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Initial{Boundary and Nonlocal Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables

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Initial–Boundary and Nonlocal Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables

By

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"Initial–Boundary and Nonlocal Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables"
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ABSTRACT

Title: *Initial–Boundary and Nonlocal Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations with Two Independent Variables*

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Boundary value problems in a characteristic rectangle for nonlinear hyperbolic equations of higher order are considered. The concept of strong well–posedness of a boundary value problem is introduced.

For initial–boundary value problems there are established:

(i) Necessary and sufficient conditions of strong well–posedness;

(ii) Unimprovable sufficient conditions of local and global solvability;

(iii) Effective sufficient conditions of solvability of two–point, multi–point, periodic and Dirichlet type problems;

(iv) Sharp a priori estimates of solutions of ill–posed initial–boundary value problems;

(v) Unimprovable conditions guaranteeing unique solvability of ill–posed initial–boundary value problems.

For nonlocal boundary value problems there are established:

(i) Necessary and sufficient conditions for a linear problem to have the Fredholm property;

(ii) Necessary and sufficient conditions of strong well–posedness;

(iii) Optimal sufficient conditions of solvability and unique solvability;

(iv) Effective sufficient conditions of solvability of periodic and Dirichlet type problems in case, where the righthand side of the equation has arbitrary growth order with respect to some phase variables.

CONTENTS

List of Notations

$\mathbb{R} = (-\infty, +\infty)$; $\Omega = [0, a] \times [0, b]$; $\Omega_x = [0, x] \times [0, b]$;

$$u^{(j,k)} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}.$$

δ_{ij} is Kronecker delta.

$(j, k) < (m, n) \Leftrightarrow j \leq m,\ k \leq n$ and $(j, k) \neq (m, n)$.

If $\mathbf{v} \in \mathbb{R}^k$, then

$$\|\mathbf{v}\| = \sum_{i=1}^{k} |v_i|.$$

$C^k([0, a])$ is the Banach space of functions $u : [0, a] \to \mathbb{R}$, having continuous partial derivatives $u^{(i)}$ $(i = 0, \ldots, k)$, endowed with the norm

$$\|u\|_{C^k([0,a])} = \sum_{i=0}^{k} \|u^{(i)}\|_{C([0,a])}.$$

$C^{m,n}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j = 0, \ldots, m;\ k = 0, \ldots, n)$, endowed with the norm

$$\|u\|_{C^{m,n}(\Omega)} = \sum_{j=0}^{m} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}.$$

$\widetilde{C}^{m,n}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(j,k)}$ $(j = 0, \ldots, m;\ k = 0, \ldots, n;\ j + k < m + n)$, endowed with the norm

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)} = \sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)}.$$

$L(\Omega)$ is the Banach space of Lebesgue integrable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \iint\limits_{\Omega} |z(x_1, x_2)|\, dA.$$

$L^\infty(\Omega)$ is the spaces of essentially bounded measurable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$|u|_{L^\infty} = \operatorname*{ess\,sup}_{(x,y)\in\Omega} |u(x, y)|.$$

$AC([0,a])$ is the Banach space of absolutely continuous functions $u : [0,a] \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC} = |u(0)| + \int\limits_0^a \left|u'(x)\right| dx.$$

$AC(\Omega)$ is the Banach space of absolutely continuous functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC} = |u(0,0)| + \int\limits_0^a \left|u^{(1,0)}(x,0)\right| dx + \int\limits_0^b \left|u^{(0,1)}(0,y)\right| dy + \iint\limits_\Omega \left|u^{(1,1)}(x,y)\right| dA.$$

$AC^{m-1,n-1}(\Omega)$ is the Banach space of functions $u \in \widetilde{C}^{(m-1,n-1)}(\Omega)$, having absolutely continuous partial derivative $u^{(m-1,n-1)}$, endowed with the norm

$$\|z\|_{AC^{m-1,n-1}} = \|u\|_{\widetilde{C}^{m-1,n-1}} + \|z^{(m-1,n-1)}\|_{AC}.$$

If $z \in C^k(\Omega)$ and $r > 0$, then

$$\mathcal{B}^k(z;r) = \{\zeta \in C^k([0,a]) : \|\zeta - z\|_{\widetilde{C}^k([0,a])} \le r\}.$$

If $z \in C^{m,n}(\Omega)$ and $r > 0$, then

$$\mathcal{B}^{m,n}(z;r) = \{\zeta \in C^{m,n}(\Omega) : \|\zeta - z\|_{\widetilde{C}^{m,n}} \le r\}.$$

If $z \in \widetilde{C}^{m,n}(\Omega)$ and $r > 0$, then

$$\widetilde{\mathcal{B}}^{m,n}(z;r) = \{\zeta \in \widetilde{C}^{m,n}(\Omega) : \|\zeta - z\|_{\widetilde{C}^{m,n}} \le r\}.$$

Introduction

In the present dissertation for the higher order nonlinear hyperbolic equation

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big), \tag{0.1}$$

we investigate the initial–boundary value problem

$$\begin{aligned} u^{(j-1,0)}(0, y) &= \varphi_j(y) \ (j = 1, \dots, m), \\ h_k(u^{(m,0)}(x, \cdot))(x) &= \psi_k^{(m)}(x) \ (k = 1, \dots, n), \end{aligned} \tag{0.2}$$

as well as the nonlocal boundary value problem

$$\begin{aligned} l_j(u(\cdot, y))(0, y) &= \varphi_j(y) \ (j = 1, \dots, m), \\ h_k(u^{(m,0)}(x, \cdot))(x) &= \psi_k^{(m)}(x) \ (k = 1, \dots, n). \end{aligned} \tag{0.3}$$

Here $f : \Omega \times \mathbb{R}^{n+m+mn} \to \mathbb{R}$ is a continuous function that is locally Lipschitz continuous with respect to the first $n + m$ phase variables, $\varphi_j \in C^n([0, b])$ $(j = 1, \dots, m)$, $\psi_k \in C^m([0, a])$ $(k = 1, \dots, n)$, and $l_j : C^{m-1}([0, a]) \to C^n([0, b])$ $(j = 1, \dots, m)$ and $h_k : C^{n-1}([0, b]) \to C([0, a])$ $(k = 1, \dots, n)$ are bounded linear operators.

By a solution of problem (0.1),(0.2) (problem (0.1),(0.3)) we understand a *classical solution*, i.e., a function $u : \Omega \to R$ having continuous partial derivatives $u^{(j,k)}$ $(j = 0, \dots, m;\ k = 0, \dots, n)$ and satisfying equation (0.1) and initialboundary conditions (0.2) (boundary conditions (0.3)) everywhere in Ω.

Problems (0.1),(0.2) and (0.1),(0.3) do not belong to the classical boundary value problems of mathematical physics, with the exception of Darboux and Goursat initial value problems for second order hyperbolic equations (the case $m = n = 1$).

Beginning from the 1960ies, problems on periodic solutions in a strip or in the large, as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see [4-15, 17, 18, 39, 43–55]). These problems naturally led to the initial–boundary value problems in a rectangle with general boundary conditions:

$$w^{(1,1)} = P_0(x, y)w + P_1(x, y)w^{(1,0)} + P_2(x, y)w^{(0,1)} + q(x, y), \tag{0.4}$$

$$w(0, y) = \varphi(y), \qquad h\big(w^{(1,0)}(x, \cdot)\big)(x) = \psi(x), \tag{0.5}$$

where $P_i \in C([0,a]\times[0,b];\mathbb{R}^{n\times n})$ $(i=0,1,2)$, $q \in C([0,a]\times[0,b];\mathbb{R}^n)$, $\varphi \in C^1([0,b];\mathbb{R}^n)$, $\varphi \in C([0,a];\mathbb{R}^n)$ and $h : C([0,b]) \to C([0,a])$ is a bounded linear operator. A complete theory of problem (0.4), (0.5) was constructed in [24].

Initial–boundary value problems with integral boundary conditions for quasi–linear systems were studied in [1–3].

The initial–periodic boundary value problems for quasi-linear and nonlinear systems

$$w^{(1,1)} = F\big(x, y, w^{(1,0)}, w^{(0,1)}, w\big), \tag{0.6}$$

$$w(0,y) = \varphi(y), \quad w^{(1,0)}(x,0) = w^{(1,0)}(x,b), \tag{0.7}$$

were studied in [25, 26, 32, 34].

Nonlocal boundary value problems, in particular the Dirichlet problem and problems on doubly periodic solutions, were studied in [22, 28, 29, 33].

The linear case of problem (0.1),(0.2), i.e. the linear problem

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} + q(x,y), \tag{0.8}$$

$$\begin{gathered} u^{(j,0)}(0,y) = \varphi_j(y) \quad (j=0,\dots,m-1), \\ h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k(x) \quad (k=1,\dots,n). \end{gathered} \tag{0.9}$$

was studied in [30] and [31].

Investigation of the nonlinear problems (0.1),(0.2) and (0.1),(0.3) required a rather substantial modification of methods developed in [30] and [31], as well as application of the theory of nonlinear boundary value problems for ordinary differential equations from [19].

CHAPTER I
Strongly Well–Posed Initial–Boundary Value Problems

1. Formulation of the Main Results

1.1. **General Initial–Boundary Value Problems.** In the rectangle $\Omega = [0,a]\times[0,b]$ consider the nonlinear hyperbolic equation

$$u^{(m,n)} = f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{1.1}$$

$$\begin{gathered} u^{(j-1,0)}(0,y) = \varphi_j(y) \ (j=1,\dots,m), \\ h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k^{(m)}(x) \ (k=1,\dots,n), \end{gathered} \tag{1.2}$$

where $f:\Omega\times\mathbb{R}^{n+m+mn}\to\mathbb{R}$ is a continuous function that is continuously differentiable with respect to the first $n+m$ phase variables, $\varphi_j\in C^n([0,b])$ $(j=1,\dots,m)$, $\psi_k\in C^m([0,a])$ $(k=1,\dots,n)$, and $h_k : C^{n-1}([0,b])\to C([0,a])$ $(k=1,\dots,n)$ are bounded linear operators.

Let $\mathbf{v}=(v_0,\dots,v_{n-1})$, $\mathbf{w}=(w_0,\dots,w_{m-1})$ and $\mathbf{z}=(z_{m-1\,n-1},\dots,z_{0\,0})$. For a function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ that is continuously differentiable with respect to $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$, set:

$$f_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = \frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial v_k} \quad (k=0,\dots,n-1),$$
$$f_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = \frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial w_j} \quad (j=0,\dots,m-1),$$
$$f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = \frac{\partial f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})}{\partial z_{j\,k}} \quad (j=0,\dots,m-1,\ k=0,\dots,n-1)$$
$$p_{jk}[u](x,y) = f_{jk}\Big(x,y,u^{(m,0)}(x,y),\dots,u^{(m,n-1)}(x,y),u^{(0,n)}(x,y),\dots,u^{(m-1,n)}(x,y),$$
$$u^{(m-1,n-1)}(x,y),\dots,u(x,y)\Big) \quad (j=0,\dots,m,\ k=0,\dots,n).$$

Definition 1.1. Let u_0 be a solution of problem (1.1),(1.2), and $r>0$. Problem (1.1),(1.2) is said to be (u_0,r)–*well-posed* if:

(i) $u_0(x,y)$ is the unique solution of the problem in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$;

(ii) For an arbitrary $\varepsilon>0$ there exists $\delta_0>0$ such that for any $\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ that is continuously differentiable with respect to $\mathbf{v}$ and $\mathbf{w}$, $\widetilde{\varphi}_j \in C^n([0,b])$ $(j=1,\dots,m)$, $\widetilde{\psi}_k \in C^m([0,a])$ $(k=1,\dots,n)$, satisfying the inequalities

$$\sum_{k=0}^{n-1} |\widetilde{f}_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \delta_0 \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \tag{1.3}$$

$$\begin{gathered} |\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \delta \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \\ \sum_{j=1}^{m} \|\widetilde{\varphi}_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\widetilde{\psi}_k\|_{C^m([0,a])} \le \delta, \end{gathered} \tag{1.4}$$

the problem

$$\begin{aligned} u^{(m,n)} &= f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \\ &+\widetilde{f}\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \end{aligned} \tag{$\widetilde{1.1}$}$$

$$\begin{gathered} u^{(j,0)}(0,y)(y) = \varphi_j(y) + \widetilde{\varphi}_j(y) \ \ (j=1,\dots,m), \\ h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k^{(m)}(x) + \widetilde{\psi}_k^{(m)}(x) \ \ (k=1,\dots,n) \end{gathered} \tag{$\widetilde{1.2}$}$$

has at least one solution in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;\varepsilon)$.

Definition 1.2. Let u_0 be a solution of problem (1.1),(1.2), and $r>0$. Problem (1.1),(1.2) is said to be *strongly* (u_0,r)*–well-posed* if:

(i) Problem (1.1),(1.2) is (u_0,r)–well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta \in (0,\delta_0)$, $\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ that is continuously differentiable with respect to $\mathbf{v}$ and $\mathbf{w}$, $\widetilde{\varphi}_j \in C^n([0,b])$ $(j=1,\dots,m)$ and $\widetilde{\psi}_k \in C^m([0,a])$ $(k=1,\dots,n)$, satisfying the inequalities (1.3),(1.4), problem $(\widetilde{1.1}),(\widetilde{1.2})$ has at least one solution in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;M_0\,\delta)$.

Definition 1.3. Problem (1.1),(1.2) is called *well–posed* (*strongly well–posed*) if it has a unique solution u_0 and it is (u_0,r)–well–posed (strongly (u_0,r)–well-posed) for every $r>0$.

In [30] the linear hyperbolic equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)} + q(x,y) \quad (1.5)$$

was considered and the necessary and sufficient conditions of well–posedness of problem (1.5),(1.2) were established.

Notice that for the linear problem (u_0, r)–well–posedness is equivalent to strong well–posedness. Furthermore, Definitions 1.1 and 1.2 for problem (1.5),(1.2) are equivalent to the Definition 1.1 from [30].

Consider the boundary value problem for the nonlinear ordinary differential equation:

$$z^{(n)} = p(t, z, \dots, z^{(n-1)}); \quad l_k(z) = c_k \ \ (k = 1, \dots, n). \quad (1.6)$$

Here $p \in C([0,a]\times\mathbb{R}^n)$, $c_k \in \mathbb{R}$ and $l_k : C^{(n-1)}([0,a]) \to \mathbb{R}$ is a bounded linear functional $(k = 1, \dots, n)$.

Definition 1.4. Let u_0 be a solution of problem (1.6), and $r > 0$. Problem (1.6) is said to be (z_0, r)*–well-posed* if:
(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathcal{B}^{n-1}(z_0; r)$;
(ii) For an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\widetilde{c}_k$, and $\widetilde{p} \in C([0,a]\times\mathbb{R}^n)$ satisfying the inequalities

$$\sum_{k=1}^{n} |c_k - \widetilde{c}_k| < \delta \quad (k = 1, \dots, n), \quad \|p - \widetilde{p}\|_{C([0,a]\times\mathbb{R}^n)} < \delta \quad (1.7)$$

the problem

$$z^{(n)} = \widetilde{p}(t, z, \dots, z^{(n-1)}); \quad l_k(z) = \widetilde{c}_k \ \ (k = 1, \dots, n), \quad (\widetilde{1.6})$$

has at least one solution in the ball $\mathcal{B}^{n-1}(z_0; r)$, and each such solution belongs to the ball $\mathcal{B}^{n-1}(z_0; \varepsilon)$.

Definition 1.4 is a slight modification of Definition 3.2 from [19]. Definition 1.1 is an adaptation of the idea of Definition 1.4 to problem (1.1),(1.2).

Definition 1.5. Let u_0 be a solution of problem (1.6), and $r > 0$. Problem (1.6) is said to be *strongly* (z_0, r)*–well-posed* if:

(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathcal{B}^{n-1}(z_0; r)$;

(ii) There exist positive numbers M and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c}_k$, and $\widetilde{p} \in C([a, b] \times \mathbb{R}^n)$ satisfying inequalities (1.7), problem $(\widetilde{1.6})$ has at least one solution in the ball $\mathcal{B}^{n-1}(z_0; r)$, and each such solution belongs to the ball $\mathcal{B}^{n-1}(z_0; M\,\delta)$.

Remark 1.1. It is obvious that strong well–posedness implies well–posedness. The converse, however, is not true. As an example, consider the problem

$$z' = z^3, \quad z(0) = z(\omega), \tag{1.8}$$

which is well–posed and has the unique solution $z_0(t) \equiv 0$. The perturbed problem

$$z' = z^3 - \delta, \quad z(0) = z(b)$$

has the unique solution $z_\delta(t) = \delta^{\frac{1}{3}}$. It is clear that there exists no positive number M such that $\delta^{\frac{1}{3}} \le M\delta$ as $\delta \to 0$. Consequently, problem (1.8) is not strongly well–posed.

Definition 1.6. A solution z_0 of problem (1.6) is said to be strongly isolated, if problem (1.6) is strongly (z_0, r)–well-posed for some $r > 0$.

Remark 1.2. The concept of a strongly isolated solution of a nonlinear boundary value problem was introduced in [19]. However, our definition of a strongly isolated solution is a modification of Definition 3.1 from [19]. Also, Corollary 3.6 from [19] implies that if the function $p(t, z_1, \ldots, z_n)$ is continuously differentiable with respect to the phase variables, then strong isolation of a solution z_0 is equivalent to the fact that the linear homogeneous problem

$$z^{(n)} = \sum_{i=0}^{n-1} p_k(t) z^{(i)}; \quad l_k(\zeta)(x) = 0 \;\; (k = 1, \ldots, n),$$

where $p_k(t) = p_{z_{k-1}}(t, z_0(t), \ldots, z_0^{n-1}(t))$ $(k = 0, \ldots, n-1)$, has only the trivial solution.

Theorem 1.1. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v}, \mathbf{w}$ and $\mathbf{z}$, and let u_0 be a solution of problem* (1.1), (1.2). *Then,* *Problem* (1.1), (1.2) *is strongly* (u_0, r)*-well–posed for some* $r > 0$*, if and only if the linear homogeneous problem*

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}, \qquad (1.5_0)$$

$$u^{(j-1,0)}(0,y) = 0 \ \ (j = 1,\dots,m), \quad h_k(u^{(m,0)}(x,\cdot))(x) = 0 \ \ (k = 1,\dots,n), \qquad (1.2_0)$$

where $p_{jk}(x,y) = p_{jk}[u_0](x,y)$ $(j = 0,\dots,m,\ k = 0,\dots,n)$*, is well–posed.*

Remark 1.3. By Theorem 1.1 from [30], problem $(1.5_0), (1.2_0)$ is well posed if and only if the problem

$$\zeta^{(n)} = \sum_{i=0}^{n-1} p_{mk}(x,y)\zeta^{(i)}; \quad h_k(\zeta)(x) = 0 \ \ (k = 1,\dots,n) \qquad (1.9)$$

has only the trivial solution for every $x \in [0,a]$.

Theorem 1.2. *Let* f *be a continuously differentiable function with respect to the phase variables* $\mathbf{v}, \mathbf{w}$ *and* $\mathbf{z}$*, and there exist functions* $P_{ik} \in C(\Omega)$ $(i = 1,2;\ k = 0,\dots,n-1)$ *and a positive constant* ρ *such that:*

(A_1) $|f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \rho$ *for* $(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}$ $(j = 0,\dots,m-1,\ k = 0,\dots,n)$;
(A_2) $P_{1k}(x,y) \le f_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \le P_{2k}(x,y)$ *for* $(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}$ $(k = 0,\dots,n-1)$;
(A_3) *for every* $x \in [0,a]$ *and arbitrary measurable functions* $p_{mk}(x,\cdot) : [0,b] \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1k}(x,y) \le p_{mk}(x,y) \le P_{2k}(x,y) \quad \text{for} \quad y \in [0,b] \ \ (k = 0,\dots,n-1), \qquad (1.10)$$

the problem (1.9) *has only the trivial solution. Then problem* (1.1), (1.2) *is strongly well–posed.*

Consider the equation

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big)$$
$$+q(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u). \tag{1.11}$$

Theorem 1.3. *Let f satisfy all of the conditions of Theorem 1.2, and $q \in C(\Omega \times \mathbb{R}^{n+m+mn})$ such that*

$$|q(x, y, \mathbf{v}_1, \mathbf{w}, \mathbf{z}) - q(x, y, \mathbf{v}_2, \mathbf{w}, \mathbf{z})| \le \varepsilon \|\mathbf{v}_1 - \mathbf{v}_2\|$$
$$\textit{for} \quad (x, y, \mathbf{v}_i, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \quad (i = 1, 2), \tag{1.12}$$
$$|q(x, y, \mathbf{v}_1, \mathbf{w}_1, \mathbf{z}) - q(x, y, \mathbf{v}_2, \mathbf{w}_2, \mathbf{z})|$$
$$\le M\|\mathbf{w}_1 - \mathbf{w}_2\| \quad \textit{for} \quad (x, y, \mathbf{v}, \mathbf{w}_i, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \quad (i = 1, 2), \tag{1.13}$$
$$|q(x, y, \mathbf{0}, \mathbf{0}, \mathbf{z})| \le M(1 + \|\mathbf{z}\|) \quad \textit{for} \quad (x, y, \mathbf{z}) \in \Omega \times \mathbb{R}^{mn}, \tag{1.14}$$

where $\varepsilon > 0$ is sufficiently small and M is an arbitrary positive number. Then problem (1.11), (1.2) *has at least one solution. Moreover, if $q(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})$ is locally Lipschitz continuous with respect to $\mathbf{z}$, then problem* (1.11), (1.2) *is strongly well–posed.*

Remark 1.4. In Theorem 1.3 Lipschitz continuity of $q(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})$ with respect to $\mathbf{v}$ and $\mathbf{w}$ is essential and it cannot be replaced by Hölder continuity with exponent $\alpha \in (0, 1)$. Indeed, in the rectangle $[0, 1]\times[0, 2]$ consider the characteristic value problem

$$u_{xy} = \frac{1}{1-\alpha}|u_y|^\alpha \operatorname{sgn}(u_y), \tag{1.15}$$
$$u(0, y) = \frac{1}{2}(y - 1)^2 \quad \text{for} \quad y \in [0, 2], \quad u_x(x, 0) = 0 \quad \text{for} \quad x \in [0, 1], \tag{1.16}$$

where $\alpha \in (0, 1)$ is an arbitrary number. Problem (1.15),(1.16) has the unique *absolutely continuous* solution

$$u(x, y) = \frac{1}{2} + \int_0^y (x + |t - 1|^{1-\alpha})^{\frac{1}{1-\alpha}} \operatorname{sgn}(t - 1)\, dt,$$

which is not a classical solution because $u_y(x, y) = (x + |y - 1|^{1-\alpha})^{\frac{1}{1-\alpha}} \operatorname{sgn}(y - 1)$ is discontinuous along the line $y = 1$.

Consider the quasi–linear equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}$$
$$+q(x,y,u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u), \tag{1.17}$$

where $p_{jk} \in C(\omega)$ $(j=0,\dots,m;\ k=0,\dots,n)$ and $q(x,y,\mathbf{w},\mathbf{z})$ is a continuous function that is Lipschitz continuous with respect to $\mathbf{w}$ and $\mathbf{z}$, i.e. there exists $M>0$ such that

$$|q(x,y,\mathbf{w}_1,\mathbf{z}_1) - q(x,y,\mathbf{w}_2,\mathbf{z}_2)| \le M(\|\mathbf{w}_1-\mathbf{w}_2\| - \|\mathbf{z}_1-\mathbf{z}_2\|)$$

for $(x,y,\mathbf{w}_i,\mathbf{z}_i) \in \Omega \times \mathbb{R}^{n+m+mn}$ $(i=1,2)$.

The following corollary is an immediate consequence of Theorem 1.3.

Corollary 1.1. *Let problem* (1.9) *have only the trivial solution for every* $x \in [0,a]$. *Then problem* (1.17), (1.2) *is strongly well–posed.*

Theorem 1.4. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v},\mathbf{w}$ and $\mathbf{z}$, and let v_0 be a strongly isolated solution of the problem*

$$v^{(n)} = p_0(y,v,\dots,v^{(n-1)}), \quad h_k(v)(0) = \psi_k(0) \ \ (k=1,\dots,n), \tag{1.18}$$

where

$$p_0(y,v_0,\dots,v_{n-1}) = f\big(0,y,v_0,\dots,v_{n-1},\varphi_1^{(n)}(y),\dots,\varphi_m^{(n)}(y),\varphi_{m-1}^{(n-1)}(y),\dots,\varphi_1(y)\big).$$

Then there exists $\alpha \in (0,a]$ such that in the rectangle $\Omega_\alpha = [0,\alpha]\times[0,b]$ problem (1.1), (1.2) *has a unique solution u satisfying the condition*

$$u^{(m,0)}(0,y) = v_0(y) \quad \textit{for} \quad y \in [0,b].$$

Remark 1.5. In Theorem 1.4 the requirement of strong isolation of the solution v_0, i.e. strong (v_0,r)–well-posedness of problem (1.18) for some $r>0$ cannot be replaced by well-posedness of the problem under consideration. In order to illustrate this, consider the problem

$$u^{(1,1)} = \omega(|u^{(1,0)}|)\,u^{(1,0)} - \omega(x)\,u^{(1,0)}, \tag{1.19}$$

$$u(0,y)=0, \quad u^{(1,0)}(x,b) - u^{(1,0)}(x,0) = x\sin\frac{1}{x}, \tag{1.20}$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is an increasing continuous function, continuously differentiable in $(0, +\infty)$ and $\omega(0) = 0$. In this case, problem (1.18) is the following one:

$$v' = \omega(|v|)\, v, \quad v(b) - v(0) = 0. \tag{1.21}$$

It is clear that problem (1.21) has only the zero solution $v_0(y) \equiv 0$. Moreover, according to Corollary 4.2 and Theorem 4.4 from [19], problem (1.21) is well–posed. However, the problem is not strongly (v_0, r)–well–posed under any $r > 0$. Indeed, for an arbitrarily small $\delta > 0$ the perturbed problem

$$v' = \omega(|v|)\, v - \delta, \quad v(b) - v(0) = 0$$

has the unique solution

$$v_\delta(y) \equiv \eta(\delta),$$

where the function $\eta(\cdot)$ is inverse to $\omega(z)z$, and

$$\frac{v_\delta(y) - v_0(y)}{\delta} \equiv \frac{1}{\omega\big(\eta(\delta)\big)} \to +\infty \quad \text{as} \quad \delta \downarrow 0.$$

In particular, if $\omega(z) \leq z^\lambda$ for sufficiently small $z > 0$, then the following estimates

$$\frac{\delta}{\omega\big(\delta^{\frac{1}{\alpha+\lambda}}\big)} \leq v_\delta(y) \leq \frac{\delta}{\omega\big(\delta\big)}$$

hold for sufficiently small $\delta > 0$.

Our goal is to show that problem (1.19),(1.20) has no solution in the rectangle Ω_α no matter how small $\alpha > 0$ is.

Assume the contrary that problem (1.19)(1.20) has a solution u in Ω_α for some $\alpha > 0$. Then for an arbitrarily fixed $x \in (0, \alpha]$, the function $v(\cdot) = u^{(1,)0}(x, \cdot)$ is a solution of the problem

$$v' = \omega(|v|)\, v - \omega(x^2)\, v, \tag{1.22}$$

$$v(b) - v(0) = x \sin \frac{1}{x}. \tag{1.23}$$

containing the parameter $x \in [0, a]$. Moreover, if problem (1.19),(1.20) has a solution, then z is a solution (1.22),(1.23) depending continuously on the parameter x.

For every fixed $x \in (0, \alpha]$ equation (1.22) has three constant solutions: $v_0(y) = 0$, $v_1(y) = x$ and $v_2(y) = -x$. Due to the Picard's existence and uniqueness theorem on unique solvability of the Cauchy problem, no nonconstant solution v of equation (1.22) intersects v_0, v_1 or v_2, and thus $v'(y) \neq 0$ for $y \in [0, b]$. Let

$$k > \frac{1}{2\pi\alpha} \quad \text{and} \quad x \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Then $v(b) > v(0)$ and $v'(y) > 0$ for $y \in [0, b]$. Therefore, either

$$v(y) > x \quad \text{for} \quad y \in [0, b],$$

or

$$v(y) \in (-x, 0) \quad \text{for} \quad y \in [0, b].$$

If $x = \frac{1}{\frac{\pi}{2} + 2\pi k}$, then $v(b) - v(0) = x$, and consequently

$$v(y) \notin (-x, 0) \quad \text{for} \quad y \in [0, b].$$

From the aforesaid, in view of continuity of $u^{(1,0)}$ in Ω_α, it follows that

$$u^{(1,0)}(x, y) > x \quad \text{for} \quad x \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Similarly, one can show that

$$u^{(1,0)}(x, y) < -x \quad \text{for} \quad x \in \Big(\frac{1}{2\pi(k+1)}, \frac{1}{\pi + 2\pi k}\Big).$$

However, the latter two inequalities imply that $u^{(1,0)}(x, y)$ is discontinuous along the lines $x = \frac{1}{\pi k}$ $(k = 1, 2, \dots)$. Thus we have proved that problem (1.19),(1.20) has no solution in Ω_α under no $\alpha > 0$.

Remark 1.6. Conditions of Theorem 1.4 do not guarantee unique solvability of problem (1.1),(1.2). Indeed, consider the problem

$$u^{(1,1)} = \sin(u^{(1,0)}) + x\, f_0(x, y, u^{(1,0)}, u^{(0,1)}, u), \tag{1.24}$$

$$u(0, y) = 0, \;\; u^{(1,0)}(x, 0) = u^{(1,0)}(x, b), \tag{1.25}$$

where $f_0 : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function. For this case problem (1.18) has the form

$$z' = \sin z, \quad z(0) = z(b).$$

The latter problem has a countable set of strongly isolated solutions $z_k = k\pi$ ($k = 0, \pm 1, \dots$). By Theorem 1.4, for every integer k there exists $\alpha_k > 0$ such that in $\Omega_{\alpha_k} = [0, \alpha_k] \times [0, b]$, problem (1.24),(1.24) has a unique solution u_k satisfying the condition

$$u_k^{(1,0)}(0, y) = k\pi \quad \text{for} \quad y \in [0, b].$$

Let Λ be a set of reals. Consider the problem

$$z^{(n)} = p_\lambda(t, z, \dots, z^{(n-1)}), \quad l_{\lambda k}(z) = c_{\lambda k} \ \ (k = 1, \dots, n), \tag{1.26_λ}$$

containing the parameter $\lambda \in \Lambda$, where $p_\lambda \in C([0, b] \times \mathbb{R}^n)$, $l_{\lambda k} : C^{n-1}[0, b] \to \mathbb{R}$ ($k = 1, \dots, n$) are bounded linear functionals, and $c_{\lambda k} \in \mathbb{R}$ ($k = 1, \dots, n$).

Let z_λ be a solution of problem (1.26_λ) for $\lambda \in \Lambda$, and $r > 0$. The family of problems (1.26_λ) ($\lambda \in \Lambda$) is said to be *uniformly strongly* (z_λ, r)*–well-posed*, if:

(i) z_λ is unique in the ball $\mathcal{B}^{n-1}(z_\lambda; r)$;

(ii) There exist positive numbers M and δ_0 *independent of* λ such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c}_{\lambda k}$, and $\widetilde{p}_\lambda \in C([a, b] \times \mathbb{R}^n)$ satisfying the inequalities

$$\sum_{k=1}^{n} |c_{\lambda k} - \widetilde{c}_{\lambda k}| < \delta \quad (k = 1, \dots, n), \quad \|p_\lambda - \widetilde{p}_\lambda\|_{C([a,b] \times \mathbb{R}^n)} < \delta,$$

the problem

$$z^{(n)} = \widetilde{p}_\lambda(t, z, \dots, z^{(n-1)}); \quad l_{\lambda k}(z) = \widetilde{c}_{\lambda k} \ \ (k = 1, \dots, n), \tag{$\widetilde{1.26_\lambda}$}$$

has at least one solution in the ball $\mathcal{B}^{n-1}(z_\lambda; r)$, and each such solution belongs to the ball $\mathcal{B}^{n-1}(z_\lambda; M\delta)$.

A family of solutions $\{z_\lambda\}_{\lambda \in \Lambda}$ is said to be *uniformly strongly isolated* if the family of problems (1.26_λ) ($\lambda \in \Lambda$) is *uniformly strongly* (z_λ, r)*–well-posed* for some $r > 0$.

Let $J = [0, \alpha)$, $\alpha \in (0, a]$, ($J = [0, \alpha]$, $\alpha \in (0, a)$), and u be a solution of problem (1.1),(1.2) in the rectangle $J \times [0, b]$. u is called *continuable*, if there exists $\alpha_1 \in [\alpha, a]$ ($\alpha_1 \in (\alpha, a]$) and a solution u_1 of problem (1.1),(1.2) in $[0, \alpha_1] \times [0, b]$ such that

$$u_1(x, y) = u(x, y) \quad \text{for} \quad (x, y) \in [0, \alpha) \times [0, b].$$

Otherwise u is called *non–continuable*.

Theorem 1.5. *Let u be a a non-continuable solution of problem* (1.1), (1.2) *defined on $J \times [0,b]$, and let for every $x_0 \in J$, $v(y) = u^{(m,0)}(x_0,y)$ be a solution of the problem*

$$v^{(n)} = p[u](x_0, y, v, v', \dots, v^{(n-1)}), \quad h_k(v)(x_0) = \psi_k(x_0) \quad (k = 1, \dots, n). \tag{1.27}$$

If the family of solutions $v(y) = u^{(m,0)}(x_0,y)$ $(x_0 \in J)$ is uniformly strongly isolated, then either $J = [0,a]$, or $J = [0,\alpha)$ and

$$\lim_{x\to\alpha} \Big(\|u^{(m,0)}(x,\cdot)\|_{C^{n-1}([0,b])} + \sum_{j=0}^{m-1} \|u^{(j,0)}(x,\cdot)\|_{C^{n}([0,b])} \Big) = +\infty. \tag{1.28}$$

Definition 1.7. Let u be a non-continuable solution of problem (1.1), (1.2) in $J \times [0,b]$ and let $\alpha = \sup J$. We say that a bounded and measurable vector function $(p_0, \dots, p_{n-1}) : [0,b] \to \mathbb{R}^n$ belongs to the set $S_f^\alpha[u]$, if there exists an increasing sequence $x_l \uparrow \alpha$ as $l \to \infty$ such that

$$\lim_{l\to\infty} \int_0^y p_{mk}[u](x_l,t)\,dt = \int_0^y p_k(t)\,dt \qquad (k = 0, 1, \dots, n-1)$$

uniformly on $[0,b]$.

Corollary 1.2. *Let u be a a non-continuable solution of problem* (1.1), (1.2) *in $J \times [0,b]$, and let $\alpha = \sup J$. If for an arbitrary $(p_0, \dots, p_{n-1}) \in S_f^\alpha[u]$ the homogeneous problem*

$$\zeta^{(n)} = \sum_{i=0}^{n-1} p_k(y)\zeta^{(i)}; \quad h_k(\zeta)(\alpha) = 0 \;\; (k = 1, \dots, n), \tag{1.29}$$

has only the trivial solution, then either $J = [0,a]$, or $J = [0,\alpha)$ and (1.28) *holds.*

Remark 1.7. In Theorem 1.5 the requirement of uniform strong isolation cannot be replaced by strong isolation. As an example, in the rectangle $\Omega = [0,2] \times [0,b]$ consider the problem

$$u^{(1,1)} = (1-x)^4\, u^4\, u^{(1,0)} + \Big(-(1-x)^4 + (1-x)^2 \pi \sin \frac{2\pi}{1-x} \Big) u^6, \tag{1.30}$$

$$u(0,y) = 1, \quad u^{(1,0)}(x,0) = u^{(1,0)}(x,b). \tag{1.31}$$

The problem has a solution

$$u(x) = \frac{1}{1 - x + \sin^2 \frac{\pi}{1-x}}.$$

It is clear that $u(x)$ is a non-continuable solution of the problem (1.30),(1.31) in the rectangle $[0, 1) \times [0, b]$. On the other hand, the limit (1.28) does not exist since

$$\lim_{k\to+\infty} \Big|u\Big(1 - \frac{1}{\frac{1}{2} + k\pi}\Big)\Big| + \Big|u'\Big(1 - \frac{1}{\frac{1}{2} + k\pi}\Big)\Big| = 2,$$

$$\lim_{k\to+\infty} \Big|u\Big(1 - \frac{1}{k\pi}\Big)\Big| + \Big|u'\Big(1 - \frac{1}{k\pi}\Big)\Big| = +\infty.$$

The reason for this is that the family of solutions

$$v(y) = u^{(1,0)}(x, y) = \frac{1 - \frac{\pi}{(1-x)^2} \sin \frac{2\pi}{1-x}}{\Big(1 - x + \sin^2 \frac{\pi}{1-x}\Big)^2}$$

of the problem

$$v' = (1 - x)^4 u^4(x, y)\, v + \Big(- (1 - x)^4 + (1 - x)^2 \pi \sin \frac{2\pi}{1 - x}\Big) u^6(x, y), \quad v(0) = v(b). \quad (1.32)$$

$(x \in [0, 1))$ is not uniformly strongly isolated.

Remark 1.8. In Corollary 1.2 the requirement that problem (1.29) has only the trivial solution is essential and cannot be weakened. As a matter of fact, if problem (1.29) has a nontrivial solution for some $\alpha \in (0, a)$, then problem (1.1),(1.2) may have a non–continuable solution in the closed rectangle Ω_α.

To see this, in the rectangle Ω consider the problem

$$u^{(1,1)} = \arctan(|u|)u^{(1,0)} + \arctan(u),$$

$$u(0, y) = \alpha, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, b).$$

In the rectangle Ω_α this problem has a solution $u(x, y) = \alpha - x$. Let us show that this solution is non–continuable. Assume the contrary, that it can be continued on the rectangle Ω_β for some $\beta > \alpha$. Without loss of generality one can assume that

$$u^{(1,0)}(x, y) < 0, \ \ u(x, y) < 0 \quad \text{for} \quad (x, y) \in (\alpha, \beta] \times [0, b].$$

Consequently,

$$u^{(1,1)}(x,y) = \arctan(|u(x,y)|)u^{(1,0)}(x,y) + \arctan(u(x,y)) < 0 \quad \text{for} \quad (x,y) \in (\alpha,\beta]\times[0,b].$$

However, the latter inequality contradicts to the periodicity of $u^{(1,0)}$ with respect to the second argument. The obtained contradiction shows that u is non–continuable. This is caused by the fact that problem (1.29), which for this case has the form

$$\zeta' = 0, \qquad \zeta(0) = \zeta(b),$$

has a nontrivial solution.

The following initial–boundary conditions are important particular cases of (1.2):

$$\begin{gathered} u^{(j-1,0)}(0,y) = \varphi_j(y) \quad (j = 1,\dots,m), \\ u^{(m,k)}(x, y_1(x)) = u^{(m,k)}(x, y_2(x)) + \psi_k(x) \ \ (k = 0,1), \end{gathered} \tag{1.33}$$

and

$$\begin{gathered} u^{(j-1,0)}(0,y) = \varphi_j(y) \quad (j = 1,\dots,m), \\ u^{(m,k)}(x, y_i(x)) = \psi_{ik}(x) \quad (k = 0,\dots,n_i - 1;\ i = 1,\dots,l). \end{gathered} \tag{1.34}$$

Here when considering problem (1.1), (1.33) it will be assumed that $y_i : [0,a] \to [0,b]$ $(i = 1,2)$ are continuous functions satisfying $0 \le y_1(x) < y_2(x) \le b$ on $[0,a]$. As for the problem (1.1),(1.34), we will assume that $n \ge 2$, $l \in \{2,\dots,n\}$, $n_i \in \{1,\dots,n-1\}$, $n_1 + \dots + n_l = n$, and $y_i : [0,a] \to [0,b]$ $(i = 1,\dots,l)$ are continuous functions such that

$$0 \le y_1(x) < \dots < y_l(x) \le b \quad \text{for} \quad x \in [0,a].$$

Corollary 1.3. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ satisfying conditions (A_1) and (A_2) of Theorem 1.2. Furthermore, let the functions P_{ik} $(k = 0,\dots,n-1;\ i = 1,2)$ satisfy the conditions*

$$\int\limits_{y_1(x)}^{y_2(x)} P_{i0}(x,t)\,dt \neq 0, \quad \sum_{k=0}^{n-1} \gamma_{ik}(x) \int\limits_{y_1(x)}^{y_2(x)} |P_{ik}(x,t)|\,dt < 1, \quad (i = 1,2)^1 \tag{1.35}$$

[1] If $n = 1$, then the second inequality is unnecessary.

hold on $[0,a]$, *where* $\Delta(x) = y_2(x) - y_1(x)$,

$$\gamma_i(x) = \int\limits_{y_1(x)}^{y_2(x)} |P_{i0}(x,t)|\,dt \Bigg/ \Bigg| \int\limits_{y_1(x)}^{y_2(x)} P_{i0}(x,t)\,dt \Bigg|,$$

$$\gamma_{i0}(x) = \frac{\Delta(x)}{\sqrt{2}} \Big(\frac{\Delta(x)}{2\pi}\Big)^{n-2} \gamma_i(x), \quad \gamma_{n-1}(x) = 1 + \Delta(x),$$

$$\gamma_{ik}(x) = \frac{\Delta(x)}{\sqrt{2}} \Big(\frac{\Delta(x)}{2\pi}\Big)^{n-k-2} (1 + \gamma_i(x)) \quad (k = 1, \dots, n-2).$$

Then problem (1.1), (1.33) *is strongly well–posed.*

Corollary 1.4. *Let* f *be a continuously differentiable function with respect to the phase variables* $\mathbf{v}, \mathbf{w}$ *and* $\mathbf{z}$ *satisfying condition* (A_1) *of Theorem* 1.2 *for some* $\rho > 0$, *and let there exist nonnegative functions* $p_k \in C([0,a])$ $(k = 0, \dots, n-1)$ *such that*

$$|f_{mk}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})| \le p_k(x) \ \text{ for } \ (x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \ (k = 0, \dots, n-1), \quad (1.36)$$

$$\sum_{k=1}^{n} \frac{(y_l(x) - y_1(x))^k}{2^k k[(k-1)/2]![k/2]!} p_{n-k}(x) < 1, \quad (1.37)$$

where $[k/2]$ *is the integer part of* $k/2$. *Then problem* (1.1), (1.34) *is strongly well–posed.*

Now consider the case, where the righthand side of equation (1.1) does not contain the derivatives $u^{(m,k)}$ $(k = 1, \dots, n-1)$, i.e., where equation (1.1) has the form

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big). \quad (1.38)$$

For equation (1.38) consider the following problems

$$\begin{gathered} u^{(j-1,0)}(0,y) = \varphi_j(y) \ (j = 1, \dots, m), \ \ u^{(m,i-1)}(x, y_1(x)) = \psi_{1i}(x) \ (i = 1, \dots, n^*), \\ u^{(m,k-1)}(x, y_2(x)) = \psi_{2k}(x) \quad (k = 1, \dots, n-n^*), \end{gathered} \quad (1.39)$$

and

$$\begin{gathered} u^{(j-1,0)}(0,y) = \varphi_j(y) \quad (j = 1, \dots, m), \\ u^{(m,k-1)}(x, y_1(x)) = u^{(m,k-1)}(x, y_2(x)) + \psi_k(x) \ \ (k = 1, \dots, n), \end{gathered} \quad (1.40)$$

where n^* is the integer part of $n/2$, $\varphi_j \in C^n([0,b])$, $\psi_k \in C([0,a])$, $\psi_{1k}, \ \psi_{2k} \in C([0,a])$, and $y_1, \ y_2 \in C([0,a])$, $0 \le y_1(x) < y_2(x) \le b$ for $x \in [0,a]$.

Corollary 1.5. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ satisfying condition (A_1) of Theorem 1.2 for some $\rho > 0$, and let there exist a nonnegative function $p_0 \in C(\Omega)$ such that*

$$|f_v(x, y, v, \mathbf{w}, \mathbf{z})| \leq p_0(x, y) \text{ for } (x, y, v, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{1+m+mn},$$

$$\int\limits_{y_1(x)}^{y_2(x)} (t - y_1(x))^{n-1}(y_2(x) - t)^{n-1} p_0(x, t)\, dt$$
$$< (n-1)(n_1-1)!(n-n_1-1)!(y_2(x) - y_1(x))^{n-1}. \tag{1.41}$$

Then problem (1.38), (1.39) *is strongly well–posed.*

Corollary 1.6. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ satisfying condition (A_1) of Theorem 1.2 for some $\rho > 0$, and let there exist nonnegative functions $p_i \in C(\Omega)$ $(i = 0, 1)$ such that*

$$\int\limits_{y_1(x)}^{y_2(x)} p_1(x, y)\, dy > 0 \quad \textit{for} \quad x \in [0, a], \tag{1.42}$$

and

$$-p_0(x, y) \leq \sigma\, f_v(x, y, v, \mathbf{w}, \mathbf{z}) \leq -p_1(x, y) \text{ for } (x, y, v, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{1+m+mn},$$

where

$$\sigma = (-1)^{n^*} \quad \textit{for} \quad n = 2n^*, \quad \text{and} \quad \sigma \in \{-1, 1\} \quad \textit{for} \quad n = 2n^* + 1.$$

Then problem (1.38), (1.40) *is strongly well–posed.*

Corollary 1.7. *Let f be a continuously differentiable function with respect to the phase variables $\mathbf{v}, \mathbf{w}$ and $\mathbf{z}$ satisfying condition (A_1) of Theorem 1.2 for some $\rho > 0$, $n = 2n^*$, and let there exist a positive number ε and a nonnegative function $p_1 \in C(\Omega)$ satisfying inequality* (1.39) *such that*

$$p_1(x, y) \leq (-1)^{n^*} f_v(x, y, v, \mathbf{w}, \mathbf{z}) \leq \Big(\frac{2\pi - \varepsilon}{y_2(x) - y_1(x)}\Big)^n \text{ for } (x, y, v, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{1+m+mn}.$$

Then problem (1.38), (1.40) *is strongly well–posed.*

1.2. **Two–Point Initial–Boundary Value Problems.** In Theorem 1.2 it is assumed that the function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ has at most linear growth with respect to the phase variables $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$. Moreover, without that assumption problem (1.1),(1.2) may not have a (global) solution at all.

In this subsection we separately study two–point boundary initial–boundary value problems, because the existence results below cover the case where the function f has an arbitrary growth order with respect to some of the phase variables.

For the equation

$$u^{(1,2n)} = f_1\big(x,y,u,\dots,u^{(0,n-1)}\big)u^{(1,0)} + f_0(u) + q(x,y,u,\dots,u^{(0,2n-1)}) \tag{1.43}$$

consider the following initial–boundary value problems

$$u(0,y) = \varphi(y), \qquad u^{(1,k-1)}(x,0) = u^{(1,k-1)}(x,b) \ \ (k=1,\dots,2n), \tag{1.44}$$

$$u(0,y) = \varphi(y), \qquad u^{(1,k-1)}(x,0) = 0, \quad u^{(1,k-1)}(x,b) = 0 \ \ (k=1,\dots,n), \tag{1.45}$$

and

$$u(0,y) = \varphi(y) \qquad u^{(1,2(k-1))}(x,0) = 0, \quad u^{(1,2(k-1))}(x,b) = 0 \ \ (k=1,\dots,n). \tag{1.46}$$

Here f_1, f_0 and q are continuous functions.

Corollary 1.8. *Let there exist $\delta > 0$ and $M > 0$ such that*

$$(-1)^{n-1} f_1(x,y,z_1,\dots,z_n) \ge \delta \quad \textit{for} \quad (x,y,z_1,\dots,z_n) \in \Omega \times \mathbb{R}^n, \tag{1.47}$$

$$(-1)^{n-1} f_0'(z) \ge 0 \quad \textit{for} \quad z \in \mathbb{R}, \tag{1.48}$$

$$|q(x,y,z_1,\dots,z_{2n})| \le M \quad \textit{for} \quad (x,y,z_1,\dots,z_{2n}) \in \Omega \times \mathbb{R}^{2n}. \tag{1.49}$$

Then problem (1.43), (1.44) *has at least one solution.*

Corollary 1.9. *Let there exist $\delta > 0$ and $M > 0$ such that*

$$(-1)^{n-1} f_1(x,y,z_1,\dots,z_n) \ge -\frac{\pi^{2n}}{b^{2n}} + \delta \quad \textit{for} \quad (x,y,z_1,\dots,z_n) \in \Omega \times \mathbb{R}^n, \tag{1.50}$$

and conditions (1.48) *and* (1.49) *hold. Then problem* (1.43), (1.45) *has at least one solution.*

Corollary 1.10. *Let the functions f_0, f_1 and q satisfy all of the conditions of Corollary* 1.9. *Then problem* (1.43), (1.46) *has at least one solution.*

2. Auxiliary Statements

For arbitrary $x \in [0,a]$, consider the linear boundary value problem

$$\zeta^{(n)} = \sum_{j=0}^{n-1} p_j(y)\zeta^{(j)} + q(y), \quad h_k(\zeta)(x) = c_k \ \ (k = 1, \dots, n), \tag{2.1}$$

where $p_j : [0,b] \to \mathbb{R}$ $(j = 0, \dots, n-1)$ are measurable functions satisfying the inequalities

$$P_{1j}(x,y) \le p_j(y) \le P_{2j}(x,y) \quad \text{for} \quad y \in [0,b] \ \ (j = 0, \dots, n-1), \tag{2.2}$$

$q \in C([0,b])$, $P_{ij} \in C(\Omega)$ $(i = 1,2;\ j = 0, \dots, n-1)$, $h_k : C^{n-1}([0,b]) \to C([0,a])$ $(k = 1, \dots, n)$ are bounded linear operators, and $c_k \in \mathbb{R}$ $(k = 1, \dots, n)$.

The following lemma is a direct consequence of Theorem 1.2 from [19].

Lemma 2.1. *Let for every $x \in [0,a]$ and arbitrary measurable functions $p_j : [0,b] \to \mathbb{R}$ $(j = 0, \dots, n-1)$ satisfying the inequalities* (2.2), *the linear homogeneous problem*

$$\zeta^{(n)} = \sum_{j=0}^{n-1} p_j(y)\zeta^{(j)}, \quad h_k(\zeta)(x) = 0 \ \ (k = 1, \dots, n)$$

have only the trivial solution. Then for every fixed $x \in [0,a]$ problem (2.1) *has unique solution ζ admitting the estimate*

$$\|\zeta\|_{C^{n-1}([0,b])} \le M\Big(\|q\|_{C([0,b])} + \sum_{k=1}^{n} |c_k|\Big),$$

where M is a positive constant independent of p_j $(j = 0, \dots, n-1)$, c_k $(k = 1, \dots, n)$, q and x.

Now consider the quasi–linear differential equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}$$
$$+q\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big) \tag{2.3}$$

with the initial–boundary conditions (1.2), and the differential inequality

$$\left|u^{(m,n)} - \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}\right|$$
$$\leq \varepsilon \sum_{k=0}^{n-1} |u^{(m,k)}| + M \sum_{j=0}^{m-1}\sum_{k=0}^{n} |u^{(j,k)}| + \delta \tag{2.4}$$

with the boundary conditions

$$\sum_{j=0}^{m-1}\sum_{k=0}^{n} |u^{(j,k)}(0,y)| + \sum_{k=1}^{n} \left|h_k(u^{(m,0)}(x,\cdot))(x)\right| \leq \delta \quad \text{for} \quad (x,y) \in \Omega. \tag{2.5}$$

Here $q \in C(\Omega \times \mathbb{R}^{n+m+mn})$ and ε, M and δ are nonnegative constants.

Lemma 2.2. *Let problem* (1.9) *have only the trivial solution for every* $x \in [0,a]$. *Then problem* (1.5), (1.2) *has a unique solution* u *admitting the representation*

$$u(x,y) = \mathcal{A}_0(\varphi_1,\dots,\varphi_m,\psi_1,\dots,\psi_n)(x,y) + \mathcal{A}(q)(x,y), \tag{2.6}$$

where $\mathcal{A}_0 : C^{n-1}([0,b];\mathbb{R}^m) \times C^{m-1}([0,a];\mathbb{R}^n) \to C^{m,n}(\Omega)$ *and* $A : C(\Omega) \to C^{m,n}(\Omega)$ *are bounded linear operators. Furthermore, there exists a positive constant* M_0 *independent of* q *such that*

$$\sum_{k=0}^{n-1} \|\mathcal{A}^{(m,k)}(q)\|_{C(\Omega_x)} \leq M_0 \|q\|_{C(\Omega_x)} \quad \textit{for} \quad x \in [0,a], \tag{2.7}$$

$$\sum_{j=0}^{m-1}\sum_{k=0}^{n} \|\mathcal{A}^{(j,k)}(q)\|_{C(\Omega_x)} \leq M_0 \int_0^x \|q\|_{C(\Omega_s)}\, ds \quad \textit{for} \quad x \in [0,a], \tag{2.8}$$

where $\Omega_x = [0,x] \times [0,b]$.

This lemma immediately follows from Theorem 1.1 and Lemma 2.1 from [30].

Lemma 2.3. *Let problem* (1.9) *have only the trivial solution for every* $x \in [0,a]$, *and* M_0 *be the number appearing in Lemma 2.2. Then there exists an increasing function* $\rho : [0,+\infty) \to (0,+\infty)$ *such that for every* $\varepsilon \in [0, \frac{1}{M_0})$ *and* $M > 0$ *an arbitrary solution of problem* (2.4), (2.5) *admits the estimate*

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)} \leq \rho(M)\, \delta. \tag{2.9}$$

Proof. Set

$$q(x,y) = u^{(m,n)} - \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)},$$

$$\varphi_j(y) = u^{(j-1,0)}(0,y) \ \ (j=1,\dots,m), \quad \psi_k^{(m)}(x) = h_k(u^{(m,0)}(x,\cdot))(x) \ \ (k=1,\dots,n).$$

Then by Lemma 2.2, representation (2.6) and inequalities (2.7), (2.8) hold. Taking into account inequalities (2.4) and (2.5), from (2.6)–(2.8) we get

$$\sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega_x)} \le (\|\mathcal{A}_0\| + M_0)\delta + \varepsilon M_0 \sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega_x)}$$
$$+ MM_0 \sum_{j=0}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_x)}, \tag{2.10}$$

$$\sum_{j=}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_x)} \le (\|\mathcal{A}_0\| + aM_0)\delta + \varepsilon M_0 \int_0^x \sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega_s)}\,ds$$
$$+ MM_0 \int_0^x \sum_{j=0}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_s)}\,ds,$$

where $\|\mathcal{A}_0\|$ is the norm of the operator $\mathcal{A}_0$. Let $\varepsilon \in [0, \frac{1}{2M_0}]$. Then (2.10) yields

$$\sum_{k=0}^{n-1} \|u^{(m,k)}\|_{C(\Omega_x)} \le 2(\|\mathcal{A}_0\| + M_0)\delta + 2MM_0 \sum_{j=0}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_x)}, \tag{2.11}$$

$$\sum_{j=}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_x)} \le (1+a)(\|\mathcal{A}_0\| + 2M_0)\delta$$
$$+ 2M_0M \int_0^x \sum_{j=0}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega_s)}\,ds.$$

From the latter inequality, by Gronwall's lemma it follows that

$$\sum_{j=0}^{m-1}\sum_{k=0}^{n} \|u^{(j,k)}\|_{C(\Omega)} \le (1+a)(\|\mathcal{A}_0\| + 2M_0)e^{2M_0M}\delta. \tag{2.12}$$

Then (2.11) and (2.12) imply (2.9), where

$$\rho(M) = 2(\|\mathcal{A}_0\| + M_0) + (2MM_0 + 1)(1+a)(\|\mathcal{A}_0\| + 2M_0)e^{2M_0M}. \quad \square$$

Lemma 2.4. *Let problem* (1.9) *have only the trivial solution for every* $x \in [0,a]$*, and let the function* $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ *be locally Lipschitz continuous with respect to* $\mathbf{z}$*. Furthermore, let* q *satisfy the inequalities* (1.12)$-$(1.14)*, where* M *is a positive constant,* $\varepsilon \in [0, \frac{1}{2M_0}]$*, and* M_0 *is the number appearing in Lemma* 2.2*. Then problem* (2.3), (1.2) *is uniquely solvable.*

Proof. First notice that in view of (1.12)–(1.14) q satisfies the inequality

$$|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \varepsilon\|\mathbf{v}\| + M(\|\mathbf{w}\|+\|\mathbf{z}\|+1) \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn}. \tag{2.13}$$

Set

$$\Delta = M + \sum_{j=1}^{m} \|\varphi_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\psi_k\|_{C^m([0,a])}. \tag{2.14}$$

Let ρ be the function appearing in Lemma 2.3, $r = \rho(M)\Delta$,

$$\Xi_r(t) = \begin{cases} 1 & \text{for} \quad t \in [0,r] \\ \frac{r}{t} & \text{for} \quad t \in (r,+\infty) \end{cases},$$

$$q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = q(x,y,\Xi_r(\|\mathbf{v}\|)\mathbf{v},\Xi_r(\|\mathbf{w}\|)\mathbf{w},\Xi_r(\|\mathbf{z}\|)\mathbf{z}).$$

By (1.12)–(1.14), the function $q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ satisfies the inequalities

$$|q_r(x,y,\mathbf{v}_1,\mathbf{w},\mathbf{z}) - q_r(x,y,\mathbf{v}_2,\mathbf{w},\mathbf{z})| \le \varepsilon\|\mathbf{v}_1-\mathbf{v}_2\|$$
$$\text{for} \quad (x,y,\mathbf{v}_i,\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2), \tag{2.15}$$

$$|q_r(x,y,\mathbf{v},\mathbf{w}_1,\mathbf{z}) - q_r(x,y,\mathbf{v},\mathbf{w}_2,\mathbf{z})| \le M\|\mathbf{w}_1-\mathbf{w}_2\|$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w}_i,\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2), \tag{2.16}$$

$$|q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \varepsilon\|\mathbf{v}\| + M(\|\mathbf{w}\|+\|\mathbf{z}\|+1)$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn}. \tag{2.17}$$

Moreover, there exists a positive constant M_1 such that

$$|q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_1) - q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_2)| \le M_1\|\mathbf{z}_1-\mathbf{z}_2\|$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}_i) \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2). \tag{2.18}$$

Along with (2.3) consider the differential equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}$$
$$+q_r\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big). \tag{2.19}$$

If the function u is a solution of problem (2.3),(1.2), then in view of (2.13) and (2.14) it is a solution of (2.4),(2.5) too, and according to the aforesaid, admits the estimate

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)} \le r. \tag{2.20}$$

On the other hand, the latter estimate implies that u is a solution of the problem (2.19),(1.2).

In view of (2.14) and (2.17), similarly one can prove that arbitrary solution of problem (2.19),(1.2) also admits the estimate (2.20), and consequently is a solution of problem (2.3),(1.2). Thus, equivalence of problems (2.19),(1.2) and (2.3),(1.2) is proved. Therefore in order to prove the lemma, it is sufficient to prove unique solvability of problem (2.19),(1.2).

Without loss of generality one can assume that $M_1 \ge M$. Then inequalities (2.16) and (2.18) yield

$$|q_r(x,y,\mathbf{v}_1,\mathbf{w}_1,\mathbf{z}_1) - q_r(x,y,\mathbf{v}_2,\mathbf{w}_2,\mathbf{z}_2)| \le \varepsilon\|\mathbf{v}_1-\mathbf{v}_2\| + M_1\big(\|\mathbf{w}_1-\mathbf{w}_2\| + \|\mathbf{z}_1-\mathbf{z}_2\|\big)$$
$$\text{for} \quad (x,y,\mathbf{v}_i,\mathbf{w}_i,\mathbf{z})_i \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2). \tag{2.21}$$

Furthermore, by Lemma 2.2, problem (2.19),(1.2) is equivalent to the operator equation

$$u(x,y) = \mathcal{A}_0(\varphi_1,\dots,\varphi_m,\psi_1,\dots,\psi_n)(x,y) + \mathcal{A}_1\big(u\big)(x,y), \tag{2.22}$$

where

$$\mathcal{A}_1\big(u\big)(x,y)$$
$$= \mathcal{A}\Big(q_r\big(u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)\Big)(x,y). \tag{2.23}$$

Also, in view of (2.7), (2.8), (2.21) and (2.23), we have

$$\sum_{k=0}^{n-1}\|\mathcal{A}_1^{(m,k)}(u_1)-\mathcal{A}_1^{(m,k)}(u_2)\|_{C(\Omega_x)}\le\frac{1}{2}\sum_{k=0}^{n-1}\|u_1^{(m,k)}-u_2^{(m,k)}\|_{C(\Omega_x)}$$

$$+M_0M_1\sum_{j=0}^{m-1}\sum_{k=0}^{n}\|u_1^{(j,k)}-u_2^{(j,k)}\|_{C(\Omega_x)},\tag{2.24}$$

$$\sum_{j=0}^{m-1}\sum_{k=0}^{n}\|\mathcal{A}_1^{(j,k)}(u_1)-\mathcal{A}_1^{(m,k)}(u_2)\|_{C(\Omega_x)}\le\frac{1}{2}\int_0^x\sum_{k=0}^{n-1}\|u_1^{(m,k)}-u_2^{(m,k)}\|_{C(\Omega_s)}\,ds$$

$$+M_0M\int_0^x\sum_{j=0}^{m-1}\sum_{k=0}^{n}\|u_1^{(j,k)}-u_2^{(j,k)}\|_{C(\Omega_s)}\,ds.\tag{2.25}$$

In the space $\widetilde{C}^{m,n}(\Omega)$ introduce the following norm

$$\|u\|_l=\max\left\{e^{-lx}\sum_{k=0}^{n-1}\|u^{(m,k)}\|_{C(\Omega_x)}:x\in[0,a]\right\}$$

$$+4M_0M_1\max\left\{e^{-lx}\sum_{j=0}^{n-1}\sum_{k=0}^{n}\|u^{(j,k)}\|_{C(\Omega_x)}:x\in[0,a]\right\},$$

where $l=8M_0M$. It is obvious that the new norm is equivalent to the standard norm of the space $\widetilde{C}^{m,n}(\Omega)$. On the other hand, from (2.24) and (2.25) one can easily deduce that

$$\|\mathcal{A}_1(u_1)-\mathcal{A}_1(u_2)\|_l\le\frac{5}{8}\|u_1-u_2\|_l.$$

Therefore, by the contraction mapping principle, equation (2.22), and consequently, problem (2.3),(1.2) has a unique solution. □

Lemma 2.5. *Let problem* (1.9) *have only the trivial solution for every* $x\in[0,a]$, *and let the function* $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ *satisfy the inequalities* (1.12) – (1.14), *where* M *is a positive constant,* $\varepsilon\in[0,\frac{1}{2M_0}]$ *and* M_0 *is the number appearing in Lemma* 2.2. *Then problem* (2.3), (1.2) *has at least one solution.*

Proof. Let ρ be the function appearing in Lemma 2.3, Δ be the number given by (2.14), $r=\rho(M)\Delta$, and let q_r be the positive constants and function introduced in the proof of Lemma 2.4. Then, as it was shown above, problem (2.3),(1.2) is equivalent to problem

(2.19),(1.2). Therefore, in order to prove the lemma, it is sufficient to show solvability of problem (2.19),(1.2).

Notice that in view of (1.12) and (1.13), q_r satisfies the inequalities

$$|q_r(x,y,\mathbf{v}_1,\mathbf{w}_1,\mathbf{z}_1)-q_r(x,y,\mathbf{v}_2,\mathbf{w}_2,\mathbf{z}_2)|\le\varepsilon\|\mathbf{v}_1-\mathbf{v}_2\|+M\|\mathbf{w}_1-\mathbf{w}_2\|+\gamma\big(\|\mathbf{z}_1-\mathbf{z}_2\|\big)$$
$$\text{for}\quad (x,y,\mathbf{v}_i,\mathbf{w}_i,\mathbf{z})_i\in\Omega\times\mathbb{R}^{n+m+mn}\quad(i=1,2), \tag{2.26}$$

$$|q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\le\eta\quad\text{for}\quad(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\in\Omega\times\mathbb{R}^{n+m+mn}, \tag{2.27}$$

where

$$\gamma(t)=\max\big\{|q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_1)-q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_2)|:(x,y)\in\Omega,$$
$$\|\mathbf{v}\|+\|\mathbf{w}\|+\|\mathbf{z}_1\|+\|\mathbf{z}_2\|\le 4r,\quad\|\mathbf{z}_1-\mathbf{z}_2\|\le t\big\}$$
$$\eta=\max\big\{|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})|:(x,y)\in\Omega,\quad\|\mathbf{v}\|+\|\mathbf{w}\|+\|\mathbf{z}\|\le 3r\big\}.$$

In view of Lemma 2.4 and condition (2.26), for arbitrarily fixed $z\in C^{m-1,n-1}(\Omega)$ the differential equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}$$
$$+q_r\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},z^{(m-1,n-1)}(x,y),\dots,z(x,y)\big). \tag{2.28}$$

has a unique solution satisfying conditions (1.2).

Let $\mathcal{K}:C^{m-1,n-1}(\Omega)\to C^{m,n}(\Omega)$ be the operator assigning to every $z\in C^{m-1,n-1}(\Omega)$ a solution of problem (2.28),(1.2).

Let

$$\Delta_1=\eta+\sum_{j=1}^{m}\|\varphi_j\|_{C^n([0,b])}+\sum_{k=1}^{n}\|\psi_k\|_{C^m([0,a])}$$

and

$$u(x,y)=\mathcal{K}(z)(x,y).$$

Then, in view of (2.27), the function u is also a solution of problem (2.4),(2.5), where $\delta=\Delta_1$. Hence, by Lemma 2.3, it follows that

$$\mathcal{K}(z)\in\widetilde{\mathcal{B}}^{m,n}(0,r_1), \tag{2.29}$$

where $r_1=\rho(M)\Delta_1$.

Now assume that

$$z_i \in C^{m-1,n-1}(\Omega) \ \ (i=1,2), \quad u(x,y) = \mathcal{K}(z_1)(x,y) - \mathcal{K}(z_1)(x,y).$$

Then, by (2.27), u is a solution of the problem (2.4),(2.5), with

$$\delta = \gamma\big(\|\mathbf{z}_1 - \mathbf{z}_2\|_{C^{m-1,n-1}(\Omega)}\big).$$

By Lemma 2.3,

$$\|\mathcal{K}(z_1) - \mathcal{K}(z_2)\|_{C^{m-1,n-1}(\Omega)} \leq \rho(M)\gamma\big(\|\mathbf{z}_1 - \mathbf{z}_2\|_{C^{m-1,n-1}(\Omega)}. \tag{2.30}$$

On the other hand, $\widetilde{\mathcal{B}}^{m,n}(0,r_1)$ is a compact subset of $\mathcal{B}^{m-1,n-1}(0,r_1)$, and $\gamma : [0,+\infty) \to [0,+\infty)$ is a continuous function such that $\gamma(0) = 0$. Therefore, (2.29) and (2.30) imply that $\mathcal{K}$ is a compact operator mapping $\mathcal{B}^{m-1,n-1}(0,r_1)$ into itself.

By Schauder's fixed point theorem, $\mathcal{K}$ has a fixed point $z_0 \in \mathcal{B}^{m-1,n-1}(0,r_1)$. Hence, it is clear that $u = \mathcal{K}(z_0)(x,y)$ is a solution of problem (2.19),(2.2). □

Lemma 2.6. *Let problem* (1.9) *have only the trivial solution for every* $x \in [0,a]$*, and let the function* $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ *be continuously differentiable with respect to* $\mathbf{v}$ *and* $\mathbf{w}$*, and locally Lipschitz continuous with respect to* $\mathbf{z}$*. Furthermore, let along with* (1.14) *the inequalities*

$$\sum_{k=0}^{n-1} |q_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq \varepsilon, \quad \sum_{j=0}^{m-1} |q_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq M$$
$$\textit{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \tag{2.31}$$

hold, where M *is a positive constant,* $\varepsilon \in [0, \frac{1}{2M_0}]$ *and* M_0 *is the number appearing in Lemma* 2.2*. Then problem* (2.3), (1.2) *is strongly well–posed.*

Proof. Let ρ be the function appearing in Lemma 2.3, Δ be the number defined by equality (2.14),

$$\delta_0 = \min\Big\{\frac{1}{2M_0} - \varepsilon, 1\Big\}, \quad r = \rho(M)(1+\Delta).$$

By virtue of (1.14) and (2.31), the inequality

$$|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq \frac{1}{2M_0}\|\mathbf{v}\| + M(\|\mathbf{w}\| + \|\mathbf{z}\|) + \Delta$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \tag{2.32}$$

holds. On the other hand, in view of local Lipschitz continuity of the function $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ with respect to $\mathbf{z}$, there exists $M_1 \geq M$ such that

$$|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_1) - q_r(x,y,\mathbf{v},\mathbf{w},\mathbf{z}_2)| \leq M_1\|\mathbf{z}_1 - \mathbf{z}_2\|$$
$$\text{for} \quad (x,y) \in \Omega, \|\mathbf{v}\| + \|\mathbf{w}\| + \|\mathbf{z})_i\| \leq r \;\; (i=1,2). \tag{2.33}$$

Along with problem (2.3),(1.2) consider the differential equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}$$
$$+\widetilde{q}\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big) \tag{$\widetilde{2.3}$}$$

with the boundary conditions $(\widetilde{1.2})$. Here $\widetilde{\varphi}_j \in C^n([0,b])$ $(j=1,\dots,m)$, $\widetilde{\psi}_k \in C^m([0,a])$ $(k=1,\dots,n)$, $\widetilde{q} \in C(\omega \times \mathbb{R}^{n+m+mn})$ and $\widetilde{q}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ is continuously differentiable with respect to $\mathbf{v}$ and $\mathbf{w}$. Furthermore,

$$|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) - \widetilde{q}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| + \sum_{k=0}^{n-1} |q_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) - \widetilde{q}_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})|$$
$$+ \sum_{j=0}^{m-1} |q_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) - \widetilde{q}_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| < \delta$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \tag{2.34}$$

and inequality (1.3) holds.

If $\delta \in [0,\delta_0]$, then in view of conditions (2.31)–(2.34), we get

$$\sum_{k=0}^{n-1} |\widetilde{q}_{mk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq \frac{1}{2M_0}, \quad \sum_{j=0}^{m-1} |\widetilde{q}_{jn}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq M+1$$
$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \tag{2.35}$$
$$|\widetilde{q}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \leq \frac{1}{2M_0}\|\mathbf{v}\| + M(\|\mathbf{w}\| + \|\mathbf{z}\|) + \Delta + 1$$
$$\text{for} \;\; (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \tag{2.36}$$
$$|q(x,y,\mathbf{v}_1,\mathbf{w}_1,\mathbf{z}_1) - \widetilde{q}(x,y,\mathbf{v}_2,\mathbf{w}_2,\mathbf{z}_2)| \leq \frac{1}{2M_0}\|\mathbf{v}_1 - \mathbf{v}_2\|$$
$$+M(\|\mathbf{w}_1 - \mathbf{w}_2\| + \|\mathbf{z}_1 - \mathbf{z}_2\|) \;\; \text{for } (x,y) \in \Omega, \;\; \|\mathbf{v}_i\| + \|\mathbf{w}_i\| + \|\mathbf{z}_i\| \leq r \;(i=1,2). \tag{2.37}$$

On the other hand, (1.3) and (2.14) imply

$$\sum_{j=1}^{m} \|\widetilde{\varphi}_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\widetilde{\psi}_k\|_{C^m([0,a])} < \Delta + 1. \tag{2.38}$$

By Lemma 2.4, the conditions of Lemma 2.6 guarantee unique solvability of problem (2.3),(1.2). On the other hand, by Lemma 2.5 and conditions (2.35) and (2.36), problem $(\widetilde{2.3}), (\widetilde{1.2})$ is solvable.

Let u_0 be a solution of problem (2.3),(1.2), $\delta \in [0, \delta_0]$, and let $\widetilde{u}$ be an arbitrary solution of problem $(\widetilde{2.3}), (\widetilde{1.2})$. By Lemma 2.3, conditions (2.14) and (2.32) guarantee validity of the estimate

$$\|u_0\|_{C^{m,n}(\Omega)} \leq \rho(M)\Delta < r, \tag{2.39}$$

while conditions (2.36) and (2.38) guarantee validity of the estimate

$$\|\widetilde{u}\|_{C^{m,n}(\Omega)} \leq r. \tag{2.40}$$

Set

$$u(x,y) = u_0(x,y) - \widetilde{u}(x,y).$$

Then, according to inequalities (2.3), (2.33), (2.39) and (2.40), the function u is a solution of the differential inequality

$$\left| u^{(m,n)} - \sum_{k=0}^{n-1} p_{mk}(x,y) u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y) u^{(j,n)} + \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} p_{jk}(x,y) u^{(j,k)} \right|$$
$$\leq \frac{1}{2M_0}|u| + M_1 \sum_{j=0}^{m-1} \sum_{k=0}^{n} |u^{(j,k)}| + \delta \tag{2.41}$$

satisfying boundary conditions (2.5). Hence, in view of Lemma 2.3, it follows the estimate

$$\|\widetilde{u}\|_{\widetilde{C}^{m,n}(\Omega)} \leq r_0 \delta,$$

where $r_0 = \rho(M_1)$ is the positive constant independent of $\widetilde{q}$ and $\widetilde{u}$. Consequently,

$$\widetilde{u} \in \widetilde{\mathcal{B}}^{m,n}(u_0, r_0\delta) \quad \text{for} \quad \delta \in [0, \delta_0].$$

Thus, strong well–posedness of problem (2.3),(1.2) is proved. $\square$

3. Proofs of the Main Results

Proof of Theorem 1.1. Let u_0 be a solution of problem (1.1), (1.2). Assume first that problem (1.1),(1.2) is (u_0, r)–well-posed for some $r > 0$. Consider the equation

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big)$$
$$+\Big(\sum_{(j,k)<(m,n)} p_{jk}(x,y)\big(u^{(j,k)} - u_0^{(j,k)}(x,y)\big) + f\big(x, y, u_0^{(m,0)}(x,y), \dots, u_0^{(m,n-1)}(x,y),$$
$$u_0^{(0,n)}(x,y), \dots, u_0^{(m-1,n)}(x,y), u_0^{(m-1,n-1)}(x,y), \dots, u_0(x,y)\big)$$
$$-f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big)\Big)$$
$$\times \Xi_\delta\Big(\sum_{(j,k)<(m,n)} \big|u^{(j,k)} - u_0^{(j,k)}(x,y)\big|\Big), \tag{3.1}$$

where $\Xi_\delta(z)$ is the function introduced in the proof of Lemma 2.4.

In view of continuous differentiability of f, if $u \in \widetilde{\mathcal{B}}^{m,n}(u_0;\delta)$, then

$$|f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big)$$
$$-\widetilde{f}\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big)| = o(\delta) \tag{3.2}$$

as $\delta \to 0$.

Therefore, in view of strong (u_0, r)–well-posedness of problem (1.1),(1.2), every solution of problem (3.1), $(\widetilde{1.2})$ admits the estimate

$$\|u - u_0\|_{\widetilde{C}^{m,n}(\Omega)} \le M o(\delta).$$

Choosing $\delta > 0$ sufficiently small, we achieve that $\|u - u_0\|_{\widetilde{C}^{m,n}(\Omega)} < \delta$, but then

$$\Xi_\delta\Big(\sum_{(j,k)<(m,n)} \big(u^{(j,k)} - u_0^{(j,k)}(x,y)\big)^2\Big) = 1,$$

and thus equation (3.1) receives the form

$$u^{(m,n)} = \sum_{(j,k)<(m,n)} p_{jk}(x,y)\big(u^{(j,k)} - u_0^{(j,k)}(x,y)\big) + f\big(x, y, u_0^{(m,0)}(x,y),$$
$$\dots, u^{(m,n-1)}(x,y), u_0^{(0,n)}(x,y), \dots, u_0^{(m-1,n)}(x,y), u_0^{(m-1,n-1)}(x,y), \dots, u_0(x,y)\big).$$

Denote $u - u_0$ by v. Thus, we have shown that the linear problem

$$v^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)v^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)v^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)v^{(j,k)}, \tag{3.3}$$

$$\begin{gathered} v^{(j-1,0)}(0,y) = \widetilde{\varphi}_j(y) \ \ (j = 1,\dots,m), \\ h_k(v^{(m,0)}(x,\cdot))(x) = \widetilde{\psi}_k^{(m)}(x) \ \ (k = 1,\dots,n) \end{gathered} \tag{3.4}$$

is solvable for arbitrary $\widetilde{\varphi}_j$ $(j = 1,\dots,m)$ and $\widetilde{\psi}_k$ $(k = 1,\dots,n)$ such that

$$\|\bar{\varphi}_j\|_{C^n([0,b])} < \delta \quad (j = 1,\dots,m), \quad \|\bar{\psi}_k\|_{C^m([0,a])} < \delta \quad (k = 1,\dots,n).$$

By Theorem 1.1 from [30], problem (3.3),(3.4) is well–posed. Consequently, problem $(1.5_0), (1.2_0)$ is well–posed too.

Now assume that problem $(1.5_0), (1.2_0)$ is well–posed. Then (u_0, r)–well-posedness of problem (1.1),(1.2) immediately follows from Lemma 2.6. □

Proof of Theorem 1.3. First notice that there exists $\varepsilon > 0$ sufficiently small such that in Theorem 1.2 the functions $P_{1mk}(x,y)$ and $P_{2mk}(x,y)$ $(k = 0,\dots,n-1)$ can be replaced by $P_{1mk}(x,y) - \varepsilon$ and $P_{2mk}(x,y) + \varepsilon$ $(k = 0,\dots,n-1)$.

Let u_0 be the solution of problem (1.1),(1.2). Then equation (1.11) can be written as follows

$$\begin{aligned} u^{(m,n)} = &\sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)} \\ &+\eta(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u) \\ &+q(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u), \end{aligned}$$

where

$$p_{jk}(x,y) = p_{jk}[u_0](x,y) \quad (j=0,\dots,m,\ k=0,\dots,n; j+k<m+n), \tag{3.5}$$

$$\begin{aligned} &\eta(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u) \\ &= f\big(x,y,u_0^{(m,0)}(x,y),\dots,u_0^{(m,n-1)}(x,y), \\ &u_0^{(0,n)}(x,y),\dots,u_0^{(m-1,n)}(x,y),u_0^{(m-1,n-1)}(x,y),\dots,u_0(x,y)\big) \\ &+\Big(f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big) \\ &-f\big(x,y,u_0^{(m,0)}(x,y),\dots,u_0^{(m,n-1)}(x,y),u_0^{(0,n)}(x,y), \\ &\dots,u_0^{(m-1,n)}(x,y),u_0^{(m-1,n-1)}(x,y),\dots,u_0(x,y)\big) \\ &-\sum_{(j,k)<(m,n)} p_{jk}(x,y)\big(u^{(j,k)}-u_0^{(j,k)}(x,y)\big)\Big)\Xi_\varepsilon\Big(\sum_{(j,k)<(m,n)}\big|u^{(j,k)}-u_0^{(j,k)}(x,y)\big|\Big). \end{aligned} \tag{3.6}$$

Hence, by Lemma 2.3, an arbitrary solution of problem (1.11),(1.2) admits estimate (2.9). Therefore, without loss of generality one can assume that

$$q(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = 0 \qquad \text{for} \quad \|\mathbf{v}\|+\|\mathbf{w}\|+\|\mathbf{z}\| \ge 3r,$$

where $r=\rho(M)\delta$. Notice that, for an arbitrary $\varepsilon>0$ the function q admits the representation

$$q(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = q_1(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) + q_2(x,y,\mathbf{v},\mathbf{w},\mathbf{z}),$$

where

$$|q_{1v_k}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \varepsilon \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \ \ (k=0,\dots,n-1),$$

$$|q_{1w_j}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le M \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \ \ (j=0,\dots,m-1),$$

$$|q_{1z_{jk}}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le M \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn}$$
$$(j=0,\dots,m-1;\ 0,\dots,n-1),$$

and

$$|q_2(x,y,\mathbf{v}_1,\mathbf{w},\mathbf{z}) - q_2(x,y,\mathbf{v}_2,\mathbf{w},\mathbf{z})| \le \varepsilon\|\mathbf{v}_1-\mathbf{v}_2\|$$

$$\text{for} \quad (x,y,\mathbf{v}_i,\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2),$$

$$|q_2(x,y,\mathbf{v},\mathbf{w}_1,\mathbf{z}) - q_2(x,y,\mathbf{v},\mathbf{w}_2,\mathbf{z})| \le M\|\mathbf{w}_1-\mathbf{w}_2\|$$

$$\text{for} \quad (x,y,\mathbf{v},\mathbf{w}_i,\mathbf{z}) \in \Omega\times\mathbb{R}^{n+m+mn} \quad (i=1,2),$$

$$|q_2(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \varepsilon \quad \text{for} \quad x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{mn}.$$

As q_1 one can use the Steklov average

$$q_\delta(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) = \frac{1}{\delta^{n+m+mn}} \int\limits_{v_1}^{v_1+\delta}\cdots\int\limits_{v_n}^{v_n+\delta}\int\limits_{w_1}^{w_1+\delta}\cdots\int\limits_{w_m}^{w_m+\delta}\int\limits_{z_1}^{z_1+\delta}\cdots\int\limits_{z_{mn}}^{v_1+\delta} q(x,y,\mathbf{s},\mathbf{t},\mathbf{r})\,d\mathbf{r}\,d\mathbf{t}\,d\mathbf{s}$$

for sufficiently small δ.

Notice that the function $f+q_1$ satisfies all of the conditions of Theorem 1.2 with the functions $P_{1mk}(x,y)-\varepsilon$ and $P_{2mk}(x,y)+\varepsilon$ $(k=0,\dots,n-1)$. Therefore, without loss of generality, one can assume that along with (1.13) and (1.14), the function q satisfies the condition

$$|q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| \le \varepsilon \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega\times\mathbb{R}^{mn}. \tag{3.7}$$

Now Theorem 1.3 immediately follows from Lemmas 2.5 and 2.6. □

Proof of Theorem 1.4. Set

$$q_0(x,y) = f(x,y,v_0(y),\dots,v_0^{(n-1)}(y),\varphi_1^{(n)}(y),\dots,\varphi_m^{(n)}(y),\phi_{m-1}^{(n-1)}(y),\dots,\varphi_1(y)).$$

and

$$p_{jk}(x,y) = f_{jk}\big(x,y,v_0(y),\dots,v_0^{(n-1)}(y),\varphi_1^{(n)}(y),\dots,\varphi_m^{(n)}(y),\phi_{m-1}^{(n-1)}(y),\dots,\varphi_1(y)\big).$$

In view of strong isolation of v_0, the linear problem

$$\zeta^{(n)} = \sum_{k=0}^{n-1} p_{mk}(0,y)\zeta^{(k)}, \quad h_k(\zeta)(0)=0 \ \ (k=1,\dots,n) \tag{3.8}$$

has only the trivial solution. Therefore, there exists $\delta_0>0$ such that the problem

$$\zeta^{(n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)\zeta^{(k)}, \quad h_k(\zeta)(x)=0 \ \ (k=1,\dots,n) \tag{3.9}$$

has only the trivial solution for every $x \in [0, \delta_0]$. In the domain Ω_{δ_0}, consider the linear equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_k(x,y)\big(u^{(m,k)} - v_0^{(k)}(y)\big) +$$
$$\sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)\big(u^{(m,k)} - \varphi_{j+1}^{(k)}(y)\big) + q_0(x,y). \tag{3.10}$$

By Theorem 1.1 from [30], we obtain that in the domain Ω_{δ_0} problem (3.9),(1.2) is well–posed and has a unique solution $u_0(x,y)$.

Consider the equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_k(x,y)\big(u^{(m,k)} - v_0^{(k)}(y)\big) + \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)\big(u^{(m,k)} - \varphi_{j+1}^{(k)}(y)\big)$$
$$+q\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{3.11}$$

where

$$q\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)$$
$$= q_0(x,y) + \Bigg(f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)$$
$$-\sum_{k=0}^{n-1} p_k(x,y)\big(u^{(m,k)} - v_0^{(k)}(y)\big) - \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)\big(u^{(m,k)} - \varphi_{j+1}(k)(y)\big)$$
$$-q(x,y)\Bigg)\Xi_\delta\Big(\sum_{(j,k)<(m,n)} \big|u^{(j,k)} - u_0^{(j,k)}(x,y)\big|\Big), \tag{3.12}$$

and Ξ_δ is the function defined in the proof of Lemma 2.4.

Notice that

$$\sum_{k=1}^{n-1} |u^{(m,k)}(x,y) - v_0^{(k)}(y)| + \sum_{j=0}^{m-1}\sum_{k=0}^{n} \big|u_0^{(m,k)} - \varphi_{j+1}^{(k)}(y)\big| \le \mu(\delta_0) \quad \text{for} \quad (x,y) \in \Omega_{\delta_0}, \tag{3.13}$$

where $\mu(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Due to continuous differentiability of the function f, in the ball $\mathcal{B}^{n-1}(u_0;\delta)$ q admits the estimate

$$|q\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)| \le \mu(\delta_0) + \beta(\delta), \tag{3.14}$$

where $\beta(\delta) = o(\delta)$. Notice that the functions μ and β depend only on

$$\sup\Big\{ \sum_{(j,k)<(m,n)} |f_{jk}\mathbf{v}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| : (x,y) \in \Omega,\ \|\mathbf{v}\| + \|\mathbf{w}\| + \|\mathbf{z}\| \le 2\rho \Big\},$$

where

$$\rho = 1 + \|v_0\|_{C^n[0,b]} + \sum_{j=1}^{m} \|\varphi_j\|_{C^n[0,b]} + \sum_{j=1}^{n} \|\psi_k\|_{C^n[0,a]}.$$

By Lemmas 2.3 and 2.5, problem (3.10),(1.2) is solvable and its every solution admits the estimate

$$\|u - u_0\|_{\widetilde{C}^{m},n(\Omega)} \le M(\alpha(\delta_0) + \beta(\delta)). \tag{3.15}$$

Choosing δ_0 and δ sufficiently small, one can achieve the estimate $\|u - u_0\|_{\widetilde{C}^{m},n(\Omega)} \le \delta$. However, then

$$\Xi_\delta\Big(\sum_{(j,k)<(m,n)} \big|u^{(j,k)} - u_0^{(j,k)}(x,y)\big|\Big) = 1$$

and every solution of equation (2.25) is a solution of equation (1.1) too. □

Proof of Theorem 1.5. Let u be a non–continuable solution of problem (1.1),(1.2) defined on the set $J \times [0,b]$. The fact that J is open in $[0,a]$ immediately follows from Theorem 1.4.

Let us assume the contrary: $J = [0,\alpha)$, and there exist $R > 0$ and a sequence $x_l \uparrow \alpha$ as $l \to +\infty$ such that

$$\|u^{(m,0)}(x_l,\cdot)\|_{C^{n-1}([0,b])} + \sum_{j=0}^{m-1} \|u^{(j,0)}(x_l,\cdot)\|_{C^n([0,b])} \le R \quad (l = 1,2,\dots). \tag{3.16}$$

Set $u^{(j-1,0)}(x_l,y) = \varphi_{lj}(y)$ and for equation (1.1) consider the initial–boundary value problem

$$\begin{aligned} u^{(j-1,0)}(x_l,y) &= \varphi_{lj}(y) \ \ (j = 1,\dots,m), \\ h_k(u^{(m,0)}(x,\cdot))(x) &= \psi_k^{(m)}(x) \ \ (k = 1,\dots,n). \end{aligned} \tag{3.17}$$

By Theorem 1.4, problem (1.1),(3.16) has a solution in the domain $[x_l, x_l+\delta] \times [0,b]$, where the number δ depends on the constant M from (3.15) and

$$\sup\Big\{ \sum_{(j,k)<(m,n)} |f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})| : (x,y) \in \Omega,\ \|\mathbf{v}\| + \|\mathbf{w}\| + \|\mathbf{z}\| \le 2\rho \Big\},$$

where $\rho = 1 + R + \sum_{j=1}^{n} \|\psi_k\|_{C^n[0,a]}$. Due to uniformly strong isolation of the family of solutions $v_l(y) = u^{(m,0)}(x_l, y)$ $(l = 1, 2, \dots)$, the constant M is independent of x_l, that makes δ independent of x_l too. But this leads to the contradiction

$$x_l + \delta < \alpha \quad (l = 1, 2, \dots).$$

The obtained contradiction proves the theorem. $\square$

Proof of Corollary 1.2. Theorem 1.2 and Corollary 3.6 from [19] imply that if $f(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})$ is continuously differentiable with respect to v, then the family of solutions $v(y) = u^{(m,0)}(x_0, y)$ $(x_0 \in J)$ is uniformly strongly isolated if and only if the homogeneous problem (1.29) has only the trivial solution for every $(p_0, \dots, p_{n-1}) \in S_J[u]$. $\square$

Proof of Theorem 1.2. By Corollary 3.6 from [19], conditions (A_2) and (A_3) guarantee existence of a strongly isolated solution of problem (1.18). Therefore, by Theorem 1.4, problem (1.1),(1.2) has a solution in the domain Ω_δ for some $\delta \in (0, a]$.

Let u be a solution of problem (1.1),(1.2). Then u is a solution of equation (1.5), where $q(x, y) = f(x, y, \mathbf{0}, \mathbf{0}, \mathbf{0})$ and

$$p_{jk}(x, y) = \int_0^1 p_{jk}[tu](x, y)\, dt \quad (j = 0, \dots, m,\ k = 0, \dots, n).$$

Due to the conditions (A_1) and (A_2), p_{jk} and q are bounded by functions P_{ik} and the constant R. Therefore, by Lemma 2.2, an arbitrary solution of problem (1.5),(1.2), and consequently, of problem (1.1),(1.2) admits the estimate

$$\|u\|_{C^{m,n}(\Omega)} \le M\Big(\sum_{j=1}^{m} \|\varphi_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\psi_k\|_{C^m([0,a])} + \|q\|_{C(\Omega)} \Big). \tag{3.18}$$

By Theorem 1.5, the latter a priori estimate guarantees that every non-continuable solution of problem (1.1),(1.2) is defined in the entire domain Ω.

Let u_0 be a solution of problem (1.1),(1.2). Equation (1.1) can be viewed as a perturbed linear equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p^*_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p^*_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk^*}(x,y)u^{(j,k)} + q^*(x,y), \tag{3.19}$$

where $p^*_{jk}(x,y) = p_{jk}[u_0](x,y)$ $(j = 0,\dots,m,\ k = 0,\dots,n)$ and

$$q^*(x,y) = f\big(x,y,u_0^{(m,0)}(x,y),\dots,u_0^{(m,n-1)}(x,y), u_0^{(0,n)}(x,y),\dots,u_0^{(m-1,n)}(x,y),u_0^{(m-1,n-1)}(x,y),\dots,u_0(x,y)\big).$$

By Lemma 2.6, problem (3.19),(1.2) and, consequently, problem (1.1),(1.2), is strongly well–posed. □

If inequality (1.35) (inequality (1.37)) holds, then by Theorem 1.1 from [21] (Theorem 2 from [41]) condition (A_3) of Theorem 1.2 holds. Hence by Theorem 1.2, there follows the validity of Corollary 1.3 (Corollary 1.4).

If inequality (1.41) holds, then by Theorem 9.8 from [16] condition (A_3) of Theorem 1.2 holds. Hence by Theorem 1.2, there follows the validity of Corollary 1.5.

Corollary 1.6 follows from Theorem 1.1 from [20] and Theorem 1.2.

Corollary 1.7 follows from Theorem 1.1 from [23] and Theorem 1.2.

Proof of Corollary 1.8. First notice that without loss of generality one can assume $f_0(0) = 0$. Let u be a solution of problem (1.43),(1.44). Multiply both sides of equation (1.43) by $u^{(1,0)}$ and integrate over $[0,x]\times[0,b]$. After integrating by parts we arrive at the equality

$$\int_0^x \int_0^b \Big(|u^{(1,n)}(s,t)|^2 - (-1)^n f_1(s,t,u(s,t),\dots,u^{(0,n-1)}(s,t))|u^{(1,0)}(s,t)|^2| \Big)\, dt\, ds$$

$$-(-1)^n \int_0^b F_0(u(x,t))\, dt = -(-1)^n \int_0^b F_0(\varphi(t))\, dt$$

$$= (-1)^n \int_0^x \int_0^b q(s,t,u(s,t),\dots,u^{(0,2n-1)})(s,t)\, dt\, ds,$$

where

$$F_0(z) = \int_0^z f_0(t)\, dt.$$

In view of (1.47)–(1.49) we get

$$\int_0^x \int_0^b \Big(|u^{(1,n)}(s,t)|^2 + \delta |u^{(1,0)}(s,t)|^2 \Big)\, dt\, ds \le M \int_0^x \int_0^b |u^{(1,0)}(s,t)|\, dt\, ds + M_1,$$

and

$$\int_0^x \int_0^b \Big(|u^{(1,n)}(s,t)|^2 + |u^{(1,0)}(s,t)|^2 \Big)\, dt\, ds \le M_2, \tag{3.20}$$

where

$$M_1 = \Big| \int_0^b F_0(\varphi(t))\, dt \Big|, \qquad M_2 = \frac{2+\delta}{\delta} \Big(\frac{M^2}{2\delta} + M_1 \Big).$$

(3.20) implies the estimate

$$\|u\|_{C^{0,n-1}(\Omega)} \le \rho, \tag{3.21}$$

where ρ is a positive number depending on a, b, δ, M, M_1 and φ only.

For an arbitrary $v \in C^{0,2n-1}(\Omega)$ consider the equations

$$u^{(1,2n)} = f_1\big(x,y,v(x,y),\dots,v^{(0,n-1)}(x,y)\big) u^{(1,0)} + f_0\big(x,y,u\big)$$
$$+ q(x,y,v(x,y),\dots,v^{(m-1,n-1)}(x,y)) \tag{3.22}$$

and

$$
\begin{aligned}
u^{(1,2n)} =& f_1\big(x, y, v(x,y), \ldots, v^{(0,n-1)}(x,y)\big)u^{(1,0)} + f_0\big(x, y, \chi_\rho(u)\big) \\
&+ q(x, y, v(x,y), \ldots, v^{(m-1,n-1)}(x,y))
\end{aligned} \tag{3.23}
$$

where $\chi_r(t)$ is a nondecreasing continuously differentiable function such that $\chi_r'(t) \le 1$ and

$$
\chi_r(t) = \begin{cases} t & \text{for} \quad t \in [0, r] \\ 2r & \text{for} \quad t \in (3r, +\infty) \end{cases} .
$$

By Theorem 1.2, problem (3.23),(1.44) is uniquely solvable. It is clear an arbitrary solution of problem (3.23),(1.44) admits estimate (3.21). Consequently, every solution of problem (3.23),(1.44) is a solution of problem (3.22),(1.44) too. Thus problem (3.22),(1.44) is uniquely solvable for every $v \in C^{0,2n-1}(\Omega)$.

Consider the operator $\mathcal{K} : v \to u$. By Lemma 2.3, $\mathcal{K} : C^{0,2n-1}(\Omega) \to C^{1,2n}(\Omega)$ is a continuous operator. In view of estimate (3.21), $\mathcal{K} : C^{0,2n-1}(\Omega) \to C^{0,2n-1}(\Omega)$ is a compact operator mapping $C^{0,2n-1}(\Omega)$ in to ball $\mathcal{B}^{0,2n-1}(0, \rho)$.

By Schauder's fixed point theorem, $\mathcal{K}$ has a fixed point $v_0 \in \mathcal{B}^{0,2n-1}(0, \rho)$. Hence $u = \mathcal{K}(v_0)(x, y) = v_0(x, y)$ is a solution of the problem (1.43),(1.44). □

CHAPTER II
Ill–Posed Initial–Boundary Value Problems

4. Formulation of the Main Results

4.1. General Initial–Boundary Value Problems. In this chapter we study the equations [2]

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=m_0+1}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} \\ + F\big(x,y,u^{(0,n)},\dots,u^{(m_0,n)},u^{(m_0,n-1)},\dots,u\big), \qquad (4.1_{m_0})$$

with the initial–boundary conditions

$$\begin{aligned} u^{(j-1,0)}(0,y) &= \varphi_j(y) \ \ (j=1,\dots,m),\\ h_k(u^{(m,0)}(x,\cdot))(x) &= \psi_k^{(m)}(x) \ \ (k=1,\dots,n). \end{aligned} \qquad (4.2)$$

Let the functions $\zeta_k : \Omega \to \mathbb{R}$ $(k=1,\dots,n)$ be such that for every $x \in [0,a]$ $\zeta_1(x,\cdot),\dots,\zeta_m(x,\cdot)$ is the fundamental set of solutions of the ordinary differential equation

$$\frac{d^n z}{dy^n} = \sum_{k=0}^{n-1} p_{mk}(x,y)\frac{d^k z}{dy^k}. \qquad (4.3)$$

Set

$$H(x) = \begin{pmatrix} h_1(\zeta_1(x,\cdot))(x) & \dots & h_1(\zeta_n(x,\cdot))(x)\\ \cdot & \dots & \cdot\\ \cdot & \dots & \cdot\\ \cdot & \dots & \cdot\\ h_n(\zeta_1(x,\cdot))(x) & \dots & h_n(\zeta_n(x,\cdot))(x) \end{pmatrix}.$$

In Chapter I, for problem (4.1_{m_0}), (4.2) we studied the case where

$$\det H(x) \neq 0 \quad \text{for} \quad x \in [0,a],$$

i.e., where equation (4.3) had only the trivial solution satisfying the boundary conditions

$$h_k(z)(x) = 0 \quad (k=0,\dots,n) \qquad (4.4)$$

for every $x \in [0,a]$.

[2] If $m_0 = m$ 1, then the double sum in the equation disappears.

In this chapter we consider the case, where there exists $n_0 \in \{1, \dots, n\}$ such that problem (4.3),(4.4) has an n_0–dimensional space of solutions for arbitrary $x \in [0, a]$, i.e.,

$$\operatorname{rank} H(x) = n_1 = n - n_0 \quad \text{for} \quad x \in [0, a].$$

By virtue of Lemmas 2.1 and 2.2 from [31], without loss of generality one can assume that if $p_{mk} \in C^{l,0}(\Omega)$ $(k = 0, \dots, n-1)$, $h_k : C^{n-1}([0, b]) \to C^l([0, a])$ $(k = 1 \dots, n)$ are bounded linear operators, then the fundamental set of solutions $\zeta_1, \dots, \zeta_n$ belongs to $C^{l,n}(\Omega)$ and

$$\text{either } H(x) \equiv \Theta_{n,n}; \ \text{ or } n_0 < n, \ H(x) = \begin{pmatrix} \Theta_{n_0,n_0} & \Theta_{n_0,n_1} \\ \Theta_{n_1,n_0} & H_0(x) \end{pmatrix}$$
$$\text{and } \ \det H_0(x) \neq 0 \quad \text{for} \quad x \in [0, a], \tag{4.5}$$

or even

$$\text{either} \quad H(x) \equiv \Theta_{n,n}; \quad \text{or } n_0 < n, \quad H(x) = \begin{pmatrix} \Theta_{n_0,n_0} & \Theta_{n_0,n_1} \\ \Theta_{n_1,n_0} & I_{n_1,n_1} \end{pmatrix}, \tag{4.5'}$$

where Θ_{n_i,n_k} is the zero $n_i \times n_k$ matrix, and I_{n_1,n_1} is the unit $n_1 \times n_1$ matrix.

For an arbitrary fixed $x \in [0, a]$ by $\zeta(\cdot, \cdot, x)$ denote the Cauchy function of equation (4.3) and set:

$$F_{jn}(x, y, \mathbf{w}, \mathbf{z}) = \frac{\partial F(x, y, \mathbf{w}, \mathbf{z})}{\partial w_j} \quad (j = 0, \dots, m-1),$$
$$F_{jk}(x, y, \mathbf{w}, \mathbf{z}) = \frac{\partial F(x, y, \mathbf{w}, \mathbf{Z})}{\partial z_{jk}} \quad (j = 0, \dots, m-1; \ k = 0, \dots, n-1),$$
$$p_{jk}[u](x, y) = F_{jk}\Big(x, y, u^{(0,n)}(x, y), \dots, u^{(m-1,n)}(x, y), u^{(m-1,n-1)}(x, y), \dots, u(x, y)\Big),$$
$$\overset{\circ}{p}_{jk}(y) = F_{jk}\Big(0, y, \varphi_0^{(n)}(y), \dots, \varphi_{m-1}^{(n)}(y), \varphi_{m-1}^{(n-1)}(y), \dots, \varphi_1(y)\Big),$$
$$\rho_{jk}[u](x, y) = p_{jk}(u)(x, y) + p_{jn}(u)(x, y) p_{mk}(x, y),$$
$$\overset{\circ}{\rho}_{jk}(y) = \overset{\circ}{p}_{jk}(y) + \overset{\circ}{p}_{jn}(y) p_{mk}(0, y) \quad (j = 0, \dots, m-1; \ k = 0, \dots, n-1),$$
$$\eta_{jk}[u](x, y; x_0) = \int_0^y \zeta(y, t, x) \sum_{l=0}^{n-1} \rho_{jl}[u](x_0, t) \zeta_k^{(0,l)}(x, t)\, dt,$$
$$\overset{\circ}{\eta}_{jk}(x, y) = \int_0^y \zeta(y, t, x) \sum_{l=0}^{n-1} \overset{\circ}{\rho}_{jl}(t) \zeta_k^{(0,l)}(x, t)\, dt \quad (j = 0, \dots, m-1; \ k = 1, \dots, n_0);$$

$$\lambda_{j_{ik}}[u](x) = h_i(\eta_{jk}[u](x,\cdot))(x) \; (i,k = 1,\dots,n),$$
$$\Lambda_j[u](x) = \big(\lambda_{j_{ik}}[u](x)\big)_{1,1}^{n_0,n_0} \; (j = 0,\dots m-1),$$
$$\overset{\circ}{\lambda}_{j_{ik}} = h_i(\overset{\circ}{\eta}_{jk}[u](\cdot))(0) \; (i,k = 1,\dots,n), \qquad \overset{\circ}{\Lambda}_j = (\overset{\circ}{\lambda}_{j_{ik}})_{l,k=1}^{n_0} \; (j = 0,\dots,m-1).$$

Theorem 4.1. *Let along with* (4.5) *the conditions*

$$p_{mk}(x,y) \equiv p_{mk}(y) \quad (k = 0,\dots,n-1), \tag{4.6}$$

hold and let there exist $m_0 \in \{0,\dots,m-2\}$ *such that*

$$p_{jk}(x,y) + p_{jn}(x,y)p_{mk}(y) \equiv 0 \quad (j = m_0+1,\dots,m-1;\; k = 0,\dots,n-1), \tag{4.7}$$
$$\det \overset{\circ}{\Lambda}_{m_0} \neq 0. \tag{4.8}$$

Furthermore, let $F(x,y,\mathbf{w},\mathbf{z})$ *be* $m - m_0$*–times continuously differentiable with respect to* x, $\mathbf{w}$ *and* $\mathbf{z}$, $\psi_k \in C^{2m-m_0}([0,a])$ $(k = 1,\dots,n)$ *and* $h_k : C^{n-1}([0,b]) \to C^{m-m_0}([0,a])$ $(k = 1,\dots,n)$ *be bounded linear operators. Then problem* $(4.1_{m_0}), (4.2)$ *has a unique solution* u *in the rectangle* $\Omega_\alpha = [0,\alpha] \times [0,b]$ *for some* $\alpha \in (0,a]$ *if and only if the equalities*

$$h_k(\theta)(0) = \psi_i(0) \quad (i = 1,\dots,n_0) \tag{4.9}$$

hold, where

$$\theta(y) = \int_0^y \zeta(y,t)\Big(\sum_{j=m_0+1}^{m-1}\sum_{k=0}^{n} p_{jk}(0,t)\varphi_j^{(k)}(t) + F\big(0,t,\varphi_1^{(n)}(t),\dots,\varphi_{m_0+1}^{(n)}(t),\varphi_{m_0+1}^{(n-1)}(t),\dots,\varphi_1(t)\big)\Big)\,dt. \tag{4.10}$$

Theorem 4.2. *Let condition* (4.5) *hold and*

$$\det \overset{\circ}{\Lambda}_{m-1} \neq 0. \tag{4.11}$$

Furthermore, let $F(x,y,\mathbf{w},\mathbf{z})$ *be continuously differentiable with respect to* x, $\mathbf{w}$ *and* $\mathbf{z}$, $\psi_k \in C^{m+1}([0,a])$ *and* $h_k : C^{n-1}([0,b]) \to C^1([0,a])$ $(k = 1,\dots,n)$ *be bounded linear operators. Then problem* $(4.1_{m-1}), (4.2)$ *has a unique solution* u *in the rectangle* $\Omega_\alpha = [0,\alpha] \times [0,b]$ *for some* $\alpha \in (0,a]$ *if and only if equalities* (4.9) *hold.*

Remark 4.1. Let $m_0 \in \{0, \ldots, m-2\}$ $(m_0 = m-1)$ and let all of the conditions of Theorem 4.1 (Theorem 4.2) hold. Then a solution u of problem $(4.1_{m_0}), (1.2)$ (problem $(4.1_{m-1}), (1.2)$) admits the estimates

$$\|u\|_{C^{m_0+j,n}(\Omega_x)} \le M\Big(\sum_{i=1}^{m} \|\varphi_i\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\psi_k\|_{C^j([0,x])} + \|q\|_{C^{j,0}(\Omega_x)}\Big) \qquad (j = 0, \ldots, m-m_0) \tag{4.12}$$

for $x \in [0, \alpha]$, where $q(x,y) = F(x, y, \mathbf{0}, \mathbf{0})$, and M is a positive constant independent of φ_i, ψ_k $(i = 1, \ldots, m;\ k = 1, \ldots, n)$ and q.

Theorem 4.3. *Let all of the conditions of Theorem* 4.1 *hold, and let u be a non-continuable solution of problem* $(4.1_{m_0}), (4.2)$ *defined on* $J \times [0, b]$ *such that*

$$\det \Lambda_{m_0}[u](x) \neq 0 \quad \textit{for} \quad x \in J. \tag{4.13}$$

Then either $J = [0, a]$, *or* $J = [0, \alpha)$ *and*

$$\lim_{x\to\alpha} \Big(\|u^{(m_0,0)}(x,\cdot)\|_{C^{n-1}([0,b])} + \sum_{j=0}^{m_0} \|u^{(j,0)}(x,\cdot)\|_{C^n([0,b])}\Big) = +\infty. \tag{4.14}$$

Theorem 4.4. *Let all of the conditions of Theorem* 4.2 *hold, and let u be a non-continuable solution of problem* $(4.1_{m-1}), (4.2)$ *defined on* $J \times [0, b]$ *such that*

$$\det \Lambda_{m-1}[u](x) \neq 0 \quad \textit{for} \quad x \in J. \tag{4.15}$$

Then either $J = [0, a]$, *or* $J = [0, \alpha)$ *and*

$$\lim_{x\to\alpha} \Big(\|u^{(m-1,0)}(x,\cdot)\|_{C^{n-1}([0,b])} + \sum_{j=0}^{m-1} \|u^{(j,0)}(x,\cdot)\|_{C^n([0,b])}\Big) = +\infty. \tag{4.16}$$

Corollary 4.1. *Let all of the conditions of Theorem* 4.1 *hold, and let there exist* $\delta > 0$ *and* $M > 0$ *such that*

$$|\det \Lambda_{m_0}[v](x)| \ge \delta \quad \textit{for} \quad x \in [0, a] \tag{4.17}$$

for any $v \in C^{m_0,n}(\Omega)$ *and*

$$|F(x, y, \mathbf{w}, \mathbf{z})| \le M\Big(1 + \|\mathbf{w}\| + \|\mathbf{z}\|\Big). \tag{4.18}$$

Then problem $(4.1_{m_0}), (4.2)$ *has a unique solution in* Ω *if and only if* (4.9) *holds.*

Corollary 4.2. *Let all of the conditions of Theorem 4.2 hold, and let there exist $\delta > 0$ and $M > 0$ such that along with (4.18) the following condition*

$$|\det \Lambda_{m-1}[v](x)| \geq \delta \quad \textit{for} \quad x \in [0, a] \tag{4.19}$$

holds for any $v \in C^{m-1,n}(\Omega)$. Then problem $(4.1_{m-1}), (4.2)$ has a unique solution in Ω if and only if (4.9) holds.

Remark 4.2. Conditions (4.13) and (4.17) in Theorem 4.3 and Corollary 4.1 (conditions (4.15) and (4.19) in Theorem 4.4 and Corollary 4.2) are essential and cannot be weakened. In the rectangle $\Omega = [0, 2] \times [0, 1]$ consider the following three problems

$$u^{(m,1)} = |u|^{2m+1} u^{(m_0,0)} + m_0!\, u^{2m+1}, \tag{4.20}$$

$$u^{(j,0)}(0, y) = c_j \quad (j = 0, \ldots, m-1), \quad u^{(m,0)}(x, 0) = u^{(m,0)}(x, 1), \tag{4.21}$$

where $c_0 = 1$, $c_{m_0} = -1$ and $c_j = 0$ for $j \in \{1, \ldots, m-1\} \setminus \{m_0\}$;

$$u^{(1,1)} = (1-x)^2 e^y - (1-x)^3 - (1-x)\sin^2 \frac{1}{1-x}, \tag{4.22}$$

$$u(0, y) = 1, \quad u^{(1,0)}(x, 0) = u^{(1,0)}(x, 1); \tag{4.23}$$

and

$$u^{(1,2)} = u^3 - x^3 \sin \frac{1}{x}, \tag{4.24}$$

$$u(0, y) = 0, \quad u^{(1,k-1)}(x, 0) = u^{(1,k-1)}(x, 1) \ (k = 1, 2). \tag{4.25}$$

By Theorem 4.1, problem (4.20),(4.21) has a unique local solution u, which is independent of y (due to uniqueness). Therefore u is a solution to the initial value problem

$$u^{(m_0)} = -m_0!\ \mathrm{sgn}(u), \quad u(0) = 1, \ u^{(j)}(0) = 0 \quad (j = 1, \ldots, m_0 - 1). \tag{4.26}$$

One can easily see that problem (4.26) has a unique non-continuable solution

$$u(x) = 1 - x^{m_0}$$

defined on $[0, 1]$ and

$$\lim_{x \to 1} \big(|u(x)| + |u'(x)|\big) = m_0! < +\infty.$$

Problem (4.22),(4.23) has the unique noncontinuable solution

$$u(x) = \ln\Big(1 - x + \sin^2\frac{1}{1-x}\Big)$$

defined on the interval $[0,1)$, and

$$\liminf_{x\to 1}\big(|u(x)| + |u'(x)|\big) < \infty, \quad \limsup_{x\to 1}\big(|u(x)| + |u'(x)|\big) = \infty.$$

Let u be a solution of problem (4.24),(4.25). It is not difficult to verify that problem (4.24),(4.25) has no other solution , and that u is independent of y. Therefore u satisfies the equation

$$u^3 - x^3\sin\frac{1}{x} = 0.$$

Consequently

$$u(x) = x\sin^{\frac{1}{3}}\frac{1}{x}$$

is the unique *absolutely continuous* but not a classical solution of problem (4.23),(4.24). Although the righthand side of equation (4.23) is continuously differentiable, problem (4.23),(4.24) does not have a classical solution in $\Omega_\alpha = [0,\alpha)\times[0,1]$ no matter how small $\alpha > 0$ is, since u is not differentiable along the points $x = \frac{1}{\pi k}$ $(k = 1,2,\dots)$.

4.2. **Two–point Initial Boundary Value Problems.** Consider the initial–periodic problem

$$u^{(m,n)} = F\big(x,y,u^{(0,n)},\dots,u^{(m_0,n)},u^{(m_0,n-1)},\dots,u\big), \tag{4.27}$$

$$\begin{gathered} u^{(j,0)}(0,y) = \varphi_j(y) \quad (j = 0,\dots,m-1),\\ u^{(m,k)}(x,0) = u^{(m,k)}(x,b) \ \ (k = 0,\dots,n-1) \end{gathered} \tag{4.28}$$

Corollary 4.3. *Let $m_0 \in \{0,\dots,m-1\}$, F be $m - m_0$–times continuously differentiable with respect to x, $\mathbf{w}$ and $\mathbf{z}$ and*

$$\int\limits_0^b \overset{\circ}{p}_{m_0 0}(t)\,dt \neq 0. \tag{4.29}$$

Then problem (4.27), (4.28) *has a unique solution* u *in the rectangle* $\Omega_\alpha = [0,\alpha]\times[0,b]$ *for some* $\alpha \in (0,a]$ *if and only if*

$$\int\limits_0^b F\big(0,t,\varphi_1^{(n)}(t),\dots,\varphi_{m_0+1}^{(n)}(t),\varphi_{m_0+1}^{(n-1)}(t),\dots,\varphi_1(t)\big)\,dt = 0. \tag{4.30}$$

Corollary 4.4. *Let* $m_0 \in \{0,\dots,m-1\}$, F *be* $m-m_0$*–times continuously differentiable with respect to* x, $\mathbf{w}$ *and* $\mathbf{z}$, *and let there exist* $\sigma \in \{-1,1\}$ *and* $\delta > 0$ *such that*

$$\sigma\int\limits_0^b p_{m_0 0}[z](x,t)\,dt \ge \delta \quad \text{for} \quad x\in[0,a] \tag{4.31}$$

for every $z \in C^{m_0,n}(\Omega)$. *Furthermore, let*

$$|F(x,y,\mathbf{w},\mathbf{z})| \le M(1+\|\mathbf{w}\|+\|\mathbf{z}\|)| \quad \text{for} \quad (x,y,\mathbf{w},\mathbf{z})\in\Omega\times\mathbb{R}^{m+mn} \tag{4.32}$$

for some $M>0$. *Then problem* (4.27), (4.28) *has a unique solution* u *in the rectangle* Ω *if and only if* (4.30) *holds.*

For the equation

$$u^{(m,2)} = -u^{(m,0)} + F\big(x,y,u^{(0,2)},\dots,u^{(m_0,2)},u^{(m_0,1)},\dots,u\big) \tag{4.33}$$

consider the initial–Dirichlet and initial–periodic problems

$$u^{(j,0)}(0,y) = \varphi_j(y)\ (j=0,\dots,m-1), \quad u^{(m,0)}(x,0)=0,\ u^{(m,0)}(x,\pi)=0; \tag{4.34}$$

$$u^{(j,0)}(0,y) = \varphi_j(y)\ (j=0,\dots,m-1), \quad u^{(m,k)}(x,0)=u^{(m,k)}(x,2\pi)\ (k=0,1). \tag{4.35}$$

Corollary 4.5. *Let* $m_0 \in \{0,\dots,m-1\}$, F *be* $m-m_0$*–times continuously differentiable with respect to* x, $\mathbf{w}$ *and* $\mathbf{z}$, *and*

$$\int\limits_0^\pi \big(\mathring{p}_{m_0 2}(t)+\mathring{p}_{m_0 1}(t)\big)\sin t\,dt \ne 0. \tag{4.36}$$

Then problem (4.33), (4.34) *has a unique solution* u *in the rectangle* $\Omega_\alpha = [0,\alpha]\times[0,b]$ *for some* $\alpha\in(0,a]$ *if and only if*

$$\int\limits_0^\pi F\big(0,t,\varphi_1''(t),\dots,\varphi_{m_0+1}''(t),\varphi_{m_0+1}'(t),\dots,\varphi_1(t)\big)\sin t\,dt = 0. \tag{4.37}$$

Corollary 4.6. *Let $m_0 \in \{0,\dots,m-1\}$, F be $m-m_0$–times continuously differentiable with respect to x, $\mathbf{w}$ and $\mathbf{z}$, and let there exist $\sigma \in \{-1,1\}$ and $\delta > 0$ such that*

$$\sigma \int\limits_0^\pi \big(p_{m_0 2}[z](x,t) + p_{m_0 1}[z](x,t)\big)\,dt \geq \delta \quad \textit{for} \quad x \in [0,a] \tag{4.38}$$

for every $z \in C^{m_0,2}(\Omega)$. Furthermore, let

$$|F(x,y,\mathbf{w},\mathbf{z})| \leq M(1 + \|\mathbf{w}\| + \|\mathbf{z}\|)| \quad \textit{for} \quad (x,y,\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{3m} \tag{4.39}$$

for some $M > 0$. Then problem (4.33), (4.34) *has a unique solution u in the rectangle Ω if and only if* (4.37) *holds.*

Corollary 4.7. *Let $m_0 \in \{0,\dots,m-1\}$, F be $m-m_0$–times continuously differentiable with respect to x, $\mathbf{w}$ and $\mathbf{z}$, and*

$$\det \begin{pmatrix} \int_0^{2\pi} (\overset{\circ}{p}_{m_0 0}(t) - \overset{\circ}{p}_{m_0 2}(t))\, \sin t\, dt & \int_0^{2\pi} \overset{\circ}{p}_{m_0 1}(t)\, \sin t\, dt \\ \int_0^{2\pi} (\overset{\circ}{p}_{m_0 0}(t) - \overset{\circ}{p}_{m_0 2}(t))\, \cos t\, dt & \int_0^{2\pi} \overset{\circ}{p}_{m_0 1}(t)\, \cos t\, dt \end{pmatrix} \neq 0. \tag{4.40}$$

Then problem (4.33), (4.35) *has a unique solution u in the rectangle $\Omega_\alpha = [0,\alpha] \times [0, 2\pi]$ for some $\alpha \in (0,a]$ if and only if*

$$\int\limits_0^\pi F\big(0,t,\varphi_1''(t),\dots,\varphi_{m_0+1}''(t),\varphi_{m_0+1}'(t),\dots,\varphi_1(t)\big)\cos t\, dt = 0, \tag{4.41}$$

$$\int\limits_0^\pi F\big(0,t,\varphi_1''(t),\dots,\varphi_{m_0+1}''(t),\varphi_{m_0+1}'(t),\dots,\varphi_1(t)\big)\sin t\, dt = 0. \tag{4.42}$$

Corollary 4.8. *Let $m_0 \in \{0,\dots,m-1\}$, F be $m-m_0$–times continuously differentiable with respect to x, $\mathbf{w}$ and $\mathbf{z}$, and let there exist $\sigma \in \{-1,1\}$ and $\delta > 0$ such that*

$$\sigma \det \begin{pmatrix} \int_0^{2\pi} (p_{m_0 0}[z](x,t) - p_{m_0 2}[z](x,t)))\, \sin t\, dt & \int_0^{2\pi} p_{m_0 1}[z](x,t)\, \sin t\, dt \\ \int_0^{2\pi} (p_{m_0 0}[z](x,t) - p_{m_0 2}[z](x,t))\, \cos t\, dt & \int_0^{2\pi} p_{m_0 1}[z](x,t)\, \cos t\, dt \end{pmatrix} \geq \delta \quad \textit{for} \quad x \in [0,a] \tag{4.43}$$

for every $z \in C^{m_0,2}(\Omega)$. Furthermore, let (4.32) *hold for some $M > 0$. Then problem* (4.33), (4.35) *has a unique solution u in the rectangle Ω if and only if* (4.41) *and* (4.42) *hold.*

Consider the equation

$$u^{(1,2n)} = F\big(x, y, u\big) \tag{4.44}$$

consider the initial-periodic problem

$$u(0,y) = \varphi(y), \qquad u^{(m,k-1)}(x,0) = u^{(m,k-1)}(x,b) \ \ (k = 1, \dots, 2n). \tag{4.45}$$

Corollary 4.9. *Let F be continuously differentiable function and let there exist $\delta > 0$ such that*

$$(-1)^{n-1} F_z(x,y,z) \geq \delta \quad \textit{for} \quad (x,y,z) \in \Omega \times \mathbb{R}.$$

Then problem (4.44), (4.45) *is uniquely solvable if and only if*

$$\int_0^b F\big(0, t, \varphi(t)\big)\, dt = 0. \tag{4.46}$$

5. Lemmas on Representation of Solutions of Problem (4.1_{m_0}), (4.2)

For every $j \in \{0, \dots, m-1\}$ and $y \in [0,b]$ by $\overset{\circ}{\xi}_j(\cdot, y)$ denote a solution of the equation

$$w^{(m)} = \sum_{j=0}^{m_0} p_{jn}(x,y) w^{(j)} + \sum_{j=m_0+1}^{m-1} p_{jn}(x,y) w^{(j)}$$

satisfying the initial conditions $w^{(i)}(0) = \delta_{ij}$ $(i = 0, \dots, m-1)$, and by $\xi(\cdot,\cdot,y)$ denote the Cauchy function of this equation.

Let

$$\eta(z)(x,y) = \int_0^y \zeta(y,t,x) z(t)\, dt,$$

where $\zeta(y,t,x)$ is the Cauchy function of equation (4.3). By Lemma 2.1_4 from [24], for each $k \in \{1, \dots, n\}$ the operator $h_k : C^{n-1}([0,b]) \to C^{m-m_0}(I)$ uniquely defines a function $T_k \in L_\infty(\Omega)$, satisfying the conditions

$$T_k^{(m-m_0,0)} \in L_\infty(\Omega), \quad \int_0^y T_k^{(m-m_0,0)}(\cdot, t)\, dt \in C([0,a]) \quad \text{for} \quad y \in [0,b],$$

such that

$$h_k(\eta(z)(x,\cdot))(x) = \int\limits_0^b T_k(x,t)z(t)\,dt.$$

If $\det \overset{\circ}{\Lambda}_j \neq 0$, then by $\overset{\circ}{\gamma}_{j_{ik}}$ denote the (i,k)–element of the matrix $\overset{\circ}{\Lambda}{}_j^{-1}$. Set

$$\tilde{\overset{\circ}{\lambda}}_{j_{kl}} = \sum_{i=1}^{n_0} \overset{\circ}{\gamma}_{j_{ki}} \overset{\circ}{\lambda}_{j_{il}}, \qquad \tilde{\overset{\circ}{T}}_{jk}(x,y) = \sum_{l=1}^{n_0} \overset{\circ}{\gamma}_{j_{kl}} T_l(x,y);$$

$$\begin{aligned}\overset{\circ}{\Upsilon}[u](x,y) = F\Big(&x,y,u^{(0,n)}(x,y),\dots,u^{(m-1,n)}(x,y),u^{(m-1,n-1)}(x,y),\dots,u(x,y)\Big)\\ &-F\Big(0,y,\varphi_1^{(n)}(y),\dots,\varphi_m^{(n)}(y),\varphi_m^{(n-1)}(y),\dots,\varphi_1(y)\Big)\\ &-\sum_{j=0}^{m-1}\sum_{k=0}^{n} \overset{\circ}{p}_{jk}(y)\big(u^{(j,k)}(x,y)-\varphi_{j+1}^{(k)}(y)\big)\big);\end{aligned}$$

$$\overset{\circ}{q}(y) = F\Big(0,y,\varphi_1^{(n)}(y),\dots,\varphi_m^{(n)}(y),\varphi_m^{(n-1)}(y),\dots,\varphi_1(y)\Big) + \sum_{j=0}^{m-1}\sum_{k=0}^{n} \overset{\circ}{p}_{jk}(y)\,\varphi_{j+1}^{(k)}(y);$$

$$\overset{\circ}{\mathcal{Q}}_{m_0}(u)(x,y) = \sum_{i=0}^{m_0}\sum_{k=0}^{n-1} \overset{\circ}{\rho}_{ik}(y)u^{(i,k)}(x,y) + \overset{\circ}{q}(y);$$

$$\overset{\circ}{\Phi}(x,y) = \sum_{j=0}^{m-1} \overset{\circ}{\xi}_j(x,y)\Big(\varphi_{j+1}^{(n)}(y) - \sum_{k=1}^{n} p_{mk}(y)\varphi_{j+1}^{(k)}(y)\Big);$$

$$\begin{aligned}\overset{\circ}{\Phi}_{m_0}(x,y) = &\int\limits_0^y \zeta(y,t)\overset{\circ}{\Phi}{}^{(m,0)}(x,t)\,dt\\ &+\int\limits_0^y \zeta(y,t)\Big(\sum_{k=0}^{n-1}\overset{\circ}{\rho}_{m_0k}(t)\int\limits_0^t \zeta^{(k,0)}(t,\tau)\overset{\circ}{\Phi}{}^{(m_0,0)}(x,\tau)\,d\tau\Big)\,dt;\end{aligned}$$

$$\begin{aligned}\overset{\circ}{K}_{m_0}(x,y,s,t) =&\zeta(y,t)\overset{\circ}{\xi}{}^{(m,0,0)}(x,s,t)\\ &+\overset{\circ}{\xi}{}^{(m_0,0,0)}(x,s,t)\int\limits_t^y \zeta(y,\tau)\sum_{k=0}^{n-1}\overset{\circ}{\rho}_{m_0k}(\tau)\zeta^{(k,0)}(\tau,t)\,d\tau;\end{aligned}$$

$$\begin{aligned}\overset{\circ}{G}_l(x,s,t) =&T_l(x,t)\overset{\circ}{\xi}{}^{(m,0,0)}(x,s,t)\\ &+\overset{\circ}{\xi}{}^{(m_0,0,0)}(x,s,t)\int\limits_t^b T_l(x,\tau)\sum_{k=0}^{n-1}\overset{\circ}{\rho}_{m_0k}(\tau)\zeta^{(k,0)}(\tau,t)\,d\tau;\end{aligned}$$

$$\overset{\circ}{f}_l(x) = \psi_l(x) - h_l(\overset{\circ}{\Phi}_{m_0}(x,\cdot))(x), \quad \overset{\circ}{\widetilde{f}}_{jl}(x) = \sum_{k=1}^{n_0} \overset{\circ}{\gamma}_{jlk} f_k(x),$$

$$\overset{\circ}{\widetilde{G}}_{jl}(x,s,t) = \sum_{k=1}^{n_0} \overset{\circ}{\gamma}_{jlk} \overset{\circ}{G}_k(x,s,t).$$

Let $u_{jl} \in C^{m-j}([0,a])$ $(j = 0,\dots,m;\ l = 1,\dots,n)$ be the functions satisfying the system of linear algebraic equations

$$\sum_{l=1}^{n} \zeta^{(0,k)}(x,0) u_{jl}(x) = u^{(j,k)}(x,0) \quad (k = 0,\dots,n-1).$$

Lemma 5.1. *Let all of the conditions of Theorem* 4.1 *hold and u be a solution of problem* $(4.1_{m_0}), (4.2)$*. Then there exists $\alpha \in (0,a]$ such that for every $j \in \{0,\dots,m-1\}$ the representation*

$$u^{(j,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y) u_{jl}(x) + \int_0^y \zeta(y,t) \overset{\circ}{\Phi}{}^{(j,0)}(x,t)\,dt + \int_0^x \int_0^y \zeta(y,t) \overset{\circ}{\xi}{}^{(j,0,0)}(x,s,t) \big(\overset{\circ}{\mathcal{Q}}_{m_0}[u](s,t) + \overset{\circ}{\Upsilon}[u](s,t)\big)\,dt\,ds. \tag{5.1_j}$$

is valid in Ω_α. Furthermore,

$$u^{(m,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y) u_{ml}(x) + \int_0^y \zeta(y,t) \overset{\circ}{\Phi}{}^{(m,0)}(x,t)\,dt + \int_0^y \zeta(y,t) \big(\overset{\circ}{\mathcal{Q}}_{m_0}[u](x,t) + \overset{\circ}{\Upsilon}[u](x,t)\big)\,dt + \int_0^x \int_0^y \zeta(y,t) \overset{\circ}{\xi}{}^{(m,0,0)}(x,s,t) \big(\overset{\circ}{\mathcal{Q}}_{m_0}[u](s,t) + \overset{\circ}{\Upsilon}[u](s,t)\big)\,dt\,ds, \tag{5.2}$$

$$u^{(m,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y) u_{ml}(x) + \sum_{l=1}^{n} \eta_{m_0 l}(x,y) u_{m_0 l}(x) + \overset{\circ}{\Phi}_{m_0}(x,y) + \int_0^y \zeta(y,t) \big(\overset{\circ}{\mathcal{Q}}_{m_0-1}[u](x,t) + \overset{\circ}{\Upsilon}[u](x,t)\big)\,dt$$

$$+\int_0^x\int_0^y K_{m_0}(x,y,s,t)\big(\mathring{\mathcal{Q}}_{m_0}[u](s,t)+\mathring{\Upsilon}[u](s,t)\big)\,dt\,ds, \tag{5.3}$$

and the functions u_{m_0l} and u_{ml} satisfy the equalities

$$u_{m_0l}(x) = -\sum_{r=n_0+1}^{n}\overset{\circ}{\widetilde{\lambda}}_{m_0lr}(x)u_{m_0r}(x)+\overset{\circ}{\widetilde{f}}_{m_0l}(x)$$

$$-\int_0^b \overset{\circ}{\widetilde{T}}_{m_0l}(x,t)\big(\mathring{\mathcal{Q}}_{m_0-1}[u](x,t)+\mathring{\Upsilon}[u](x,t)\big)\,dt$$

$$-\int_0^x\int_0^b \overset{\circ}{\widetilde{G}}_{m_0l}(x,s,t)\big(\mathring{\mathcal{Q}}_{m_0}[u](s,t)+\mathring{\Upsilon}[u](s,t)\big)\,dt\,ds \quad (l=1,\dots,n_0), \tag{5.4}$$

$$u_{ml}(x)=\psi_l(x)-\int_0^b T_l(x,t)\mathring{\Phi}^{(m,0)}(x,t)\,dt$$

$$-\int_0^b T_l(x,t)\big(\mathring{\mathcal{Q}}_{m_0}[u](x,t)+\mathring{\Upsilon}[u](x,t)\big)\,dt$$

$$-\int_0^x\int_0^b T_l(x,t)\xi^{(m,0,0)}(x,s,t)\big(\mathring{\mathcal{Q}}_{m_0}[u](s,t)+\mathring{\Upsilon}[u](s,t)\big)\,dt\,ds$$

$$(l=n_0+1,\dots,n). \tag{5.5}$$

Proof. Let u be a solution of problem $(4.1_{m_0}), (4.2)$. Set $v(x,y)=u^{(m,0)}(x,y)$,

$$w(x,y)=u^{(0,n)}(x,y)-\sum_{k=0}^{n-1}p_{mk}(y)u^{(0,k)}(x,y).$$

Then

$$v^{(0,n)}(x,y)=\sum_{k=0}^{n-1}p_{mk}(y)v^{(0,k)}(x,y)+\sum_{j=0}^{m_0}\sum_{k=0}^{n}\mathring{p}_{jk}(y)u^{(j,k)}$$

$$\sum_{j=m_0+1}^{m-1}\sum_{k=0}^{n}p_{jk}(x,y)u^{(j,k)}+\mathring{q}(y)+\mathring{\Upsilon}[u](x,y),$$

$$w^{(m,0)}(x,y)=\sum_{j=0}^{m_0}\mathring{p}_{jn}(y)w^{(j,0)}(x,y)+\sum_{j=m_0+1}^{m-1}p_{jn}(x,y)w^{(j,0)}(x,y)$$

$$+\mathring{\mathcal{Q}}_{m_0}[u](x,y)+\mathring{\Upsilon}[u](x,y),$$

$$w^{(j-1,0)}(0,y) = \varphi_j^{(n)}(y) - \sum_{k=0}^{n-1} p_{mk}(y)\varphi_j^{(k)}(y) \quad (j = 1,\dots,m-1).$$

From these representation we get

$$u^{(m,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y)u_{ml}(x) + \int_0^y \zeta(y,t)\Big(\sum_{j=0}^{m-1}\sum_{k=0}^{n} \overset{\circ}{p}_{jk}(t)u^{(j,k)}(x,t)$$

$$+ \sum_{j=m_0+1}^{m-1}\sum_{k=0}^{n} p_{jk}(x,t)u^{(j,k)}(x,t) + \overset{\circ}{q}(t) + \overset{\circ}{\Upsilon}[u](x,t)\Big)\,dt, \tag{5.6}$$

$$u^{(j,n)}(x,y) = \sum_{k=0}^{n-1} p_{mk}(y)u^{(j,k)}(x,y) + \overset{\circ}{\Phi}{}^{(j,0)}(x,y)$$

$$+ \int_0^x \overset{\circ}{\xi}{}^{(j,0,0)}(x,s,y)\big(\overset{\circ}{\mathcal{Q}}_{m_0}[u](s,y) + \overset{\circ}{\Upsilon}[u](s,y)\big)\,ds \;\; (j = 0,\dots,m-1). \tag{5.7}$$

Equalities (5.1_j) ($j = 0,\dots,m-1$) and (5.2) immediately follow from equalities (5.6) and (5.7). As for (5.3), it follows from (5.1_{m_0}) and (5.2).

Applying operators h_l to equalities (5.2) and (5.3) and utilising conditions (4.5′) and (4.8), we obtain equalities (5.4) and (5.5). $\square$

Introduce the functions

$$\Psi(x,y) = \sum_{j=0}^{m-1} \overset{\circ}{\xi}_j(x,y)\varphi_{j+1}^{(n)}(y);$$

$$\overset{\circ}{\rho}_{m-1k}(x,y) = \overset{\circ}{p}_{m-1k}(y) + \overset{\circ}{p}_{m-1n}(y)p_{mk}(x,y) \quad (k = 0,\dots,n-1);$$

$$\overset{\circ}{d}_k(x,s,y) = \overset{\circ}{\xi}{}^{(m-1,0,0)}(x,s,y)\big(p_{m-1k}(s,y) - p_{mk}^{(1,0)}(s,y)\big) - \overset{\circ}{\xi}{}^{(m-1,1,0)}(x,s,y)p_{mk}(s,y);$$

$$\overset{\circ}{\beta}_k(x,y,s,t) = \zeta(y,t,x)\overset{\circ}{d}{}_k^{(1,0,0)}(x,s,t)$$

$$+\overset{\circ}{d}_k(x,s,t)\int_t^y \zeta(y,\tau,x)\sum_{l=0}^{n-1}\overset{\circ}{\rho}_{m-1l}(x,\tau)\zeta^{(l,0,0)}(\tau,t,x)\,d\tau;$$

$$\overset{\circ}{K}(x,y,s,t) = \zeta(y,t,x)\overset{\circ}{\xi}{}^{(m,0,0)}(x,s,t)$$

$$+\overset{\circ}{\xi}{}^{(m-1,0,0)}(x,s,t)\int_t^y \zeta(y,\tau,x)\sum_{l=0}^{n-1}\overset{\circ}{\rho}_{m-1l}(x,\tau)\zeta^{(l,0,0)}(\tau,t,x)\,d\tau,$$

$$\overset{\circ}{\widetilde{\Psi}}(x,y)=\int\limits_0^y \zeta(y,t,x)\Big(\overset{\circ}{\Psi}{}^{(m,0)}(x,t)$$

$$+\sum_{k=0}^{n-1}\overset{\circ}{\rho}_{m-1k}(x,t)\int\limits_0^t \zeta^{(k,0,0)}(t,\tau,x)\overset{\circ}{\Psi}{}^{(m-1,0)}(x,\tau)\,d\tau\Big)\,dt$$

and the operator

$$\overset{\circ}{\mathcal{Q}}[u]=\sum_{j=0}^{m-2}\sum_{k=0}^{n-1}\overset{\circ}{p}_{jk}(y)u^{(j,k)}+\overset{\circ}{q}(y).$$

The following lemma can be proved similarly to Lemma 5.1.

Lemma 5.2. *Let all of the conditions of Theorem* 4.2 *hold, and let* u *be a solution of problem* $(4.1_{m-1}),(4.2)$. *Then there exists* $\alpha\in(0,a]$ *such that in* Ω_α u *admits the following representations:*

$$u^{(m-1,0)}(x,y)=\sum_{l=1}^n \zeta_l(x,y)u_{m-1l}(x)+\int\limits_0^y \zeta(y,t,x)\overset{\circ}{\Psi}{}^{(m-1,0)}(x,t)\,dt$$

$$+\int\limits_0^x\int\limits_0^y \zeta(y,t,x)\Big(\sum_{k=0}^{n-1}\overset{\circ}{d}_k(x,s,t)u^{(m-1,k)}(s,t)$$

$$+\overset{\circ}{\xi}{}^{(m-1,0,0)}(x,s,t)\Big(\overset{\circ}{\mathcal{Q}}[u](s,t)+\overset{\circ}{\Upsilon}[u](s,t)\Big)\Big)\,dt\,ds,$$

$$u^{(m,0)}(x,y)=\sum_{l=1}^n \zeta_l(x,y)u_{ml}(x)+\int\limits_0^y \zeta(y,t,x)\overset{\circ}{\Psi}{}^{(m,0)}(x,t)\,dt$$

$$+\int\limits_0^y \zeta(y,t,x)\Big(\sum_{k=0}^{n-1}\overset{\circ}{\rho}_{m-1k}(x,t)u^{(m-1,k)}(x,t)+\overset{\circ}{\mathcal{Q}}[u](x,t)+\overset{\circ}{\Upsilon}[u](x,t)\Big)\,dt$$

$$+\int\limits_0^x\int\limits_0^y \zeta(y,t,x)\sum_{k=0}^{n-1}\Big(\overset{\circ}{d}{}_k^{(1,0,0)}(x,s,t)u^{(m-1,k)}(s,t)$$

$$+\overset{\circ}{\xi}{}^{(m,0,0)}(x,s,t)\Big(\overset{\circ}{\mathcal{Q}}[u](s,t)+\overset{\circ}{\Upsilon}[u](s,t)\Big)\Big)\,dt\,ds,$$

$$u^{(m,0)}(x,y)=\sum_{l=1}^n \zeta_l(x,y)u_{ml}(x)+\sum_{l=1}^n \eta_{m-1l}(x,y)u_{m-1l}(x)$$

$$+\widetilde{\Psi}(x,y)+\int\limits_0^y \zeta(y,t,x)\Big(\mathring{\mathcal{Q}}[u](x,t)+\mathring{\Upsilon}[u](x,t)\Big)\,dt$$
$$+\int\limits_0^x\int\limits_0^y\sum_{k=0}^{n-1}\mathring{\beta}_k(x,y,s,t)u^{(m-1,k)}(s,t)\,dt\,ds$$
$$+\int\limits_0^x\int\limits_0^y \mathring{K}(x,y,s,t)\Big(\mathring{\mathcal{Q}}[u](s,t)+\mathring{\Upsilon}[u](s,t)\Big)\,dt\,ds.$$

Lemmas 5.1 and 5.2 are about representation of a solution of problem $(4.1_{m_0}),(4.2)$ in the rectangle $\Omega_\alpha=[0,\alpha]\times[0,b]$ for sufficiently small $\alpha>0$. Similar representations in the rectangle $[x_0,x_0+\alpha]\times[0,b]$ are given in the Lemmas 5.1_{x_0} and 5.2_{x_0} below. Before formulating the lemmas introduce the following notation:

For every $j\in\{0,\dots,m-1\}$ and $y\in[0,b]$ by $\xi_j[u](\cdot,y;x_0)$ denote a solution of the equation

$$w^{(m)}=\sum_{j=0}^{m_0}p_{jn}(x,y)w^{(j)}+\sum_{j=m_0+1}^{m-1}p_{jn}[u](x_0,y)w^{(j)}$$

satisfying the initial conditions $w^{(i)}(x_0)=\delta_{ij}$ $(i=0,\dots,m-1)$, and by $\xi[u](\cdot,\cdot,y;x_0)$ denote the Cauchy function of this equation.

If $\det\Lambda_j[u](x)\neq 0$, by $\gamma_{j_{ik}}[u](x)$ denote the (i,k)–element of the matrix $\Lambda_j^{-1}[u](x)$. Set

$$\widetilde{\lambda}_{j_{kl}}[u](x,y)=\sum_{i=1}^{n_0}\gamma_{j_{ki}}[u](x)\lambda_{j_{il}}[u](x),\quad \widetilde{T}_{jk}[u](x,y,;x_0)=\sum_{l=1}^{n_0}\gamma_{j_{kl}}[u](x_0)T_l(x,y);$$
$$\Upsilon[u](x,y;x_0)=F\Big(x,y,u^{(0,n)}(x,y),\dots,u^{(m-1,n)}(x,y),u^{(m-1,n-1)}(x,y),\dots,u(x,y)\Big)$$
$$-F\Big(x_0,y,u^{(0,n)}(x_0,y),\dots,u^{(m-1,n)}(x_0,y),u^{(m-1,n-1)}(x_0,y),\dots,u(x_0,y)\Big)$$
$$-\sum_{j=0}^{m-1}\sum_{k=0}^{n}p_{jk}[u](x_0,y)\big(u^{(j,k)}(x,y)-u^{(j,k)}(x_0,y)\big)$$
$$q[u](x_0,y)=F\Big(x_0,y,u^{(0,n)}(x_),y),\dots,u^{(m-1,n)}(x_0,y),u^{(m-1,n-1)}(x_0,y),\dots,u(x_0,y)\Big)$$
$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n}p_{jk}[u](x_0,y)u^{(j,k)}(x_0,y);$$

$$\mathcal{Q}_{m_0}[u](x,y;x_0)=\sum_{j=0}^{m_0}\sum_{k=0}^{n-1}\rho_{jk}[u](x_0,y)u^{(j,k)}(x,y)+q[u](x_0,y);$$

$$\begin{aligned}
\Phi[u](x,y;x_0)=&\sum_{j=0}^{m-1}\xi_j[u](x,y)\Big(u^{(j,n)}(x_0,y)-\sum_{k=1}^{n}p_{mk}(y)u^{(j,k)}(x_0,y)\Big);\\
\Phi_{m_0}[u](x,y;x_0)=&\int_0^y\zeta(y,t)\Phi^{(m,0)}[u](x,t;x_0)(x,t)\,dt\\
&+\int_0^y\zeta(y,t)\Big(\sum_{k=0}^{n-1}\rho_{m_0k}[u](x_0,t)\int_0^t\zeta^{(k,0)}(t,\tau)\Phi^{(m_0,0)}[u](x,\tau;x_0)\,d\tau\Big)\,dt;\\
K_{m_0}[u](x,y,s,t;x_0)=&\zeta(y,t)\xi^{(m,0,0)}(x,s,t;x_0)\\
&+\xi^{(m_0,0,0)}[u](x,s,t;x_0)\int_t^y\zeta(y,\tau)\sum_{k=0}^{n-1}\rho_{m_0k}[u](x_0,\tau)\zeta^{(k,0)}(\tau,t)\,d\tau;\\
G_l[u](x,s,t;x_0)=&T_l(x,t)\xi^{(m,0,0)}[u](x,s,t;x_0)\\
&+\xi^{(m_0,0,0)}[u](x,s,t;x_0)\int_t^b T_l(x,\tau)\sum_{k=0}^{n-1}\rho_{m_0k}(x_0,\tau)\zeta^{(k,0)}(\tau,t)\,d\tau;
\end{aligned}$$

$$f_l[u](x;x_0)=\psi_l(x)-h_l(\Phi_{m_0}[u](x,\cdot;x_0))(x),\quad \widetilde{f}_{jl}[u](x;x_0)=\sum_{k=1}^{n_0}\gamma_{j_{lk}}[u](x_0)f_k[u](x;x_0),$$

$$\widetilde{G}_{jl}[u](x,s,t;x_0)=\sum_{k=1}^{n_0}\gamma_{j_{lk}}[u](x)G_k[u](x,s,t;x_0).$$

Lemma 5.1$_{\mathbf{x_0}}$. *Let all of the conditions of Theorem 4.1 hold and u be a solution of problem $(4.1_{m_0}), (4.2)$. Then there exists $\alpha\in(0,a]$ such that for every $j\in\{0,\dots,m-1\}$ the representation*

$$\begin{aligned}
u^{(j,0)}(x,y)=&\sum_{l=1}^{n}\zeta_l(x,y)u_{jl}(x)+\int_0^y\zeta(y,t)\Phi^{(j,0)}[u](x,t;x_0)\,dt\\
&+\int_{x_0}^x\int_0^y\zeta(y,t)\overset{\circ}{\xi}{}^{(j,0,0)}(x,s,t)\big(\mathcal{Q}_{m_0}[u](s,t;x_0)+\Upsilon[u](s,t;x_0)\big)\,dt\,ds.
\end{aligned}$$

is valid in $[x_0, x_0+\alpha] \times [0,b]$. *Furthermore,*

$$u^{(m,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y) u_{ml}(x) + \int\limits_0^y \zeta(y,t) \Phi^{(m,0)}[u](x,t;x_0)\,dt$$

$$+ \int\limits_0^y \zeta(y,t) \big(\mathcal{Q}_{m_0}[u](x,t;x_0) + \Upsilon[u](x,t;x_0)\big)\,dt$$

$$+ \int\limits_{x_0}^x \int\limits_0^y \zeta(y,t) \xi^{(m,0,0)}[u](x,s,t;x_0) \big(\mathcal{Q}_{m_0}[u](s,t;x_0) + \Upsilon[u](s,t;x_0)\big)\,dt\,ds,$$

$$u^{(m,0)}(x,y) = \sum_{l=1}^{n} \zeta_l(x,y) u_{ml}(x) + \sum_{l=1}^{n} \eta_{m_0 l}(x,y) u_{m_0 l}(x) + \Phi_{m_0}[u](x,y;x_0)$$

$$+ \int\limits_0^y \zeta(y,t) \big(\mathcal{Q}_{m_0-1}[u](x,t;x_0) + \Upsilon[u](x,t;x_0)\big)\,dt$$

$$+ \int\limits_{x_0}^x \int\limits_0^y K_{m_0}[u](x,y,s,t;x_0) \big(\mathcal{Q}_{m_0}[u](s,t;x_0) + \Upsilon[u](s,t;x_0)\big)\,dt\,ds,$$

and the functions $u_{m_0 l}$ *and* u_{ml} *satisfy the equalities*

$$u_{m_0 l}(x) = -\sum_{r=n_0+1}^{n} \widetilde{\lambda}_{m_0 l r}(x) u_{m_0 r}(x) + \widetilde{f}_{m_0 l}[u](x;x_0)$$

$$- \int\limits_0^b \widetilde{T}_{m_0 l}(x,t) \big(\mathcal{Q}_{m_0-1}[u](x,t;x_0) + \Upsilon[u](x,t;x_0)\big)\,dt$$

$$- \int\limits_{x_0}^x \int\limits_0^b \widetilde{G}_{m_0 l}[u](x,s,t;x_0) \big(\mathcal{Q}_{m_0}[u](s,t;x_0) + \Upsilon[u](s,t;x_0)\big)\,dt\,ds \quad (l=1,\ldots,n_0),$$

$$u_{ml}(x) = \psi_l(x) - \int\limits_0^b T_l(x,t) \overset{\circ}{\Phi}{}^{(m,0)}(x,t)\,dt$$

$$- \int\limits_0^b T_l(x,t) \big(\mathcal{Q}_{m_0}[u](x,t;x_0) + \Upsilon[u](x,t;x_0)\big)\,dt$$

$$- \int\limits_{x_0}^x \int\limits_0^b T_l(x,t) \xi^{(m,0,0)}[u](x,s,t;x_0) \big(\mathcal{Q}_{m_0}[u](s,t;x_0) + \Upsilon[u](s,t;x_0)\big)\,dt\,ds$$

$(l = n_0 + 1, \dots, n)$.

Introduce the following operators:

$$\mathcal{Q}[u](x,y;x_0) = \sum_{j=0}^{m-2}\sum_{k=0}^{n-1} p_{jk}[u](x_0,y)u^{(j,k)}(x,y) + q[u](x_0,y);$$

$$\Psi[u](x,y;x_0) = \sum_{j=0}^{m-1} \xi_j[u](x,y)u^{(j,n)}(x_0,y);$$

$$\rho_{m-1k}[u](x,y) = p_{m-1k}[u](x,y) + p_{m-1n}[u](x,y)p_{mk}(x,y) \quad (k = 0,\dots,n-1);$$

$$d_k[u](x,s,y;x_0) = \xi^{(m-1,0,0)}[u](x,s,y;x_0)(p_{m-1k}(s,y) - p_{mk}^{(1,0)}(s,y))$$

$$-\xi^{(m-1,1,0)}[u](x,s,y;x_0)p_{mk}(s,y);$$

$$\beta_k[u](x,y,s,t;x_0) = \zeta(y,t,x)d_k^{(1,0,0)}[u](x,s,t;x_0)$$

$$+d_k[u](x,s,t;x_0)\int_t^y \zeta(y,\tau,x)\sum_{l=0}^{n-1}\rho_{m-1l}[u](x_0,\tau)\zeta^{(l,0,0)}(\tau,t,x)\,d\tau;$$

$$K[u](x,y,s,t;x_0) = \zeta(y,t,x)\xi^{(m,0,0)}[u](x,s,t;x_0)$$

$$+\xi^{(m-1,0,0)}(x,s,t;x_0)\int_t^y \zeta(y,\tau,x)\sum_{l=0}^{n-1}\rho_{m-1l}[u](x_0,\tau)\zeta^{(l,0,0)}(\tau,t,x)\,d\tau,$$

$$\widetilde{\Psi}[u](x,y;x_0) = \int_0^y \zeta(y,t,x)\Big(\Psi^{(m,0)}[u](x,t;x_0)$$

$$+\sum_{k=0}^{n-1}\rho_{m-1k}[u](x_0,t)\int_0^t \zeta^{(k,0,0)}(t,\tau,x)\Psi^{(m-1,0)}[u](x,\tau;x_0)\,d\tau\Big)\,dt$$

Lemma 5.2$_{\mathbf{x_0}}$. *Let all of the conditions of Theorem 4.2 hold, and let u be a solution of problem $(4.1_{m-1}), (4.2)$. Then there exists $\alpha \in (0,a]$ such that in $[x_0, x_0+\alpha]\times[0,b]$ u admits the following representations:*

$$u^{(m-1,0)}(x,y) = \sum_{l=1}^{n}\zeta_l(x,y)u_{m-1l}(x) + \int_0^y \zeta(y,t,x)\Psi^{(m-1,0)}[u](x,t;x_0)\,dt$$

$$
\begin{aligned}
&+\int\limits_{x_0}^{x}\int\limits_{0}^{y}\zeta(y,t,x)\Big(\sum_{k=0}^{n-1}d_k[u](x,s,t;x_0)u^{(m-1,k)}(s,t)\\
&+\xi^{(m-1,0,0)}[u](x,s,t;x_0)\Big(\mathcal{Q}[u](s,t;x_0)+\Upsilon[u](s,t;x_0)\Big)\Big)\,dt\,ds,\\
u^{(m,0)}(x,y)=&\sum_{l=1}^{n}\zeta_l(x,y)u_{ml}(x)+\int\limits_{0}^{y}\zeta(y,t,x)\Psi^{(m,0)}[u](x,t;x_0)\,dt\\
&+\int\limits_{0}^{y}\zeta(y,t,x)\Big(\sum_{k=0}^{n-1}\rho_{m-1k}[u](x_0,t)u^{(m-1,k)}(x,t)\\
&+\mathcal{Q}[u](x,t;x_0)+\Upsilon[u](x,t;x_0)\Big)\,dt\\
&+\int\limits_{x_0}^{x}\int\limits_{0}^{y}\zeta(y,t,x)\sum_{k=0}^{n-1}\Big(d_k^{(1,0,0)}[u](x,s,t;x_0)u^{(m-1,k)}(s,t)\\
&+\xi^{(m,0,0)}[u](x,s,t;x_0)\Big(\mathcal{Q}[u](s,t;x_0)+\Upsilon[u](s,t;x_0)\Big)\Big)\,dt\,ds,\\
u^{(m,0)}(x,y)=&\sum_{l=1}^{n}\zeta_l(x,y)u_{ml}(x)+\sum_{l=1}^{n}\eta_{m-1l}(x,y)u_{m-1l}(x)\\
&+\widetilde{\Psi}(x,y)+\int\limits_{0}^{y}\zeta(y,t,x)\Big(\mathring{\mathcal{Q}}[u](x,t)+\mathring{\Upsilon}[u](x,t)\Big)\,dt\\
&+\int\limits_{x_0}^{x}\int\limits_{0}^{y}\sum_{k=0}^{n-1}\beta_k(x,y,s,t)u^{(m-1,k)}(s,t)\,dt\,ds\\
&+\int\limits_{x_0}^{x}\int\limits_{0}^{y}K(x,y,s,t)\Big(\mathcal{Q}[u](s,t;x_0)+\Upsilon[u](s,t;x_0)\Big)\,dt\,ds.
\end{aligned}
$$

Lemmas 5.1_{x_0} and 5.2_{x_0} can be proved similarly to Lemma 5.1.

The following lemmas are borrowed from [31].

Lemma 5.3. *Let* $m_0 \in \{0,\dots,m-2\}$ *and conditions* (4.5), (4.6) *and* (4.8) *hold* $\big(m_0 = m-1$ *and conditions* (4.5) *and* (4.1) *hold* $\big)$. *Then there exist* $\varepsilon_0 > 0$ *and* $\alpha > 0$

such that the problem

$$z^{(n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)z^{(k)} + \varepsilon \sum_{k=0}^{n-1} \mathring{\rho}_{m_0k}(y)z^{(k)}, \quad h_k(z)(x) = 0 \quad (k = 1, \dots, n)$$

has only a trivial solution for any $\varepsilon \in (0, \varepsilon_0]$ *and* $x \in [0, \alpha]$.

Lemma 5.4. *Let* l *be a nonnegative integer,* m *be a natural number,* d_i $(i = 0, \dots, m)$ *be the natural numbers defined from the identity* $(t+1)(t+2)\dots(t+m) \equiv \sum_{i=0}^{m} d_i t^i$, $q \in C^l([0, a])$, *and* z_ε *be a solution of the differential equation*

$$\sum_{i=0}^{m} d_i \varepsilon^i z^{(i)} = m! q(x)$$

for arbitrary $\varepsilon \in (0, 1]$. *Then there exists a positive constant* M *independent of* ε *and* q *such that the inequalities*

$$|\varepsilon^j z_\varepsilon^{(r+j)}(x)| \le M \exp(-\varepsilon^{-1} x) \Big(\sum_{i=0}^{l-1} \varepsilon^i |z_\varepsilon^{(r+i)}(0)|$$

$$+ \varepsilon^{-1} \int_0^x \exp(\varepsilon^{-1} s) |q^{(r)}(s)| \, ds \Big) \quad \textit{for} \quad x > 0 \quad (j = 0, \dots, m),$$

$$|z_\varepsilon^{(k)}(x) - q^{(k)}(x)| \le M \exp(-\varepsilon^{-1} x) \Big(|z_\varepsilon^{(k)}(0) - q^{(k)}(x)| + \sum_{i=1}^{m-1} x^i |z_\varepsilon^{(k+i)}(0)| \Big)$$

$$+ M \Big(\exp(-\varepsilon^{-1}\delta) \|q^{(k)}\|_{C([0,x])} + \max\{|q^{(r)}(s) - q^{(r)}(x)| : s \in [x - \delta, x] \Big) \quad \textit{for} \quad x > \delta$$

hold for arbitrary $k \in \{0, \dots, l\}$.

6. Proofs of the Main Results

Proof of Theorem 1.1. For the sake of simplicity assume that conditions (4.5′) hold. Let problem $(4.1_{m_0}), (4.2)$ has a solution u. Then, as it was shown in the proof of Lemma 5.1, u admits representation (5.6) and, consequently,

$$u^{(m,0)}(0, y) = \sum_{l=1}^{n} \zeta_l(0, y) u_{ml}(0) + \theta(y),$$

where θ is the function given by (4.10). Hence, in view of conditions (4.2) and (4.5′), we arrive at equalities (4.9).

Consider the equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(y)u^{(m,k)}+\sum_{j=m_0+1}^{m-1}\sum_{k=0}^{n}p_{jk}(x,y)u^{(j,k)}+\frac{1}{m_1!}\sum_{j=1}^{m_1}\sum_{k=0}^{n-1}c_j\,\varepsilon^j\,\overset{\circ}{\rho}_{m_0k}(y)u^{(m_0+j,k)}$$
$$+F\big(x,y,u^{(0,n)},\dots,u^{(m_0,n)},u^{(m_0,n-1)},\dots,u\big). \qquad (6.1_\varepsilon)$$

Notice that due to continuous differentiability of the function F with respect to $x,\mathbf{w}$ and $\mathbf{z}$, for arbitrary $\varepsilon_1>0$ and $r>0$ there exists $\alpha=\alpha(\varepsilon_1,r)>0$ such that

$$|\overset{\circ}{\Upsilon}[u_1](x,y)-\overset{\circ}{\Upsilon}[u_2](x,y)|\le\varepsilon_1\sum_{j=0}^{m_0}\sum_{k=0}^{n}|u_1^{(j,k)}(x,y)-u_2^{(j,k)}(x,y)|\ \text{ for }(x,y)\in\Omega_\alpha \quad (6.2_{\varepsilon_1})$$

for any u_1 and $u_2\in\mathcal{B}_\circ^{m_0,n}\big(r;\Omega_\delta\big)$, where

$$\mathcal{B}_\circ^{m_0,n}\big(r;\Omega_\delta\big)=\Big\{\zeta\in C^{m_0,n}(\Omega_\delta):\sum_{j=0}^{m_0}\sum_{k=0}^{n}|\zeta^{(j,k)}(x,y)-\varphi_{j+1}^{(k)}(y)|\le r\text{ for }(x,y)\in\Omega_\delta\Big\}.$$

Furthermore, by Lemma 5.1, representations (5.1_j) $(j=0,\dots,m-1)$,(5.4) and (5.5) are valid. Choosing $r=1$ and ε_1 sufficiently small, from (6.1_{ε_1}), (5.4) and (5.5) by means of Gronwall's lemma we obtain estimates (4.12) in Ω_{α_1}, where $\alpha_1=\alpha(\varepsilon_1,1)$ and M is a positive constant independent of q,φ_j,ψ_k $(j=0,\dots,m-1;\ k=1,\dots,n)$. These estimates yield that problem $(4.1_{m_0}),(4.2)$ has at most one solution.

To complete the proof of the theorem we have to show that problem $(4.1_{m_0}),(4.2)$ is solvable if equalities (4.9) hold.

Let $m_1=m-m_0$ and d_j $(j=0,\dots,m_1)$ be the natural numbers defined by the identity

$$(z+1)(z+2)\dots(z+m_1)=\sum_{j=0}^{m_1}d_jz^j.$$

By Lemma 5.3, there exists $\varepsilon_2>0$ sufficiently small such that the problem

$$z^{(n)}=\sum_{k=0}^{n-1}\big(p_{mk}(y)+\varepsilon\overset{\circ}{\rho}_{m_0k}(y)\big)z^{(k)},\quad h_k(z)(x)=0\quad(k=1,\dots,n)$$

has only the trivial solution for any $\varepsilon\in(0,\varepsilon_2]$ and $x\in[0,\alpha_1]$. Hence, By Theorem 1.4, conditions (4.5)–(4.8) guarantee that for an arbitrary $\varepsilon\in(0,\varepsilon_2]$ problem $(6.1_\varepsilon),(4.2)$ has a unique solution u_ε in a rectangle Ω_{α_1}. Our goal is to show that

$u_\varepsilon^{(j,k)}(x,y)\to u^{(j,k)}(x,y)$ uniformly on $[\delta,\alpha_1]\times[0,b]$ as $\varepsilon\downarrow 0$ $(j=0,\dots,m;\ k=0,\dots,n)$

for any $\delta \in (0, \alpha_1]$, where u is a solution of problem $(4.1_{m_0}), (4.2)$.

Let $u_{\varepsilon jl} \in C^{m-j}([0, a])$ $(j = 0, \dots, m;\ l = 1, \dots, n)$ be functions satisfying the system of linear algebraic equations

$$\sum_{l=1}^{n} \zeta^{(0,k)}(x, 0) u_{\varepsilon jl}(x) = u_{\varepsilon}^{(j,k)}(x, 0) \quad (k = 0, \dots, n-1) \tag{6.3}$$

for arbitrary $j \in \{0, \dots, m\}$ and $x \in [0, a]$, and let $\overset{\circ}{\widetilde{f}}_{m_0 l}, \overset{\circ}{\widetilde{\lambda}}_{m_0 kl}, \overset{\circ}{\widetilde{T}}_{m_0 l}, \overset{\circ}{\widetilde{G}}_{m_0 l}$ $(l = 1, \dots, n_0;$ $k = n_0 + 1, \dots, n)$, $\overset{\circ}{\Phi}$, $\overset{\circ}{\widetilde{\Phi}}_{m_0}$, $\overset{\circ}{K}_{m_0}$ and $\overset{\circ}{\mathcal{Q}}_{m_0}$ be functions and operators introduced in Section 5. Below by $M_1, M_2, M_3, \dots$ we will understand positive constants independent of φ_j $(j = 1, \dots, m)$, ψ_k $(k = 1 \dots, n)$, $\overset{\circ}{q}$ and ε.

Set

$$\begin{aligned}
\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](x, y) =& \overset{\circ}{\mathcal{Q}}_{m_0}[u_\varepsilon](x, y) + \frac{1}{m_1!} \sum_{i=1}^{m_1} \sum_{k=0}^{n-1} d_i \varepsilon^i \overset{\circ}{\rho}_{m_0 k}(y) u_\varepsilon^{(m_0+i,k)}(x, y), \\
\mathcal{F}_l(u_\varepsilon)(x) =& m_1! \Big(\overset{\circ}{\widetilde{f}}_{m_0 l}(x) - \sum_{r=n_0+1}^{n} \overset{\circ}{\widetilde{\lambda}}_{m_0 l r} u_{\varepsilon m_0 r}(x) - \int_0^b \widetilde{T}_{m_0 l}(x, t) \Big(\overset{\circ}{\mathcal{Q}}_{m_0 - 1}[u_\varepsilon](x, t) \\
& + \overset{\circ}{\Upsilon}[u_\varepsilon](x, t) \Big)\, dt - \int_0^x \int_0^b \overset{\circ}{\widetilde{G}}_{m_0 l}(x, s, t) \Big(\overset{\circ}{\mathcal{Q}}_{m_0}[u_\varepsilon](s, t) + \overset{\circ}{\Upsilon}[u_\varepsilon](s, t) \Big)\, dt\, ds \Big), \\
\mathcal{F}_{\varepsilon l}(u_\varepsilon)(x) =& \sum_{i=1}^{m_1} d_i \varepsilon^i (u_{\varepsilon m_0 l}^{(i)}(x) - u_{\varepsilon (m_0+i) l}(x)) \\
& - \int_0^b \widetilde{T}_{m_0 l}(x, t) \sum_{i=1}^{m_1} \alpha_i \varepsilon^i \sum_{k=0}^{n-1} \overset{\circ}{\rho}_{m_0 k}(t) \Big(u_\varepsilon^{(m_0+i,k)}(x, t) \\
& - \sum_{l=1}^{n} \zeta_l^{(0,k)}(x, t) u_{\varepsilon (m_0+i) l}(x) \Big)\, dt \\
& - \int_0^x \int_0^b \widetilde{G}_{m_0 l}(x, s, t) \sum_{i=1}^{m_1} d_i \varepsilon^i \sum_{k=0}^{n-1} \overset{\circ}{\rho}_{m_0 k}(t) u_\varepsilon^{(m_0+i,k)}(s, t)\, ds\, dt.
\end{aligned}$$

Then by Lemma 2.5, we have

$$u_\varepsilon^{(j,0)}(x, y) = \sum_{l=1}^{n} \zeta_l(x, y) u_{\varepsilon jl}(x) + \int_0^y \zeta(y, t) \overset{\circ}{\Phi}^{(j,0)}(x, t)\, dt$$

$$+\int_0^x\int_0^y \zeta(y,t)\overset{\circ}{\xi}{}^{(j,0,0)}(x,s,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](s,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](s,t)\Big)\,ds\,dt$$

$$(j=0,\dots,m-1), \tag{6.4}$$

$$u_\varepsilon^{(m,0)}(x,y)=\sum_{l=1}^n \zeta_l(x,y)u_{\varepsilon ml}(x)+\int_0^y \zeta(y,t)\overset{\circ}{\Phi}{}^{(m,0)}(x,t)\,dt$$

$$+\int_0^y \zeta(y,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](x,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](x,t)\Big)\,dt$$

$$+\int_0^x\int_0^y \zeta(y,t)\xi^{(m,0,0)}(x,s,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](s,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](s,t)\Big)\,ds\,dt, \tag{6.5}$$

$$u_\varepsilon^{(m,0)}(x,y)=\sum_{l=1}^n \zeta_l(x,y)u_{\varepsilon ml}(x)+\frac{1}{m_1!}\sum_{i=0}^{m_1}\sum_{l=1}^n \alpha_i\varepsilon^i\overset{\circ}{\eta}_{m_0l}(x,y)u_{\varepsilon m_0l}(x)+\overset{\circ}{\widetilde{\Phi}}_{m_0}(x,y)$$

$$+\int_0^y \zeta(y,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](x,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](x,t)\Big)\,dt$$

$$+\int_0^x\int_0^y K_{m_0}(x,y,s,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](s,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](s,t)\Big)\,dt\,ds$$

$$+\frac{1}{m_1!}\int_0^y \zeta(y,t)\sum_{i=1}^{m_1}\sum_{l=1}^n d_i\varepsilon^i\rho_{m_0k}(x,t)\Big(u_\varepsilon^{(m_0+i,k)}(x,t)$$

$$-\sum_{l=1}^n \zeta^{(0,k)}(x,t)u_{\varepsilon(m_0+i)l}(x)\Big)\,dt. \tag{6.6}$$

Applying operators h_l $(l=1,\dots,n)$ to identities (6.5) and (6.6), in view of (4.5′) and (4.8), we get

$$\sum_{i=0}^{m_1} d_i\varepsilon^i u_{\varepsilon m_0l}^{(i)}(x)=\mathcal{F}_l(u_\varepsilon)(x)+\mathcal{F}_{\varepsilon l}(u_\varepsilon)(x)\quad (l=1,\dots,n_0), \tag{6.7}$$

$$u_{\varepsilon ml}(x)=\psi_l(x)-\int_0^b T_l(x,t)\overset{\circ}{\Phi}{}^{(m,0)}(x,t)\,dt$$

$$-\int\limits_0^b T_l(x,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](x,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](x,t)\Big)\,dt$$

$$-\int\limits_0^x\int\limits_0^b T_l(x,t)\overset{\circ}{\xi}^{(m,0,0)}(x,s,t)\Big(\overset{\circ}{\mathcal{Q}}_{\varepsilon m_0}[u_\varepsilon](s,t)+\overset{\circ}{\Upsilon}[u_\varepsilon](s,t)\Big)\,dt\,ds$$

$$(l=n_0+1,\dots,n). \tag{6.8}$$

From (6.4) and (6.5) we have

$$\sum_{l=1}^n \zeta_l(x,y)u_{\varepsilon jl}(x)=\Big(\sum_{l=1}^n \zeta_l(x,y)u_{\varepsilon 0l}(x)\Big)^{(j,0)}\quad (j=1,\dots,m). \tag{6.9}$$

(6.9) yields the estimates

$$|u^{(i+j)}_{\varepsilon m_0 l}(x)-u^{(j)}_{\varepsilon(m_0+i)l}(x)|\le M_1\sum_{l=1}^n\|u_{\varepsilon m_0 l}\|_{C^{i+j-1}([0,x])}\quad (i,\,j=0,\dots,m_1),$$

$$\Big|\sum_{i=1}^{m_1} d_i\varepsilon^i\Big(u^{(i+j)}_{\varepsilon m_0 l}(x)-u^{(j)}_{\varepsilon(m_0+i)l}(x)\Big)\Big|$$

$$\le M_2\sum_{i=0}^{m_1-1}\sum_{l=1}^n|\varepsilon|^{i+1}\|u_{\varepsilon m_0 l}\|_{C^{j+i}([0,x])}\ (j=0,\dots,m_1). \tag{6.10}$$

On the other hand, taking into account (6.10), from (6.4) and (6.5) we get

$$\Big|u^{(j,k)}_\varepsilon(x,y)-\sum_{l=1}^n\zeta_l^{(0,k)}(x,y)u_{\varepsilon jl}(x)\Big|\le M_3\Big(\sum_{l=0}^{m-1}\|\varphi_l\|_{C^n([0,b])}+\|q\|_{C(\Omega_x)}$$

$$+\|u_\varepsilon\|_{C^{m_0,n-1}(\Omega_x)}+\sum_{i=1}^{m_1}\sum_{r=1}^n|\varepsilon|^i\|u_{\varepsilon m_0 r}\|_{C^i([0,x])}\Big)$$

$$(j=0,\dots,m-1;\ k=0,\dots,n-1), \tag{6.11}$$

$$\Big|\Big(u^{(m,k)}_\varepsilon(x,y)-\sum_{l=1}^n\zeta_l^{(0,k)}(x,y)u_{\varepsilon ml}(x)\Big)^{(j,0)}\Big|$$

$$\le M_3\Big(\sum_{l=0}^{m-1}\|\varphi_l\|_{C^n([0,b])}+\|q\|_{C^{j,0}(\Omega_x)}$$

$$+\|u_\varepsilon\|_{C^{m_0+j,n-1}(\Omega_x)}+\sum_{i=1}^{m_1}\sum_{r=1}^n|\varepsilon|^i\|u_{\varepsilon m_0 r}\|_{C^{j+i}([0,x])}\Big)$$

$$(j=0,\dots,m_1;\ k=0,\dots,n-1). \tag{6.12}$$

Representation (6.8) and inequalities (6.2_{ε_1}), (6.10)–(6.12) guarantee the validity of the estimates

$$\sum_{l=n_0+1}^{n} |u^{(j)}_{\varepsilon ml}(x)| \leq M_4\Big(\Gamma_j(x) + \|u_\varepsilon\|_{C^{m_0+j,n-1}(\Omega_x)} + \sum_{i=1}^{m_1}\sum_{r=1}^{n_0} |\varepsilon|^i \|u_{\varepsilon m_0 r}\|_{C^{j+i}([0,x])}\Big) \quad (j = 0,\dots,m_1), \tag{6.13}$$

$$\sum_{l=1}^{n_0} |\mathcal{F}^{(j)}_l(u_\varepsilon)(x)| \leq M_5\Big(\Gamma_j(x) + \varepsilon_1 \sum_{i=0}^{m_0+j}\sum_{k=0}^{n-1} |u^{(i,k)}_\varepsilon(x,y)| + \int\limits_0^b \sum_{i=0}^{m_0+j-1}\sum_{k=0}^{n-1} |u^{(i,k)}_\varepsilon(x,t)|\, dt + \int\limits_0^x\int\limits_0^b \sum_{i=0}^{m_0}\sum_{k=0}^{n-1} |u^{(i,k)}_\varepsilon(s,t)|\, ds\, dt\Big) \quad (j = 0,\dots,m_1), \tag{6.14}$$

$$\sum_{l=1}^{n_0} |\mathcal{F}^{(j)}_{\varepsilon l}(u_\varepsilon)(x)| \leq M_5\Big(\varepsilon\Gamma_j(x) + \varepsilon\|u_\varepsilon\|_{C^{m_0,n-1}(\Omega_x)} + \sum_{i=1}^{m_1}\sum_{r=1}^{n_0} |\varepsilon|^{i+1} \|u_{\varepsilon m_0 r}\|_{C^{j+i}([0,x])}\Big) \quad (j = 0,\dots,m_1), \tag{6.15}$$

where

$$\Gamma_j(x) = \sum_{l=1}^{n} \|\psi\|_{C^j([0,x])} + \sum_{j=0}^{m-1} \|\varphi_j\|_{C^n([0,b])} + \|q\|_{C^{j,0}(\Omega_x)}.$$

In view of equalities (6.3) and (6.9), the numbers $u^{(j)}_{\varepsilon m_0 l}(0)$ $(j = 0,\dots,m-1;\ l = 1,\dots,n)$ are independent of ε, since $u^{(i,k)}_\varepsilon(0,0) = \varphi^{(k)}_i(0)$ $(i = 0,\dots,m-1;\ k = 0,\dots,n-1)$. Moreover, according to the definition of the operator $\mathcal{F}_l$ and equality (4.9) we have

$$u_{\varepsilon m_0 l}(0) = \mathcal{F}_l(u_\varepsilon)(0) \quad (l = 1,\dots,n). \tag{6.16}$$

On the basis of these equalities from (6.7) we obtain

$$|u^{(m_1+j)}_{\varepsilon m_0 l}(0)| \leq M_6 \varepsilon^{-(m_1+j-1)}\Gamma_j(0) \quad (j = 0,\dots,m_1-1;\ l = 1,\dots,n). \tag{6.17}$$

On the other hand, assuming $\varepsilon_1 > 0$ sufficiently small, in view of (6.14),(6.15) and (6.16), by Lemma 5.4, we get

$$\sum_{j=0}^{m_1}\sum_{r=1}^{n_0} \varepsilon^j |u^{(j)}_{\varepsilon m_0 r}(x)| \leq M_7\Big(\Gamma_0(x) + \int\limits_0^x\int\limits_0^b \sum_{i=0}^{m_0}\sum_{r=0}^{n-1} |u^{(i,r)}_\varepsilon(s,t)|\, ds\, dt\Big) \quad \text{for} \quad x \in (0,\alpha_1].$$

Therefore from representation (3.2) and estimates (3.9)–(3.11) it follows that

$$\sum_{i=0}^{m_0}\sum_{k=0}^{n-1}|u_\varepsilon^{(i,k)}(x,y)| \le M_8\Big(\Gamma_0(x)+\int\limits_0^x\int\limits_0^b\sum_{i=0}^{m_0}\sum_{r=0}^{n-1}|u_\varepsilon^{(i,r)}(s,t)|\,ds\,dt\Big) \quad \text{for} \quad x\in(0,\alpha_1].$$

Hence by Gronwall's lemma there follows the estimate

$$\sum_{i=0}^{m_0}\sum_{k=0}^{n-1}|u_\varepsilon^{(i,k)}(x,y)| \le M_9\Gamma_0(x) \quad \text{for} \quad x\in(0,\alpha_1]. \tag{6.18}$$

The inequality

$$\sum_{i=0}^{m_0}\sum_{k=0}^{n-1}|u_{\varepsilon_1}^{(i,k)}(x,y)-u_{\varepsilon_2}^{(i,k)}(x,y)| \le M_{10}\Gamma_0(x)|\varepsilon_1-\varepsilon_2| \ \text{ for } x\in(0,\alpha_1],\ \varepsilon_1,\varepsilon_2>0$$

can be proved similarly to (6.18). In view of these inequalities and formula (6.4), there exists a function $u\in C^{m_0,n}(\Omega_{\alpha_1})$ such that

$$\lim_{\varepsilon\to 0+}\|u_\varepsilon-u\|_{C^{m_0,n}(\Omega_{\alpha_1})}=0. \tag{6.19}$$

Moreover, u satisfies equalities (5.1_{m_0}),(5.4) and (5.5) on Ω_{α_1}. These equalities yield that u belongs not only to $C^{m_0,n}(\Omega_{\alpha_1})$ but to $C^{m,n}(\Omega_{\alpha_1})$ as well.

By Lemma 5.4 and estimates (6.14),(6.15) and (6.18), we have

$$\begin{aligned}\sum_{i=0}^{m_1}\sum_{l=1}^{n_0}\varepsilon^i|u_{\varepsilon m_0 l}^{(i+j)}(x)| \le\, & M_{11}\Big(\varepsilon^{-(j-1)}\Gamma_j(0)\exp(-\varepsilon^{-1}x)+\Gamma_j(x)\\ &+\|u\|_{C^{(m_0+j-1,n-1)}(\Omega_x)}+\sum_{i=0}^{m_1}\sum_{l=1}^{n_0}\varepsilon^{i+1}|u_{\varepsilon m_0 l}^{(i+j)}(x)|\Big).\end{aligned}$$

Therefore

$$\sum_{i=0}^{m_1}\sum_{l=1}^{n_0}\varepsilon^i|u_{\varepsilon m_0 l}^{(i+j)}(x)| \le M_{12}\Big(\varepsilon^{-(j-1)}\Gamma_j(0)\exp(-\varepsilon^{-1}x)+\Gamma_j(x)+\|u\|_{C^{m_0+j-1,n-1}(\Omega_x)}\Big).$$

By virtue of these inequalities and estimates (6.11),(6.12), for every $x\in[0,\alpha_1]$ and $j\in\{1,\dots,m_1\}$ we get

$$\sum_{l=1}^{n_0}\|u_{\varepsilon m_0 l}\|_{C^j([0,x])} \le M_{13}\Big(\varepsilon^{-(j-1)}\Gamma_j(0)\exp(-\varepsilon^{-1}x)+\Gamma_j(x)+\sum_{l=1}^{n_0}\|u_{\varepsilon m_0 l}\|_{C^{j-1}([0,x])}\Big).$$

In view of estimates (6.18), it follows that

$$\sum_{l=1}^{n_0}\|u_{\varepsilon m_0 l}\|_{C^j([0,x])} \le M_{14}\Big(\varepsilon^{-(j-1)}\Gamma_j(0)\exp(-\varepsilon^{-1}x)+\Gamma_j(x)\Big) \ \ (j=0,\dots,m_1), \tag{6.20}$$

$$\|u\|_{C^{m_0+j,n-1}(\Omega_x)} \le M_{15}\Big(\varepsilon^{-(j-1)}\Gamma_j(0)\exp(-\varepsilon^{-1}x) + \Gamma_j(x)\Big) \quad (j = 0,\dots,m_1). \quad (6.21)$$

By Lemma 5.4, we have

$$|u^{(j)}_{\varepsilon m_0 l}(x) - \mathcal{F}^{(j)}_l(u_\varepsilon)(x) - \mathcal{F}^{(j)}_{\varepsilon l}(u_\varepsilon)(x)|$$
$$\le M_{16}\exp(-\varepsilon^{-1}x)\Big(|u^{(r)}_{\varepsilon m_0 l}(0) - \mathcal{F}^{(r)}_l(u_\varepsilon)(0) - \mathcal{F}^{(r)}_{\varepsilon l}(u_\varepsilon)(0)|$$
$$+ \sum_{i=1}^{m_1-1} |x|^i |u^{(r+i)}_{\varepsilon m_0 l}(0)|\Big) + M_{16}\Big(\exp(-|\varepsilon|^{-1}\delta)(|\mathcal{F}^{(j)}_l(u_\varepsilon)(x)| + |\mathcal{F}^{(j)}_{\varepsilon l}(u_\varepsilon)(x)|)$$
$$+ \max\{|\mathcal{F}^{(j)}_l(u_\varepsilon)(s)) - \mathcal{F}^{(j)}_l(u_\varepsilon)(x)| + |\mathcal{F}^{(j)}_{\varepsilon l}(u_\varepsilon)(s)) - \mathcal{F}^{(j)}_{\varepsilon l}(u_\varepsilon)(x)| : s \in [x-\delta, x]\}\Big)$$
$$\text{for} \quad x \in [\delta, \alpha_1], \quad (l = 1,\dots,n;\ j = 0,\dots,m_1).$$

Hence in view of (6.14),(6.15),(6.120) and (6.21) we obtain

$$|u^{(j)}_{\varepsilon m_0 l}(x) - \mathcal{F}^{(j)}_l(u_\varepsilon)(x)| \le M_{17}\Big(\exp(-\varepsilon^{-1}x)\varepsilon^{-(j-1)}\Gamma_j(0)\big(1 + \exp(-\varepsilon^{-1}\delta) + \delta + \varepsilon\big)$$
$$+\big(\exp(-\varepsilon^{-1}\delta) + \delta + \varepsilon\big)\Gamma_j(x)\Big) \quad \text{for} \quad \varepsilon\, x \ge \delta > 0 \quad (l = 1,\dots,n;\ j = 0,\dots,m_1).$$

From these estimates and equalities (6.4),(6.5), by virtue of conditions (6.19) and the definition of the operator $\mathcal{F}_l$, it follows that

$$\lim_{\varepsilon\to 0+} \|u_\varepsilon - u\|_{C^{m,n}([\delta,\alpha_1]\times[0,b])} = 0$$

for an arbitrary small $\delta > 0$. Hence we get that u is a solution of problem (4.1_{m_0}), (4.2) in the rectangle Ω_{α_1}. □

Theorem 4.2 can be proved similarly. The only difference is that Lemma 5.2 should be applied instead of Lemma 5.1.

Theorems 4.3 and 4.4 follow from Theorem 1.5 and Lemmas 5.1_{x_0} and Lemmas 5.3_{x_0}, respectively.

Corollaries 4.1 and 4.2 immediately follow from Theorem 1.3 and Theorems 4.3 and 4.4, respectively.

Proof of Corollary 4.3. For problem (4.27),(4.28) problem (4.3),(4.4) receives the form

$$\frac{d^n z}{dy^n} = 0, \quad (6.22)$$

$$z^{(k-1)}(0) - z^{(k-1)}(b) = 0 \ \ (k = 1, \dots, n). \tag{6.23}$$

Problem (6.22),(6.23) has 1–dimensional space of solutions. Equation (6.22) has the fundamental set of solutions $\zeta_k(x, y) = y^{k-1}$ $(k = \overline{1, n})$. Therefore condition (4.5) holds, where $n_0 = 1$. On the other hand, conditions (1.1) and (1.12) follow from conditions (4.29) and (4.30).Also, it is cleat that if $m_0 < m - 1$, then equalities (4.7) hold. Corollary 4.3 immediately follows from Theorems 4.1 and 4.2. □

Corollary 4.4 is a direct consequence of Corollaries 4.1, 4.2 and 4.3.

Corollaries 4.5 and 4.7 can be proved similarly to Corollary 4.3.

Corollary 4.6 (Corollary 4.8) immediately follows from Corollaries 4.1, 4.2 and 4.5 (Corollaries 4.1, 4.2 and 4.7).

Corollary 4.9 follows from Corollaries 4.1, 4.2 and 4.5 and 1.8.

CHAPTER III
Nonlocal Boundary Value Problems

7. Formulation of the Main Results

7.1. **General Initial–Boundary Value Problems.** In the rectangle Ω consider the boundary value problem

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big), \qquad (7.1)$$

$$\begin{gathered} l_j(u(\cdot, y)(y) = \varphi_j(y) \ \ (j = 1, \dots, m), \\ h_k(u^{(m,0)}(x, \cdot))(x) = \psi_k^{(m)}(x) \ \ (k = 1, \dots, n), \end{gathered} \qquad (7.2)$$

$\varphi_j \in C^n([0, b])$ $(j = 1, \dots, m)$, $\psi_k \in C^m([0, a])$ $(k = 1, \dots, n)$, $l_j : C^{m-1}([0, a] \to C^n([0, b]))$ $(j = 1, \dots, m)$ and $h_k : C^{n-1}[0, b] \to C([0, a])$ $(k = 1, \dots, n)$ are bounded linear operators.

Definition 7.1. Let u_0 be a solution of problem (1.1),(1.2), and $r > 0$. Problem (7.1),(7.2) is said to be (u_0, r)–*well-posed* if:
(i) $u_0(x, y)$ is the unique solution of the problem in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0; r)$;
(ii) For an arbitrary $\varepsilon > 0$ there exists $\delta_0 > 0$ such that for any $\widetilde{f}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})$ that is continuously differentiable with respect to $\mathbf{v}$ and $\mathbf{w}$, $\widetilde{\varphi}_j \in C^n([0, b])$ $(j = 1, \dots, m)$, $\widetilde{\psi}_k \in C^m([0, a])$ $(k = 1, \dots, n)$, satisfying the inequalities

$$\begin{gathered} \sum_{k=0}^{n-1} |\widetilde{f}_{mk}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})| \le \delta_0 \quad \text{for} \quad (x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn} \\ \sum_{j=0}^{m-1} |\widetilde{f}_{jn}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})| \le \delta_0 \quad \text{for} \quad (x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \end{gathered} \qquad (7.3)$$

$$\begin{gathered} |\widetilde{f}(x, y, \mathbf{v}, \mathbf{w}, \mathbf{z})| < \delta \quad \text{for} \quad (x, y, \mathbf{v}, \mathbf{w}, \mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}, \\ \sum_{j=1}^{m} \|\widetilde{\varphi}_j\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\widetilde{\psi}_k\|_{C^m([0,a])} \le \delta, \end{gathered} \qquad (7.4)$$

the problem

$$\begin{aligned} u^{(m,n)} = {} & f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big) \\ & + \widetilde{f}\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big), \end{aligned} \qquad (\widetilde{7.1})$$

$$\begin{aligned} l_j(u(\cdot,y))(y) &= \varphi_j(y) + \widetilde{\varphi}_j(y) \ \ (j=1,\dots,m), \\ h_k(u^{(m,0)}(x,\cdot))(x) &= \psi_k^{(m)}(x) + \widetilde{\psi}_k^{(m)}(x) \ \ (k=1,\dots,n) \end{aligned} \tag{$\widetilde{7.2}$}$$

has at least one solution in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;\varepsilon)$.

Definition 7.2. Let u_0 be a solution of problem (1.1),(1.2), and $r>0$. Problem (7.1),(7.2) is said to be *strongly* (u_0,r)*–well-posed* if:

(i) Problem (7.1),(7.2) is (u_0,r)–well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta\in(0,\delta_0)$, $\widetilde{f}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ that is continuously differentiable with respect to $\mathbf{v}$ and $\mathbf{w}$, $\widetilde{\varphi}_j\in C^n([0,b])$ $(j=1,\dots,m)$ and $\widetilde{\psi}_k\in C^m([0,a])$ $(k=1,\dots,n)$, satisfying the inequalities (7.3),(7.4), problem $(\widetilde{7.1}),(\widetilde{7.2})$ has at least one solution in the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathcal{B}}^{m,n}(u_0;M_0\,\delta)$.

Definition 7.3. Problem (7.1),(7.2) is called well–posed if it is (u_0,r)–well-posed for every $r>0$.

First consider the linear case, i.e., the equation

$$u^{(m,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)u^{(m,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}+q(x,y). \tag{7.5}$$

Theorem 7.1. *The linear problem* (7.5), (7.2) *is well–posed if and only if:*

(F_1) *the problem*

$$\zeta^{(n)}=\sum_{i=0}^{n-1}p_{mi}(x,y)\zeta^{(i)};\quad h_k(\zeta)(x)=0 \ \ (k=1,\dots,n) \tag{7.6}$$

has only the trivial solution for every $x\in[0,a]$*;*

(F_2)

$$\xi^{(m)}=\sum_{i=0}^{m-1}p_{in}(x,y)\xi^{(i)};\quad l_j(\xi)(y)=0 \ \ (j=1,\dots,m) \tag{7.7}$$

has only the trivial solution for every $y \in [0, b]$*;*

(F_3) *the homogeneous problem*

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}, \qquad (7.5_0)$$

$$l_j(u(\cdot,y)(y) = 0 \ \ (j = 1,\dots,m), \quad h_k(u^{(m,0)}(x,\cdot))(x) = 0 \ \ (k = 1,\dots,n) \qquad (7.2_0)$$

has only the trivial solution.

Theorem 7.2. *The* f *be a continuously differentiable function with respect to the phase variables* $\mathbf{v}, \mathbf{w}$ *and* $\mathbf{z}$*, and let problem* $(7.1), (7.2)$ *be strongly* (u_0, r)*-well–posed for some* $r > 0$*. Then problem* $(7.5_0), (7.2_0)$ *is well–posed, where*

$$p_{jk}(x,y) = p_{jk}[u_0](x,y) \quad (j = 0,\dots,m,\ k = 0,\dots,n;\ j+k < m+n).$$

Theorem 7.3. *Let* f *be a continuously differentiable function with respect to the phase variables* v, w *and* z*, and let there exist functions* $P_{ijk} \in C(\Omega)$ $(j = 0,\dots,m;$ $k = 0,\dots,n;\ j+k < m+n;\ i = 1,2)$ *such that:*

(A_0)

$$P_{1jk}(x,y) \le f_{jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \le P_{2jk}(x,y) \quad \text{for} \quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z}) \in \Omega \times \mathbb{R}^{n+m+mn}$$
$$(j = 0,\dots,m,\ k = 0,\dots,n;\ j+k < m+n); \qquad (7.8)$$

(A_1) *for every* $x \in [0, a]$ *and arbitrary measurable functions* $p_{mk} : \Omega \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1mk}(x,y) \le p_{mk}(x,y) \le P_{2mk}(x,y) \quad \text{for} \quad (x,y) \in \Omega \ \ (k = 0,\dots,n-1), \qquad (7.9)$$

problem (7.6) *has only the trivial solution;*

(A_2) *for every* $y \in [0,b]$ *and arbitrary measurable functions* $p_{jn} : \Omega \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1jn}(x,y) \le p_{jn}(x,y) \le P_{2jn}(x,y) \quad \textit{for} \quad (x,y) \in \Omega \ \ (j = 0,\dots,m-1), \tag{7.10}$$

problem (7.7) *has only the trivial solution;*

(A_3) *for arbitrary measurable functions* $p_{jk} : \Omega \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1jk}(x,y) \le p_{jk}(x,y) \le P_{2jk}(x,y) \quad \textit{for} \quad (x,y) \in \Omega$$
$$(j = 0,\dots,m, \ k = 0,\dots,n; j+k < m+n), \tag{7.11}$$

problem $(7.5_0), (7.2_0)$ *has only the trivial solution.*

Then problem $(7.5), (7.2)$ *is well–posed.*

Remark 7.1. It is rather obvious that for the linear problem (7.5),(7.2) (u_0, r)–well–posedness is equivalent to strong well–posedness. In fact, if problem (7.5),(7.2) is well–posed, its solution u admits the estimate

$$\|u\|_{C^{m,n}(\Omega)} \le M\Big(\sum_{i=1}^{m} \|\varphi_i\|_{C^n([0,b])} + \sum_{k=1}^{n} \|\psi_k\|_{C([0,a])} + \|q\|_{C(\Omega)}\Big), \tag{7.12}$$

where M is a positive constant independent of φ_i, ψ_k $(i = 1,\dots,m; \ k = 1,\dots,n)$ and q.

Consider the equation

$$u^{(m,n)} = f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)$$
$$+q(x,y,u^{(m-1,n-1)},\dots,u). \tag{7.13}$$

Theorem 7.4. *Let* f *satisfy all of the conditions of Theorem* 7.3, *and* $q(x,y,\mathbf{z})$ *be an arbitrary continuous function such that*

$$\lim_{\|\mathbf{z}\| \to +\infty} \frac{|q(x,y,\mathbf{z})|}{\|\mathbf{z}\|} = 0 \tag{7.14}$$

uniformly on Ω. *Then problem* $(7.13), (7.2)$ *has at least one solution.*

Consider the quasi–linear equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)}$$
$$+ \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)} + q(x,y,u^{(m-1,n-1)},\dots,u). \qquad (7.15)$$

Theorem 7.4 immediately implies

Corollary 7.1. *Let problem* $(7.5_0),(7.2_0)$ *be well–posed, and* $q(x,y,\mathbf{z})$ *be an arbitrary continuous function satisfying condition* (7.14) *uniformly on* Ω. *Then problem* $(7.15),(7.2)$ *has at least one solution.*

7.2. **Two–Point Boundary Value Problems.** In this subsection we separately study two–point boundary value problems, because the existence results below cover the case where the function $f(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ has an arbitrary growth order with respect to some of the phase variables.

For the equation

$$u^{(2m,2n)} = \Big(f_1(x,y,u)u^{(m,0)}\Big)^{(m,0)} + \Big(f_2(x,y,u)u^{(0,n)}\Big)^{(0,n)}$$
$$+ f_0\big(x,y,u\big) + q(x,y,u,\dots,u^{(m-1,n-1)}) \qquad (7.16)$$

consider the following boundary value problems

$$\begin{aligned} &u^{(j-1,0)}(0,y) = u^{(j-1,0)}(a,y) \quad (j=1,\dots,2m),\\ &u^{(m,k-1)}(x,0) = u^{(m,k-1)}(x,b) \ \ (k=1,\dots,2n); \end{aligned} \qquad (7.17)$$

$$\begin{aligned} &u^{(j-1,0)}(0,y) = 0, \quad u^{(j-1,0)}(a,y) = 0, (j=1,\dots,m),\\ &u^{(m,k-1)}(x,0) = 0, \quad u^{(m,k-1)}(x,b) = 0 \ \ (k=1,\dots,n) \end{aligned} \qquad (7.18)$$

and

$$\begin{aligned} &u^{(2(j-1),0)}(0,y) = 0, \quad u^{(2(j-1),0)}(a,y) = 0, (j=1,\dots,m),\\ &u^{(m,2(k-1))}(x,0) = 0, \quad u^{(m,2(k-1))}(x,b) = 0 \ \ (k=1,\dots,n). \end{aligned} \qquad (7.19)$$

Here f_1 is m–times continuously differentiable function, f_2 is n–times continuously differentiable function, f_0 is a continuously differentiable function, while q is a continuous function.

Corollary 7.2. *Let there exist* $\delta > 0$ *and* $M > 0$ *such that*

$$(-1)^{n-1} f_1(x, y, z) \geq \delta \quad for \quad (x, y, z) \in \Omega \times \mathbb{R}, \tag{7.20}$$

$$(-1)^{m-1} f_2(x, y, z) \geq \delta \quad for \quad (x, y, z) \in \Omega \times \mathbb{R}, \tag{7.21}$$

$$(-1)^{m+n-1} f_{0z}(x, y, z) \geq \delta \quad for \quad (x, y, z) \in \Omega \times \mathbb{R}, \tag{7.22}$$

$$|q_1(x, y, z_0, \dots, z_{mn})| \leq M \quad for \quad (x, y, z_0, \dots, z_{mn}) \in \Omega \times \mathbb{R}^{mn+1}. \tag{7.23}$$

Then problem (7.16), (7.17) *has at least one solution.*

Remark 7.2. In Theorem 7.3 conditions (A_1) and (A_2) are essential and cannot be weakened. In Corollary 7.2 inequalities (7.20) and (7.21) guarantee that conditions (A_1) and (A_2) of Theorem 7.3 hold. The following examples demonstrates what may happen if either of conditions (7.20) and (7.21) are violated.

In the rectangle $[0, 2\pi] \times [0, 1]$ consider the problem

$$u^{(2,2)} = \big(3u^2 u^{(1,0)}\big)^{(1,0)} + u^{(0,2)} + \sin x, \tag{7.24}$$

$$u(0, y) = 0, \quad u(2\pi, y) = 0, \quad u^{(2,k-1)}(x, 0) = u^{(2,k-1)}(x, 1) \ \ (k = 1, 2). \tag{7.25}$$

Problem (7.24),(7.25) satisfies all of the conditions of Corollary 7.2 except the condition (7.20). For problem (7.24),(7.25) instead of (7.20) we have

$$(-1)^{n-1} f_1(x, y, z) = 3z^2 \geq 0 \quad \text{for} \quad (x, y, z) \in \Omega \times \mathbb{R}.$$

As a result, problem (7.24),(7.25) has no solution. Indeed, one can easily show that every solution of problem (7.24), (7.25) is independent of y. Therefore problem has the unique weak solution $u(x) = \sin^{\frac{1}{3}} x$, which is not a classical solution since u is not differentiable at π.

Corollary 7.3. *Let there exist* $\delta > 0$ *and* $M > 0$ *such that*

$$(-1)^{n-1} f_1(x, y, z) \geq -\frac{\pi^{2n}}{b^{2n}} + \delta \quad for \quad (x, y, z) \in \Omega \times \mathbb{R}, \tag{7.26}$$

$$(-1)^{m-1} f_2(x, y, z) \geq -\frac{\pi^{2m}}{b^{2m}} + \delta \quad for \quad (x, y, z) \in \Omega \times \mathbb{R}, \tag{7.27}$$

$$(-1)^{m+n-1} f_{0z}(x, y, z) \geq \Big(-\frac{\pi^{2(m+n)}}{b^{2(m+n)}} + \delta\Big) \quad for \quad (x, y, z) \in \Omega \times \mathbb{R} \tag{7.28}$$

and condition (7.23) *holds. Then problem* (7.16), (7.18) *has at least one solution.*

Corollary 7.4. *Let all of the conditions of Corollary* 7.3 *hold. Then problem* (7.16), (7.19) *has at least one solution.*

8. Auxiliary Statements

Along with (7.7) consider the problem

$$\xi^{(m)} = \sum_{i=0}^{m-1} \widetilde{p}_{in}(x,y)\xi^{(i)}, \quad l_j(\xi)(y) = 0 \ \ (j = 1,\dots,m), \tag{$\widetilde{7.7}$}$$

where $\widetilde{p}_{jn} \in C^{0,n}(\omega)$ $(j = 0,\dots,m-1)$.

By Theorem 1.1 from [19], if problems (7.7) and ($\widetilde{7.7}$) have only the trivial solutions for every $y \in [0,b]$, and problem (7.6) has only the trivial solution for every $x \in [0,a]$, then the solutions ξ, $\widetilde{\xi}$ and ζ of the problems

$$\xi^{(m)} = \sum_{i=0}^{m-1} p_{in}(x,y)\xi^{(i)} + q(x), \quad l_j(\xi)(y) = \varphi_j(y) \ \ (j = 1,\dots,m),$$

$$\xi^{(m)} = \sum_{i=0}^{m-1} \widetilde{p}_{in}(x,y)\xi^{(i)} + q(x), \quad l_j(\xi)(y) = \varphi_j(y) \ \ (j = 1,\dots,m)$$

and

$$\zeta^{(n)} = \sum_{i=0}^{n-1} p_{mi}(x,y)\zeta^{(i)} + q(y); \quad h_k(\zeta)(x) = \psi_k(x) \ \ (k = 1,\dots,n)$$

admit the representation

$$\xi(x) = \int_0^a G_1(x,s;y)q(s)\,ds + \mathcal{A}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y), \tag{8.1}$$

$$\widetilde{\xi}(x) = \int_0^a \widetilde{G}_1(x,s;y)q(s)\,ds + \widetilde{\mathcal{A}}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y), \tag{8.2}$$

and

$$\zeta(y) = \int_0^b G_2(y,t;x)q(t)\,dt + \mathcal{A}_2[\psi_1(x),\dots,\psi_n(x)](x,y). \tag{8.3}$$

Here for every $y \in [0,b]$, $G_1(\cdot,\cdot,y)$ and $\widetilde{G}_1(\cdot,\cdot,y)$ are the Green's functions of problems (7.7) and $(\widetilde{7.7})$, respectively; for every $x \in [0,a]$, $G_2(\cdot,\cdot,x)$ is the Green's function of problem (7.6), and

$$\mathcal{A}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y) = \sum_{j=1}^{m} \xi_j(x,y)\varphi_j(y),$$

$$\widetilde{\mathcal{A}}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y) = \sum_{j=1}^{m} \widetilde{\xi}_j(x,y)\varphi_j(y),$$

$$\mathcal{A}_1[\psi_1(x),\dots,\psi_n(x)](x,y) = \sum_{k=1}^{n} \zeta_k(x,y)\psi_k(x),$$

where and for every ξ_j, $\widetilde{\xi}_j$ $(j=1,\dots,m)$ and ζ_k $(k=1,\dots,n)$, respectively, are solutions of the problems

$$\xi^{(m)} = \sum_{i=0}^{m-1} p_{in}(x,y)\xi^{(i)}, \quad l_i(\xi)(y) = \delta_{ij} \ \ (i=1,\dots,m);$$

$$\xi^{(m)} = \sum_{i=0}^{m-1} \widetilde{p}_{in}(x,y)\xi^{(i)}, \quad l_i(\xi)(y) = \delta_{ij} \ \ (i=1,\dots,m);$$

$$\zeta^{(n)} = \sum_{i=0}^{m-1} p_{mi}(x,y)\zeta^{(i)}, \quad h_i(\zeta)(x) = \delta_{ik} \ \ (i=1,\dots,n).$$

Lemma 8.1. *Let conditions (F_1) and (F_2) of Theorem 7.1 hold, and let problem $(\widetilde{7.7})$ have only the trivial solution for every $x \in [0,a]$. Then an arbitrary solution u of problem (7.5), (7.2) admits the following representations:*

$$u^{(m,0)}(x,y) = \int_0^b G_2(y,t;x)\Bigg(\sum_{j=0}^{m-1} p_{jn}(x,t)u^{(j,n)}(x,t)$$
$$+ \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,t)u^{(j,k)}(x,t) + q(x,t)\Bigg)\,dt + \mathcal{A}_2(\psi_1^{(m)}(x),\dots,\psi_n^{(m)}(x))(x,y); \tag{8.4}$$

$$u^{(0,n)}(x,y) = \int_0^a G_1(x,s;y)\Bigg(\sum_{k=0}^{n-1} p_{mk}(s,y)u^{(m,k)}(s,y)$$
$$+ \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(s,y)u^{(j,k)}(s,y) + q(s,y)\Bigg)\,ds + \mathcal{P}_1[u,\varphi_1^{(n)},\dots,\varphi_m^{(n)}](x,y); \tag{8.5}$$

$$u(x,y)=\int\limits_0^a \widetilde{G}_1(x,s;y)\int\limits_0^b G_2(y,t;s)\Big(\sum_{j=0}^{m-1}\Big(p_{jn}(s,t)-\widetilde{p}_{jn}(s,t)\Big)u^{(j,n)}(x,t)$$

$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\rho_{jk}(s,t)u^{(j,k)}(s,t)+q(s,t)\Big)\,dt\,ds$$

$$+\widetilde{\mathcal{P}}_2[u,\psi_1^{(m)},\dots,\psi_n^{(m)}](x,y)+\widetilde{\mathcal{A}}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y), \tag{8.6}$$

where

$$\rho_{jk}(x,y)=p_{jk}(x,y)+\sum_{i=k}^{n}\frac{i!}{k!(i-k)!}p_{mi}(x,y)\widetilde{p}_{jn}^{(0,i-k)}(x,y)$$

$$-\frac{n!}{k!(n-k)!}\widetilde{p}_{jn}^{(0,n-k)}(x,y)\quad (j=0,\dots,m-1;\ k=0,\dots n-1), \tag{8.7}$$

$$\mathcal{P}_1[u,\varphi_1^{(n)},\dots,\varphi_m^{(n)}](x,y)=\mathcal{A}_1\Big[\varphi_1^{(n)}(y)-\sum_{i=0}^{n-1}\frac{n!}{i!(n-i)!}l_1^{(n-i)}\big(u^{(0,i)}(\cdot,y)\big),\dots,$$

$$\varphi_m^{(n)}(y)-\sum_{i=0}^{n-1}\frac{n!}{i!(n-i)!}l_n^{(n-i)}\big(u^{(0,i)}(\cdot,y)\big)\Big](x,y), \tag{8.8}$$

$$\widetilde{\mathcal{P}}_2[u,\psi_1^{(m)},\dots,\psi_n^{(m)}](x,y)=\int\limits_0^a \widetilde{G}_1(x,s;y)\mathcal{A}_2\Big[\psi_1^{(m)}(s)-h_1\Big(\sum_{j=0}^{m-1}\widetilde{p}_{jn}(s,\cdot)u^{(j,n)}(s,\cdot)\Big),$$

$$\dots,\psi_n^{(m)}(s)-h_n\Big(\sum_{j=0}^{m-1}\widetilde{p}_{jn}(s,\cdot)u^{(j,n)}(s,\cdot)\Big)\Big](s,y)\,ds. \tag{8.9}$$

Proof. Let u be a solution of problem (7.5),(7.2). Set

$$v(x,y)=u^{(m,0)}(x,y);\quad w(x,y)=u^{(0,n)};$$

$$\widetilde{v}(x,y)=u^{(m,0)}(x,y)-\sum_{j=0}^{m-1}\widetilde{p}_{jn}(x,y)u^{(j,0)}(x,y).$$

Then v, w and $\widetilde{v}$ are solution to the following boundary value problems:

$$v^{(0,n)}=\sum_{k=0}^{n-1}p_{mk}(x,y)v^{(0,k)}+\sum_{j=0}^{m-1}p_{jn}(x,y)u^{(j,n)}(x,y)$$

$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,y)u^{(j,k)}(x,y)+q(x,y),$$

$$h_k(v(x,\cdot))(x)=\psi_k^{(m)}(x)\quad (k=1,\dots,n);$$

$$w^{(m,0)} = \sum_{j=0}^{m-1} p_{jn}(x,y) w^{(j,0)} + \sum_{k=0}^{n-1} p_{mk}(x,y) u^{(m,k)}$$
$$+ \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} p_{jk}(x,y) u^{(j,k)}(x,y) + q(x,y),$$
$$l_j(w(x,\cdot))(y) = \varphi_j^{(n)}(y) - \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} l_j^{(n-i)}\big(u^{(0,i)}(\cdot,y)\big) \quad (j = 1,\dots,m);$$

$$\widetilde{v}^{(0,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y) \widetilde{v}^{(0,k)} + \sum_{j=0}^{m-1} (p_{jn}(x,y) - \widetilde{p}_{jn}(x,y)) u^{(j,n)}(x,y)$$
$$+ \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \rho_{jk}(x,y) u^{(j,k)}(x,y) + q(x,y), \tag{8.10}$$
$$h_k(\widetilde{v}(x,\cdot))(x) = \psi_k^{(m)}(x) - h_k\Big(\sum_{j=0}^{m-1} \widetilde{p}_{jn}(s,\cdot) u^{(j,n)}(s,\cdot)\Big) \quad (k = 1,\dots,n). \tag{8.11}$$

Hence, (8.4) follows from (8.3), (8.5) follows from (8.1) and (8.8), while (8.6) follows from (8.2), (8.7) and (8.9). □

Lemma 8.2. *Let conditions* (F_1) *and* (F_2) *of Theorem* 7.1 *hold. Then problem* (7.5), (7.2) *has the Fredholm property, i.e. problem* $(7.5_0), (7.2_0)$ *has a finite dimensional space of solutions and problem* (7.5), (7.2) *is uniquely solvable if and only if problem* $(7.5_0), (7.2_0)$ *has only the trivial solution.*

Proof. In view of Lemma 2.1, problem (7.5), (7.2) is equivalent to the following system of integral equations

$$v(x,y) = \mathcal{F}_1(u,w)(x,y); \tag{8.12}$$
$$w(x,y) = \mathcal{F}_2(u,v)(x,y); \tag{8.13}$$
$$u(x,y) = \mathcal{F}(u,w)(x,y), \tag{8.14}$$

where

$$\mathcal{F}_1(u,w)(x,y) = \int_0^b G_2(y,t;x)\Big(\sum_{j=0}^{m-1} p_{jn}(x,t) w^{(j,0)}(x,t)$$

$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(x,t)u^{(j,k)}(x,t)+q(x,t)\Big)\,dt+\mathcal{A}_2(\psi_1^{(m)}(x),\dots,\psi_n^{(m)}(x))(x,y); \quad (8.15)$$

$$\mathcal{F}_2(u,v)(x,y)=\int_0^a G_1(x,s;y)\Big(\sum_{k=0}^{n-1}p_{mk}(s,y)v^{(0,k)}(s,y)$$

$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}p_{jk}(s,y)u^{(j,k)}(s,y)+q(s,y)\Big)\,ds+\mathcal{P}_1[u,\varphi_1^{(n)},\dots,\varphi_m^{(n)}](x,y); \quad (8.16)$$

$$\mathcal{F}(u,v)(x,y)=\int_0^a \widetilde{G}_1(x,s;y)\int_0^b G_2(y,t;s)\Big(\sum_{j=0}^{m-1}\Big(p_{jn}(s,t)-\widetilde{p}_{jn}(s,t)\Big)w^{(j,0)}(x,t)$$

$$+\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\rho_{jk}(s,t)u^{(j,k)}(s,t)+q(s,t)\Big)\,dt\,ds$$

$$+\widetilde{\mathcal{P}}_2[u,\psi_1^{(m)},\dots,\psi_n^{(m)}](x,y)+\widetilde{\mathcal{A}}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y). \quad (8.17)$$

On the other hand, system (8.12)–(8.14) is equivalent to the system

$$w(x,y)=\mathcal{K}(u,w)(x,y), \quad (8.18)$$

$$u(x,y)=\mathcal{F}(u,w)(x,y). \quad (8.19)$$

where $\mathcal{K}(u,w)(x,y)=\mathcal{F}_2(\mathcal{F}(u,w),\mathcal{F}_1(u,w))(x,y)$. From (8.15)–(8.17) it is obvious that $K: C^{m-1,n-1}(\Omega)\times C^{0,n-1}(\Omega)\to C^{0,n-1}(\Omega)$ and $F: C^{m-1,n-1}(\Omega)\times C^{0,n-1}(\Omega)\to C^{m-1,n-1}(\Omega)$ are linear compact integral operators. Therefore, (8.18),(8.19) is the system of Fredholm integral operators. Hence system (8.18),(8.19), and consequently, problem (7.5),(7.2) have the Fredholm property. □

Remark 8.1. By Lemma 8.2, problem (7.5),(7.2) has the Fredholm property in $C^{m,n}(\Omega)$, i.e. problem $(7.5_0), (7.2_0)$ has a finite dimensional space of solutions and problem (7.5), (7.2) has a unique solution in $C^{m,n}(\Omega)$ if and only if problem $(7.5_0), (7.2_0)$ has only the trivial solution in $C^{m,n}(\Omega)$.

It is not difficult to notice that if the coefficients $p_{jk}\in L^\infty(\Omega)$ $(j=0,\dots,m;\ k=0,\dots,n;\ j+k<m+n)$, and conditions (A_1) and (A_2) of Theorem 7.3 hold, then problem (7.5),(7.2) has the Fredholm property $AC^{m-1,n-1}(\Omega)$, i.e. problem $(7.5_0), (7.2_0)$ has a finite dimensional space of solutions in $AC^{m-1,n-1}(\Omega)$, and for any $q\in L(\Omega)$,

$\varphi_j \in AC^{n-1}([0,b])$ $(j = 1,\dots,m)$ and $\psi_k \in AC^{m-1}([0,a])$ $(k = 1,\dots,n)$, problem $(7.5),(7.2)$ has a unique solution in $AC^{m-1,n-1}(\Omega)$ if and only if problem $(7.5_0),(7.2_0)$ has only the trivial solution in $AC^{m-1,n-1}(\Omega)$.

Remark 8.2. Notice that if $p_{jn} \in C^{0,n}(\Omega)$ $(j = 0,\dots,m-1)$, then problem problem (7.5) is equivalent to the Fredholm integral equation

$$u(x,y) = \mathcal{F}_0(u)(x,y), \tag{8.20}$$

where

$$\mathcal{F}_0(u)(x,y) = \int_0^a G_1(x,s;y) \int_0^b G_2(y,t;x)\Big(\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\rho_{jk}(s,t)u^{(j,k)}(s,t) + q(s,t)\Big)\,dt\,ds$$
$$+\mathcal{P}_2[u,\psi_1^{(m)},\dots,\psi_n^{(m)}](x,y)+\widetilde{\mathcal{A}}_1[\varphi_1(y),\dots,\varphi_m(y)](x,y), \tag{8.21}$$

and

$$\mathcal{P}_2[u,\psi_1^{(m)},\dots,\psi_n^{(m)}](x,y)=\int_0^a G_1(x,s;y)\mathcal{A}_2\Big[\psi_1^{(m)}(s)-h_1\Big(\sum_{j=0}^{m-1}p_{jn}(s,\cdot)u^{(j,n)}(s,\cdot)\Big),$$
$$\dots,\psi_n^{(m)}(s) - h_n\Big(\sum_{j=0}^{m-1}p_{jn}(s,\cdot)u^{(j,n)}(s,\cdot)\Big)\Big](s,y)\,ds. \tag{8.22}$$

Lemma 8.3. *Let functions $P_{ijk} \in C(\Omega)$ $(j = 0,\dots,m;\ k = 0,\dots,n;\ j+k < m+n;\ i = 1,2)$ satisfy conditions $(A_1)-(A_3)$ of Theorem* 7.3. *Then for arbitrary measurable functions $p_{jk} : \Omega \to \mathbb{R}$ satisfying the inequalities* (7.10)*, a solution of problem* (7.5), (1.2) *admits the estimate*

$$\|u\|_{AC^{m-1,n-1}(\Omega)} \le M\Big(\sum_{i=1}^{m}\|\varphi_i\|_{AC^{n-1}([0,b])} + \sum_{k=1}^{n}\|\psi_k\|_{AC^{m-1}([0,a])} + \|q\|_{L(\Omega)}\Big), \tag{8.23}$$

where M is a positive constant independent of p_{jk},$(j = 0,\dots,m;\ k = 0,\dots,n;\ j+k < m+n)$, φ_j $(j = 1,\dots,m)$, ψ_k $(k = 1,\dots,n)$ and q.

The proof of Lemma 8.3 is based on Lemmas 8.1 and 8.2 and is similar to the proof of Lemma 2.1.

Now consider the quasi–linear differential equation

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}$$
$$+q\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big) \tag{8.24}$$

with the initial–boundary conditions (7.2), and the differential inequality

$$\left|u^{(m,n)} - \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} p_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)}\right|$$
$$\le \varepsilon \sum_{k=0}^{n-1} |u^{(m,k)}| + \varepsilon \sum_{j=0}^{m-1}\sum_{k=0}^{n} |u^{(j,k)}| + \delta \tag{8.25}$$

with the boundary conditions

$$\sum_{j=1}^{m}\sum_{k=0}^{n} \left|\Big(l_j(u(\cdot,y)(y)\Big)^{(k)}\right| + \sum_{k=1}^{n} \left|h_k(u^{(m,0)}(x,\cdot))(x)\right| \le \delta \quad \text{for} \quad (x,y)\in\Omega, \tag{8.26}$$

where ε and δ are nonnegative constants.

Lemma 8.4. *Let conditions* (F_1)*–*(F_3) *of Theorem* 7.1 *hold. Then there exists* $\varepsilon_0 > 0$ *and* $M > 0$ *such that for every* $\varepsilon \in [0,\varepsilon_0)$ *an arbitrary solution of problem* (8.25), (8.26) *admits the estimate*

$$\|u\|_{\widetilde{C}^{m,n}(\Omega)} \le M\,\delta. \tag{8.27}$$

The proof of Lemma 8.4 is similar to the proof of Lemma 2.3.

Lemma 8.5. *Let conditions* (F_1)*–*(F_3) *of Theorem* 7.1 *hold, and let the function* $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ *be locally Lipschitz continuous with respect to* $\mathbf{z}$*. Furthermore, let* q *satisfy the inequality*

$$|q(x,y,\mathbf{v}_1,\mathbf{w}_1,\mathbf{z}_1) - q(x,y,\mathbf{v}_2,\mathbf{w}_2,\mathbf{z}_2)| \le \varepsilon\big(\|\mathbf{v}_1-\mathbf{v}_2\| + \|\mathbf{w}_1-\mathbf{w}_2\| + \|\mathbf{z}_1-\mathbf{z}_2\|\big)$$
$$\textit{for} \quad (x,y,\mathbf{v}_i,\mathbf{w}_i,\mathbf{z}_i) \in \Omega\times\mathbb{R}^{n+m+mn}, \tag{8.28}$$

where $\varepsilon\in[0,\varepsilon_0)$ *and* ε_0 *is the number appearing in Lemma* 8.4*. Then problem* (8.24), (7.2) *is strongly well–posed.*

The proof of Lemma 8.5 is similar to the proof of Lemmas 2.4 and 2.6.

Lemma 8.6. *Let conditions* (F_1)–(F_3) *of Theorem* 7.1 *hold, and let the function* $q(x,y,\mathbf{v},\mathbf{w},\mathbf{z})$ *satisfy the inequality* (8.28), *where* $\varepsilon \in [0,\varepsilon_0)$ *and* ε_0 *is the number appearing in Lemma* 8.4. *Then problem* (8.24), (7.2) *is uniquely solvable.*

The proof of Lemma 8.6 is similar to the proof of Lemma 2.5.

9. Proofs of the Main Results

Proof of Theorem 7.1. Sufficiency immediately follows from Lemma 8.2.

Let us prove necessity. Let problem (7.5),(7.2) be strongly well–posed. Assume the contrary: let problem (7.7) have a nontrivial solution $\xi_0(x)$ for some $y_0 \in [0,b]$. Then due to well–posedness of problem (7.5),(7.2) there exist $\delta > 0$, $\widehat{p}_{jn} \in C^{(0,n)}(\Omega)$ $(j = 0,\dots,m;\ k=0,\dots,n;\ j+k < m+n)$ and $\widehat{l}_j : C^{m-1}([0,a]) \to C^n([0,b])$ $(j=1,\dots,m)$ such that

$$\begin{aligned}\widehat{p}_{jn}(x,y) =& p_{jn}(x,y_0) \quad \text{for} \quad y \in [y_0-\delta, y_0+\delta] \cap [0,b] \quad (j,k) < (m,n)\\ \widehat{l}_j(z)(y) =& l_j(z)(y_0) \quad \text{for} \quad y \in [y_0-\delta, y_0+\delta] \cap [0,b],\\ & \qquad\qquad z \in C^{m-1}([0,a])\ (j=1,\dots,m),\end{aligned}$$

and the problem

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} \widehat{p}_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p_{jk}(x,y)u^{(j,k)},$$
$$\widehat{l}_j(u(\cdot,y)(y) = 0 \ \ (j=1,\dots,m), \qquad h_k(u^{(m,0)}(x,\cdot))(x) = 0 \ \ (k=1,\dots,n),$$

is well–posed. From Remark 8.2 and representation (8.20) it follows that the problem

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1} \widehat{p}_{jn}(x,y)u^{(j,n)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \widehat{p}_{jk}(x,y)u^{(j,k)}, \tag{$\widehat{7.5}$}$$

$$\widehat{l}_j(u(\cdot,y)(y) = 0 \ \ (j=1,\dots,m),$$
$$h_k\Big(u^{(m,0)}(x,\cdot) - \sum_{j=0}^{m-1} \widehat{p}_{jn}(s,\cdot)u^{(j,n)}(s,\cdot)\Big) = 0 \ \ (k=1,\dots,n), \tag{$\widehat{7.2}$}$$

where

$$\widehat{p}_{jk}(x,y) = -\sum_{i=k}^{n} \frac{i!}{k!(i-k)!}\, p_{mi}(x,y)\, \widehat{p}_{jn}^{(0,i-k)}(x,y)$$
$$+\frac{n!}{k!(n-k)!}\, \widehat{p}_{jn}^{(0,n-k)}(x,y) \quad (j=0,\dots,m-1;\ k=0,\dots n-1),$$

has the Fredholm property. On the other hand an arbitrary solution u of problem $(\widehat{7.5}), (\widehat{7.2})$ is a solution of the problem

$$\Big(u^{(m,0)} - \sum_{j=0}^{m-1} \widehat{p}_{jn}(x,y) u^{(j,0)}\Big)^{(0,n)}$$
$$= \sum_{k=0}^{n-1} p_{mk}(x,y)\Big(u^{(m,0)} - \sum_{j=0}^{m-1} \widehat{p}_{jn}(x,y) u^{(j,0)} u^{(j,0)}\Big)^{(0,k)},$$
$$h_k\Big(u^{(m,0)}(x,\cdot) - \sum_{j=0}^{m-1} \widehat{p}_{jn}(s,\cdot) u^{(j,n)}(s,\cdot)\Big) = 0 \quad (k=1,\dots,n).$$

Hence, every solution $u \in C^{m,n}(\Omega)$ of the problem

$$u^{(m,0)} = \sum_{j=0}^{m-1} \widehat{p}_{jn}(x,y) u^{(j,0)}, \tag{9.1}$$

$$\widehat{l}_j(u(\cdot,y)(y) = 0 \ \ (j=1,\dots,m) \tag{9.2}$$

is a solution of problem $(\widehat{7.5}), (\widehat{7.2})$.

Let $\gamma \in C^\infty([0,b])$, $\operatorname{supp}\gamma \subset [y_0-\delta, y_0+\delta] \cap [0,b]$ be an arbitrary function. Then

$$u(x,y) = \xi_0(x)\gamma(y)$$

is a solution of problem (9.1),(9.2), and, consequently, is a solution of the problem $(\widehat{7.5}), (\widehat{7.2})$. Thus problem $(\widehat{7.5}), (\widehat{7.2})$ has an infinite dimensional space of solutions, which contradicts to the fact that is has the Fredholm property. The obtained contradiction completes the proof of the theorem. □

The Proof of Theorem 7.2. is similar to the proof of Theorem 1.1.

Proof of Theorem 7.3. Consider the equation

$$u^{(m,n)} = f_\lambda\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big)$$

$$= (1-\lambda)\sum_{(j,k)<(m,n)} p_{jk}(x,y)u^{(j,k)}$$

$$+\lambda f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{9.3}$$

where $p_{jn}(x,y)=P_{1jk}(x,y)$ $(j=0,\dots,m;\ k=0,\dots,n;\ j+k<m+n)$ and $\lambda\in[0,1]$. Notice that

$$P_{1jk}(x,y)\le f_{\lambda jk}(x,y,\mathbf{v},\mathbf{w},\mathbf{z})\le P_{2jk}(x,y)\quad\text{for}\quad (x,y,\mathbf{v},\mathbf{w},\mathbf{z})\in\Omega\times\mathbb{R}^{n+m+mn}$$

$$(j=0,\dots,m,\ k=0,\dots,n;\ j+k<m+n).$$

Therefore, by Lemma 8.3 and Remark 8.1, an arbitrary solution u_λ of problem (9.3),(7.2) admits the estimate

$$\|u\|_{C^{m,n}(\Omega)}\le M\Big(\sum_{i=1}^{m}\|\varphi_i\|_{C^n([0,b])}+\sum_{k=1}^{n}\|\psi_k\|_{C^m([0,a])}+\|q\|_{C(\Omega)}\Big),$$

where $M>0$ is independent of φ_i, ψ_k, q and λ.

If for some $\lambda_0\in[0,1)$ problem (9.3),(7.2) has a solution, then by Lemma 8.5, there exists $\varepsilon_0>0$ independent of λ such that problem (9.3),(7.2) is strongly well–posed for every $\lambda\in(\lambda_0-\varepsilon_0,\lambda_0+\varepsilon_0)\cap[0,1]$.

By Theorem 7.1, problem (9.3),(7.2) is uniquely solvable for $\lambda=0$. Therefore according to the aforesaid, problem (9.3),(7.2) is solvable for every $\lambda\in[0,1]$, in particular for $\lambda=1$. □

The proof of Theorem 7.4 is similar to the proof of Theorem 1.3. The only difference is that Lemmas 8.4, 8.5 and 8.6 should be used instead of Lemmas 2.3, 2.5 and 2.6.

Proof of Corollary 7.2. First notice that without loss of generality one can assume $f_0(x,y,0)\equiv 0$. Let u be a solution of problem (7.16),(7.18). Multiply both sides of equation (7.16) by u and integrate over Ω. After integrating by parts we arrive at the equality

$$\int_0^a\int_0^b\Big(|u^{(m,n)}(x,y)|^2-(-1)^n f_1(x,y,u(x,y))|u^{(m,0)}(x,y)|^2$$

$$-(-1)^m f_2(x,y,u(x,y))|u^{(0,n)}(x,y)|^2-(-1)^{m+n}f_0(x,y,u(x,y))u(x,y)\Big)\,dy\,dx$$

$$= (-1)^{m+n} \int_0^a \int_0^b q(x, y, u(x, y), \dots, u^{(m-1,n-1)}(x, y))\, u(x, y)\, dy\, dx$$

In view of (7.21)–(7.24) we get

$$\int_0^a \int_0^b \Big(|u^{(m,n)}(x, y)|^2 + \delta |u^{(m,0)}(x, y)|^2 + \delta |u^{(0,n)}(x, y)|^2 + \delta u^2(x, y) \Big)\, dy\, dx$$

$$\leq M \int_0^a \int_0^b |u(x, y)|\, dy\, dx + Mab,$$

and

$$\int_0^a \int_0^b \Big(|u^{(m,n)}(x, y)|^2 + |u^{(m,0)}(x, y)|^2 + |u^{(0,n)}(x, y)|^2 + u^2(x, y) \Big)\, dy\, dx \leq M_1, \quad (9.4)$$

where

$$M_1 = \frac{2 + \delta}{\delta} \Big(\frac{M^2}{2\delta} + Mab \Big).$$

(9.4) implies the estimate

$$\|u\|_{C^{m-1,n-1}(\Omega)} \leq \rho, \quad (9.5)$$

where ρ is a positive number depending on a, b, δ and M only.

For an arbitrary $v \in C^{m-1,n-1}(\Omega)$ consider the equations

$$\begin{aligned} u^{(2m,2n)} =& \Big(f_1(x, y, v(x, y)) u^{(m,0)} \Big)^{(m,0)} + \Big(f_2(x, y, v(x, y)) u^{(0,n)} \Big)^{(0,n)} \\ &+ f_0\big(x, y, u\big) + q(x, y, v(x, y), \dots, v^{(m-1,n-1)}(x, y)) \end{aligned} \quad (9.6)$$

and

$$\begin{aligned} u^{(2m,2n)} =& \Big(f_1(x, y, v(x, y)) u^{(m,0)} \Big)^{(m,0)} + \Big(f_2(x, y, v(x, y)) u^{(0,n)} \Big)^{(0,n)} \\ &+ (-1)^{m+n-1}\, \delta u + f_0\big(x, y, \chi_\rho(u)\big) - (-1)^{m+n-1} \delta\, \chi_\rho(u) \\ &+ q\big(x, y, v(x, y), \dots, v^{(m-1,n-1)}(x, y)\big), \end{aligned} \quad (9.7)$$

where $\chi_r(t)$ is a nondecreasing continuously differentiable function such that $\chi_r'(t) \leq 1$ and

$$\chi_r(t) = \begin{cases} t & \text{for} \quad t \in [0, r] \\ 2r & \text{for} \quad t \in (3r, +\infty) \end{cases} .$$

By Theorem 7.3, problem (9.7),(7.2) is uniquely solvable. It is clear an arbitrary solution of problem (9.7),(7.2) admits estimate (9.5). Consequently, every solution of problem (9.7),(7.2) is a solution of problem (9.6),(7.2) too. Thus problem (9.6),(7.2) is uniquely solvable for every $v \in C^{m-1,n-1}(\Omega)$.

Consider the operator $\mathcal{K} : v \to u$. By Lemma 8.4, $\mathcal{K} : C^{m-1,n-1}(\Omega) \to C^{2m,2n}(\Omega)$ is a continuous operator. In view of estimate (9.5), $\mathcal{K} : C^{m-1,n-1}(\Omega) \to C^{m-1,n-1}(\Omega)$ is a compact operator mapping $C^{m-1,n-1}(\Omega)$ in to ball $\mathcal{B}^{m-1,n-1}(0,\rho)$.

By Schauder's fixed point theorem, $\mathcal{K}$ has a fixed point $v_0 \in \mathcal{B}^{m-1,n-1}(0,\rho)$. Hence $u = \mathcal{K}(v_))(x,y) = v_0(x,y)$ is a solution of the problem (7.16),(7.18). $\square$

$\square$

Corollaries 7.3 and 7.4 can be proved similarly.

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