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On Logconcavity of Multivariate Discrete Distributions

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by

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ABSTRACT

Title:

On Logconcavity of Multivariate Discrete Distributions

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The contribution of this dissertation to the literature is twofold. First, we use a geometric perspective to present all possible subdivisions of $\mathbb{R}^3$ into tetrahedra with disjoint interiors and adopt a combinatorial approach to obtain a special subdivision of $\mathbb{R}^n$ into simplices with disjoint interiors, where two simplices are called neighbors if they share a common facet. We then use the neighborhood relationship of the simplices in each subdivision to fully describe the sufficient conditions for the strong unimodality/logconcavity of the trivariate discrete distributions and further extend these results to present a new sufficient condition for the strong unimodality/logconcavity of multivariate discrete distributions defined on $\mathbb{Z}^n$. We show that the multivariate Pólya-Eggenberger distribution, multivariate Poisson distribution, and multivariate Ewens distribution are strongly unimodal, and hence logconcave.

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Acknowledgements

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made them the best years of my life. I am deeply thankful to them for their love, support, and sacrifices. There are no words that can express my gratitude and appreciation for all you have done and been for me.

Dedication

To my parents, my wife Manal, my son Moath, and my daughter Shahad.

Chapter 1

Introduction

In the field of optimization, convex analysis plays a crucial role in both theory and practice (Boyd and Vandenberghe (2004) [11]). While some continuous multivariate functions enjoy a number of useful properties such as convexity, concavity, logconcavity, and other generalized convexity, most of these properties do not directly carry over to the discrete case. In the area of discrete optimization, an analogous theory has been developed and several different types of discrete convexity have been introduced: (*i*) Miller (1971) [27] introduced discretely convex functions. However, the class of discretely convex functions is not closed under addition. (*ii*) Favati and Tardella (1990) [20] introduced integrally convex functions and investigated connections between the convexity of a function on $\mathbb{R}^n$ and the integer convexity of its restriction to $\mathbb{Z}^n$. They presented a polynomial time algorithm to find the minimum of a submodular integrally convex function. Murota and Shioura (2001) [32] showed that the class of integrally convex functions is not closed under addition in general. (*iii*) Murota (1998, 2000, 2015) [29, 31, 34] and Murota and Shioura (1999, 2001, 2003) [30, 32, 33] has introduced L-convex and M-convex functions and advocated the theory of discrete convex analysis that aims to establish a general theoretical framework for solvable discrete optimization problems by integrating the ideas in continuous optimization and combinatorial optimization. Local minima of the discretely convex functions, integrally convex functions and L/M-convex functions are also global minima, however, according to Ui (2006) [45] the definition of locality depends on the type of discrete convexity.

Another concept which lies at the very heart of optimization is logconcavity. Logconcavity of continuous multivariate distributions has been extensively studied in literature and a variety of important results has been obtained (see, e.g., Prékopa (1995) [40] and Boyd and Vandenberghe (2004) [11]).

A nonnegative function f defined on a convex subset A of the space $\mathbb{R}^n$ is said to be *logconcave* if for every pair $\mathbf{x}, \mathbf{y} \in A$ and $0 < \lambda < 1$, we have the inequality

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq [f(\mathbf{x})]^{\lambda}[f(\mathbf{y})]^{(1-\lambda)}.$$

If f is positive valued, then $\log f$ is a concave function on A. If the inequality holds strictly for $\mathbf{x} \neq \mathbf{y}$, then f is said to be strictly logconcave.

The notion of logconcave probability measures was introduced by Prékopa (1971, 1973) [38, 39]. Before giving the definition of a logconcave probability measure, let us recall a few definitions we shall use. A class S of subsets of $\mathbb{R}^n$ is called n *algebra* if (i) $\mathbb{R}^n \in S$, (ii) $A \in S$ implies $\bar{A} \in S$, and (iii) $A, B \in S$ implies $A \cup B \in S$. If S is an algebra and $A_1, A_2, ... \in S$ implies $\bigcup_{i=1}^{\infty} A_i \in S$, then S is called a σ-*algebra*. All finite unions of all finite or infinite n-dimensional intervals form an algebra and the smallest σ-algebra that contains this algebra is the collection of Borel measurable sets and is designated by $\mathscr{B}_n$ (see, e.g., [40]).

A set function $P(A), A \in S$ is called a *measure* if S is a σ-algebra and

1. $P(A) \geq 0$ for every $A \in S$,
2. $P(\emptyset) = 0$,
3. $A_1, A_2, ... \in S$ and $A_i \cap A_j = 0$ for $i \neq j$ implies

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

A measure P is called a *probability measure* if $P(\mathbb{R}^n) = 1$. Let A probability measure P, defined on $\mathscr{B}_n$, is called a *logconcave probability measure* if for every pair of nonempty convex sets $A, B \subset \mathbb{R}^n$ (any convex set is Borel measurable, see, e.g., [40]), we have the inequality

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{(1-\lambda)},$$

where the + sign refers to Minkowski addition of sets, i.e.,

$$\lambda A + (1-\lambda)B = \{\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \mid \mathbf{x} \in A, \mathbf{y} \in B\}.$$

The above notion generalizes in a natural way to nonnegative valued measures. In this case we require the logconcavity inequality to hold for finite $P(A), P(B)$.

The notion of logconcave probability measures has revolutionized the theory of logconcavity and led to a wide range of applications in different areas including optimization theory, statistic, probability theory, and economics. Prékopa (1971, 1973, 1995) [38, 39, 40] showed that the convexity of stochastic optimization problems frequently depends on the logconcavity of the underlying distribution. Dentcheva, Prékopa, Ruszczynski (2000) [14] and Dentcheva, Lai, Ruszczynski (2004) [15] discussed the use of logconcavity property in probabilistic constrained optimization problems. Ninh and Prékopa (2013) [35] proved the logconcavity of compound distributions and presented an application to bond-portfolio optimization.

Bagnoli and Bergstrom (2005) [5] presented several applications of logconcavity in economics, where they reported that the logconcavity of a continuously differentiable density function implies the logconcavity of the corresponding reliability function. They also showed the connection between the logconcavity of a density function and the failure rate of an object modeled as a random variable. They noted that it is easier to determine the monotonicity of the failure rate and the mean residual life time function if the density function is logconcave. They used the fact that the logconcavity of a density function implies the logconcavity of any truncation of the probability distribution. Other economics-related applications of logconcavity can be found in monopoly theory, see for example, Buyers' and Sellers' agents in the housing market by Bagnoli and Khanna (1991) [4] and Pre-tender offer share acquisition strategy in takeovers by Chowdhry and Jegadeesh (1994) [12]. Logconcavity has also attracted considerable attention in the statistical theory, including reliability, truncated distributions, hypothesis testing, and maximum likelihood estimations (see, for example, Barlow and Proschan (1975) [8], Mu (2015) [28] and the references therein). Theoretical properties and applications of logarithmic maximum likelihood estimator can be found in Cule and Samworth (2009) [13] and Balabdaoui et al. (2013) [3].

Yu (2009) [47] showed that, under a logconcavity assumption, two compound distributions are ordered in terms of Shannon entropy if both the number of claims and the claim sizes are ordered accordingly in the convex order and proved maximum entropy property of compound Poisson and compound binomial distributions. Johnson (2007) [24] used a semigroup approach to show that the Poisson distribution has maximal entropy among all ultra-log-concave distributions with fixed mean. Johnson et al. (2013) [25] showed via a non-trivial extension of this semigroup approach that the natural analog of the Poisson maximum entropy property remains valid if the compound Poisson distributions under consideration are log-concave, but that it fails in general. Johnson et al. (2013) [25] also presented the sufficient conditions for compound distributions to be logconcave and applications to combinatorics. Wild and Gilks (1993) [46] proposed an algorithm for rejection sampling from univariate logconcave density functions. Saumard and Wellner (2014) [41] presented a review of logconcavity and strong logconcavity as well as their connections with concepts such as concentration of measure, log-Sobolev inequalities, convex geometry, Monte-Carlo Markov Chain algorithms, Laplace approximations, and machine learning.

While the logconcavity of probability density functions has attracted considerable attention, little attention has been given to the logconcavity of discrete distributions. The classical result in this respect is due to Fekete (1912) [21] who introduced the notion of an *r-times positive sequence*. The sequence of nonnegative elements $\ldots, a_{-2}, a_{-1}, a_0, \ldots$ is said to be *r-times positive* if the matrix

$$A = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \ldots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ \ldots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & \ldots \\ \ldots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

has no negative minor of order smaller than or equal to r.

Twice-positive sequences are those for which we have

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0 \tag{1.1}$$

for every $i < j$ and $t \geq 1$. This holds if and only if $a_i^2 \geq a_{i-1}a_{i+1}$. Fekete (1912) [21] also proved that the convolution of two r-times positive sequences is r-times positive. Twice-positive sequences are also called *logconcave sequences*. Hence, Fekete's theorem states that the convolution of two logconcave sequences is logconcave. A discrete probability distribution, defined on the real line, is said to be logconcave if the corresponding probability mass function (p.m.f) is logconcave.

Following Barndorff-Nielsen (1973) [7] a joint p.m.f p of a random vector $X \in \mathbb{Z}^n$ is called *strongly unimodal* if there exists a convex function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ such that

$$f(\mathbf{x}) = -\log p(\mathbf{x}) \text{ if } \mathbf{x} \in \mathbb{Z}^n.$$

If $p(\mathbf{x}) = 0$, then $f(\mathbf{x}) = \infty$ by definition. This notion is not a direct generalization of that of the one-dimensional case, i.e., of formula (1.1). However, the two notions are the same when $n = 1$ (see, e.g., Prékopa (1995) [40]). Note that if p is strongly unimodal, then it is logconcave. In general, the convolution of two logconcave p.m.f's defined on $\mathbb{Z}^n$ is no longer logconcave if $n \geq 2$.

Pedersen (1975) [37] used the strong unimodality definition of Barndorff-Nielsen and presented two sufficient conditions for a bivariate discrete distribution to be strongly unimodal. Pedersen [37] also proved that the trinomial distribution is logconcave and the convolution of any finite number of these distributions with possibly different parameter sets is also logconcave.

Subasi et al. (2009) [44] presented six sufficient conditions for a trivariate discrete distribution to be strongly unimodal and one sufficient condition for multivariate discrete distributions defined on $\mathbb{Z}^n$. Subasi et al. (2009) [44] also showed that negative multinomial, multivariate hypergeometric, multivariate negative hypergeometric, and Dirichlet (or beta)-compound multinomial distributions are strongly unimodal.

The goal of this dissertation is to extend the results of Subasi et al. (2009) [44] by providing new sufficient conditions that ensure the strong unimodality of multivariate discrete distributions. The organization of the dissertation is as follows. In Chapter 2 we present all possible subdivisions of a cube into simplices with disjoint interiors and use them to obtain all possible sufficient conditions for a trivariate discrete distribution to be strongly unimodal and hence, logconcave. In Chapter 3 we adopt a combinatorial approach to obtain a special subdivision of $\mathbb{R}^n$ into simplices with disjoint interiors, where two simplices are called neighbors if they have a common facet. We then use the neighborhood relationship of the simplices in the special subdivision to obtain a new sufficient condition that ensures the strong unimodality, and hence, logconcavity of multivariate discrete distributions. In Chapter 4, we show the multivariate Pólya-Eggenberger distribution, multivariate Poisson distribution, and multivariate Ewens distribution are logconcave. Finally, in Appendix A we give a collection of continuous and discrete convexity and generalized convexity notions and theorems that previously studied in literature.

Chapter 2

Sufficient Conditions for Strong Unimodality of Trivariate Discrete Distributions

In this chapter we shall present sufficient conditions for a discrete distribution defined on $\mathbb{Z}^3$ to be strongly unimodal (see Alharbi, Subasi, Subasi (2017) [1]). In order to fully describe the sufficient conditions that ensure strong unimodality of trivariate distributions we first investigate all possible subdivisions of a cube into simplices (tetrahedra) with disjoint interiors. We shall call two tetrahedra "neighbors" if they share a common face.

2.1 Subdivisions of a Cube in $\mathbb{R}^3$

Subasi et al. (2009) [44] gave six different ways of subdividing a cube in $\mathbb{R}^3$ into six simplices (tetrahedra) with pairwise disjoint interiors such that the vertices of the simplices are the lattice points in $\mathbb{Z}^3$ and are those of the cube.

We use a combinatorial approach to describe all possible tetrahedral subdivisions of a cube in $\mathbb{R}^3$. We observe that there are only 18 tetrahedral subdivisions where the subdividing simplices are pairwise disjoint and may share a common face. The neighborhood relation of a pair of tetrahedra in a subdivision is used to obtain two new simplices out of them by changing one vertex in both simplices in a way that allow the new simplices to cover the same area of the cube that was covered by the original two tetrahedra. These two new simplices with the remaining four simplices from the original subdivision form a new subdivision of the cube.

In the following theorem and its proof we present all possible tetrahedral subdivisions of a cube in $\mathbb{R}^3$ and describe our combinatorial approach that uses the neighborhood relationship of two simplices in a subdivision to obtain a different subdivision of the cube.

Theorem 1. *There are exactly eighteen different ways to subdivide a unit cube in* $\mathbb{R}^3$ *into six tetrahedra with disjoint interiors such that the vertices of the tetrahedra are the lattice points and are those of the cube. The subdivisions (see Figures 2.1-2.18) are as follows:*

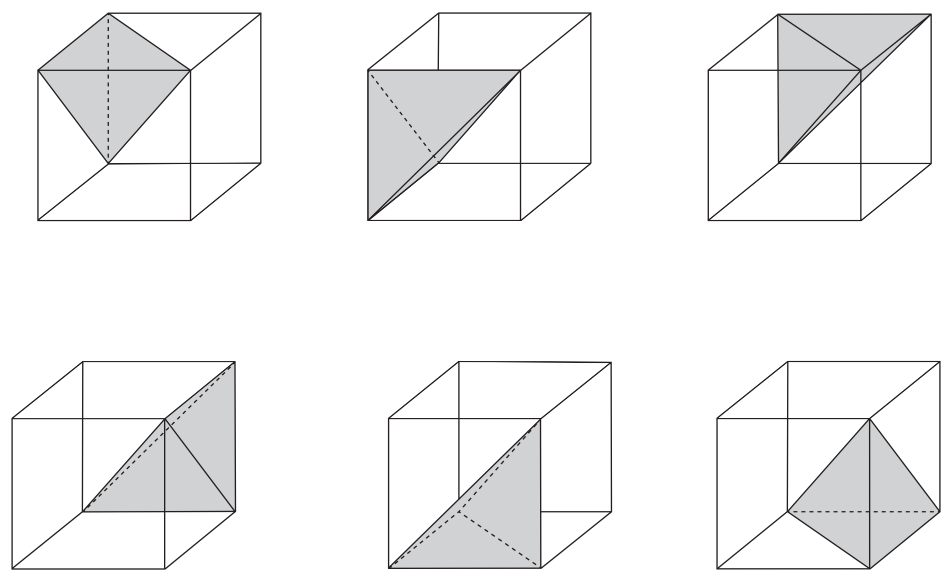

Figure 2.1: Subdivision 1

<u>*Subdivision 1.*</u> $T_{1,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{1,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{1,2}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{1,3}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{1,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{1,5}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{1,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

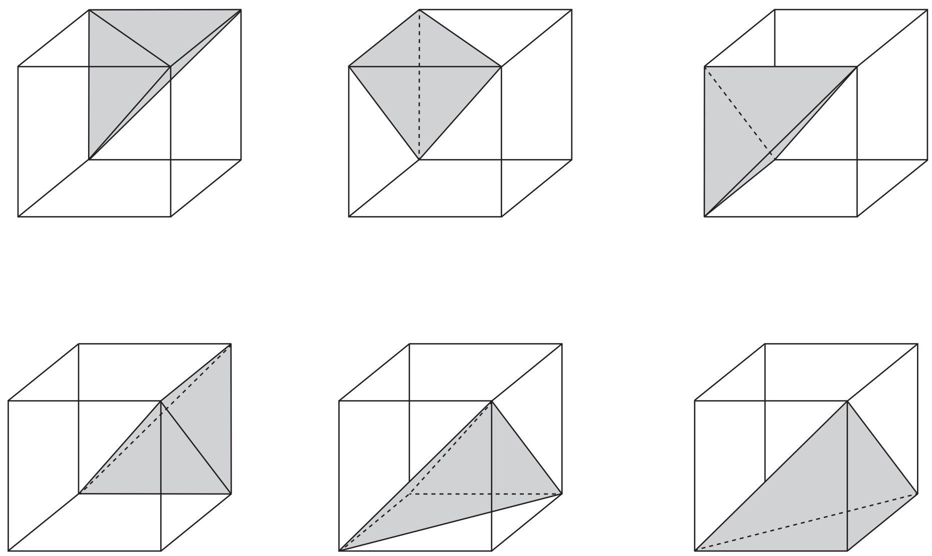

Figure 2.2: Subdivision 2

<u>Subdivision 2.</u> $T_{2,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{2,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{2,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{2,3}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{2,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{2,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{2,6}(i,j,k) = conv\{(i+1,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

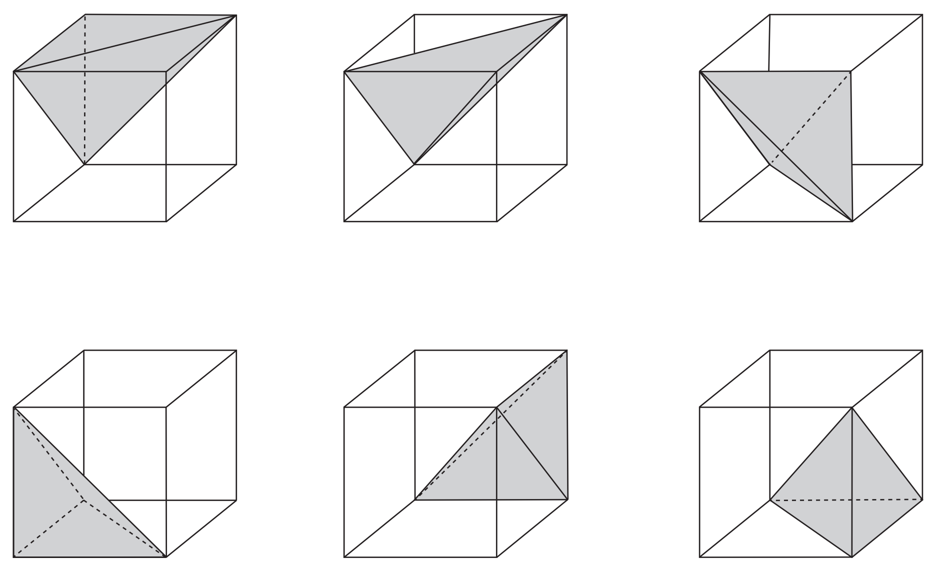

Figure 2.3: Subdivision 3

Subdivision 3. $T_{3,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{3,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},$$

$$T_{3,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{3,3}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{3,4}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i,j+1,k+1)\},$$

$$T_{3,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k+1),(i+1,j,k),(i+1,j+1,k+1)\},$$

$$T_{3,6}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k),(i,j+1,k+1)\}.$$

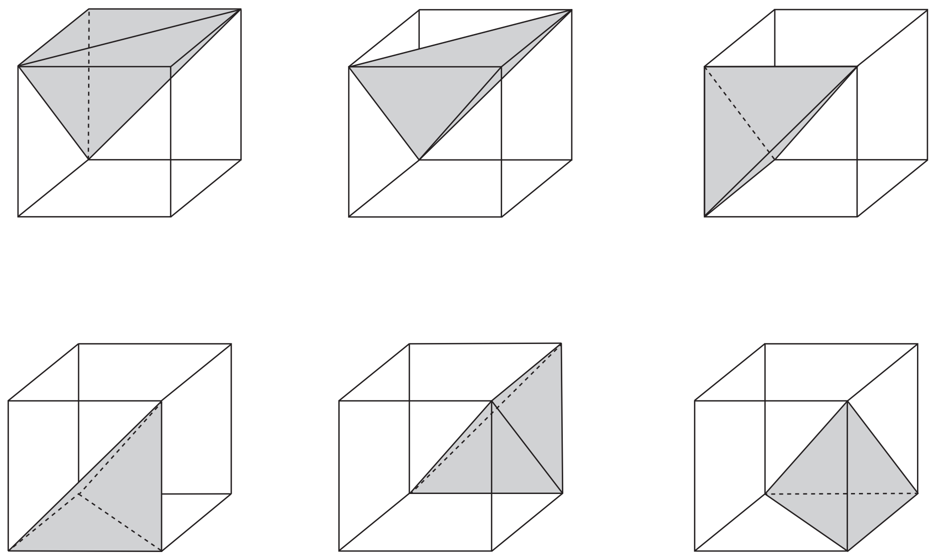

Figure 2.4: Subdivision 4

Subdivision 4. $T_{4,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{4,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},$$

$$T_{4,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{4,3}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{4,4}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{4,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{4,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

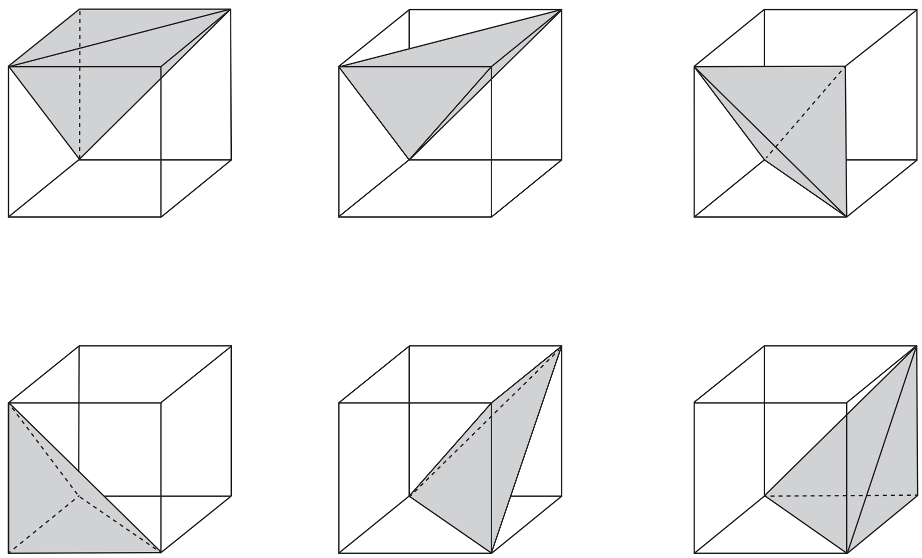

Figure 2.5: Subdivision 5

<u>*Subdivision 5.*</u> $T_{5,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{5,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},$$

$$T_{5,2}(i,j,k) = conv\{(i,j,k),(i+!,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{5,3}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{5,4}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k)\},$$

$$T_{5,5}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{5,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\}.$$

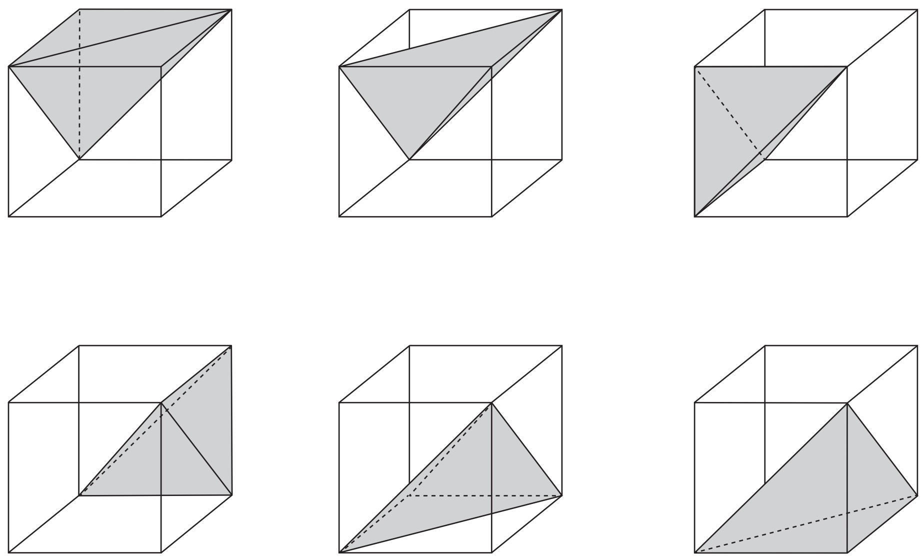

Figure 2.6: Subdivision 6

<u>*Subdivision 6.*</u> $T_{6,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{6,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},$$

$$T_{6,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{6,3}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{6,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{6,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{6,6}(i,j,k) = conv\{(i+1,j,k),(i+1,j+1,k),(i,j+1,k),(i+1,j+1,k+1)\}.$$

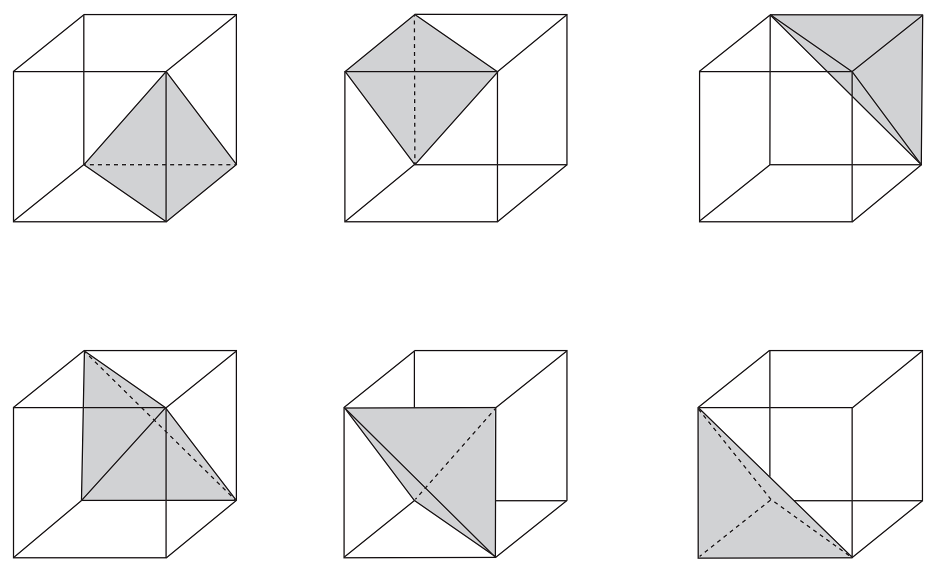

Figure 2.7: Subdivision 7

Subdivision 7. $T_{7,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{7,1}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{7,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{7,3}(i,j,k) = conv\{(i+1,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{7,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{7,5}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{7,6}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i,j+1,k+1)\}.$$

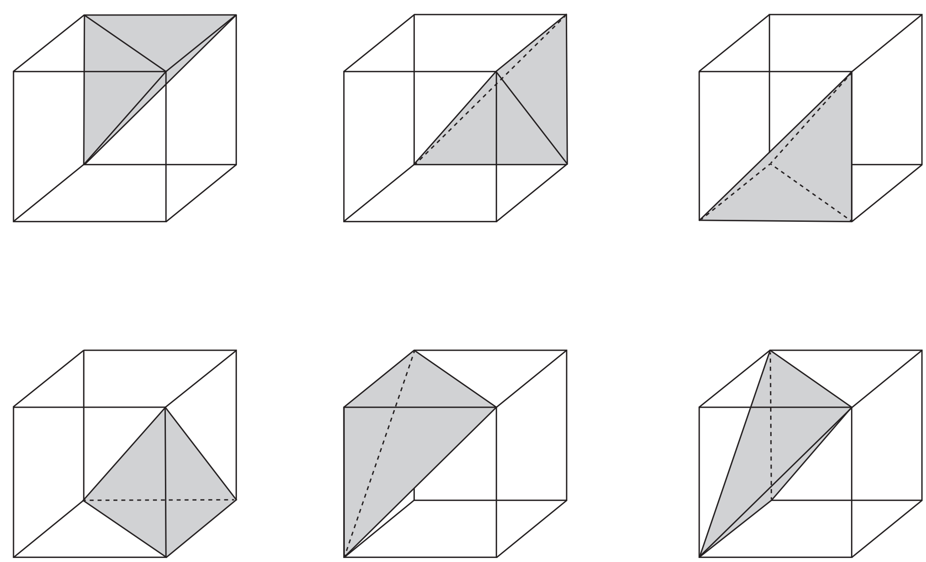

Figure 2.8: Subdivision 8

<u>*Subdivision 8.*</u> $T_{8,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{8,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{8,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{8,3}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{8,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{8,5}(i,j,k) = conv\{(i,j,k+1),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{8,6}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k),(i+1,j+1,k+1)\}.$$

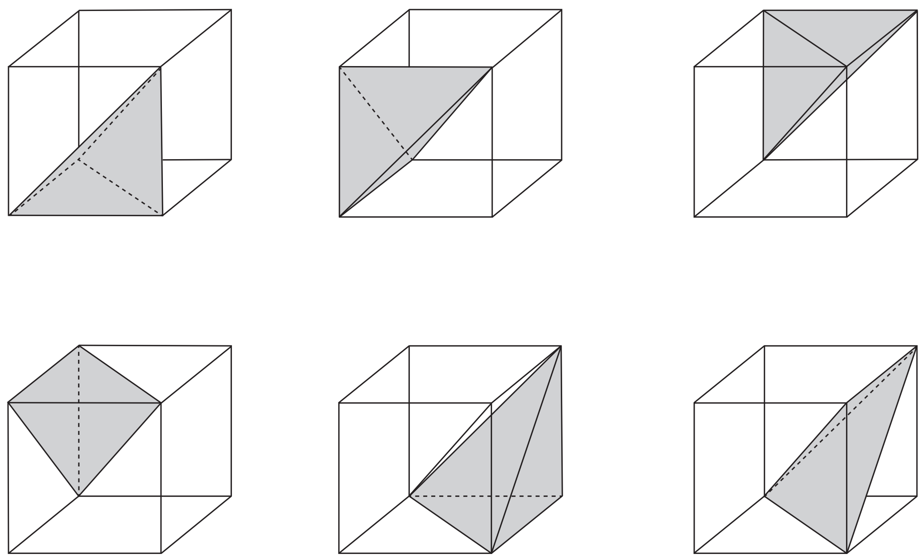

Figure 2.9: Subdivision 9

Subdivision 9. $T_{9,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{9,1}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{9,2}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{9,3}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{9,4}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{9,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\},$$

$$T_{9,6}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\}.$$

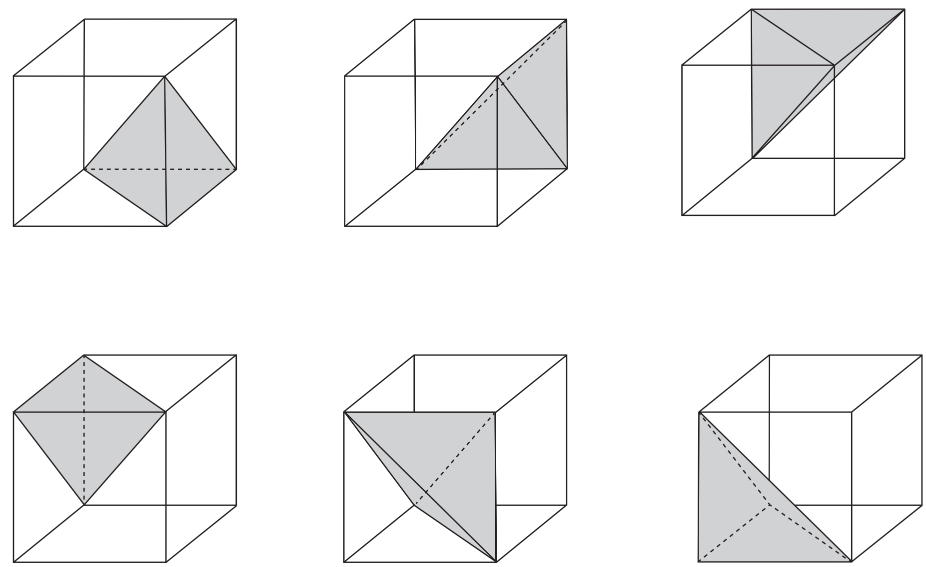

Figure 2.10: Subdivision 10

<u>*Subdivision 10.*</u> $T_{10,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{10,1}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{10,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{10,3}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{10,4}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{10,5}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{10,6}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i,j+1,k+1)\}.$$

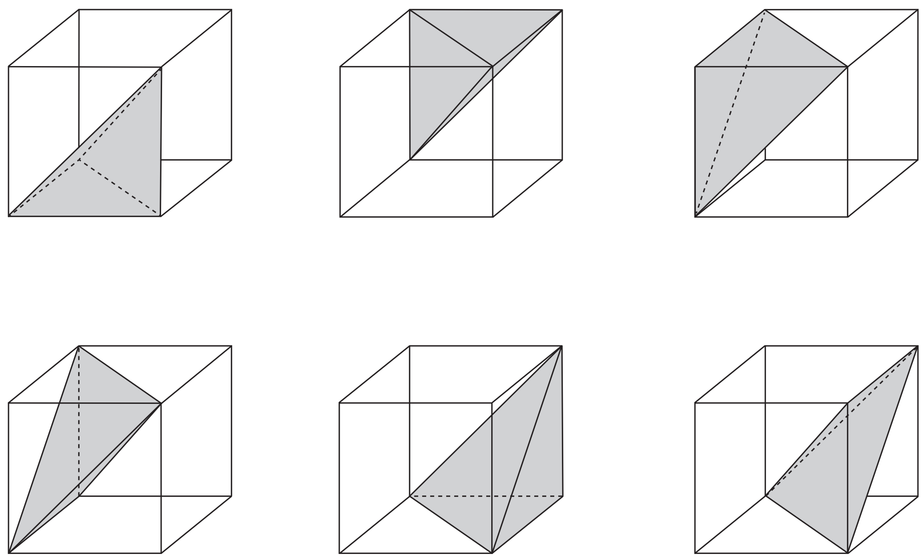

Figure 2.11: Subdivision 11

Subdivision 11. $T_{11,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{11,1}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{11,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{11,3}(i,j,k) = conv\{(i,j,k+1),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{11,4}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{11,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\},$$

$$T_{11,6}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\}.$$

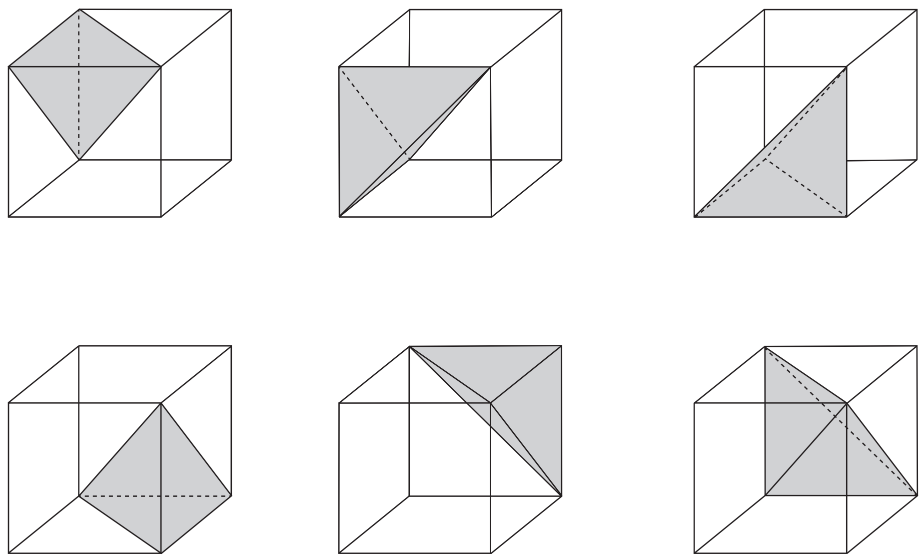

Figure 2.12: Subdivision 12

Subdivision 12. $T_{12,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{12,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{12,2}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{12,3}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{12,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{12,5}(i,j,k) = conv\{(i+1,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{12,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\}.$$

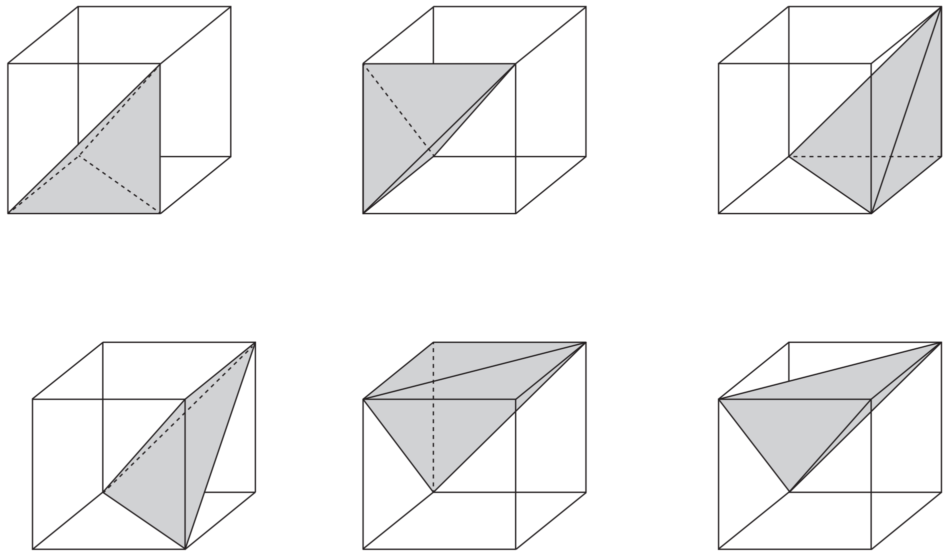

Figure 2.13: Subdivision 13

<u>Subdivision 13.</u> $T_{13,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{13,1}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{13,2}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{13,3}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\},$$

$$T_{13,4}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{13,5}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i,j+1,k+1)\},$$

$$T_{13,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\}.$$

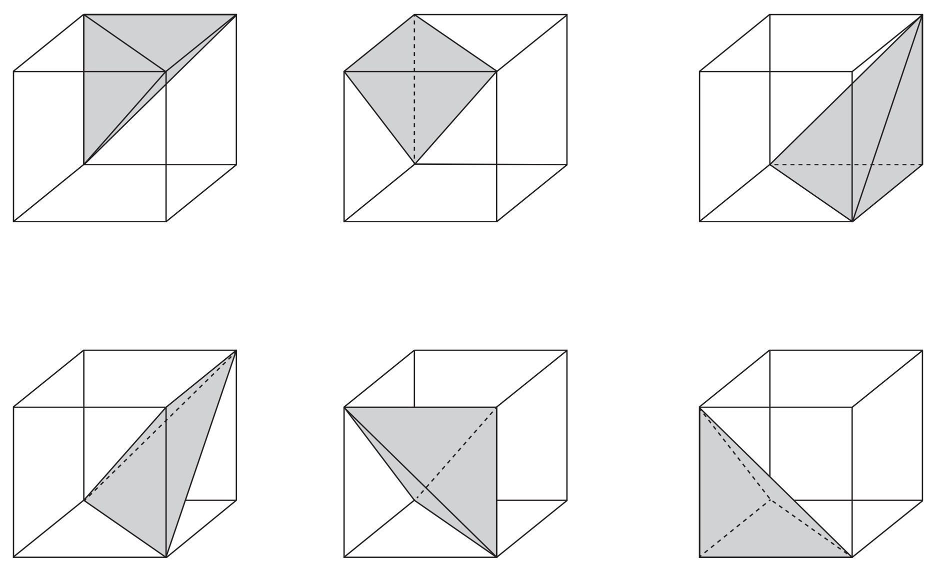

Figure 2.14: Subdivision 14

Subdivision 14. $T_{14,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{14,1}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{14,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{14,3}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j,k+1)\},$$

$$T_{14,4}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{14,5}(i,j,k) = conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{14,6}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i,j+1,k+1)\}.$$

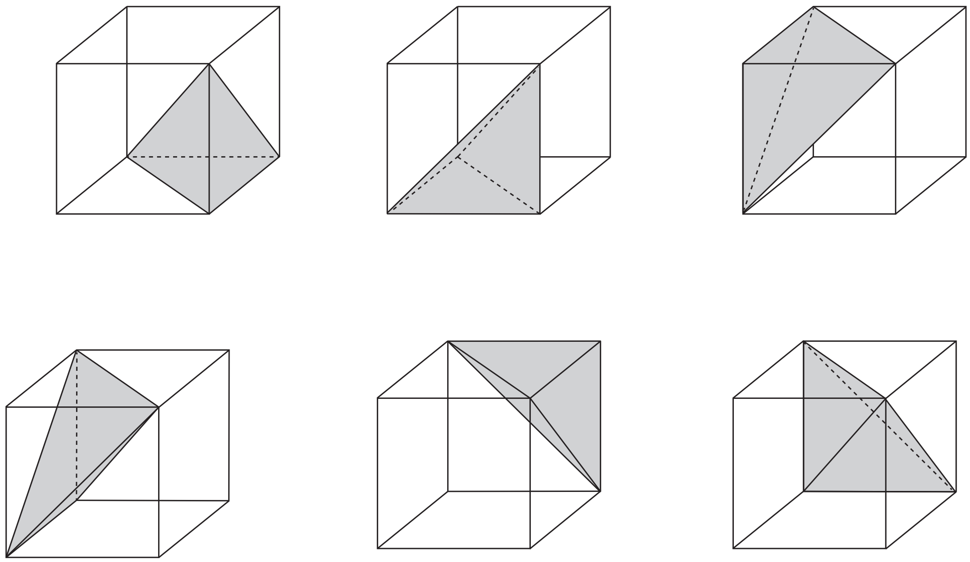

Figure 2.15: Subdivision 15

<u>Subdivision 15.</u> $T_{15,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{15,1}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{15,2}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{15,3}(i,j,k) = conv\{(i,j,k+1),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{15,4}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{15,5}(i,j,k) = conv\{(i+1,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{15,6}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\}.$$

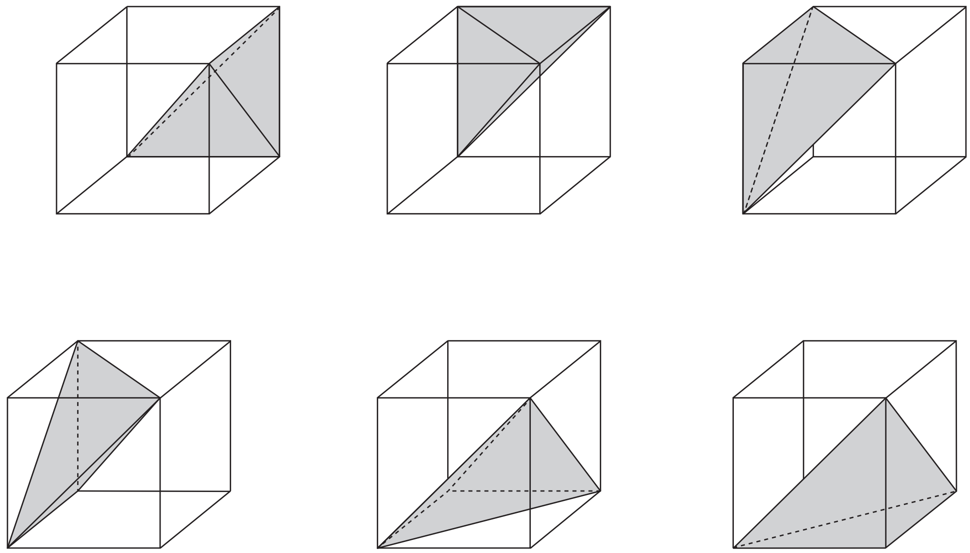

Figure 2.16: Subdivision 16

<u>Subdivision 16.</u> $T_{16,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{16,1}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{16,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{16,3}(i,j,k) = conv\{(i,j,k+1),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{16,4}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{16,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{16,6}(i,j,k) = conv\{(i+1,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

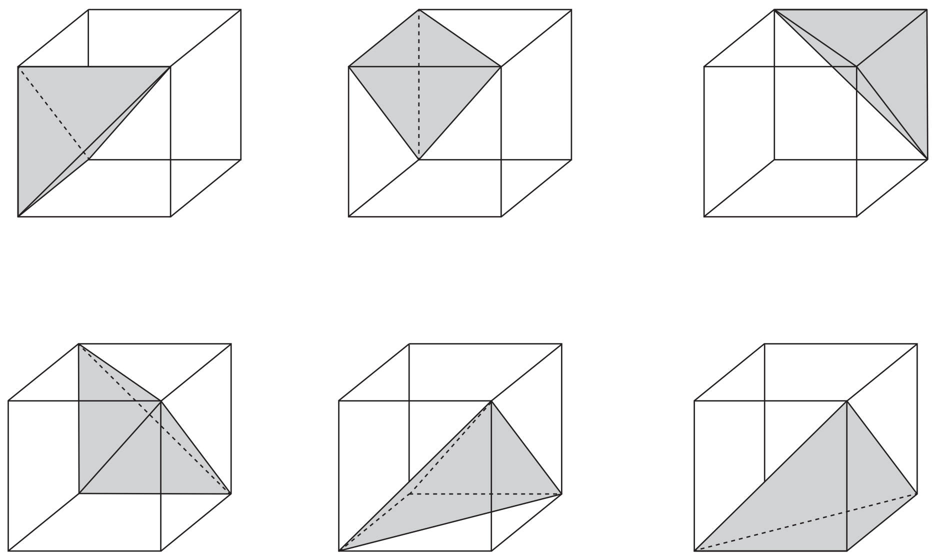

Figure 2.17: Subdivision 17

Subdivision 17. $T_{17,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{17,1}(i,j,k) = conv\{(i,j,k),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{17,2}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{17,3}(i,j,k) = conv\{(i+1,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{17,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{17,5}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{17,6}(i,j,k) = conv\{(i+1,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

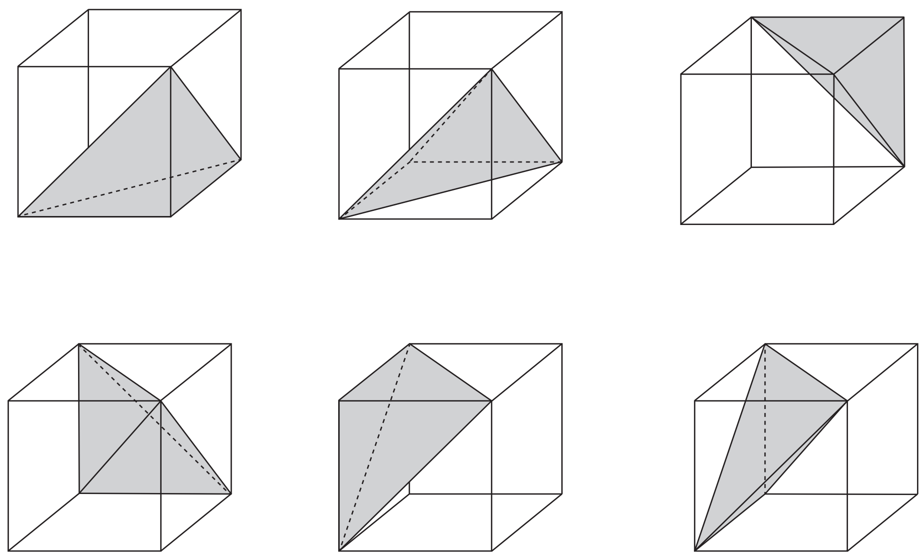

Figure 2.18: Subdivision 18

Subdivision 18. $T_{18,c}(i,j,k), c = 1, ..., 6$ *are the simplices in* $\mathbb{R}^3$ *defined by*

$$T_{18,1}(i,j,k) = conv\{(i+1,j,k),(i,j+1,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{18,2}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j+1,k),(i+1,j+1,k+1)\},$$

$$T_{18,3}(i,j,k) = conv\{(i+1,j,k),(i,j,k+1),(i+1,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{18,4}(i,j,k) = conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\},$$

$$T_{18,5}(i,j,k) = conv\{(i,j,k+1),(i,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$

$$T_{18,6}(i,j,k) = conv\{(i,j,k),(i,j,k+1),(i,j+1,k),(i+1,j+1,k+1)\}.$$

Proof. Let us consider Subdivision 1. For the sake of simplicity let us assume that vertices of tetrahedra are binary, i.e., $i, j, k \in \{0, 1\}$. Each tetrahedron of Subdivision 1 has two different neighbors (sharing one face) within the same cube as listed below:

$$T_{1,1}\&T_{1,2} \quad T_{1,1}\&T_{1,3} \quad T_{1,2}\&T_{1,5} \quad T_{1,3}\&T_{1,4} \quad T_{1,4}\&T_{1,6} \quad T_{1,5}\&T_{1,6} \tag{2.1}$$

In order to obtain a new subdivision of the cube, we take any neighboring tetrahedra (simplices) and make two new simplices out of them by changing one vertex in both simplices in a way that allow the new simplices to cover the same area of the cube that was covered by the original simplices. The technique is very simple: We replace one of the vertices of the first simplex by the vertex of the second simplex which is not included in the first simplex and vice versa such that exactly 12 (i.e., $n(n+1)$, where $n = 3$) components of the vertices of both simplices are 1 (zero). These two new simplices with the remaining four simplices from the original subdivision (Subdivision 1) form a new subdivision of a cube.

To be more explicit, let us consider the first two tetrahedra $T_{1,1}$ and $T_{1,2}$ in Subdivision 1:

$$T_{1,1} = conv\{(0,0,0),(0,0,1),(0,1,1),(1,1,1)\},$$
$$T_{1,2} = conv\{(0,0,0),(0,1,0),(0,1,1),(1,1,1)\}.$$

Note that these two tetrahedra are neighboring simplices since they share three common vertices (also can be seen in Figure 2.1). We need to get two new simplices out of $T_{1,1}\&T_{1,2}$ by replacing one vertex of $T_{1,1}$ by one of $T_{1,2}$'s vertices and one vertex of $T_{1,2}$ by one of $T_{1,1}$'s vertices such that the new simplices satisfy the condition of having 12 ones (12 zeros).

The following two simplices are the only ones that can be made out of $T_{1,1}$ and $T_{1,2}$ that satisfy this condition:

$$T^*_{1,1} = conv\{(0,0,0),(0,0,1),(0,1,0),(1,1,1)\},$$
$$T^*_{1,2} = conv\{(0,0,1),(0,1,0),(0,1,1),(1,1,1)\}.$$

Note that $T^*_{1,1}$ and $T^*_{1,2}$ along with the remaining simplices of Subdivision 1 (i.e., $T_{1,3}$, $T_{1,4}$, $T_{1,5}$, and $T_{1,6}$) form a new tetradral subdivision of the cube which, in this case, is Subdivision 8. If we repeat this process for any pair of neighboring simplices in Subdivision 1 listed in (2.1), we obtain six new subdivisions.

Now, we consider two pairs of neighboring simplices in Subdivision 1 such that a simplex is not contained in both pairs. There are nine such pairs:

$$\begin{array}{lll} T_{1,1}\&T_{1,2};\ T_{1,3}\&T_{1,4} & T_{1,1}\&T_{1,2};\ T_{1,4}\&T_{1,6} & T_{1,1}\&T_{1,2};\ T_{1,5}\&T_{1,6} \\ T_{1,1}\&T_{1,3};\ T_{1,2}\&T_{1,5} & T_{1,1}\&T_{1,3};\ T_{1,4}\&T_{1,6} & T_{1,1}\&T_{1,3};\ T_{1,5}\&T_{1,6} \\ T_{1,2}\&T_{1,5};\ T_{1,3}\&T_{1,4} & T_{1,2}\&T_{1,5};\ T_{1,4}\&T_{1,6} & T_{1,3}\&T_{1,4};\ T_{1,5}\&T_{1,6} \end{array} \tag{2.2}$$

From each one of these nine combinations, we can obtain four new simplices using the technique explained above. Taking these simplices along with the remaining two simplices from the original subdivision will result in a new subdivision. All nine combinations listed in (2.2) provide us with nine new tetrahedral subdivisions of the cube.

Finally, we take combinations of three pairs of neighboring simplices from the original subdivision such that a simplex is contained in only one of them. There are only two such combinations:

$$T_{1,1}\&T_{1,3};\ T_{1,2}\&T_{1,5};\ T_{1,4}\&T_{1,6} \qquad T_{1,1}\&T_{1,2};\ T_{1,3}\&T_{1,4};\ T_{1,5}\&T_{1,6} \tag{2.3}$$

The simplices in (2.3) provide us with two new subdivisions, namely, Subdivisions 5 and 18, respectively. One can show that no other combinations of simplices are possible to form a tetrahedral subdivision of a cube. To summarize we note that six subdivisions are obtained from those in (2.1), nine subdivisions from the combinations of two pairs in (2.2), and two from the combinations of three in (2.3). Together with Subdivision 1 the technique used above gives us eighteen subdivisions of a cube in $\mathbb{R}^3$, where each subdivision contains six tetrahedra with disjoint interiors and the vertices of the tetrahedra are those of the cube. □

We shall use Subdivisions 1-18 to obtain all possible sufficient conditions for a trivariate discrete distribution to be strongly unimodal. Each tetrahedral subdivision enables us to define a function f which is a collection of linear pieces defined by the vertices of a tetrahedron and coincides with $-\log p(.)$ on each vertex, where p is a joint p.m.f defined on $\mathbb{Z}^3$. We then ensure f is convex on any pair of neighboring tetrahedra sharing a common face. The corresponding conditions provide us with sufficient conditions for f to be convex on $\mathbb{R}^3$, and equivalently, p to be strongly unimodal.

While investigating the neighborhood relationship of Subdivisions 1-18, we observe that there are only five types of graphs that represent this relationship. We draw a graph $G = (V, E)$, where V is the set of subdividing tetrahedra $T_{t,1}, ..., T_{t,6}$ in some t, $(t = 1, ..., 18)$, and an edge $e = (q, r) \in E$ is added if tetrahedra $T_{t,q}$ and $T_{t,r}$, $(1 \leq q, r \leq 6, q \neq r)$, are neighbors, i.e., they share a common face. Below we give the graphical representations of Subdivisions 1-18, where the graphs are grouped together based on their types:

Note that there are only five different graphs representing the neighborhood relationship of simplices within a subdivision.

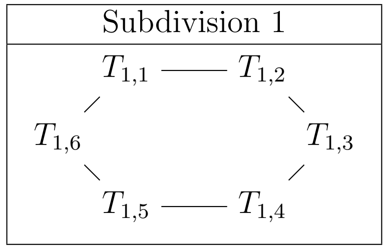

Figure 2.19: Graphical Representation of Subdivision 1

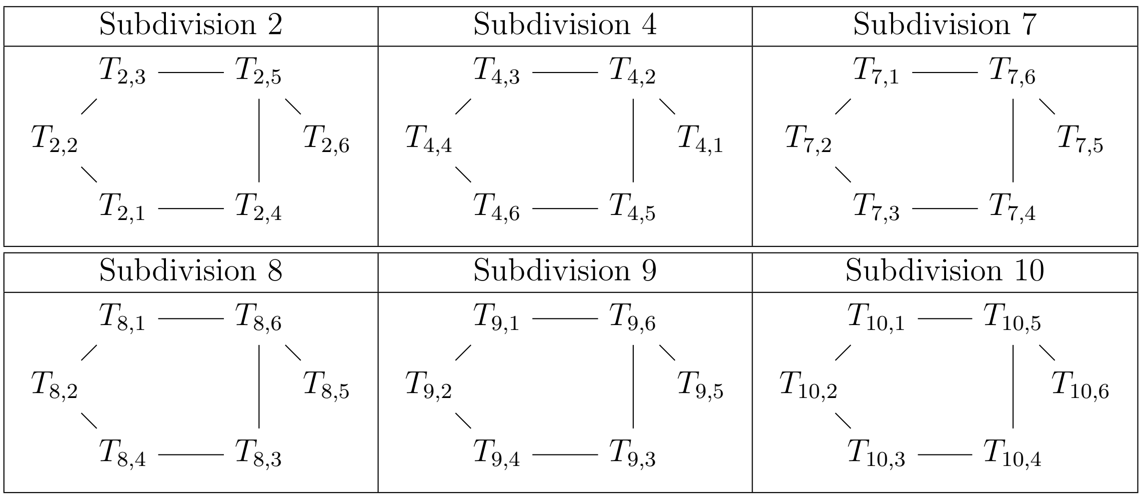

Figure 2.20: Graphical Representation of Subdivisions 2, 3, 7-10

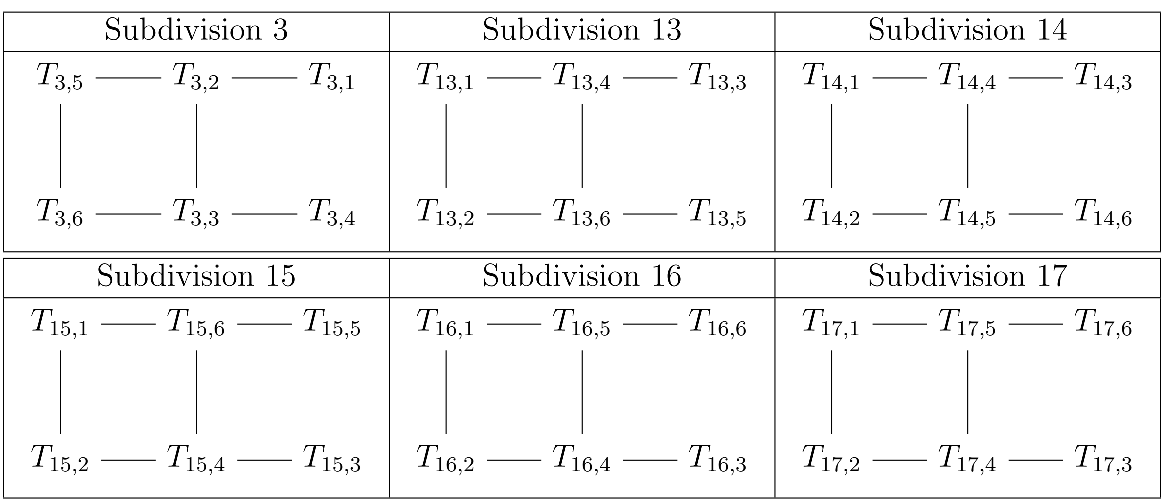

Figure 2.21: Graphical Representation of Subdivisions 3, 13-17

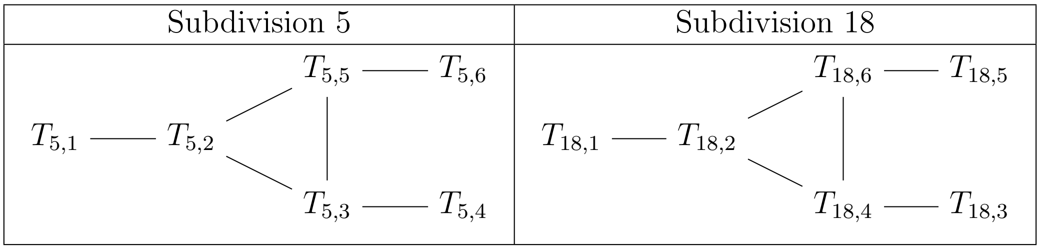

Figure 2.22: Graphical Representation of Subdivisions 5 and 18

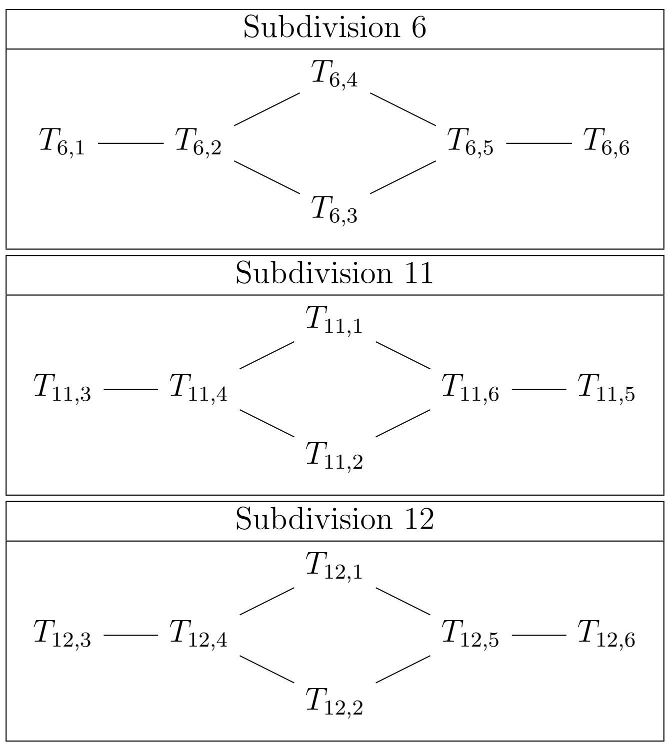

Figure 2.23: Graphical Representation of Subdivisions 6, 11, and 12

2.2 Sufficient Conditions for Strong Unimodality of Discrete Distributions Defined on $\mathbb{Z}^3$

We make use of the tetrahedral subdivisions of a cube presented in Section 2.1 and provide sufficient conditions that ensure the strong unimodality of a trivariate discrete distribution.

Let p be the joint p.m.f of a random vector $X \in \mathbb{Z}^3$ and $p_{i,j,k}$ the value of p at $(i,j,k) \in \mathbb{Z}^3$. Let S denote the support set of p. Define $C_t, t = 1, \dots, 6$, as the collection of the simplices $T_{t,c}(i,j,k), c = 1, \dots, 6, (i,j,k) \in \mathbb{Z}^3$, all vertices of which belong to S. In order to ensure the strong unimodality of the joint p.m.f p, we must find a convex function $f : \mathbb{R}^3 \to \mathbb{R}$ such that the value of f coincides with $-\log p(.)$ on lattice points, that is, $f(\mathbf{x}) = -\log p(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^3$. The function f can be obtained as a piecewise linear function that is convex on every pair of neighboring simplices in $\mathbb{R}^3$, i.e., f is a simplicial (polyhedral) function. The construction of the convex function f provides us with sufficient conditions for trivariate discrete distributions to be strongly unimodal as presented in the following theorem.

Theorem 2. *Let p be the joint p.m.f of a random vector $X \in \mathbb{Z}^3$ and $p_{i,j,k}$ the value of p at $(i,j,k) \in \mathbb{Z}^3$. Let S denote the support set of p. Define $C_t, t = 1, \dots, 6$, as the collection of the simplices $T_{t,c}(i,j,k), c = 1, \dots, 6, (i,j,k) \in \mathbb{Z}^3$, all vertices of which belong to S. Then p is strongly unimodal if at least one of the following conditions $(1), (2), ..., (18)$ is satisfied:*

Condition 1. *C_1 is the collection of the simplices $T_{1,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C1.1) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C1.2) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i,j+1,k}\ p_{i+1,j,k}$,

(C1.3) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j,k+1}\ p_{i+1,j,k}$,

(C1.4) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+2,j+1,k+1}\ p_{i,j,k}$,

(C1.5) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i,j,k-1}\ p_{i+1,j+1,k+1}$,

(C1.6) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j-1,k}\ p_{i+1,j+1,k+1}$.

Condition 2. *C_2 is the collection of the simplices $T_{2,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C2.1) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j,k+1}$,

(C2.2) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i,j,k+1}$,

(C2.3) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i,j,k+1}$,

(C2.4) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j+1,k+1}$,

(C2.5) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k+1}$,

(C2.6) $p_{i+1,j,k}\ p_{i,j+1,k} \geq p_{i+1,j+1,k}\ p_{i,j,k}$,

(C2.7) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j-1,k}\ p_{i+1,j+1,k+1}$,

(C2.8) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i-1,j,k}\ p_{i+1,j+1,k+1}$,

(C2.9) $p_{i,j+1,k}\ p_{i,j+1,k+1} \geq p_{i-1,j+1,k}\ p_{i+1,j+1,k+1}$,

(C2.10) $p_{i+1,j,k}\ p_{i+1,j,k+1} \geq p_{i+1,j-1,k}\ p_{i+1,j+1,k+1}$.

Condition 3. *C_3 is the collection of the simplices $T_{3,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C3.1) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k+1}$,

(C3.2) $p_{i,j,k}\ p^2_{i+!,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1}\ p_{i,j+1,k+1}$,

(C3.3) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j+1,k+1}$,

(C3.4) $p_{i+1,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j+1,k+1}$,

(C3.5) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C3.6) $p_{i,j,k}\ p_{i,j,k+1} \geq p_{i-1,j,k}\ p_{i+1,j,k+1}$,

(C3.7) $p_{i,j,k}\ p_{i,j+1,k} \geq p_{i-1,j,k}\ p_{i+1,j+1,k}$.

Condition 4. *C_4 is the collection of the simplices $T_{4,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C4.1) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k+1}$,

(C4.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k+1}$,

(C4.3) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i,j+1,k+1}$,

(C4.4) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i,j+1,k+1}$,

(C4.5) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k}\ p_{i,j+1,k}$,

(C4.6) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C4.7) $p_{i,j,k}\ p_{i,j,k+1} \geq p_{i-1,j,k}\ p_{i+1,j,k+1}$,

(C4.8) $p_{i,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+2,k+1}\ p_{i,j,k}$,

(C4.9) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i-1,j,k}\ p_{i+1,j+1,k+1}$,

(C4.10) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+2,k+1}\ p_{i,j,k}$.

Condition 5. *C_5 is the collection of the simplices $T_{5,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C5.1) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k+1}$,

(C5.2) $p_{i,j,k}\ p^2_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1} p_{i,j+1,k+1}$,

(C5.3) $p_{i+1,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j+1,k+1}$,

(C5.4) $p_{i+1,j+1,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i+1,j+1,k+1}$.

Condition 6. *C_6 is the collection of the simplices $T_{6,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C6.1) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k+1}$,

(C6.2) $p_{i,j,k}\ p_{i,j,k+1} \geq p_{i-1,j,k}\ p_{i+1,j,k+1}$,

(C6.3) $p_{i,j,k}\ p_{i,j,k+1} \geq p_{i,j-1,k}\ p_{i,j+1,k+1}$,

(C6.4) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k}\ p_{i+1,j+1,k+2}$,

(C6.5) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k+1}$,

(C6.6) $p_{i,j,k}p_{i+1,j+1,k+1} \geq p_{i+1,j,k}p_{i,j+1,k+1}$.

Condition 7. *C_7 is the collection of the simplices $T_{7,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C7.1) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k}$,

(C7.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+,j,k}$,

(C7.3) $p_{i+1,j,k}\ p_{i+1,j+1,k} \geq p_{i+2,j+1,k}\ p_{i,j,k}$,

(C7.4) $p_{i+1,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k-1}\ p_{i+1,j+1,k+1}$,

(C7.5) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j,k}\ p_{i+1,j,k+1}$,

(C7.6) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j-1,k}\ p_{i+1,j+1,k+1}$.

Condition 8. *C_8 is the collection of the simplices $T_{8,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C8.1) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i,j,k+1}$,

(C8.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k+1}$,

(C8.3) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i,j,k}\ p_{i+1,j+1,k+2}$,

(C8.4) $p_{i,j,k+1}\ p_{i+1,j,k+1} \geq p_{i,j-1,k+1}\ p_{i+1,j+1,k+1}$,

(C8.5) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C8.6) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j-1,k}\ p_{i+1,j+1,k+1}$,

(C8.7) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k}\ p_{i,j+1,k}$,

(C8.8) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k}$,

(C8.9) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i,j+1,k-1}\ p_{i+1,j+1,k+1}$,

(C8.10) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i,j+1,k+1}$.

Condition 9. *C_9 is the collection of the simplices $T_{9,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C9.1) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j+1,k}$,

(C9.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k+1}\ p_{i,j+1,k}$,

(C9.3) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i+1,j+2,k}\ p_{i,j,k}$,

(C9.4) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i,j,k-1}\ p_{i+1,j+1,k+1}$,

(C9.5) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i,j+1,k}$,

(C9.6) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+2,k+1}\ p_{i,j,k}$,

(C9.7) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j,k+1}$,

(C9.8) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i,j,k+1}$,

(C9.9) $p_{i,j,k+1}\ p_{i+1,j,k+1} \geq p_{i+1,j,k+2}\ p_{i,j,k}$,

(C9.10) $p_{i+1,j+1,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i+1,j+1,k+1}$.

Condition 10. *C_{10} is the collection of the simplices $T_{10,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C10.1) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k+1}\ p_{i+1,j+1,k}$,

(C10.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j,k}$,

(C10.3) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i,j,k-1}\ p_{i+1,j+1,k+1}$,

(C10.4) $p_{i+1,j,k}\ p_{i+1,j+1,k} \geq p_{i+2,j+1,k}\ p_{i,j,k}$,

(C10.5) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j,k+1}\ p_{i+1,j,k}$,

(C10.6) $p_{i+1,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+2,j+1,k+1}\ p_{i,j,k}$,

(C10.7) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j,k+1}$,

(C10.8) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq\ p_{i+1,j+1,k}p_{i,j,k+1}$,

(C10.9) $p_{i,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j+1,k+2}\ p_{i,j,k}$,

(C10.10) $p_{i+1,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j+1,k+1}$.

<u>Condition 11.</u> *C_{11} is the collection of the simplices $T_{11,c}(i,j,k), c = 1,...,6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C11.1) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k}$,

(C11.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k+1}\ p_{i,j+1,k}$,

(C11.3) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i+1,j+2,k}\ p_{i,j,k}$,

(C11.4) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i,j+1,k-1}\ p_{i+1,j+1,k+1}$,

(C11.5) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i,j+1,k+1}$,

(C11.6) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i-1,j,k}\ p_{i+1,j+1,k+1}$.

<u>Condition 12.</u> *C_{12} is the collection of the simplices $T_{12,c}(i,j,k), c = 1,...,6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C12.1) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i,j,k+1}$,

(C12.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j+1,k+1}$,

(C12.3) $p_{i,j,k+1}\ p_{i,j+1,k+1} \geq p_{i-1,j,k+1}\ p_{i+1,j+1,k+1}$,

(C12.4) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k+2}\ p_{i,j,k}$,

(C12.5) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i,j+1,k+1}$,

(C12.6) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i-1,j,k}\ p_{i+1,j+1,k+1}$,

(C12.7) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k}\ p_{i,j+1,k}$,

(C12.8) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k}$,

(C12.9) $p_{i+1,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k-1}\ p_{i+1,j+1,k+1}$,

(C12.10) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j,k}\ p_{i+1,j,k+1}$.

Condition 13. *C_{13} is the collection of the simplices $T_{13,c}(i,j,k), c = 1, ..., 6$ and and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C13.1) $p_{i,j+1,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j+1,k}$,

(C13.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k+1}\ p_{i,j+1,k}$,

(C13.3) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i+1,j+2,k}\ p_{i,j,k}$,

(C13.4) $p_{i,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+2,k+1}\ p_{i,j,k}$,

(C13.5) $p_{i+1,j+1,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i+1,j+1,k+1}$,

(C13.6) $p_{i,j,k}\ p^2_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C13.7) $p_{i+1,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k+1}$.

Condition 14. *C_{14} is the collection of the simplices $T_{14,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C14.1) $p_{i,j,k+1}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j,k+1}$,

(C14.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j+1,k}\ p_{i,j,k+1}$,

(C14.3) $p_{i,j,k+1}\ p_{i+1,j,k+1} \geq p_{i+1,j,k+2}\ p_{i,j,k}$,

(C14.4) $p_{i,j,k+1}\ p_{i,j+1,k+1} \geq p_{i,j+1,k+2}\ p_{i,j,k}$,

(C14.5) $p_{i+1,j+1,k}\ p_{i+1,j,k+1} \geq p_{i+1,j,k}\ p_{i+1,j+1,k+1}$,

(C14.6) $p_{i,j,k}\ p^2_{i+1,j+1,k+1} \geq p_{i,j+1,k+1}\ p_{i+1,j+1,k}\ p_{i+1,j,k+1}$,

(C14.7) $p_{i+1,j+1,k}\ p_{i,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j+1,k+1}$.

Condition 15. C_{15} *is the collection of the simplices* $T_{15,c}(i,j,k), c = 1, ..., 6$ *and for all* $(i,j,k) \in \mathbb{Z}^3$ *we have*

(C15.1) $p_{i,j,k}\ p_{i+1,j+1,k} \geq p_{i,j+1,k}\ p_{i+1,j,k}$,

(C15.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j+1,k}$,

(C15.3) $p_{i+1,j,k}\ p_{i+1,j+1,k} \geq p_{i+1,j,k-1}\ p_{i+1,j+1,k+1}$

(C15.4) $p_{i,j+1,k}\ p_{i+1,j+1,k} \geq p_{i,j+1,k-1}\ p_{i+1,j+1,k+1}$,

(C15.5) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i,j+1,k+1}$,

(C15.6) $p_{i,j,k}^2\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j,k+1} p_{i,j+1,k}$,

(C15.7) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j,k}\ p_{i+1,j,k+1}$.

Condition 16. C_{16} *is the collection of the simplices* $T_{16,c}(i,j,k), c = 1, ..., 6$ *and for all* $(i,j,k) \in \mathbb{Z}^3$ *we have*

(C16.1) $p_{i,j,k}\ p_{i+1,j,k+1} \geq p_{i,j,k+1}\ p_{i+1,j,k}$,

(C16.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k+1}$,

(C16.3) $p_{i+1,j,k}\ p_{i+1,j,k+1} \geq p_{i+1,j-1,k}\ p_{i+1,j+1,k+1}$,

(C16.4) $p_{i,j,k+1}\ p_{i+1,j,k+1} \geq p_{i,j-1,k+1}\ p_{i+1,j+1,k+1}$,

(C16.5) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i,j+1,k+1}$,

(C16.6) $p_{i,j,k}^2\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j,k+1}\ p_{i,j+1,k}$,

(C16.7) $p_{i+1,j,k} p_{i,j+1,k} \geq p_{i+1,j+1,k} p_{i,j,k}$.

Condition 17. C_{17} *is the collection of the simplices* $T_{17,c}(i,j,k), c = 1, ..., 6$ *and for all* $(i,j,k) \in \mathbb{Z}^3$ *we have*

(C17.1) $p_{i,j,k}\ p_{i,j+1,k+1} \geq p_{i,j,k+1}\ p_{i,j+1,k}$,

(C17.2) $p_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i+1,j,k}\ p_{i,j+1,k+1}$,

(C17.3) $p_{i,j+1,k}\ p_{i,j+1,k+1} \geq p_{i-1,j+1,k}\ p_{i+1,j+1,k+1}$,

(C17.4) $p_{i,j,k+1}\ p_{i,j+1,k+1} \geq p_{i-1,j,k+1}\ p_{i+1,j+1,k+1}$,

(C17.5) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j,k}\ p_{i+1,j,k+1}$,

(C17.6) $p^2_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j+1,k}\ p_{i+1,j,k}\ p_{i,j,k+1}$,

(C17.7) $p_{i+1,j,k}\ p_{i,j+1,k} \geq p_{i+1,j+1,k}\ p_{i,j,k}$.

Condition 18. *C_{18} is the collection of the simplices $T_{18,c}(i,j,k), c = 1, ..., 6$ and for all $(i,j,k) \in \mathbb{Z}^3$ we have*

(C18.1) $p_{i+1,j,k}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i+1,j+1,k}$,

(C18.2) $p^2_{i,j,k}\ p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\ p_{i+1,j,k} p_{i,j+1,k}$,

(C18.3) $p_{i+1,j,k}\ p_{i,j,k+1} \geq p_{i,j,k}\ p_{i+1,j,k+1}$,

(C18.4) $p_{i,j,k+1}\ p_{i,j+1,k} \geq p_{i,j,k}\ p_{i,j+1,k+1}$.

Proof. We remark that Conditions 1-6 are the same as those presented by Subasi et al. (2009) [44] and are obtained from Subdivisions 2.1-2.6. Below we outline the proof of Condition 7 following the proof of Theorem 1 of Subasi et al. (2009) [44].

First, let us subdivide $\mathbb{R}^3$ into unit cubes and then subdivide each cube into six simplices with disjoint interiors using Subdivision 7.

Let $L_c(i,j,k), (i,j,k) \in \mathbb{Z}^3, c = 1, ..., 6$ designate the linear function on $\mathbb{R}^3$ such that $L_c(i,j,k) = -\log p(i,j,k)$, where (i,j,k) are the vertices of tetrahedron $T_{7,c}(i,j,k), c = 1, ..., 6$. Define

$$f(x) = \begin{cases} L_c(i,j,k) & \text{if } x \in T_{7,c}(i,j,k), (i,j,k) \in \mathbb{Z}^3 \\ \infty & \text{otherwise.} \end{cases}$$

Note that f coincides with $-\log p(.)$ on the support set S.

The function f is convex on any two neighboring simplices with a common face if for any lattice points

$$\mathbf{z}_t = (z_{t1}, z_{t2}, z_{t3}), t = 1, 2, 3, 4 \text{ and } \mathbf{y} = (y_1, y_2, y_3)$$

such that $\mathbf{z}_t$ are the vertices of a simplex in Subdivision 7 and $\mathbf{y}$ is the vertex of a neighboring simplex which does not belong to the current one, we have the relation

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) & f(\mathbf{z}_4) \\ 1 & 1 & 1 & 1 & 1 \\ y_1 & z_{11} & z_{21} & z_{31} & z_{41} \\ y_2 & z_{12} & z_{22} & z_{32} & z_{42} \\ y_3 & z_{13} & z_{23} & z_{33} & z_{43} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ z_{11} & z_{21} & z_{31} & z_{41} \\ z_{12} & z_{22} & z_{32} & z_{42} \\ z_{13} & z_{23} & z_{33} & z_{43} \end{vmatrix}} \geq 0 \ . \tag{2.4}$$

First note that any simplex of type $T_{7,c}(i, j, k), c = 1, ..., 6$ has four neighbors: two in the same cube and two in neighboring cubes. We prove that inequalities (C7.1), (C7.2), and (C7.5) ensure the convexity of f within a cube and inequalities (C7.3), (C7.4), and (C7.6) ensure the convexity of f in two neighboring simplices that are in different cubes. We consider simplex $T_{7,1}(i, j, k) = conv\{(i, j, k), (i+1, j, k), (i+1, j+1, k), (i+1, j+1, k+1)\}$ whose neighbors are

$$conv\{(i,j,k),(i+1,j,k),(i,j,k+1),(i+1,j+1,k+1)\},$$
$$conv\{(i,j,k),(i+1,j+1,k),(i,j+1,k+1),(i+1,j+1,k+1)\},$$
$$conv\{(i+2,j+1,k),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\},$$
$$conv\{(i+1,j,k-1),(i+1,j,k),(i+1,j+1,k),(i+1,j+1,k+1)\}.$$

We also note that the first two simplices and $T_{7,1}(i,j,k)$ are in the same cube whereas the last two are in two different, but neighboring cubes.

Let $\mathbf{z}_t, t = 1,2,3,4$ be the vertices of simplex $T_{7,1}$ and $\mathbf{y}$ the vertex of its first neighbor that does not belong to $T_{7,1}$. In this case inequality (2.4) can be written as

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) & f(\mathbf{z}_4) \\ 1 & 1 & 1 & 1 & 1 \\ i & i & i+1 & i+1 & i+1 \\ j & j & j & j+1 & j+1 \\ k+1 & k & k & k & k+1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ i & i+1 & i+1 & i+1 \\ j & j & j+1 & j+1 \\ k & k & k & k+1 \end{vmatrix}} \geq 0, \tag{2.5}$$

where $f = -\log p(.)$. Since the denominator in (2.5) is equal to 1, the convexity of f is satisfied if the numerator is nonnegative. Therefore, we need to ensure that

$$\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) & f(\mathbf{z}_4) \\ 1 & 1 & 1 & 1 & 1 \\ i & i & i+1 & i+1 & i+1 \\ j & j & j & j+1 & j+1 \\ k+1 & k & k & k & k+1 \end{vmatrix} \geq 0$$

which is equivalent to $f(\mathbf{y}) + f(\mathbf{z}_3) \geq f(\mathbf{z}_1) + f(\mathbf{z}_4)$. Substituting $f(.) = -\log p(.)$ we obtain

$$p(\mathbf{z}_1)\, p(\mathbf{z}_4) \geq p(\mathbf{y})\, p(\mathbf{z}_3) \Longrightarrow p_{i,j,k}\; p_{i+1,j+1,k+1} \geq p_{i,j,k+1}\; p_{i+1,j+1,k}$$

which is inequality (C7.1) in Condition 7.

Taking $\mathbf{y} = (i, j+1, k+1)$ provides us with inequality (C7.2). Inequality (C7.5) can be obtained by considering simplex $T_{7,3}$ and $\mathbf{y} = (i, j, k)$ as a vertex of one of its neighbors that does not belong to $T_{7,3}$. Now, let $\mathbf{y} = (i+2, j+1, k)$ and $\mathbf{z}_i, i = 1, 2, 3, 4$ defined as before. In this case (2.4) provides us with inequality (C7.3). If we take $\mathbf{y} = (i+1, j, k-1)$, then we obtain inequality (C7.4). Inequality (C7.6) can be obtained by the use of simplex $T_{7,3}$ and $\mathbf{y} = (i, j-1, k)$.

Note that, in case of Subdivision 7, there are 12 possible layouts of neighboring simplices in the same cube and 12 in different cubes. However, they only provide us with six different inequalities (C7.1)-(C7.6).

Since f is convex on any two neighboring simplices of type $T_{7,c}(i, j, k), (i, j, k) \in \mathbb{Z}^3, c = 1, ..., 6$, it is convex on the entire space. Thus, Condition 7 is a sufficient condition for the joint p.m.f p to be strongly unimodal. The proof of other conditions does not require new ideas and is left to the reader. □

In the following chapter we present a subdivision of $\mathbb{R}^n$ into simplices with pairwise disjoint interiors and obtain a sufficient condition that ensures the strong unimodality of multivariate discrete distributions defined on $\mathbb{Z}^n$.

Chapter 3

A New Sufficient Condition for Strong Unimodality of Multivariate Discrete Distributions

Based on the assertions of Section 2 and those presented by Subasi et al. (2009) [44] we first obtain a special simplicial (polyhedral) subdivision of $\mathbb{R}^n$ with non-overlapping convex simplices. On each simplex we define a linear function determined by the vertices of the simplex and the corresponding values of $-\log p(.)$. We then form a function $f : \mathbb{R}^n \to \mathbb{R}$ as the collection of these linear pieces. The conditions that ensure the convexity of f provide us with a new sufficient condition for the strong unimodality of multivariate discrete distributions defined on $\mathbb{Z}^n$ (see Alharbi, Subasi, Subasi (2017) [1]).

3.1 A Special Subdivision of $\mathbb{R}^n$

Given a lattice point $(i_1, i_2, ..., i_n)$ there are at most $n!$ possibilities in connection with the values of its components. Subasi et al. (2009) [44] considered the following special ordering of the components of $(i_1, i_2, ..., i_n)$:

(1) $i_1 \leq i_2 \leq \ldots \leq i_n$

(2) $i_1 \leq i_2 \leq \ldots \leq i_n \leq i_{n-1}$

(3) $i_1 \leq i_2 \leq \ldots \leq i_{n-1} \leq i_{n-2} \leq i_n$

(4) $i_1 \leq i_2 \leq \ldots \leq i_{n-1} \leq i_n \leq i_{n-2}$

(5) $i_1 \leq i_2 \leq \ldots \leq i_n \leq i_{n-2} \leq i_{n-1}$

(6) $i_1 \leq i_2 \leq \ldots \leq i_n \leq i_{n-1} \leq i_{n-2}$

(7) $i_1 \leq i_2 \leq \ldots \leq i_{n-2} \leq i_{n-3} \leq i_{n-1} \leq i_n$

(8) $i_1 \leq i_2 \leq \ldots \leq i_{n-2} \leq i_{n-3} \leq i_n \leq i_{n-1}$

$\vdots$

(n!-1) $i_n \leq i_{n-1} \leq i_{n-2} \leq \ldots \leq i_2 \leq i_1$

(n!) $i_n \leq i_{n-1} \leq i_{n-2} \leq \ldots \leq i_1$.

Subasi et al. (2009) [44] showed that these $n!$ relations provide us with the vertices of the following subdividing simplices, $S_{1,1}, ..., S_{1,n!}$, of a hypercube:

$$S_{1,1} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_n + 1), (i_1, \dots, i_{n-1} + 1, i_n + 1), \dots,$$
$$(i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,2} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-1} + 1, i_n), (i_1, \dots, i_{n-1} + 1, i_n + 1), \dots,$$
$$(i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,3} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-1}, i_n + 1),$$
$$(i_1, \dots, i_{n-2} + 1, i_{n-1}, i_n + 1), (i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n + 1),$$
$$\dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,4} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-2} + 1, i_{n-1}, i_n),$$
$$(i_1, \dots, i_{n-2} + 1, i_{n-1}, i_n + 1), (i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n + 1),$$
$$\dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,5} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-1} + 1, i_n), (i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n),$$
$$(i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,6} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-2} + 1, i_{n-1}, i_n),$$
$$(i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n), (i_1, \dots, i_{n-2} + 1, i_{n-1} + 1, i_n + 1),$$
$$\dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,7} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_n + 1), (i_1, \dots, i_{n-1} + 1, i_n + 1),$$
$$(i_1, \dots, i_{n-3}+1, i_2, i_{n-1}+1, i_n+1), (i_1, \dots, i_{n-3}+1, i_2+1, i_{n-1}+1, i_n+1),$$
$$\dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$S_{1,8} = conv\{(i_1, \dots, i_n), (i_1, \dots, i_{n-1} + 1, i_n), (i_1, \dots, i_{n-1} + 1, i_n + 1),$$
$$(i_1, \dots, i_{n-3}+1, i_2, i_{n-1}+1, i_n+1), (i_1, \dots, i_{n-3}+1, i_2+1, i_{n-1}+1, i_n+1),$$
$$\dots, (i_1 + 1, \dots, i_n + 1)\},$$
$$\vdots$$

$$S_{1,n!-1} = conv\{(i_1, \ldots, i_n), (i_1, i_2+1, i_3, \ldots, i_n), (i_1+1, i_2+1, i_3, \ldots, i_n),$$
$$\ldots, (i_1+1, i_2+1, \ldots, i_n+1)\}.$$
$$S_{1,n!} = conv\{(i_1, \ldots, i_n), (i_1+1, i_2 \ldots, i_n), (i_1+1, i_2+1, i_3, \ldots, i_n),$$
$$\ldots, (i_1+1, i_2+1, \ldots, i_n+1)\}.$$

Note that $|S_{1,1}| = \ldots = |S_{1,n!}| = n+1$ and simplices $S_{1,j}, S_{1,k}, j \neq k$, have a common facet if they share n common vertices. We also remark that the pairs of simplices, $S_{1,1}\&S_{1,2}$, $S_{1,3}\&S_{1,4}$, ... , $S_{1,n!-1}\&S_{1,n!}$, have a common facet and hence, are neighboring simplices of the above subdivision. Subdivision 1 presented in Section 2.1 is a special case of this subdivision where $n = 3$.

Subasi et al. (2009) [44] used the neighborhood relationship between the pairs of simplices in subdivision $S_{1,1}, S_{1,2}, ..., S_{1,n!}$ and obtained the following sufficient condition for a multivariate discrete distribution.

Theorem 3. *(Subasi et al., 2009) Let p be the probability function of a discrete distribution on $\mathbb{Z}^n$ and $p(i_1, ..., i_n)$ the value of p at $(i_1, ..., i_n) \in \mathbb{Z}^n$. Let C denote the collection of simplices $S_{1,1}, S_{1,2}, ..., S_{1,n!}$. Then p is strongly unimodal, and hence, logconcave if the following conditions are satisfied*

Condition C-I.

$$p(\mathbf{x})p(\mathbf{y}) \;\leq\; p(\mathbf{x} \vee \mathbf{y})p(\mathbf{x} \wedge \mathbf{y}) \tag{3.1}$$

where $\mathbf{x} = (i_1+\varepsilon_1, ..., i_n+\varepsilon_n)$, $\mathbf{y} = (i_1+\delta_1, ..., i_n+\delta_n)$ *and*

$$\mathbf{x} \vee \mathbf{y} = (\max(i_1+\varepsilon_1, i_1+\delta_1), ..., \max(i_n+\varepsilon_n, i_n+\delta_n))$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(i_1+\varepsilon_1, i_1+\delta_1), ..., \min(i_n+\varepsilon_n, i_n+\delta_n))$$

and $\varepsilon_j, \delta_j \in \{0,1\},\ j = 1, ..., n$ *defined such that for* $k = 2, ..., n$

$$\sum_{j=1}^{n} \varepsilon_j = \sum_{j=1}^{n} \delta_j = k - 1 \ , \quad \sum_{j=1}^{n} \varepsilon_j \delta_j = k - 2 \ .$$

Condition C-II.

$$p(i_1+\gamma_1+1, ..., i_n+\gamma_n+1)p(i_1, ..., i_n) \ \leq \ p(i_1+1, ..., i_n+1)p(i_1+\gamma_1, ..., i_n+\gamma_n) \ , \tag{3.2}$$

where $\gamma_j \in \{0,1\},\ j = 1, ..., n$ *and* $\sum_{j=1}^{n} \gamma_j = 1.$

Condition C-III.

$$p(i_1-\alpha_1, ..., i_n-\alpha_n)p(i_1+1, ..., i_n+1) \ \leq \ p(i_1, ..., i_n)p(i_1-\alpha_1+1, ..., i_n-\alpha_n+1), \tag{3.3}$$

where $\alpha_j \in \{0,1\},\ j = 1, ..., n$ *and* $\sum_{j=1}^{n} \alpha_j = 1.$

The goal of this Chapter is to present a new sufficient condition that ensure the strong unimodality of multivariate discrete distributions. In order to achieve this goal we first find a new subdivision of $\mathbb{R}^n$ into simplices with pairwise disjoint interiors.

A different subdivision of a hypercube can be obtained from $S_{1,1}, \ldots, S_{1,n!}$ as follows: Let $\mathbf{x}_1$ be a vertex of the first simplex, $S_{1,1}$, which does not belong to the second simplex and $\mathbf{x}_2$ the vertex of the second simplex, $S_{1,2}$, which does not belong to the first simplex. Substitute $\mathbf{x}_2$ with any vertex of $S_{1,1}$ other than $\mathbf{x}_1$ and substitute $\mathbf{x}_1$ with any vertex of $S_{1,2}$ other than $\mathbf{x}_2$ such that there are exactly $n(n+1)$ components of the vertices of both simplices are 1 as we discussed in the proof of Theorem 1. Applying this procedure to every pair of consecutive simplices,

$$S_{1,1}\&S_{1,2}, S_{1,3}\&S_{1,4}, \ldots , S_{1,n!-1}\&S_{1,n!},$$

results in a new subdivision, denoted by $S_{2,1}, ..., S_{2,n!}$:

$S_{2,1} = conv\{(i_1, \ldots , i_n), (i_1, \ldots , i_n + 1), (i_1, \ldots , i_{n-1} + 1, i_n),$
$(i_1, \ldots , i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \ldots , (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,2} = conv\{(i_1, \ldots , i_n + 1), (i_1, \ldots , i_{n-1} + 1, i_n), (i_1, \ldots , i_{n-1} + 1, i_n + 1),$
$\ldots , (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,3} = conv\{(i_1, \ldots , i_n), (i_1, \ldots , i_{n-1}, i_n + 1), (i_1, \ldots , i_{n-2} + 1, i_{n-1}, i_n),$
$(i_1, \ldots , i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \ldots , (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,4} = conv\{(i_1, \ldots , i_{n-1}, i_n + 1), (i_1, \ldots , i_{n-2} + 1, i_{n-1}, i_n),$
$(i_1, \ldots , i_{n-2} + 1, i_{n-1}, i_n + 1), (i_1, \ldots , i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \ldots ,$
$(i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,5} = conv\{(i_1, \ldots , i_n), (i_1, \ldots , i_{n-1} + 1, i_n), (i_1, \ldots , i_{n-2} + 1, i_{n-1}, i_n),$
$(i_1, \ldots , i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \ldots , (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,6} = conv\{(i_1, \ldots , i_{n-1} + 1, i_n), (i_1, \ldots , i_{n-2} + 1, i_{n-1}, i_n), (i_1, \ldots , i_{n-2} + 1,$
$i_{n-1} + 1, i_n), (i_1, \ldots , i_{n-2} + 1, i_{n-1} + 1, i_n + 1), \ldots (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,7} = conv\{(i_1, \ldots , i_n), (i_1, \ldots , i_n + 1), (i_1, \ldots , i_{n-1} + 1, i_n), (i_1, \ldots ,$
$i_{n-3} + 1, i_{n-2}, i_{n-1} + 1, i_n + 1), (i_1, \ldots , i_{n-3} + 1, i_{n-2} + 1, i_{n-1} + 1,$
$i_n + 1), \ldots , (i_1 + 1, \ldots , i_n + 1)\},$

$S_{2,8} = conv\{(i_1, \ldots , i_n + 1), (i_1, \ldots , i_{n-1} + 1, i_n), (i_1, \ldots , i_{n-1} + 1, i_n + 1),$
$(i_1, \ldots , i_{n-3} + 1, i_{n-2}, i_{n-1} + 1, i_n + 1), (i_1, \ldots , i_{n-3} + 1,$

$i_{n-2}+1, i_{n-1}+1, i_n+1), \dots, (i_1+1, \dots, i_n+1)\},$

$\vdots$

$$S_{2,n!-1} = conv\{(i_1, \dots, i_n), (i_1, i_2+1, i_3, \dots, i_n), (i_1+1, i_2 \dots, i_n),$$
$$(i_1+1, i_2+1, i_3+1, \dots, i_n), \dots, (i_1+1, i_2+1, \dots, i_n+1)\},$$
$$S_{2,n!} = conv\{(i_1, i_2{+}1, i_3, \dots, i_n), (i_1{+}1, i_2 \dots, i_n), (i_1{+}1, i_2{+}1, i_3, \dots, i_n), \dots,$$
$$(i_1+1, i_2+1, \dots, i_n+1)\}.$$

3.2 A Sufficient Condition for Strong Unimodality of Discrete Distributions Defined on $\mathbb{Z}^n$

Below we make use of the subdividing simplices, $S_{2,1}, ..., S_{2,n!}$, to obtain a sufficient condition that ensures the strong unimodality of a multivariate discrete distribution.

Theorem 4. *Let p be the joint p.m.f of a discrete random vector $X \in \mathbb{Z}^n$ and $p(i_1, \dots, i_n)$ the value of p at $(i_1, \dots, i_n) \in \mathbb{Z}^n$. Let C denote the collection of simplices $S_{2,1}, \dots, S_{2,n!}$. Then p is strongly unimodal if the following conditions are met:*

Condition I.

$$p(\mathbf{x})\, p(\mathbf{y}) \;\geq\; p(\mathbf{x} \vee \mathbf{y})\, p(\mathbf{x} \wedge \mathbf{y}), \tag{3.4}$$

where

$$\mathbf{x} = (i_1+\alpha_1, \dots, i_n+\alpha_n), \mathbf{y} = (i_1+\beta_1, \dots, i_n+\beta_n)$$

$$\mathbf{x} \vee \mathbf{y} = (max(i_1+\alpha_1, i_1+\beta_1), \dots, max(i_n+\alpha_n, i_n+\beta_n))$$

$$\mathbf{x} \wedge \mathbf{y} = (min(i_1+\alpha_1, i_1+\beta_1), \dots, min(i_n+\alpha_n, i_n+\beta_n))$$

and $\alpha_j, \beta_j \in \{0,1\}, j = 1, \ldots, n,$ *such that*

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 1, \quad \textit{and} \quad \alpha_j + \beta_j \leq 1 \quad \textit{for all} \quad j = 1, \ldots, n.$$

Condition II.

$$\begin{aligned} p^2(i_1, \ldots, i_n)\, p(i_1 + m_1, \ldots, i_n + m_n) \;\geq\; & p(i_1 + \gamma_1, \ldots, i_n + \gamma_n) \\ & p(i_1 + \delta_1, \ldots, i_n + \delta_n)\, p(i_1 + \epsilon_1, \ldots, i_n + \epsilon_n) \end{aligned} \tag{3.5}$$

where $\gamma_j, \delta_j, \epsilon_j \in \{0,1\}, j = 1, \ldots, n,$ *such that*

$$\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \epsilon_j = 1$$

and

$$m_j = \gamma_j + \delta_j + \epsilon_j \leq 1 \quad \textit{for all} \quad j = 1, \ldots, n.$$

Proof. The proof of this assertion is similar to the proof of Theorem 2: We assume that $\mathbb{R}^n$ is subdivided into hypercubes and each hypercube is then divided into simplices of type $S_{2,c}, c = 1, ..., n!$. On each simplex we define a linear function

$$L_c(i_1, i_2, \ldots, i_n), (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n, c = 1, 2, \ldots, n!$$

whose value coincides with $-\log p(\mathbf{x})$ on all vertices $\mathbf{x} \in \mathbb{Z}^n$ of simplex $S_{2,c}$. Let

$$f(\mathbf{x}) = \begin{cases} L_c(i_1, i_2, \ldots, i_n) & \text{if } \mathbf{x} \in S_{2,c}(i_1, \ldots, i_n), (i_1, \ldots, i_n) \in \mathbb{Z}^n, c = 1, ..., n! \\ \infty & \text{otherwise.} \end{cases}$$

Then $f(\mathbf{x}) = -\log p(\mathbf{x}), \mathbf{x} \in \mathbb{Z}^n$. The function f is convex on any two neighboring simplices with a common facet if

$$\frac{\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & \dots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ y_1 & z_{01} & z_{11} & z_{21} & \dots & z_{n1} \\ y_2 & z_{02} & z_{12} & z_{22} & \dots & z_{n2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y_n & z_{0n} & z_{1n} & z_{2n} & \dots & z_{nn} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ z_{01} & z_{11} & z_{21} & \dots & z_{n1} \\ z_{02} & z_{12} & z_{22} & \dots & z_{n2} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{0n} & z_{1n} & z_{2n} & \dots & z_{nn} \end{vmatrix}} \geq 0 \ , \tag{3.6}$$

where $\mathbf{z}_t, t = 0, 1, ..., n$ are the vertices of a simplex in the subdivision and $\mathbf{y}$ is the vertex of a neighboring simplex which does not belong to the one under consideration.

For the sake of simplicity let $(i_1, \dots, i_n) = (0, \dots, 0)$. Similar to the proof of Theorem 2 in Section 2.2 and Theorem 3 of Subasi et al. [44] we show that conditions (3.4) and (3.5) are directly obtained from inequality (3.6) by considering all possible pairs of neighboring simplices and therefore, they ensure the convexity of f on $\mathbb{R}^n$ and provide us with a sufficient condition for the strong unimodality of the joint p.m.f p of a random vector $X \in \mathbb{Z}^n$.

Let us first consider simplex $S_{2,1}$:

$$conv\{(0,\dots,0),(0,0,\dots,1),(0,0,\dots,1,0),(0,0,\dots,1,1,1),$$
$$(0,0,\dots,1,1,1,1),\dots,(1,1,\dots,1)\}$$

whose neighbors are:

$$conv\{(0,0,\dots,1),(0,0,\dots,1,0),(0,0,\dots,1,1),(0,0,\dots,1,1,1),$$
$$(0,0,\dots,1,1,1,1),\dots,(1,1,\dots,1)\},$$
$$conv\{(0,\dots,0),(0,0,\dots,1),(0,0,\dots,1,0,0),(0,0,\dots,1,1,1),$$
$$(0,0,\dots,1,1,1,1),\dots,(1,1,\dots,1)\},$$
$$conv\{(0,\dots,0),(0,0,\dots,1,0),(0,0,\dots,1,0,0),(0,0,\dots,1,1,1),$$
$$(0,0,\dots,1,1,1,1),\dots,(1,1,\dots,1)\},$$
$$conv\{(0,\dots,0),(0,0,\dots,1),(0,0,\dots,1,0),(0,0,\dots,1,0,1,1),$$
$$(0,0,\dots,1,1,1,1),\dots,(1,1,\dots,1)\},$$
$$\vdots$$

Let $\mathbf{z}_i, i = 0,\dots,n$ be the vertices of $S_{2,1}$ and $\mathbf{y}$ the vertex of the of the neighboring simplex, that does not belong to $S_{2,1}$. Since the denominator of (3.6) is equal to 1, the convexity of f is ensured if its numerator is nonnegative.

We consider the following cases:

*Case*1. Let $\mathbf{y} = (0, 0, \ldots, 1, 1)$. The numerator of (3.6) is equivalent to

$$\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) & \ldots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & 0 & 1 & \ldots & 1 \end{vmatrix}$$

It follows that

$$f(\mathbf{y}) + f(\mathbf{z}_0) \geq f(\mathbf{z}_1) + f(\mathbf{z}_2).$$

Since $f = -\log p(.)$, we have

$$p(0, \ldots, 1)p(0, \ldots, 1, 0) \geq p(0, \ldots, 1, 1)p(0, \ldots, 0).$$

However, this is the same as

$$p(i_1, \ldots, i_n + 1)p(i_1, \ldots, i_{n-1} + 1, i_n) \geq p(i_1, \ldots, i_{n-1} + 1, i_n + 1)p(i_1, \ldots, i_n).$$

Similarly, we can obtain all possible conditions of type (3.4) for the case of simplex $S_{2,1}$ by considering its all other neighbors in the subdivision.

Case 2. Now, let $\mathbf{y} = (0, 0, \ldots, 1, 0, 0)$. In this case the numerator of 3.6 is

$$\begin{vmatrix} f(\mathbf{y}) & f(\mathbf{z}_0) & f(\mathbf{z}_1) & f(\mathbf{z}_2) & f(\mathbf{z}_3) & \ldots & f(\mathbf{z}_n) \\ 1 & 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & 1 & \ldots & 1 \\ 0 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & 0 & 1 & \ldots & 1 \end{vmatrix}$$

which provide us with the result

$$f(y) + f(\mathbf{z}_1) + f(\mathbf{z}_2) \geq 2f(\mathbf{z}_0) + f(\mathbf{z}_3).$$

Substituting $f(.) = -\log p(.)$, we obtain

$$p^2(0, \ldots, 0)p(0, \ldots, 1, 1, 1) \geq p(0, 0, \ldots, 1, 0, 0)p(0, \ldots, 1)p(0, \ldots, 1, 0).$$

Considering any neighboring simplices $S_{2,k}$ and $S_{2,k+2}$ such that k is an odd number gives us all other condition in (3.5).

Hence, conditions (3.4), and (3.5) ensure the convexity of f on any two neighboring simplices. Since $f(\mathbf{x}) = -\log p(\mathbf{x})$ for all $x \in \mathbb{Z}^n$, the joint p.m.f p of the random vector X is logconcave.

$\square$

In the following chapter we present three well-known discrete multivariate distributions that satisfy the sufficient conditions in Theorem 4.

Chapter 4

Examples of Strongly Unimodal Multivariate Discrete Distributions

Negative multinomial distribution, multivariate hypergeometric distribution, multivariate negative hypergeometric distribution, and Dirichlet (or beta)-compound multinomial distribution satisfy the sufficient conditions for strong unimodality obtained by Subasi et al. (2009) [44]. Below we present three other multivariate discrete distributions which satisfy the strong unimodality conditions presented in Chapter 3.

4.1 Multivariate Pólya-Eggenberger Distribution

Pólya-Eggenberger distribution was introduced by Eggenberger and Pólya (1923, 1928) [16, 17] through an urn model, where a ball is removed at random from an urn containing a type 1 balls and b type 2 balls. The color of the removed ball is noted and then the ball is returned to the urn along with c additional balls of the same color. Note that if $c = 0$, the scheme becomes sampling with replacement and if $c = -1$, then it is sampling without replacement. Then the distribution of the number of type 1 balls observed at the end of n trials has a Pólya-Eggenberger distribution.

A multivariate version of the above described scheme allows for $a_1, a_2, ..., a_k$ balls of k different colors denoted by $C_1, C_2, ..., C_k$. Suppose that a ball is removed from the urn and is returned into the urn along with c additional balls of the same color. Then the joint distribution of the number of balls of colors $C_1, C_2, ..., C_k$, denoted by $X_1, X_2, ..., X_k$, respectively, at the end of n trials is a Pólya-Eggenberger distribution with the joint p.m.f

$$p(x_1, x_2, \dots, x_k) = \frac{n!}{a^{[n,c]}} \prod_{i=1}^{k} \frac{a_i^{[x_i,c]}}{x_i!} \tag{4.1}$$

where $\sum_{i=1}^{k} x_i = n$, $\sum_{i=1}^{k} a_i = a$, and

$$a^{[x,c]} = a(a+c)\dots(a+(x-1)c)$$

with $a^{[0,c]} = 1$. Note that if $c = 0$, the scheme becomes sampling with replacement and if $c = -1$, then it is sampling without replacement (see, e.g., Steyn

(1951) [43] and Johnson, Kotz, and Balakrishnan (1997) [23]).

<u>Claim</u>: The joint p.m.f (4.1) of Pólya-Eggenberger distribution does not satisfy the sufficient conditions for strong unimodality in Theorem 3 presented by Subasi et al. (2009) [44] unless $c \leq a_i$ for all $i = 1, ..., k$.

<u>Proof of the Claim</u>: Once can easily show that the joint Pólya-Eggenberger p.m.f given in (4.1) satisfies Condition C-I of Theorem 3 of Subasi et al. (2009) [44].

In order to prove our claim let us recall Condition C-II of Theorem 3:

$$p(x_1+\gamma_1+1, ..., x_k+\gamma_k+1)p(x_1, ..., x_k) \ \leq \ p(x_1+1, ..., x_k+1)p(x_1+\gamma_1, ..., i_k+\gamma_k)\ ,$$

where $\gamma_j \in \{0, 1\},\ j = 1, ..., k$ and $\sum_{j=1}^{k} \gamma_j = 1$.

Without loss of generality, we assume $\gamma_1 = 1$, $\gamma_j = 0$ for all $j \neq 1$ and consider the point $(x_1, x_2, ..., x_k) \in \mathbb{Z}^n$. Then we have

$$p(x_1+2, x_2+1, ..., x_k+1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+2,c]} a_2^{[x_2+1,c]} \cdots a_k^{[x_k+1,c]}}{(x_1+2)!(x_2+1)! \cdots (x_k+1)!} \right]$$

$$p(x_1, x_2, ..., x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2,c]} \cdots a_k^{[x_k,c]}}{x_1! x_2! \cdots x_k!} \right]$$

$$p(x_1+1, x_2+1, ..., x_k+1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2+1,c]} \cdots a_k^{[x_k+1,c]}}{(x_1+1)!(x_2+1)! \cdots (x_k+1)!} \right]$$

$$p(x_1+1, x_2, ..., x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2,c]} \cdots a_k^{[x_k,c]}}{(x_1+1)! x_2! \cdots x_k!} \right]$$

Let us assume that the above condition C-II is satisfied, i.e.,

$$p(x_1{+}2, x_2{+}1, ..., x_k{+}1)p(x_1, ..., x_k) \ \le \ p(x_1{+}1, x_2{+}1, ..., x_k{+}1)p(x_1{+}1, x_2, ..., x_k).$$

Equivalently, we have

$$\frac{a_1^{[x_1+2,c]}a_1^{[x_1,c]}}{(x_1+2)!x_1!} \le \frac{a_1^{[x_1+1,c]}a_1^{[x_1+1,c]}}{(x_1+1)!(x_1+1)!}$$

$$\Downarrow$$

$$\frac{a_1^{[x_1+2,c]}a_1^{[x_1,c]}}{x_1+2} \le \frac{a_1^{[x_1+1,c]}a_1^{[x_1+1,c]}}{x_1+1}$$

$$\Downarrow$$

$$\frac{a_1(a_1+c)...(a_1+(x_1+1)c)a_1(a_1+c)...(a_1+(x_1-1)c)}{x_1+2}$$
$$\le \frac{a_1(a_1+c)...(a_1+x_1c)a_1(a_1+c)...(a_1+x_1c)}{x_1+1}$$

$$\Downarrow$$

$$\frac{a_1+(x_1+1)c}{x_1+2} \le \frac{a_1+x_1c}{x_1+1} \Longrightarrow c \le a_1$$

Hence,

$$p(x_1{+}2, x_2{+}1, ..., x_k{+}1)p(x_1, ..., x_k) \ \le \ p(x_1{+}1, x_2{+}1, ..., x_k{+}1)p(x_1{+}1, x_2, ..., x_k)$$

is satisfied if $c \le a_1$. Similarly, when we choose $\gamma_i = 1$ for some i and $\gamma_j = 0$ for all $j \ne i$, Condition C-II of Theorem 3 is satisfied provided that $c \le a_i$ for all $i = 1, ..., k$.

Now, recall Condition C-III of Theorem 3 of Subasi et al. (2009) [44]:

$$p(x_1-\alpha_1,...,x_k-\alpha_k)p(x_1+1,...,x_k+1) \ \le \ p(x_1,...,x_k)p(x_1-\alpha_1+1,...,x_k-\alpha_k+1),$$

where $\alpha_j \in \{0,1\}$, $j = 1,...,n$ and $\displaystyle\sum_{j=1}^{n} \alpha_j = 1$.

Without loss of generality, assume that $\alpha_1 = 1$ and $\alpha_j = 0, j = 2,...,k$. Then

$$p(x_1-1,x_2,...,x_k) = \frac{n!}{a^{[n,c]}}\left[\frac{a_1^{[x_1-1,c]}a_2^{[x_2,c]}\cdots a_k^{[x_k,c]}}{(x_1-1)!x_2!\cdots x_k!}\right]$$

$$p(x_1+1,x_2+1,...,x_k+1) = \frac{n!}{a^{[n,c]}}\left[\frac{a_1^{[x_1+1,c]}a_2^{[x_2+1,c]}\cdots a_k^{[x_k+1,c]}}{(x_1+1)!(x_2+1)!\cdots(x_k+1)!}\right]$$

$$p(x_1,x_2,...,x_k) = \frac{n!}{a^{[n,c]}}\left[\frac{a_1^{[x_1,c]}a_2^{[x_2,c]}\cdots a_k^{[x_k,c]}}{x_1!x_2!\cdots x_k!}\right]$$

$$p(x_1,x_2+1,...,x_k+1) = \frac{n!}{a^{[n,c]}}\left[\frac{a_1^{[x_1,c]}a_2^{[x_2+1,c]}\cdots a_k^{[x_k+1,c]}}{x_1!(x_2+1)!\cdots(x_k+1)!}\right]$$

Assume that Condition C-III of Theorem 3 is satisfied, i.e.,

$$p(x_1-1,x_2,...,x_k)p(x_1+1,x_2+1,...,x_k+1) \ \le \ p(x_1,x_2,...,x_k)p(x_1,x_2+1,...,x_k+1).$$

Substituting the corresponding probabilities, we obtain

$$\frac{a_1^{[x_1-1,c]}a_1^{[x_1+1,c]}}{(x_1-1)!(x_1+1)!} \le \frac{a_1^{[x_1,c]}a_1^{[x_1,c]}}{x_1!x_1!}$$

$$\Downarrow$$

$$\frac{a_1^{[x_1-1,c]}a_1^{[x_1+1,c]}}{x_1+1} \le \frac{a_1^{[x_1,c]}a_1^{[x_1,c]}}{x_1}$$

$$\Downarrow$$

$$\frac{a_1(a_1+c)...(a_1+(x_1-2)c)a_1(a_1+c)...(a_1+x_1c)}{x_1+1}$$
$$\leq \frac{a_1(a_1+c)...(a_1+(x_1-1)c)a_1(a_1+c)...(a_1+(x_1-1)c)}{x_1}$$

$$\Downarrow$$

$$\frac{a_1+x_1c}{x_1+1} \leq \frac{a_1+(x_1-1)c}{x_1} \Longrightarrow c \leq a_1$$

Therefore,

$$p(x_1-1, x_2, ..., x_k)p(x_1+1, x_2+1, ..., x_k+1) \ \leq \ p(x_1, x_2, ..., x_k)p(x_1, x_2+1, ..., x_k+1)$$

is satisfied if $c \leq a_1$. Similarly, when we choose $\alpha_i = 1$ for some i and $\alpha_j = 0$ for all $j \neq i$, Condition C-III of Theorem 3 is satisfied provided that $c \leq a_i$ for all $i = 1, ..., k$.

Thus, the Pólya-Eggenberger p.m.f (4.1) satisfies Theorem 3 of Subasi et al. (2009) [44] under the assumption that the parameter c, that is, the number of balls returned into the urn, is at most the initial number of balls of color $C_i,\ i = 1, ..., k$ in the urn: $c \leq a_i,\ i = 1, ..., k$.

Below we prove that the joint p.m.f (4.1) satisfies the strong unimodality conditions (3.4) and (3.5) without any additional assumptions on the parameters of the distribution.

Theorem 5. *The joint Pólya-Eggenberger p.m.f* (4.1) *is strongly unimodal, and hence, logconcave.*

Proof. Without loss of generality, let us prove that the joint p.m.f (4.1) satisfies conditions (3.4) and (3.5) for some fixed values of the parameters $\alpha_j, \beta_j, \gamma_j, \delta_j$, and $\epsilon_j, j = 1, ..., k$.

In case of Condition I, let $\alpha_1 = 1$ and $\beta_k = 1$. Then $\alpha_j = 0$ for all $j \neq 1$, $\beta_j = 0$ for all $j \neq k$, and we have

$$\mathbf{x} = (x_1 + 1, x_2, \ldots, x_{k-1}, x_k),$$

$$\mathbf{y} = (x_1, x_2, \ldots, x_{k-1}, x_k + 1)$$

$$\mathbf{x} \vee \mathbf{y} = (x_1 + 1, x_2, \ldots, x_{k-1}, x_k + 1),$$

$$\mathbf{x} \wedge \mathbf{y} = (x_1, x_2, \ldots, x_{k-1}, x_k),$$

with the corresponding probabilities

$$p(\mathbf{x}) = p(x_1 + 1, x_2, \ldots, x_{k-1}, x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k,c]}}{(x_1 + 1)! x_2! \cdots x_{k-1}! x_k!} \right]$$

$$p(\mathbf{y}) = p(x_1, x_2, \ldots, x_{k-1}, x_k + 1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k+1,c]}}{x_1! x_2! \cdots x_{k-1}! (x_k + 1)!} \right]$$

$$p(\mathbf{x} \vee \mathbf{y}) = p(x_1 + 1, x_2, \ldots, x_{k-1}, x_k + 1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k+1,c]}}{(x_1 + 1)! x_2! \cdots x_{k-1}! (x_k + 1)!} \right]$$

$$p(\mathbf{x} \wedge \mathbf{y}) = p(x_1, x_2, \ldots, x_{k-1}, x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k,c]}}{x_1! x_2! \cdots x_{k-1}! x_k!} \right]$$

where $n, a^{[n,c]}, a_1, ..., a_k$, and c are as defined before. Note that

$$p(\mathbf{x})p(\mathbf{y}) = p(\mathbf{x} \vee \mathbf{y})p(\mathbf{x} \wedge \mathbf{y})$$

and therefore, condition (3.4) is satisfied.

As for the case of Condition II, let $\gamma_1 = \delta_2 = \epsilon_k = 1$. Then $m_1 = m_2 = m_k = 1$ and we have $\gamma_j = 0$ for all $j \neq 1$, $\delta_j = 0$ for all $j \neq 2$, and $\epsilon_j = 0$ for all $j \neq k$.

Given a point $(x_1, x_2, ..., x_k) \in \mathbb{Z}^k$, we have the following probabilities in connection with condition (3.5):

$$p(x_1, x_2, \ldots, x_{k-1}, x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k,c]}}{x_1! x_2! \cdots x_{k-1}! x_k!} \right]$$

$$p(x_1+1, x_2+1, \ldots, x_{k-1}, x_k+1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2+1,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k+1,c]}}{(x_1+1)!(x_2+1)! \cdots x_{k-1}!(x_k+1)!} \right]$$

$$p(x_1+1, x_2, \ldots, x_{k-1}, x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1+1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k,c]}}{(x_1+1)! x_2! \cdots x_{k-1}! x_k!} \right]$$

$$p(x_1, x_2+1, \ldots, x_{k-1}, x_k) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2+1,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k,c]}}{x_1!(x_2+1)! \cdots x_{k-1}! x_k!} \right]$$

$$p(x_1, x_2, \ldots, x_{k-1}, x_k+1) = \frac{n!}{a^{[n,c]}} \left[\frac{a_1^{[x_1,c]} a_2^{[x_2,c]} \cdots a_{k-1}^{[x_{k-1},c]} a_k^{[x_k+1,c]}}{x_1! x_2! \cdots x_{k-1}!(x_k+1)!} \right]$$

where the first two are the probabilities appear on the left hand side and the last three on the right hand side of condition (3.5).
One can easily observe that

$$p(x_1, x_2, \ldots, x_{k-1}, x_k)^2 p(x_1+1, x_2+1, \ldots, x_{k-1}, x_k+1)$$

$$= p(x_1+1, x_2, \ldots, x_{k-1}, x_k) p(x_1, x_2+1, \ldots, x_{k-1}, x_k) p(x_1, x_2, \ldots, x_{k-1}, x_k+1)$$

Using a similar approach we can prove that Conditions I and II of Theorem 4 are satisfied for any $\alpha_j, \beta_j \in \{0, 1\}, j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} \alpha_j = \sum_{j=1}^{k} \beta_j = 1, \quad \text{and} \quad \alpha_j + \beta_j \leq 1 \quad \text{for all} \quad j = 1, \ldots, k$$

and $\gamma_j, \delta_j, \epsilon_j \in \{0, 1\}, j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} \gamma_j = \sum_{j=1}^{k} \delta_j = \sum_{j=1}^{k} \epsilon_j = 1 \quad \text{and} \quad m_j = \gamma_j + \delta_j + \epsilon_j \leq 1,\ j = 1, \ldots, k.$$

Thus, the joint p.m.f (4.1) of the multivariate Pólya-Eggenberger distribution is logconcave.

□

In the following section we turn our attention to another well-known multivariate discrete distribution and prove that its p.m.f is logconcave.

4.2 Multivariate Poisson Distribution

Let $E_1, ..., E_k$ be mutually exclusive random events and $p_i = P(E_i)$ be the probability that event $E_i, i = 1, ..., k$, occurs such that $\sum_{i=1}^{k} p_i = 1$.

Let $X_i, i = 1, ..., k$, denote the number of occurrences of event $E_i, i = 1, ..., k$, in n trials where only one of the events is observed. Then the joint distribution of $X_1, ..., X_k$ is a multinomial distribution with probability function

$$p(x_1, ..., x_k) = P(X_1 = x_1, ..., X_k = x_k) = n! \prod_{i=1}^{k} \frac{{p_i}^{x_i}}{x_i!}, \tag{4.2}$$

where $\sum_{i=1}^{k} x_i = n$.

Note that if $np_i \to \lambda_i$ for all $i = 1, ..., k-1$ as $n \to \infty$, i.e., parameters $p_1, ..., p_{k-1}$ tend to 0 and p_k tends to 1 as the number of trials approaches infinity, then the joint probability function of random variables $X_1, ..., X_{k-1}$ can be obtained as

$$p(x_1, ..., x_{k-1}) = e^{-\sum_{i=1}^{k-1} \lambda_i} \prod_{i=1}^{k-1} \frac{\lambda_i^{x_i}}{x_i!}, \tag{4.3}$$

which is simply the joint probability function of $k-1$ mutually independent Poisson random variables with means $\lambda_1, ..., \lambda_{k-1}$.

Probability function (4.3) can also be obtained as a limiting form of negative multinomial distributions (see, e.g., [23]). Patil and Bildikar (1967) [36] named (4.3) as Multiple Poisson Distribution and Banerjee (1959) [6] and Sibuya, Yoshimura, and Shimizu (1964) [42] gave formulas for its moments. The reader is referred to [23] for different types of multivariate Poisson distributions.

Below we prove that the joint p.m.f of independent Poisson random variables satisfies the strong unimodality conditions (3.4) and (3.5).

Theorem 6. *The joint p.m.f* (4.3) *of independent Poisson random variables is strongly unimodal, and hence, logconcave.*

Proof. Let $p(x_1, ..., x_{k-1}) = e^{-\sum_{i=1}^{k-1} \lambda_i} \prod_{i=1}^{k-1} \frac{\lambda_i^{x_i}}{x_i!}$ be the joint p.m.f of independent Poisson random variables $X_1, ..., X_{k-1}$.

Similar to the proof of Theorem 5 we shall prove that the joint p.m.f (4.3) satisfies conditions (3.4) and (3.5) for some fixed values of the parameters $\alpha_j, \beta_j, \gamma_j, \delta_j$, and $\epsilon_j, j = 1, ..., k-1$.

In case of Condition I, let $\alpha_1 = 1$ and $\beta_{k-1} = 1$. Then $\alpha_j = 0$ for all $j \neq 1$, $\beta_j = 0$ for all $j \neq k-1$, and we have

$$\mathbf{x} = (x_1 + 1, x_2, \ldots, x_{k-2}, x_{k-1}),$$

$$\mathbf{y} = (x_1, x_2, \ldots, x_{k-2}, x_{k-1} + 1),$$

$$\mathbf{x} \vee \mathbf{y} = (x_1 + 1, x_2, \ldots, x_{k-2}, x_{k-1} + 1),$$

$$\mathbf{x} \wedge \mathbf{y} = (x_1, x_2, \ldots, x_{k-2}, x_{k-1})$$

with the corresponding probabilities

$$p(\mathbf{x}) = p(x_1 + 1, x_2, \dots, x_{k-2}, x_{k-1})$$
$$= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1+1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}}}{(x_1+1)! x_2! \cdots x_{k-2}! x_{k-1}!} \right],$$

$$p(\mathbf{y}) = p(x_1, x_2, \dots, x_{k-2}, x_{k-1} + 1)$$
$$= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}+1}}{x_1! x_2! \cdots x_{k-2}! (x_{k-1}+1)!} \right],$$

$$p(\mathbf{x} \vee \mathbf{y}) = p(x_1 + 1, x_2, \dots, x_{k-2}, x_{k-1} + 1)$$
$$= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1+1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}+1}}{(x_1+1)! x_2! \cdots x_{k-2}! (x_{k-1}+1)!} \right],$$

$$p(\mathbf{x} \wedge \mathbf{y}) = p(x_1, x_2, \dots, x_{k-2}, x_{k-1})$$
$$= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}}}{x_1! x_2! \cdots x_{k-2}! x_{k-1}!} \right].$$

Note that

$$p(\mathbf{x})p(\mathbf{y}) = p(\mathbf{x} \vee \mathbf{y})p(\mathbf{x} \wedge \mathbf{y})$$

and therefore, condition (3.4) is satisfied.

In order to prove that p satisfies Condition II, let $\gamma_1 = \delta_2 = \epsilon_{k-1} = 1$. Then $m_1 = m_2 = m_{k-1} = 1$ and we have $\gamma_j = 0$ for all $j \neq 1$, $\delta_j = 0$ for all $j \neq 2$, and $\epsilon_j = 0$ for all $j \neq k-1$.

By the above selection of the parameters, given a point $\mathbf{x} = (x_1, x_2, ..., x_k - 1) \in \mathbb{Z}^{k-1}$, we have the following points in connection with condition (3.5):

$$
\begin{aligned}
\mathbf{x} &= (x_1, x_2, ..., x_{k-2}, x_{k-1}) \\
\mathbf{x}^1 &= (x_1 + 1, x_2 + 1, \ldots, x_{k-2}, x_{k-1} + 1) \\
\mathbf{x}^2 &= (x_1 + 1, x_2, \ldots, x_{k-2}, x_{k-1}) \\
\mathbf{x}^3 &= (x_1, x_2 + 1, \ldots, x_{k-2}, x_{k-1}) \\
\mathbf{x}^4 &= (x_1, x_2, \ldots, x_{k-2}, x_{k-1} + 1)
\end{aligned}
$$

and the corresponding probabilities are

$$
\begin{aligned}
p(\mathbf{x}) &= p(x_1, x_2, \ldots, x_{k-2}, x_{k-1}) \\
&= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}}}{x_1! x_2! \cdots x_{k-2}! x_{k-1}!} \right]
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}^1) &= p(x_1 + 1, x_2 + 1, \ldots, x_{k-2}, x_{k-1} + 1) \\
&= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1+1} \lambda_2^{x_2+1} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}+1}}{(x_1 + 1)!(x_2 + 1)! \cdots x_{k-2}!(x_{k-1} + 1)!} \right]
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}^2) &= p(x_1 + 1, x_2, \ldots, x_{k-2}, x_{k-1}) \\
&= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1+1} \lambda_2^{x_2} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}}}{(x_1 + 1)! x_2! \cdots x_{k-2}! x_{k-1}!} \right]
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}^3) &= p(x_1, x_2 + 1, \ldots, x_{k-2}, x_{k-1}) \\
&= e^{-\sum_{i=1}^{k-1} \lambda_i} \left[\frac{\lambda_1^{x_1} \lambda_2^{x_2+1} \cdots \lambda_{k-2}^{x_{k-2}} \lambda_{k-1}^{x_{k-1}}}{x_1!(x_2 + 1)! \cdots x_{k-2}! x_{k-1}!} \right]
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}^4) &= p(x_1, x_2, \dots, x_{k-2}, x_{k-1}+1) \\
&= e^{-\sum_{i=1}^{k-1}\lambda_i}\left[\frac{\lambda_1^{x_1}\lambda_2^{x_2}\cdots\lambda_{k-2}^{x_{k-2}}\lambda_{k-1}^{x_{k-1}+1}}{x_1!x_2!\cdots x_{k-2}!(x_{k-1}+1)!}\right]
\end{aligned}
$$

where the first two are the probabilities appear on the left hand side and the last three on the right hand side of condition (3.5).

Then we have

$$
p^2(x_1, x_2, \dots, x_{k-2}, x_{k-1})p(x_1+1, x_2+1, \dots, x_{k-2}, x_{k-1}+1)
$$

$$
= e^{-3\sum_{i=1}^{k-1}\lambda_i}\left[\frac{\lambda_1^{3x_1+1}\lambda_2^{3x_2}\cdots\lambda_{k-2}^{3x_{k-2}}\lambda_{k-1}^{3x_{k-1}+1}}{x_1!^2(x_1+1)!x_2!^2(x_2+1)!\cdots x_{k-2}!^3x_{k-1}!^2(x_{k-1}+1)!}\right]
$$

and

$$
p(x_1+1, x_2, \dots, x_{k-2}, x_{k-1})p(x_1, x_2+1, \dots, x_{k-2}, x_{k-1})p(x_1, x_2, \dots, x_{k-2}, x_{k-1}+1)
$$

$$
= e^{-3\sum_{i=1}^{k-1}\lambda_i}\left[\frac{\lambda_1^{3x_1+1}\lambda_2^{3x_2}\cdots\lambda_{k-2}^{3x_{k-2}}\lambda_{k-1}^{3x_{k-1}+1}}{x_1!^2(x_1+1)!x_2!^2(x_2+1)!\cdots x_{k-2}!^3x_{k-1}!^2(x_{k-1}+1)!}\right]
$$

Note that

$$
p(x_1, x_2, \dots, x_{k-2}, x_{k-1})^2 p(x_1+1, x_2+1, \dots, x_{k-2}, x_{k-1}+1)
$$

$$
= p(x_1+1, x_2, \dots, x_{k-2}, x_{k-1})p(x_1, x_2+1, \dots, x_{k-2}, x_{k-1})p(x_1, x_2, \dots, x_{k-2}, x_{k-1}+1)
$$

Similarly, we can prove that Conditions I and II of Theorem 4 are satisfied for any $\alpha_j, \beta_j \in \{0,1\}, j = 1, \ldots, k-1$, such that

$$\sum_{j=1}^{k-1} \alpha_j = \sum_{j=1}^{k-1} \beta_j = 1, \quad \text{and} \quad \alpha_j + \beta_j \leq 1 \quad \text{for all} \quad j = 1, \ldots, k-1$$

and $\gamma_j, \delta_j, \epsilon_j \in \{0,1\}, j = 1, \ldots, k-1$, such that

$$\sum_{j=1}^{k-1} \gamma_j = \sum_{j=1}^{k-1} \delta_j = \sum_{j=1}^{k-1} \epsilon_j = 1 \quad \text{and} \quad m_j = \gamma_j + \delta_j + \epsilon_j \leq 1,\ j = 1, \ldots, k-1.$$

Thus, the joint p.m.f of multivariate Poisson distribution given in (4.3) is strongly unimodal, and hence, logconcave.

□

We remark that the joint p.m.f (4.3) also satisfies the sufficient conditions in Theorem 3 of Subasi et al. (2009) [44], where conditions C-I and C-III are satisfied as equalities and condition C-II is satisfied as a strict inequality.

4.3 Multivariate Ewens Distribution

The multivariate Ewens distribution, also known as "Ewens sampling formula" in genetics, describes the probability for a specific partition of a positive integer $n \in \mathbb{Z}^+$ into parts such that

$$n = x_1 + 2x_2 + ... + nx_n,$$

where $x_i,\ i = 1, ..., n$, are nonnegative integers.

The motivation for the derivation of the multivariate Ewens distribution was the non-Darwinian theory of evolution [23]. The distribution was discovered by Ewens (1972) [19] to provide the probability of the partition of a sample of n selectively equivalent genes into a number of allelic types, either exactly in some models of genetic evolution or in others as a limiting distribution when the population size becomes infinitely large. Antoniak (1974) [2] independently discovered the multivariate Ewens distribution in the context of Bayesian statistics.

The multivariate Ewens distribution uses a parameter θ that is related to the rate of mutations of the genes to new allelic types, the population size, and the evolutionary models. Johnson, Kotz, and Balakrishnan (1997) [23] describes the distribution in terms of sequential sampling of animals from an infinite collection of distinguishable species - an example considered by Fisher et al (1943) [22], McCloskey (1965) [26], and Engen (1978) [18].
Let $P = (P_1, P_2, ...)$ be the random frequencies of the species such that

$$\sum_{i=1}^{\infty} P_i = 1 \qquad \text{and} \qquad 0 < P_i < 1,\ i = 1, 2, ...$$

Let $\nu_i, i = 1, 2, ...$ denote the species of type i animal sampled. Assume that conditional on the frequency distribution P, the ν_i's are independent identically distributed random variables with

$$P(\nu_i = k \mid P) = P_k,\ i = 1, 2, ...$$

Then the sequence $I_1, I_2, ...$ of distinct values observed in $\nu_1, \nu_2, ...$ gives a random size-biased permutation $P^* = (P_{I_1}, P_{I_2}, ...)$ of the frequency distribution P (see Johnson, Kotz, and Balakrishnan (1997) [23]).

Consider the sample of n individuals determined by $\nu_i,\ i = 1, 2, \cdots$. Let $A_i(n),\ i = 1, 2, ...$ denote the number of animals from species i to appear and $C_j(n),\ j = 1, ..., n$ denote the number of species represented by j animals in the sample. Then the random vector $C(n) = (C_1(n), ..., C_n(n))$ satisfies

$$\sum_{j=1}^{n} jC_j(n) = n$$

and if K_n is the number of distinct species to appear in the sample, we have

$$K_n = \sum_{j=1}^{n} C_j(n).$$

Now, consider the case where the frequency distribution P is given by

$$P_1 = W_1 \qquad \text{and} \qquad P_i = W_i \prod_{r=1}^{i-1} 1 - W_r, \quad i = 2, 3, ...,$$

where for some $0 < \theta < \infty$ and $W_1, W_2, ...$ are independent identically distributed random variables with density

$$P(W_i = w) = \theta(1 - w)^{\theta - 1}, \quad 0 < w < 1.$$

Then the joint distribution of the random number of species represented by j animals in the sample is called the multivariate Ewens distribution with the p.m.f

$$P(C_1(n) = x_1, ..., C_n(n) = x_n) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \frac{(\theta/j)^{x_j}}{x_j!}, \tag{4.4}$$

where $\theta^{[n]} = \theta(\theta+1)...(\theta+n-1)$ and $x_j \in \mathbb{Z}^+, j = 1, ..., n$, satisfying $\sum_{j=1}^{n} jx_j = n$.

Theorem 7. *The joint p.m.f* (4.4) *of the multivariate Ewens distribution is strongly unimodal, and hence, logconcave.*

Proof. Let $P(C_1(n) = x_1, ..., C_n(n) = x_n) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \frac{(\theta/j)^{x_j}}{x_j!}$ be the joint p.m.f of $C_j(n)$, the number of species represented by j animals in a sample of n individuals.

As in Sections 4.1 and 4.2 we first prove that the joint p.m.f (4.4) satisfies conditions (3.4) and (3.5) for some fixed values of the parameters $\alpha_j, \beta_j, \gamma_j, \delta_j$, and $\epsilon_j, j = 1, ..., n$.

In what follows we use the alternative notation

$$P(C_1(n) = x_1, ..., C_n(n) = x_n) = P(x_1, ..., x_n).$$

For Condition I, let us choose $\alpha_1 = 1$ and $\beta_n = 1$. Then $\alpha_j = 0$ for all $j \neq 1$, $\beta_j = 0$ for all $j \neq n$ and we have

$$\mathbf{x} = (x_1 + 1, x_2, \ldots, x_{n-1}, x_n),$$

$$\mathbf{y} = (x_1, x_2, \ldots, x_{n-1}, x_n + 1),$$

$$\mathbf{x} \vee \mathbf{y} = (x_1 + 1, x_2, \ldots, x_{n-1}, x_n + 1),$$

$$\mathbf{x} \wedge \mathbf{y} = (x_1, x_2, \ldots, x_{n-1}, x_n)$$

with the corresponding probabilities

$$\begin{aligned} p(\mathbf{x}) &= p(x_1+1, x_2, \dots, x_{n-1}, x_n) \\ &= \frac{n!}{\theta^{[n]}}\left[\frac{\theta^{x_1+1}[\theta/2]^{x_2}\cdots[\theta/(n-1)]^{x_{n-1}}[\theta/n]^{x_n}}{(x_1+1)!x_2!\cdots x_{n-1}!x_n!}\right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{y}) &= p(x_1, x_2, \dots, x_{n-1}, x_n+1) \\ &= \frac{n!}{\theta^{[n]}}\left[\frac{\theta^{x_1}[\theta/2]^{x_2}\cdots[\theta/(n-1)]^{x_{n-1}}[\theta/n]^{x_n+1}}{x_1!x_2!\cdots x_{n-1}!(x_n+1)!}\right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}\vee\mathbf{y}) &= p(x_1+1, x_2, \dots, x_{n-1}, x_n+1) \\ &= \frac{n!}{\theta^{[n]}}\left[\frac{\theta^{x_1+1}[\theta/2]^{x_2}\cdots[\theta/(n-1)]^{x_{n-1}}[\theta/n]^{x_n+1}}{(x_1+1)!x_2!\cdots x_{n-1}!(x_n+1)!}\right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}\wedge\mathbf{y}) &= p(x_1, x_2, \dots, x_{n-1}, x_n) \\ &= \frac{n!}{\theta^{[n]}}\left[\frac{\theta^{x_1}[\theta/2]^{x_2}\cdots[\theta/(n-1)]^{x_{n-1}}[\theta/n]^{x_n}}{x_1!x_2!\cdots x_{n-1}!x_n!}\right]. \end{aligned}$$

Note that

$$p(\mathbf{x})p(\mathbf{y}) = p(\mathbf{x}\vee\mathbf{y})p(\mathbf{x}\wedge\mathbf{y})$$

and therefore, condition (3.4) is satisfied.

In order to prove that p satisfies Condition II, let us choose $\gamma_1 = \delta_2 = \epsilon_n = 1$. Then $m_1 = m_2 = m_n = 1$ and we have $\gamma_j = 0$ for all $j \neq 1$, $\delta_j = 0$ for all $j \neq 2$, and $\epsilon_j = 0$ for all $j \neq n$.

By the selection of above parameters, given a point $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{Z}^n$, we have the following points in connection with condition (3.5):

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$
$$\mathbf{x}^1 = (x_1 + 1, x_2 + 1, \dots, x_{n-1}, x_n + 1)$$
$$\mathbf{x}^2 = (x_1 + 1, x_2, \dots, x_{n-1}, x_n)$$
$$\mathbf{x}^3 = (x_1, x_2 + 1, \dots, x_{n-1}, x_n)$$
$$\mathbf{x}^4 = (x_1, x_2, \dots, x_{n-1}, x_n + 1)$$

The probabilities corresponding to these points are

$$\begin{aligned} p(\mathbf{x}) &= p(x_1, x_2, \dots, x_{n-1}, x_n) \\ &= \frac{n!}{\theta^{[n]}} \left[\frac{\theta^{x_1} [\theta/2]^{x_2} \cdots [\theta/(n-1)]^{x_{n-1}} [\theta/n]^{x_n}}{x_1! x_2! \cdots x_{n-1}! x_n!} \right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}^1) &= p(x_1 + 1, x_2 + 1, \dots, x_{n-1}, x_n + 1) \\ &= \frac{n!}{\theta^{[n]}} \left[\frac{\theta^{x_1+1} [\theta/2]^{x_2+1} \cdots [\theta/(n-1)]^{x_{n-1}} [\theta/n]^{x_n+1}}{(x_1+1)!(x_2+1)! \cdots x_{n-1}!(x_n+1)!} \right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}^2) &= p(x_1 + 1, x_2, \dots, x_{n-1}, x_n) \\ &= \frac{n!}{\theta^{[n]}} \left[\frac{\theta^{x_1+1} [\theta/2]^{x_2} \cdots [\theta/(n-1)]^{x_{n-1}} [\theta/n]^{x_n}}{(x_1+1)! x_2! \cdots x_{n-1}! x_n!} \right], \end{aligned}$$

$$\begin{aligned} p(\mathbf{x}^3) &= p(x_1, x_2 + 1, \dots, x_{n-1}, x_n) \\ &= \frac{n!}{\theta^{[n]}} \left[\frac{\theta^{x_1} [\theta/2]^{x_2+1} \cdots [\theta/(n-1)]^{x_{n-1}} [\theta/n]^{x_n}}{x_1!(x_2+1)! \cdots x_{n-1}! x_n!} \right], \end{aligned}$$

$$p(\mathbf{x}^4) = p(x_1, x_2, \dots, x_{n-1}, x_n + 1)$$
$$= \frac{n!}{\theta^{[n]}} \left[\frac{\theta^{x_1}[\theta/2]^{x_2} \cdots [\theta/(n-1)]^{x_{n-1}}[\theta/n]^{x_n+1}}{x_1! x_2! \cdots x_{n-1}!(x_n+1)!} \right].$$

Therefore, the left hand side of condition (3.5) is

$$p^2(x_1, x_2, \dots, x_{n-1}, x_n) p(x_1+1, x_2+1, \dots, x_{n-1}, x_n+1)$$

$$= \frac{n!}{\theta^{[n]}} \left[\frac{\theta_1^{3x_1+1}[\theta/2]^{3x_2} \cdots [\theta/(n-1)]^{3x_n}[\theta/(n)]^{3x_n+1}}{x_1!^2(x_1+1)! x_2!^2(x_2+1)! \cdots x_{n-1}!^3 x_n!^2(x_n+1)!} \right]$$

and the right hand side is

$$p(x_1+1, x_2, \dots, x_{k-2}, x_{k-1}) p(x_1, x_2+1, \dots, x_{k-2}, x_{k-1}) p(x_1, x_2, \dots, x_{k-2}, x_{k-1}+1)$$

$$= \frac{n!}{\theta^{[n]}} \left[\frac{\theta_1^{3x_1+1}[\theta/2]^{3x_2} \cdots [\theta/(n-1)]^{3x_n}[\theta/(n)]^{3x_n+1}}{x_1!^2(x_1+1)! x_2!^2(x_2+1)! \cdots x_{n-1}!^3 x_n!^2(x_n+1)!} \right]$$

Note that

$$p(x_1, x_2, \dots, x_{n-1}, x_n)^2 p(x_1+1, x_2+1, \dots, x_{n-1}, x_n+1)$$

$$= p(x_1+1, x_2, \dots, x_{n-1}, x_n) p(x_1, x_2+1, \dots, x_{n-1}, x_n) p(x_1, x_2, \dots, x_{n-1}, x_n+1)$$

Similarly, we can prove that the joint p.m.f (4.4) satisfies Conditions I and

II of Theorem 4 for any $\alpha_j, \beta_j \in \{0, 1\}, j = 1, \ldots, k$, such that

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j = 1, \quad \text{and} \quad \alpha_j + \beta_j \leq 1 \quad \text{for all} \quad j = 1, \ldots, n$$

and $\gamma_j, \delta_j, \epsilon_j \in \{0, 1\}, j = 1, \ldots, n$, such that

$$\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \epsilon_j = 1 \quad \text{and} \quad m_j = \gamma_j + \delta_j + \epsilon_j \leq 1, \ j = 1, \ldots, n.$$

Thus, the joint p.m.f (4.4) of multivariate Ewens distribution is strongly unimodal, and hence, logconcave.

□

Chapter 5

Conclusion

In this work we present all possible subdivisions of R^3 into tetrahedra with disjoint interiors and adopt a combinatorial approach to obtain a special subdivision of Rn into simplices with disjoint interiors, where two simplices are called neighbors if they have a common facet. The neighborhood relationship of the simplices in a subdivision enables us to extend the results of Subasi et al. (2009) [44], where we fully describe the sufficient conditions for the strong unimodality of the joint probability mass functions defined on Z^3 and provide a new sufficient condition for the strong unimodality of multivariate discrete distributions defined on Z^n. We show that multivariate Pólya-Eggenberger, multivariate Poisson, and multivariate Ewens distributions satisfy the sufficient conditions for strongly unimodality, and hence are logconcave.

Bibliography

[1] M.G. Alharbi, E. Subasi, M.M. Subasi, New sufficient conditions for strong unimodality of multivariate discrete distributions, Discrete Applied Mathematics (2017). Accepted for publication.

[2] C.E. Antoniak, Miztures of Dirichlet processes with applications to Bayesian nonparametric problems, Annals of Statistics 2 (1974), 1152–1174.

[3] F. Balabdaoui, H. Jankowski, K. Rufibach, M. Pavlides, Asymptotics of the discrete log-concave maximum likelihood estimator and related applications, Statistical Methodology 75(4) (2013), 769–790.

[4] M. Bagnoli, N. Khanna, Buyers' and sellers' agents in the housing market, The Journal of Real Estate Finance and Economics 4(2) (1991), 147–156.

[5] M. Bagnoli, T. Bergstrom, Log-concave probability and its applications, Economic Theory 26 (2), Rationality and Equilibrium: A Symposium in Honor of Marcel K. Richter (Aug., 2005), pp. 445–469.

[6] D.P. Banerjee, On some theorems on Poisson distributions, Proceedings of the National Academy of Science, India, Section A, 28 (1959), 30–33.

[7] O. Barndorff-Nielsen, Unimodality and exponential families, Comm. Statist. 1 (1973), 189–216.

[8] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing Probability Models, Holt, Rinehart and Winston, New York, 1975.

[9] M.S. Bazaraa, H.D. Sherali, C.M. Shetty, Nonlinear Programming: Theory and Algorithms, John Wiley & Sons, Inc., New York, NY, 1993.

[10] D. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA, 1999.

[11] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, New York, 2004.

[12] B. Chowdhry, N. Jegadeesh, Pre-tender offer share acquisition strategy in takeovers, The Journal of Financial and Quantitative Analysis 29(1) (1994), 117–129.

[13] M. Cule, R. Samworth, Theoretical properties of the log-concave maximum likelihood estimator of a multidimensional density, Electron. J. Statist. 4 (2010), 254–270.

[14] D. Dentcheva, A. Prékopa, A. Ruszczynski, Concavity and efficient points of discrete distributions in probabilistic programming, Mathematical Programming 89(1) (2000), 55–77.

[15] D. Dentcheva, B. Lai, A. Ruszczynski, Efficient point methods for probabilistic optimization problems, Mathematical Methods of Operations Research 60 (2004), 331–346.

[16] F. Eggenberger, G. Pólya, Über die Statistik verketteter Vorgange, Zeitschhrift für Angewandte Mathematik und Mechanik 1 (1923), 279–289.

[17] F. Eggenberger, G. Pólya, Sur I'interpretation de certaines courbes de fréquence, Comptes Rendus de I'Académie des Sciences, Paris 187 (1978), 870–872.

[18] S. Engen, Stochastic abundance models with emphasis on biological communities and species diversity, Chapman and Hall, London, 1978.

[19] W.J. Ewens, Mathematical population genetics, Berlin: Springer-Verlag, 1972.

[20] P. Favati, F. Tardella, Convexity in nonlinear integer programming, Ricerca Operativa 53 (1990), 3–44.

[21] M. Fekete, G. Pólya, Uberein ein Problem von Laguerre, Rediconti del Circolo Matematico di Palermo 23 (1912), 89–120.

[22] R.A. Fisher, A theoretical distribution for the apparent abundance of different species, Journal of Animal Ecology 12 (1943), 54–57.

[23] N.L. Johnson, S. Kotz, N. Balakrishnan, Discrete Multivariate Distributions, Wiley, New York, 1997.

[24] O. Johnson, Log-concavity and the maximum entropy property of the Poisson distribution, Stochastic Process. Appl. 117(6) (2007) 791–802.

[25] O. Johnson, I. Kontoyiannis, M. Madiman, Log-concavity, ultra-log-concavity, and a maximum entropy property of discrete compound Poisson measures, Discrete Applied Mathematics 161(9) (2013), 1232–1250.

[26] J.W. McCloskey, A model for the distribution of individuals by species in an evironment, Ph.D. Thesis, Michigan State University, 1965.

[27] B.L. Miller, On minimizing nonseparable functions defined on the integers with an inventory application, SIAM J. Appl. Math. 21 (1971), 166–185.

[28] X. Mu, Logconcavity of a mixture of beta distributions, Statistics and Probability Letters 99 (2015) 125–130.

[29] K. Murota, Discrete convex analysis, Mathematical Programming 83 (1998), 313–371.

[30] K. Murota, A. Shioura, M-convex function on generalized polymatroid, Mathematics of Operations Research, 24 (1999), 95–105.

[31] K. Murota, Algorithms in discrete convex analysis, IEICE Transactions on Systems and Information, E83-D (2000), 344–352.

[32] K. Murota, A. Shioura, Relationship of M/L-convex functions with discrete convex functions by Miller and Favati-Tardella. Discrete Applied Mathematics 115 (2001), 151–176.

[33] K. Murota, A. Shioura, Quasi M-convex and L-convex functions: Quasiconvexity in discrete optimization. Discrete Applied Mathematics, 131 (2003), 467–494.

[34] K. Murota, Discrete convex analysis, Hausdorff Institute of Mathematics, Summer School (September 21-25, 2015) `http://www.comp.tmu.ac.jp/kzmurota/paper/HIMSummerSchool15Murota.pdf`

[35] A. Ninh, A. Prékopa, Log-concavity of compound distributions with applications in stochastic optimization. Discrete Applied Mathematics 161(18) (2013), 3017–3027.

[36] G.P. Patil, S. Bildikar, Multivariate logarithmic series distribution as a probability model in population and community ecology and some of its statistical properties, Journal of the American Statistical Association 62 (1967), 655–674.

[37] J.G. Pedersen, On strong unimodality of two-dimensional discrete distributions with applications to M-ancillarity, Scand J. Statist. 2 (1975), 99–102.

[38] A. Prékopa, Logarithmic concave functions with applications to stochastic programming, Acta. Sci. Math. 32 (1971), 301–316.

[39] A. Prékopa, On logarithmic concave measures and functions, Acta. Sci. Math. 34 (1973), 335–343.

[40] A. Prékopa, Stochastic Programming, Kluwer Academic Publishers, Dordtecht, Boston, 1995.

[41] A. Saumard, J.A. Wellner, Logconcavity and strong logconcavity: a review, Statist. Surv. 8 (2014), 45–114.

[42] M. Sibuya, I. Yoshimura, R. Shimizu, Negative multinomial distribution, Annals of the Institute of Statistical Mathematics 16 (1964), 409–426.

[43] H.S. Steyn, On discrete multivariate probability functions, Proceedings, Koninklijke Nederlandse Akademie Van Wetenschappen, Series A, 54 (1951), 23–30.

[44] E. Subasi, M.M. Subasi, A. Prékopa, On strong unimodality of multivariate discrete distributions, Discrete Applied Mathematics 157(2) (2009), 234–246.

[45] T. Ui, A Note on discrete convexity and local optimality, Japan J. Indust. Appl. Math., 23 (2006), 21–29.

[46] P. Wild, W.R. Gilks, Algorithm AS 287: Adaptive rejection sampling from log-concave Density, Applied Statistics 42(4) (1993), 701–709.

[47] Y. Yu, On the entropy of compound distributions on nonnegative integers, IEEE Transactions on Information Theory 55(8) (2009), 3645–3650.

Appendix A

Convexity, Logconcavity, and Generalized Convexity

We give a collection of continuous and discrete convexity and generalized convexity notions and theorems that previously studied in literature [9, 10, 40].

A.1 Convexity of Sets

Definition: A set $A \subseteq \mathbb{R}^n$ is said to be *convex* if for any $x, y \in A$ and $\lambda \in [0, 1]$, we have

$$\lambda x + (1 - \lambda) y \in A.$$

If A is a convex set and $x_1, \ldots x_n$ are any points in A, then

$$x = \sum_{i=1}^{n} \lambda_i x_i \in A$$

for any $\lambda_i > 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. The intersection of any collection of convex sets is convex.

Definition: Let $x_1, \ldots x_k$ be vectors in $A \subseteq \mathbb{R}^n$ and $\lambda_1, \ldots \lambda_k$ be nonnegative scalars such that $\sum_{i=1}^{k} \lambda_i = 1$. Then the vector $\sum_{i=1}^{k} \lambda_i x_i \in A$ is said to be a *convex combination* of $x_1, \ldots, x_k$ and the set of all convex combinations of these vectors is said to be a *convex hull*.

A.2 Convex Functions

Definition: A function f defined on a convex subset A of $\mathbb{R}^n$ is said to be *convex* if for every pair $x, y \in A$ and $0 < \lambda < 1$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

f is called *strictly convex* if $\forall x \neq y \in A$, and $\forall \lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

A function f is said to be *(strictly) concave* if $-f$ is (strictly) convex.

Proposition: (*Jensen's Inequality*) Let f be a convex function. Then for any convex combination $\sum_{i=1}^{N} \lambda_i x_i$ we have

$$f\left(\sum_{i=1}^{N} \lambda_i x_i\right) \leq \sum_{i=1}^{N} \lambda_i f(x_i).$$

Below we list a few properties of convex functions:

- If the functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex, then so is the function $f_1 + f_2$.
- If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\lambda \geq 0$, then also the function λf is convex.
- Every linear (affine) function is convex.
- If both f and $-f$ are convex. Then the function f is affine, that is, $f(x) = a^T x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- If the function $f : A \to \mathbb{R}$ is convex. Then the level set

$$S_\alpha = \{x \in A : f(x) \leq \alpha\}$$

  is convex set for all $\alpha \in \mathbb{R}$.

- If the functions $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, then the function f defined by $f(x) = \max_{i=1,\ldots,m} f_i(x)$ is also convex function.

Proposition: Assume that the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then f is convex if and only if for every $x, y \in \mathbb{R}^n$ the inequality

$$f(y) \geq f(x) + \bigtriangledown f(x)^T (y - x)$$

is satisfied.

It is also well-known that if f be a convex and differentiable function, then x^* is a global minimum of f if and only if $\bigtriangledown f(x^*) = 0$.

Proposition: Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function. Then f is convex if and only if the Hessian $\bigtriangledown^2 f(x)$ is positive semi-definite for all $x \in \mathbb{R}^n$.

There is a relation between the strict convexity of a function and positive definiteness of its Hessian. A function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if its Hessian $\bigtriangledown^2 f(x)$ is positive definite for all $X \in \mathbb{R}^n$, but the converse is not correct.

Theorem: Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function. Then the following are equivalent:

(i) f is convex.

(ii) $f(y) \geq f(x) + \bigtriangledown f(x)^T (y - x)$ for every $x, y \in \mathbb{R}^n$.

(iii) $\bigtriangledown^2 f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

A.3 Logconcave Functions

Logconcave distributions and many of the properties of logconcavity play important role in probability, statistics, econometrics, and optimization theory. In the following we summarize the basic facts and properties concerning logconcavity.

Definition: A nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$ defined on a convex set $A \subseteq \mathbb{R}^n$ is *logconcave* if it satisfies the inequality

$$f(\lambda x + (1 - \lambda) y) \geq f(x)^{\lambda} f(y)^{1-\lambda}$$

for all $x, y \in A$ and $0 < \lambda < 1$.

This is equivalent to saying that the logarithm of f is concave, i.e.,

$$\log f(\lambda x + (1-\lambda)y) \geq \lambda \log f(x) + (1-\lambda) \log f(y)$$

for all $x, y \in A$ and $0 < \lambda < 1$.

If the inequality holds strictly for $x \neq y$, then f is said to be *strictly logconcave*. A function is *logconvex* if it satisfies the reserve inequality.

Definition: A class S of subsets of $\mathbb{R}^n$ is called an *algebra* if (i) $\mathbb{R}^n \in S$, (ii) $A \in S$ implies $\bar{A} \in S$, and (iii) $A, B \in S$ implies $A \cup B \in S$. If S is an algebra and $A_1, A_2, ... \in S$ implies $\bigcup_{i=1}^{\infty} A_i \in S$, then S is called a *σ-algebra*. All finite unions of all finite or infinite n-dimensional intervals form an algebra and the smallest σ-algebra that contains this algebra is the collection of Borel measurable sets and is designated by $\mathscr{B}_n$ (see, e.g., [40]).

Definition: A set function $P(A), A \in S$ is called a *measure* if S is a σ-algebra and

1. $P(A) \geq 0$ for every $A \in S$,

2. $P(\emptyset) = 0$,

3. $A_1, A_2, ... \in S$ and $A_i \cap A_j = 0$ for $i \neq j$ implies

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

A measure P is called a *probability measure* if $P(\mathbb{R}^n) = 1$.

Definition: Let A probability measure P, defined on $\mathscr{B}_n$, is called a *logconcave probability measure* if for every pair of nonempty convex sets $A, B \subset \mathbb{R}^n$ (any convex set is Borel measurable, see, e.g., [40]), we have the inequality

$$P(\lambda A + (1-\lambda)B) \geq [P(A)]^{\lambda}[P(B)]^{(1-\lambda)},$$

where the + sign refers to Minkowski addition of sets, i.e.,

$$\lambda A + (1-\lambda)B = \{\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \mid \mathbf{x} \in A, \mathbf{y} \in B\}.$$

Below we present basic theorems of logconcave measures. The reader is referred to Prékopa (1995) [40] for the proofs of these theorems.

Theorem: Let $f(x), x \in \mathbb{R}^n$ be a logconcave probability density function and P be the probability measure generated by f. Then P is a logconcave measure [38, 39].

Theorem: If $f(x, y)$ is a logconcave function of $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, then

$$\int_{\mathbb{R}^m} f(x, y)dy$$

is a logconcave function of $x \in R^n$.

Theorem: The convolution of two logconcave functions in $\mathbb{R}^n$ is also logconcave.

Theorem: Assume that the probability measure P is generated by a logconcave probability density function $f(x), x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is a convex set. Then the following statements hold

(i) $P(A + x)$ is a logconcave function of $x \in \mathbb{R}^n$.

(ii) The probability distribution function $F(x) = \int_{t \leq x} f(t)dt,\ x \in \mathbb{R}^n$ is logconcave.

(iii) If $n = 1$, then $1 - F(x)$ is logconcave.

Definition: Logconcavity of f on (a, b) is equivalent to each of the following conditions:

(i) $\dfrac{f'(x)}{f(x)}$ is monotone decreasing on (a, b).

(ii) $(\ln f(x))'' < 0$.

Definition: A probability $f(x)$ is said to be *unimodal* with mode m if f is convex on $(-\infty, m)$ and concave on (m, ∞). We say f is unimodal about m. $f(x)$ is *strongly unimodal* if f is unimodal and the convolution of f with any unimodal $g(y)$ is unimodal.

Note that the above notion of the strong unimodality is different from Barndorff-Nielsen's strong unimodality definition for discrete multivariate distributions (see Chapter 1 and [7]).

Proposition: Let x be a random variable whose density function $f(x)$ is logconcave. Then for $a \neq 0$, the random variable $y = ax + b$ is logconcave.

A.4 Quasiconvexity, Quasiconcavity, and Pseudoconvexity

Definition: Let f be a function defined on a convex subset of A of the space $\mathbb{R}^n$. f is said to be *quasiconvex* if for all $x, y \in A$ and $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

i.e., if the value of the function f at any point that is directly between two other points is not higher than the value of f at both points.

The function f is said to be *quasiconcave* if

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}.$$

f is called strictly quasiconvex if for all $x \neq y \in A$, and $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}.$$

f is quasiconcave if and only if $-f$ is quasiconvex. f is strictly quasiconcave if and only if $-f$ is strictly quasiconvex.

Theorem: Let f be a function defined on a convex subset A of the space $\mathbb{R}^n$. Then the following are equivalent:

(i) f is quasiconvex.

(ii) For all $x, y \in A$ and $\lambda \in (0,1)$,

$$f(x) \geq f(y) \Rightarrow f(x) \geq f(\lambda x + (1-\lambda)y).$$

(iii) For all $x, y \in A$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}.$$

Theorem: Let f be a function defined on a convex subset A of $\mathbb{R}^n$. Then f is *quasiconvex* if and only if one of the following is satisfied

(i) The level set $A_\alpha = \{x \in A : f(x) \leq \alpha\}$ is convex for every $\alpha \in \mathbb{R}$.

(ii) The function $g(y) = f(\lambda x + (1-\lambda)y)$ is qusiconvex for any $x, y \in A$ and $\lambda \in [0, 1]$.

Proposition: If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ is nondecreasing function. Then $h = g \circ f$ is quasiconvex.

Theorem: Let f be a function defined on a convex $A \subseteq \mathbb{R}^n$. Then

(i) If f is convex (concave) on A, then it is also quasiconvex (quasiconcave) on A.

(ii) if f is strictly convex (strictly concave) on A, then it is also strictly quasiconvex (strictly quasiconcave) on A.

(iii) if f is strictly quasiconvex (strictly quasiconcave) on A, then then it is also quasiconvex (quasiconcave) on A.

Theorem: Let f be a quasiconvex function on a subset A of $\mathbb{R}^n$. Then any local minimum of f is a global minimum.

Theorem: Let f be a quasiconvex function defined on a convex set $A \subseteq \mathbb{R}^n$ and let A^* be the set of all global minimum points of f. Then A^* is convex.

Theorem: Suppose that f is a ho,ogeneous function of degree $k \geq 1$ defined on a convex subset of $\mathbb{R}^n$ and $f(x) > 0$ for all $x \in A$. Then f is quasiconvex of and only if it is convex.

Definition: A function f defined on a convex set $A \subseteq \mathbb{R}^n$ is said to be *semistrictly quasiconvex* if for all $x, y \in A$ with $f(x) \neq f(y)$, and for all $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}.$$

If f is strictly quasiconvex function on $A \subseteq \mathbb{R}^n$, then it is semistrictly quasiconvex on A. If f is strictly convex function on $A \subseteq \mathbb{R}^n$, then it is semistrictly quasiconvex on A.

Theorem: Suppose that f and g are functions defined on a subset A of the space $\mathbb{R}^n$ and

$$h(x) = \frac{f(x)}{g(x)}$$

then, the following statements hold:

(i) If f is non-negative and convex, and g is positive and concave, then h is semistrictly quasiconvex.

(ii) If f is non-positive and convex, and g is positive and convex, then h is semistrictly quasiconvex

(iii) If f is convex, and g is positive and affine, then h is semistrictly quasiconvex.

A.4.1 Differentiable Quasiconvex Functions

Definition: A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ defined on a convex subset $A \subseteq \mathbb{R}^n$ is *quasiconvex* if and only if, for all $x, y \in A$, we have

$$f(y) \leq f(x) \Rightarrow \nabla f(x)^T (y - x) \leq 0$$

A twice differentiable function defined on $A \subseteq \mathbb{R}^n$ is quasiconvex if for all $x \in A$ and $y \in \mathbb{R}^n$ we have

$$y^T \nabla f(x) = 0 \Rightarrow y^T \nabla^2 f(x) y \geq 0.$$

Definition: Let f be a differentiable function defined on a convex subset $A \subseteq \mathbb{R}^n$. Then f is said to be *pseudoconvex* if for all $x, y \in A$ with $\nabla f(x)(y - x) \geq 0$, then $f(y) \geq f(x)$.

The function f is strictly pseudoconvex if for all $x, y \in A$ with $x \neq y$ such that $\nabla f(x)(y - x) \geq 0$, then $f(y) > f(x)$. The function f is (strictly) pseudoconcave if $-f$ is pseudoconvex.

A.5 Discrete Convexity

Below we include a few discrete convexity notions. For more details the reader is referred to [29]-[34].

A.5.1 L$^\natural$-convex functions

We first note that a convex function f on $\mathbb{R}^n$ satisfies the following inequality

$$f(u) + f(v) \geq f\left(\frac{u+v}{2}\right)\left(\frac{u+v}{2}\right) \tag{A.1}$$

where $u, v \in \mathbb{R}^n$.

For a discrete function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, the midpoint of $\frac{u+v}{2}$ of two integer vectors u and v may not be integer. Instead A.1 can be simulated by discrete midpoint convexity as follows

$$f(u) + f(v) \geq f\left(\lceil\frac{u+v}{2}\rceil\right)\left(\lfloor\frac{u+v}{2}\rfloor\right) \tag{A.2}$$

where $u, v \in \mathbb{Z}^n$ and for $x \in \mathbb{R}$ in general, $\lceil x \rceil$ denotes the smallest integer that is larger than$x$ and $\lfloor x \rfloor$ the largest integer that is smaller than x. This can be extended to a vector by componentwise applications.

Definition: The function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *$L^\natural$-convex* if it satisfies (A.2).

Definition: A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to *convex-extensible* if there exists a convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $g(x) = f(x)$ for all $x \in \mathbb{Z}^n$.

Theorem: Any L$^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is convex extensible.

Definition: A function f is said to be *L-convex* function if it is L$^\natural$-convex function that satisfies

$$f(u + \mathbf{1}) = f(u) + r \tag{A.3}$$

where $\mathbf{1} = (1, 1, \cdots, 1), r \in \mathbb{R}$ and independent of u.

A.5.2 M$^\natural$-convex functions

Another kind of discrete convexity is called M-convexity. It is defined through discretization of midpoint convexity just as L-convexity.

We note that a convex function on $\mathbb{R}^n$ satisfies

$$f(x) + f(y) \geq f(x - \lambda(x - y)) + f(y + \lambda(x - y)) \tag{A.4}$$

for every $0 \leq \lambda \leq 1$. This property is called *equidistance convexity.*

In the discrete case, equidistance convexity can be simulated by moving a pair of points (x, y) to another pair $(x\prime, y\prime)$ as follows:

$$(x\prime, y\prime) = (x - \chi_i + \chi_j, y + \chi_i - \chi_j)$$

with indices i and $j$such that $x_i > y_i$ and $x_j < y_j$.

Define the positive and negative supports of $z \in \mathbb{R}^n$ as

$$supp^+(z) = \{i \mid z_i > 0\}, \quad supp^-(z) = \{j \mid z_j < 0\}$$

Definition: A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *M*$^\natural$*-convex* if for any $x, y \in dom_{\mathbb{Z}} f$ and any $i \in supp^+(x-y)$, there exists $j \in supp^-(x-y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j). \tag{A.5}$$

Theorem: An M$^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is convex-extensible.

Definition: A function f is said to be *M-convex function* if it is M$^\natural$-convex that satisfies (A.5) with $j \in supp^-(x-y)$.

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