/2-1} \end{aligned}$$

and since $\gamma > \frac{2^*}{2}$, we can solve for t and get

$$t = \left[\frac{S^{-2^*/2}\,(2^*/2-1)}{b(\gamma-1)}\right]^{1/(\gamma-2^*/2)} \tag{A.1.1}$$

In order to have $f(t) > 0$, this is equivalent to the following

$$\begin{aligned} a + bt^{\gamma-1} &> S^{-2^*/2}\, t^{2^*/2-1} \\ at^{-(2^*/2-1)} + bt^{\gamma-2^*/2} &> S^{-2^*/2} \end{aligned}$$

Plugging in (A.1.1) in the last expression we get

$$\begin{aligned} a\left[\frac{S^{-2^*/2}\,(2^*/2-1)}{b(\gamma-1)}\right]^{-\frac{(2^*/2-1)}{\gamma-2^*/2}} + b\left[\frac{S^{-2^*/2}\,(2^*/2-1)}{b(\gamma-1)}\right] &> S^{-2^*/2} \\ ab^{\frac{(2^*/2-1)}{\gamma-2^*/2}}\left[\frac{S^{-2^*/2}\,(2^*/2-1)}{(\gamma-1)}\right]^{-\frac{(2^*/2-1)}{\gamma-2^*/2}} &> S^{-2^*/2}\left(1 - \frac{2^*/2-1}{\gamma-1}\right) \end{aligned}$$

from which we get

$$
\begin{aligned}
ab^{\frac{(2^*/2-1)}{\gamma-2^*/2}} &> S^{-2^*/2}\left(\frac{\gamma-2^*/2}{\gamma-1}\right)\left[\frac{S^{-2^*/2}\left(2^*/2-1\right)}{(\gamma-1)}\right]^{\frac{2^*/2-1}{\gamma-2^*/2}} \\
a^{\gamma-2^*/2}b^{2^*/2-1} &> S^{-2^*/2(\gamma-2^*/2)}\left(\frac{\gamma-2^*/2}{\gamma-1}\right)^{\gamma-2^*/2}\left[\frac{S^{-2^*/2}\left(2^*/2-1\right)}{(\gamma-1)}\right]^{2^*/2-1} \\
&= \frac{\left(\gamma-2^*/2\right)^{\gamma-2^*/2}\left(2^*/2-1\right)^{2^*/2-1}}{(\gamma-1)^{\gamma-1}}S^{-(2^*/2)(\gamma-1)}
\end{aligned}
$$

To see that $f''(t)>0$, it is the same as showing that

$$
b(\gamma-1)(\gamma-2)t^{\gamma-2^*/2}-S^{-2^*/2}(2^*/2-1)(2^*/2-2)>0
$$

which holds from (A.1.1) and the fact that $\gamma>2^*/2$.

A.2 Computations for Corollary 3.2.5

To show that the minimum of

$$
\lambda\alpha t^2+\mu\beta|t|^{2\gamma}-\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu|t|^q, \quad t\in\mathbb{R}
$$

is nonnegative is the same as showing the minimum of

$$
f(t):=\lambda\alpha+\mu\beta|t|^{2\gamma-2}-\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu|t|^{q-2}, \quad t\in\mathbb{R}
$$

is nonnegative. Taking $f'(t)=0$, we get the following for t

$$
|t|=\left[\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu\frac{q-2}{\mu\beta(2\gamma-2)}\right]^{1/(2\gamma-q)}
$$

Plugging back in $f(t)$ and requiring $f(t)\geq 0$, we get

$$
\begin{aligned}
\lambda\alpha &\geq \left(\frac{1}{q}-\frac{1}{2^*}\right)\nu|t|^{q-2}-\mu\beta|t|^{2\gamma-2} \\
&= \left(\frac{1}{q}-\frac{1}{2^*}\right)\nu\left[\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu\frac{q-2}{\mu\beta(2\gamma-2)}\right]^{\frac{q-2}{2\gamma-q}}-\mu\beta\left[\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu\frac{q-2}{\mu\beta(2\gamma-2)}\right]^{\frac{2\gamma-2}{2\gamma-q}} \\
&= \left[\left(\frac{q-2}{\mu\beta(2\gamma-2)}\right)^{\frac{q-2}{2\gamma-q}}-\mu\beta\left(\frac{q-2}{\mu\beta(2\gamma-2)}\right)^{\frac{2\gamma-2}{2\gamma-q}}\right]\left[\left(\frac{1}{q}-\frac{1}{2^*}\right)\nu\right]^{\frac{2\gamma-2}{2\gamma-q}}
\end{aligned}
$$

That is

$$\lambda\alpha \geq \left[\left(\frac{q-2}{\mu\beta(2\gamma-2)} \right)^{\frac{q-2}{2\gamma-q}} \frac{2\gamma-q}{2\gamma-2} \right] \left[\left(\frac{1}{q} - \frac{1}{2^*} \right) \nu \right]^{\frac{2\gamma-2}{2\gamma-q}}$$

$$= \left(\frac{q-2}{\mu\beta} \right)^{\frac{q-2}{2\gamma-q}} (2\gamma-q)(2\gamma-2)^{\frac{2\gamma-q}{2\gamma-2}} \left[\left(\frac{1}{q} - \frac{1}{2^*} \right) \nu \right]^{\frac{2\gamma-2}{2\gamma-q}}$$

and so

$$(\lambda\alpha)^{\frac{2\gamma-q}{2\gamma-2}} \geq \left(\frac{q-2}{\mu\beta} \right)^{\frac{q-2}{2\gamma-2}} (2\gamma-q)^{\frac{2\gamma-q}{2\gamma-2}} (2\gamma-2)^{-1} \left(\frac{1}{q} - \frac{1}{2^*} \right) \nu$$

Bringing everything to the right hand side but ν, we get the desired result.