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by

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ABSTRACT

Title:

Analysis of Interfaces for the Nonlinear Degenerate Second Order Parabolic Equations Modeling Diffusion-Convection Processes

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Dissertation pursues analysis of the short-time evolution of interfaces or free boundaries for the non-negative solutions of the nonlinear degenerate second order parabolic partial differential equation (PDE)

$$u_t = (u^m)_{xx} + b(u^\gamma)_x, x \in \mathbb{R}, t > 0;\ m > 1, \gamma > 0, b \in \mathbb{R} \tag{1}$$

modeling diffusion-convection processes arising in fluid or gas flow in a porous media, plasma physics, population dynamics in mathematical biology and other applications. Due to the implicit degeneration ($m > 1$), PDE (1) it possesses a property of the finite speed of propagation and develops interfaces or free boundaries separating the region where a solution is positive from the region where it vanishes. The direction of the movement of interfaces and their asymptotic properties depends on the relative strength of the diffusion and convection terms (m vs. γ), direction of the movement of the convection (*sign b*), asymptotics of the initial function near its support, and whether left or right interface curve is under consideration. Classification of the direction of the movement of the interfaces is presented by *Alvarez, Diaz & Kersner, 1986*. Dissertation presents a classification of the short-time asymptotics of the interfaces and local solutions near the

interfaces depending on the parameters of the model. Proof methods are based on the general theory of the nonlinear reaction-diffusion equations in non-cylindrical and non-smooth domains (*Abdulla, J. Diff. Eq., 164, 2, 2000*), scaling laws for the identification of the asymptotics of solutions along with construction of local barriers using special comparison theorems as it is developed in *Abdulla & King, SIAM J. Math. Anal., 32, 2, 2000; Abdulla, Nonlinear Analysis, 50, 4, 2002*.

Table of Contents

List of Figures

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Finally, my appreciation extends to my fellow graduate students and faculty at the Florida Institute of Technology. I would also like to thank the Ministry of Higher Education and Scientific Research in Iraq for granting me a scholarship to pursue my goals.

Dedication

I dedicate this dissertation to beloved people who have meant and continue to mean extremely much to me. First, I am grateful to my wonderful father Kadhim, who loved me exceedingly. Although he is no longer alive, I will never forget him, and his memories will continue to guide me. Next, I would like to thank my mother, the role model in my life, for raising me and teaching me the value of hard work. Last, I appreciate my sisters and brothers for their spiritual support throughout my academic process. Their advice and encouragement have enabled me to achieve my goals and to be successful.

Chapter 1

Introduction

1.1 Diffusion-Convection Equation in Physical Applications

Diffusion-Convection equation is a widely used model in many applications in fluid and gas mechanics, mathematical biology, plasma physics and other fields. For example, consider the flow of gas in a porous medium in

$$\mathbb{R}^{N+1}_+ = \{(x,t) : x \in \mathbb{R}^N,\ 0 < t < +\infty\},$$

where x is a space variable, and t is a time variable. In physical applications usually $N \leq 3$. Let

$$u, p : \mathbb{R}^{N+1}_+ \to \mathbb{R}_+,\ \mathbf{v} : \mathbb{R}^{N+1}_+ \to \mathbb{R}^N$$

are respectively density, pressure and velocity of the gas. The following physical laws hold for the triple $(u, p, \mathbf{v})$ [39, 40]:

- A conservation of mass equation

$$f u_t + \operatorname{div}(u\mathbf{v}) = 0 \tag{1.1}$$

  where $f \in (0, 1]$ is an average fraction of pores in a fixed volume, where actual flow occurs.

- The Darcy's law

$$\mathbf{v} = -\frac{\kappa}{\mu}\nabla p, \tag{1.2}$$

  where physical constants $\kappa, \mu \in \mathbb{R}^+$ characterize permeability of the media and viscosity of the gas respectively.

- The phenomenological state equation which expresses relation between the density of the gas and a pressure

$$u = u_0 p^\gamma, \tag{1.3}$$

  where u_0 is a constant density under unit pressure, and experimentally identified constant γ satisfies $0 < \gamma \leq 1$.

Combining (1.1)-(1.3), after rescaling a time variable we derive the nonlinear diffusion equation for the density u:

$$u_t = \Delta u^m \tag{1.4}$$

where $m = 1 + \frac{1}{\gamma} \geq 1$. In the presence of the nonlinear convective flows, PDE (1.4) is replaced with the following nonlinear diffusion-convection equation

$$u_t = \Delta u^m + \mathbf{b} \cdot \nabla u^\gamma \tag{1.5}$$

where $\mathbf{b} \in \mathbb{R}^N$ and $\gamma > 0$ are physical parameters characterizing the spacial direction and strength of the convective force. PDE (1.4) (or (1.5)) is a second-order nonlinear degenerate parabolic equation with implicit degeneration. Equation loses uniform parabolicity on the zero-level set of the solution, which has a crucial effect on the smoothness and qualitative properties of the solution. Both phenomena can be seen in a typical particular example of the instantaneous point source type solution of the nonlinear diffusion equation (1.4) constructed in [39, 40, 111]. Instantaneous point-source problem is the Cauchy problem for (1.4) in $\mathbb{R}^{N+1}_+$ under the conditions

$$u(x,0) = \delta(x),\ x \in \mathbb{R}^N;\ \int_{\mathbb{R}^N} u(x,t)dx = 1,\ t > 0,$$

where $\delta(\cdot)$ is Dirac's point mass with support at the origin. The solution of this problem is ([39, 111]):

$$u_*(x,t) = t^{-\frac{N}{2+N(m-1)}}\left[\frac{m-1}{2m(2+N(m-1))}\left(\eta_0^2 - |x|^2 t^{-\frac{2}{2+N(m-1)}}\right)_+\right]^{\frac{1}{m-1}}$$

where $(\chi)_+ = \{\chi, \quad if\ \chi > 0;\ 0, \quad if\ \chi \le 0\}$. Barenblatt's solution demonstrate the vital "finite speed of propagation property" similar to linear wave equation: for $\forall t \ge 0$ support of the solution is compact

$$spt\ u := \overline{\{(x,t) : u(x,t) > 0\}} = \{|x| \le \eta_0 t^{\frac{1}{2+N\sigma}}\}.$$

It is well-known that the linear diffusion equation has an infinite speed of propagation property. This contrast is remarkable, for it demonstrates that the nonlinear degenerate parabolic PDEs are more relevant for real-world applications than their linear predecessors. The second remarkable feature of Barenblatt's solution is that it is not a classical solution with smooth derivatives. Second-order x-derivates and first-order t-derivatives doesn't exist along the boundary surfaces of the support, called interfaces or free boundaries. Described two features of the Barenblatt's solution became the driving force for the development of the fascinating theory of the second-order degenerate parabolic equations with many important applications in science and engineering.

1.2 Review of the Theory of the Second Order Degenerate Parabolic Equations

The theory of nonlinear degenerate parabolic equations began with the paper [99] on the porous medium equation ((1.4) with $n = 1$), where existence and uniqueness of weak solutions and comparison theorems are proved for the main boundary-value problems and Cauchy problem for the one-dimensional porous medium equation. To clarify the concept of the weak solution, consider a Dirichlet problem for the nonlinear PDE (1.4) in a cylinder $Q = D \times (0,T]$, where $D \subset \mathbb{R}^N$ be open domain, under the initial-boundary conditions

$$u(x,0) = u_0(x), \ x \in D \tag{1.6}$$

$$u(x,t) = 0, \ (x,t) \in S = \partial D \times (0,T) \tag{1.7}$$

Definition 1.2.1. We say that a non-negative function $u = u(x,t)$ is a weak solution of the Dirichlet problem (1.4),(1.6),(1.7) if

- $u^m \in L_2\Big(0,T;H_0^1(D)\Big)$

- u satisfies the integral identity

$$\iint\limits_{Q_T} (\nabla u^m \cdot \nabla \phi - u\phi_t) dxdt = \int_D u_0(x)\phi(x,0)dx$$

for any $\phi \in C^1(\overline{Q_T})$ satisfying $\phi(x,T) = \phi|_{S_T} = 0$, where,

$$\mathrm{L}_2(0,\mathrm{T}; H_0^1(D)) = \{u = u(t) : [0,T] \to H_0^1(D)\}$$

is an Hilbert space with the norm

$$||u||_{\mathrm{L}_2\left(0,\mathrm{T}; H_0^1(D)\right)} = \Big(\int_0^T ||u||^2_{H_0^1(D)} dt\Big)^{1/2} = \Big(\int_0^T \int_D (|u|^2 + |\nabla u|^2) dxdt\Big)^{1/2}$$

In fact, the instantaneous point source solution is weak in the sense of the Definition 1.2.1. Existence and uniqueness of the weak solution, and comparison theorem for the weak solution of the problem (1.4),(1.6),(1.7) was proved in [36, 37]. Question about the interior continuity of the weak solutions was proved in [49]. Hölder regularity of the weak solutions for the general second-order nonlinear degenerate parabolic equations is proved in [58, 59]. Currently, there is a well established general theory of the nonlinear degenerate parabolic equations [1]-[111]. The general theory of nonlinear degenerate second-order parabolic PDEs in non-cylindrical non-smooth domains was developed in [2, 3, 5, 8, 9]. We refer to [109] for a complete list of references on the general theory of nonlinear diffusion type degenerate parabolic equations, and to monography [59] for interior regularity properties of the weak solutions to the general second-order nonlinear degenerate parabolic equations.

As to the qualitative theory of the nonlinear degenerate parabolic equations, there are many fundamental open problems. The goal of the dissertation is to analyze the open problem about the evolution of the interfaces for a class of diffusion-convection equations (1.5) in space dimension $N = 1$.

1.3 Statement of the open problem

Consider the Cauchy problem (CP)

$$Lu \equiv u_t - (u^m)_{xx} - b(u^\gamma)_x = 0, \; x \in \mathbb{R}, 0 < t < T, \tag{1.8}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.9}$$

where $m > 1$, $b \in R, b \neq 0$, $\gamma > 0$, $0 < T \leq +\infty$ and u_0 is non-negative, continuous and compactly supported in $\mathbb{R}$. Define interface curves as

$$\eta(t) = \sup\,\{x : u(x,t) > 0\}, \quad \xi(t) = \inf\,\{x : u(x,t) > 0\}$$

The goal of dissertation is to analyze the short-time asymptotics of the interface curves $\eta(t), \xi(t)$, as well as local structure of solution near the interfaces depending on the parameters m, γ, b and the behaviour of the initial function near its support. Note that by changing the variable $x \to -x$, our problem is replaced with the same problem with b replaced with $-b$, and interface curves $\eta(t)$ and $\xi(t)$ are replaced with $-\xi(t)$ and $-\eta(t)$ respectively. Therefore, it is satisfactory to solve the problem for the right interface curve $\eta(t)$ for the full range of b. Due to invariance of (1.8) with respect to translation, without loss of generality we will assume that $\eta(0) = 0$. We shall assume that

$$u_0 \sim C(-x)_+^\alpha \;\text{ as } x \to 0- \quad \text{for some} \quad C > 0, \; \alpha > 0. \tag{1.10}$$

Since the main results are local in nature, without loss of generality we may suppose that u_0 either is bounded or satisfies some restriction on its growth rate as $x \to -\infty$ which is suitable for existence, uniqueness, and comparison results. The special global case

$$u_0(x) = C(-x)_+^{\alpha}, \quad x \in \mathbb{R}, \tag{1.11}$$

will be considered when the solution to the problem (1.8), (1.11) is of self-similar form. Our estimations are global in time in these special cases.

It should be noted that the nature of the problem near $\eta(t)$ is significantly different depending on *sign b*. If $b > 0$, then convection force is directed in $-x$ direction, or opposite to the diffusion force, while if $b < 0$, then convection force is directed in x direction, which is alongside with the diffusion force.

Initial development of interfaces and structure of local solution near the interfaces is very well understood in the case of the reaction-diffusion equations

$$u_t - (u^m)_{xx} + cu^{\beta} = 0 \; x \in \mathbb{R}, 0 < t < T, \tag{1.12}$$

Full classification of the evolution of interfaces and the local behavior of solutions near the interfaces for the reaction-diffusion equations (1.12) was presented in [13] for the case of slow diffusion ($m > 1$) case, and in [6] for the fast diffusion ($0 < m < 1$) case. As it is mentioned in [13] "the major obstacle in solving interface development problem for nonlinear degenerate parabolic equations is a problem of non-uniform asymptotic in the sense of singular perturbations theory, namely that the dominant balance as $t \to 0^+$ between the terms in (1.12) on curves which approach the boundary of the support on the initial line depending on how they do so". The general theory, including existence, boundary regularity, uniqueness and comparison theorems, for the reaction-diffusion equations of type (1.12) in general non-cylindrical and non-smooth domains is developed in [3] in one-dimensional case, and in [5, 8, 9] in the multi-dimensional case. Comparison theorems proved in [3] were essential tool in developing rigorous proof method in [6, 13] for solving interface problem for the reaction-diffusion equation (1.12). The rigorous proof method developed in [6, 13] is based on a barrier technique using special comparison theorems in irregular domains with characteristic boundary curves. Recently, the methods of [3, 6, 13] are developed and applied to solve the interface problem for the reaction-diffusion equations with p-Laplacian type diffusion term

$$u_t - (|u_x|^{p-1}u_x)_x + cu^\beta = 0, \tag{1.13}$$

where $p > 1, \beta > 0$. The solution of the interface problem for (1.13) is presented in [11] for the slow diffusion case ($p > 2$), and in [12] for the fast diffusion regime ($1 < p < 2$). The methods developed in [3, 6, 13] are developed to solve the interface problem for the reaction-diffusion equation with double-degenerate diffusion

$$u_t - (|(u^m)_x|^{p-1}(u^m)_x)_x + cu^\beta = 0. \tag{1.14}$$

The solution of the interface problem for (1.14) is presented in [10] for the slow diffusion case ($mp > 1$), and in [14] for the fast diffusion regime ($1 < mp < 2$).

The interface problem for the diffusion-convection equation (1.5) was investigated in many papers since 1960s (see [27, 28, 31, 63, 64, 65, 66, 67, 69, 80, 82, 84, 85, 86] etc.) The most complete results are presented in [27, 28]. The authors presented the classification of the direction of movement of the interfaces depending on the parameters $m, \gamma, \beta, b, C, \alpha$. As to precise asymptotics of the interfaces, and estimations on the local solutions near the interfaces, the results of [27, 28] are not sharp. The dissertation aims is to develop and applied the methods of papers [3, 6, 13] to prove sharp estimations of the interfaces and local solutions near the interfaces.

In Chapter 2 we present classification of the asymptotic properties of $\eta(t)$ and local solution near it in the case when diffusion and convection forces are opposing each other ($b > 0$). In Chapter 3 we present classification of the asymptotic properties of $\eta(t)$ and local solution near it in the case when convection force is acting alongside diffusion force in x-direction ($b < 0$).

Chapter 2

Classification of Interfaces and Local Solutions for the Nonlinear Degenerate Diffusion-Convection Equations: Convection versus Diffusion

In this chapter, we present a classification of the short-time behavior of the interfaces and local solutions near the interfaces in Cauchy problem (1.8)-(1.10) when $b > 0$. Throughout this chapter, we are going to assume that u is a weak solution of the problem (1.8)-(1.10). It will be specifically mentioned when u is a global solution to the problem (1.8),(1.11).

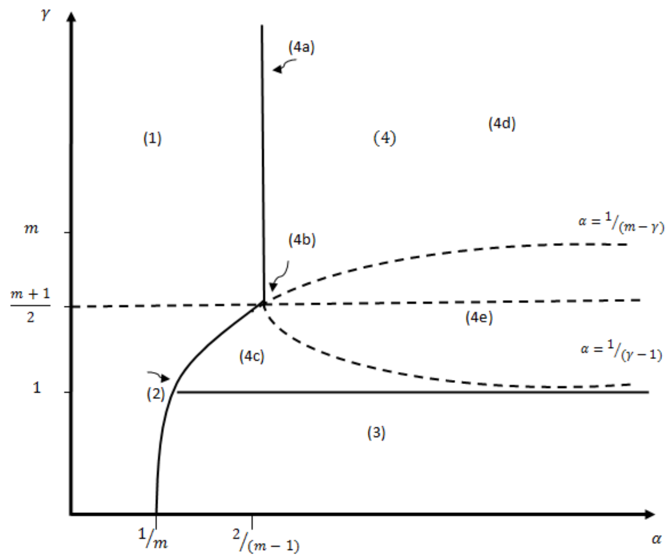

Figure 2.1: Classification of different cases in the (α,γ) plane for interface development in problem (1.8)-(1.10) when $b > 0$

2.1 Description of Main Results

Diagram in Figure 2.1 presents a classification of different cases in the (α,γ) plane for the initial interface development in problem (1.8)-(1.10) if $b > 0$.

- **Region (1):** $\alpha < 1/(m - \min\{\gamma, (m+1)/2\})$; Diffusion strongly dominates over the convection and interface expands.

- **Region (2):** $\alpha = 1/(m-\gamma), \gamma < (m+1)/2$; Diffusion and convection are in balance in this borderline case. There is a critical constant C_* such that the interface expands for $C > C_*$, shrinks for $C < C_*, \gamma \leq 1$ and has a waiting time for $C < C_*, \gamma > 1$.

- **Region (3):** $\alpha > 1/(m-\gamma), \gamma < 1$; Convection strongly dominates over the diffusion and interface shrinks.

- **Region (4):** $1 \leq \gamma < (m+1)/2,\ \alpha > 1/(m-\gamma)$ or $\gamma \geq (m+1)/2,\ \alpha \geq 2/(m-1)$; Both diffusion and convection are weak, and interface has initial ”waiting time”.

The following are the main results on the asymptotic properties of the interface curve $\eta(t)$, and local solution near the interface in respective regions (1)-(4).

Region (1)

Theorem 2.1.1. *Consider* $\alpha < 1/(m - min\{\gamma,(m+1)/2\})$ *Then interface initially expands and*

$$\eta(t) \sim \xi_* t^{1/(2-\alpha(m-1))} \ \ as\ t \to 0+, \tag{2.1}$$

where $\xi_* = \xi_*(m,C,\alpha) > 0.$ *For arbitrary* $\rho > \xi_*$ *there exists* $f(\rho) > 0$ *depending on* C, m *and* α *such that*

$$u(x,t) \sim f(\rho) t^{(\alpha/2-\alpha(m-1))} \quad as \quad t \to 0+ \tag{2.2}$$

along the curve $x = \xi_\rho(t) = \rho t^{1/(2-\alpha(m-1))}$.

Further Details of Theorem 2.1.1: Theorem 2.1.1 proves that the diffusion strongly dominates over the convection, and local behaviour of the interface $\eta(t)$ and local solution coincides with that of the Cauchy problem (1.8)-(1.10) with $b = 0$. The latter is fully described in [13]. Precise values of the constant ξ_* and the function f are associated with the one-dimensional Cauchy Problem (1.8),(1.11), which has a unique solution of self-similar form

$$w(x,t) = t^{\alpha/(2+\alpha(1-m))} f(\xi), \text{ where } \xi = \frac{x}{t^{1/(\alpha(1-m)+2)}}, \tag{2.3}$$

and the shape function f solves nonlinear ODE problem

$$\begin{cases} \frac{d^2 f^m}{d\xi^2} + \frac{1}{2+\alpha(1-m)}\xi\frac{df}{d\xi} - \frac{\alpha}{2+\alpha(1-m)}f = 0, \quad \xi \in \mathbb{R} \\ f \sim C(-\xi)^\alpha \qquad as\ \xi \to -\infty,\ f(\xi) = o(|\xi|^\alpha)\ \ as \quad \xi \downarrow +\infty, \end{cases} \tag{2.4}$$

with finite interface $\xi_* = \xi_*(C,\alpha,m) > 0$ such that

$$f(\xi) > 0,\ -\infty < \xi < \xi_*;\ f(\xi) \equiv 0,\ \xi \geq \xi_* \tag{2.5}$$

Through rescaling one can find dependence of f and ξ_* on C [13]:

$$f(\rho) = C^{\frac{2}{2-\alpha(m-1)}} f_1(C^{\frac{m-1}{2-\alpha(m-1)}}\rho) \tag{2.6}$$

$$f_1(\rho) = w_1(\rho,1),\ \xi_*^{'} = inf\{x : f_1(\rho) > 0\} < 0 \tag{2.7}$$

$$\xi_* = C^{\frac{m-1}{2-\alpha(m-1)}}\xi_*^{'} \tag{2.8}$$

where w_1 and f_1 are solutions of (1.8),(1.11), and (2.4), respectively, with the constant $C = 1$; $\xi_*^{'}$ is a positive number depending on m and α only.

Region (2)

Theorem 2.1.2. *Let* $\gamma < (m+1)/2,\ \alpha = 1/(m-\gamma)$, *and*

$$C_* \equiv \Big[\frac{b(m-\gamma)}{m}\Big]^{\frac{1}{m-\gamma}} \tag{2.9}$$

Then interface initially

(1) expands if $C > C_*$

(2) shrinks if $\gamma \le 1$, $C < C_*$

(3) has a waiting time if $\gamma > 1$, $C < C_*$

In cases (1),(2) *we have*

$$\eta(t) \sim \zeta_* t^{\frac{m-\gamma}{m+1-2\gamma}} \quad as \;\; t \to 0+ \tag{2.10}$$

where $\zeta_* \lesseqgtr 0$ if $C \lesseqgtr C_*$, and for arbitrary $\rho < \zeta_*$ there exists $h(\rho) > 0$ such that

$$u(\zeta_\rho(t), t) \sim t^{\frac{1}{m+1-2\gamma}} h(\rho) \text{ as } t \downarrow 0, \text{ along } \; \zeta_\rho(t) := \rho t^{\frac{m-\gamma}{m+1-2\gamma}} \tag{2.11}$$

For case (3), (2.11) *is true for any $\rho < 0$ and for $\forall \epsilon > 0$ $\exists$ $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that*

$$C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{\gamma-1}} \le u(x,t) \le (C+\epsilon)\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{m-\gamma}}, \quad x_\epsilon \le x < +\infty, \quad 0 \le t \le \delta_\epsilon$$

Further Details of Theorem 2.1.2: Precise values of the constant ζ_* and the function h are associated with the one-dimensional Cauchy Problem (1.8),(1.11), which has a unique solution of self-similar form

$$u(x,t) = t^{\frac{1}{m+1-2\gamma}} h(\zeta), \text{ where } \zeta = \frac{x}{t^{\frac{m-\gamma}{m+1-2\gamma}}}, \tag{2.12}$$

and the shape function h solves nonlinear ODE problem

$$\begin{cases} \frac{d^2h^m}{d\zeta^2} + b\frac{dh^\gamma}{d\zeta} + \frac{m-\gamma}{m+1-2\gamma}\zeta\frac{dh}{d\zeta} - \frac{1}{m+1-2\gamma}h = 0, \quad \zeta \in \mathbb{R} \\ h \sim C(-\zeta)^{\frac{1}{m-\gamma}} \qquad as\; \zeta \to -\infty, \; h(\zeta) = o(|\zeta|^{\frac{1}{m-\gamma}}) \;\; as \quad \zeta \downarrow +\infty, \end{cases} \tag{2.13}$$

with finite interface $\zeta_* = \zeta_*(m, \gamma, C, b)$ such that

$$h(\zeta) > 0,\ -\infty < \zeta < \zeta_*;\ h(\zeta) \equiv 0,\ \zeta \geq \zeta_* \tag{2.14}$$

$$\eta(t) = \zeta_* t^{\frac{m-\gamma}{m+1-2\gamma}},\ 0 \leq t < +\infty \tag{2.15}$$

If $\gamma = 1$ then the problem has an explicit solution

$$u = C\,(-\zeta_* t - x)_+^{\frac{1}{m-1}} \tag{2.16}$$

$$h(\zeta) = C(-\zeta_* - \zeta)_*^{\frac{1}{m-1}} \tag{2.17}$$

$$\zeta_* = b\gamma C^{\gamma-1}\left[1 - \left(\frac{C}{C_*}\right)^{m-\gamma}\right],$$

$$\eta(t) = \zeta_* t,\ 0 \leq t < +\infty.$$

If $C = C_*$, then the initial function presents explicit stationary solution

$$u(x,t) = C_*(-x)_+^{\frac{1}{m-\gamma}},\ h(\zeta) = C_*(-\zeta)_*^{\frac{1}{m-\gamma}}$$

In general we have

- $\zeta_* > 0$, if $C > C_*$
- $\zeta_* < 0$, if $C < C_*,\ \gamma \leq 1$
- $\zeta_* = 0$, if $C < C_*,\ \gamma > 1$

In particular, if $C > C_*$ we have

$$u(0,t) = A_1 t^{\frac{1}{m+1-2\gamma}},\ 0 \leq t \leq +\infty, \tag{2.18}$$

$$A_1 = A_1(m, \gamma, C, b) := h(0) > 0 \tag{2.19}$$

If $C > C_*$ and $\gamma < 1$, then we have the following global estimate for the solution of (1.8),(1.11):

$$C_1(\lambda_1 - \zeta)_+^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}} \le u(x,t) \le C_*(\lambda_2 - \zeta)_+^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \le x, t \le +\infty, \tag{2.20}$$

$$\lambda_2 = \Big(\frac{A_1}{C_*}\Big)^{m-\gamma},\ \lambda_1 = \bar{z}^{-1} A_1^{m-\gamma},\ C_1 = \bar{z}^{\frac{1}{m-\gamma}}, \tag{2.21}$$

where $\bar{z}$ is a positive root of the quadratic equation

$$z^2 - C_*^{m-\gamma} z - \frac{(m-\gamma)^2 A_1^{1+m-2\gamma}}{m\gamma(m+1-2\gamma)} = 0 \tag{2.22}$$

If $C > C_*$ and $\gamma > 1$, then we have the following global estimate for the solution of (1.8),(1.11):

$$C_1(\lambda_1 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}} \le u(x,t) \le C_2(\lambda_2 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \le x, t \le +\infty, \tag{2.23}$$

$$\lambda_2 = \Big(\frac{A_1}{C_2}\Big)^{m-1},\ C_2 = \Big(\frac{(m-\gamma)(m-1)}{m(m+1-2\gamma)}\Big)^{\frac{1}{2(m-1)}} A_1^{\frac{1}{2}},\ \lambda_1 = \bar{z}^{-1} A_1^{m-1},\ C_1 = \bar{z}^{\frac{1}{m-1}}, \tag{2.24}$$

where $\bar{z}$ is a positive root of the quadratic equation

$$z^2 - b\gamma m^{-1}(m-1) A_1^{\gamma-1} z - \frac{(m-1)^2 A_1^{m-1}}{m(m+1-2\gamma)} = 0 \tag{2.25}$$

Both estimations (2.20) and (2.23) imply that if $C > C_*$, then we have

$$\lambda_1 \le \zeta_* \le \lambda_2, \tag{2.26}$$

If $\gamma = 1$, then upper and lower bounds in both estimations (2.20) and (2.23) coincide with the explicit solution (2.16).

If $C < C_*$ and $\gamma < 1$, then we have the following estimate for the solution of (1.8),(1.11):

$$C_*\left(-\zeta_3 t^{\frac{m-\gamma}{m+1-2\gamma}} - x\right)_+^{\frac{1}{m-\gamma}} \leq u(x,t) \leq C_4\left(-\zeta_4 t^{\frac{m-\gamma}{m+1-2\gamma}} - x\right)_+^{\frac{1}{m-\gamma}}, \quad 0 \leq t < +\infty, \tag{2.27}$$

where the left-hand side is valid for $x \geq -\ell_0 t^{\frac{m-\gamma}{m+1-2\gamma}}$, while the right-hand side is valid for $x \geq -\ell_1 t^{\frac{m-\gamma}{m+1-2\gamma}}$, and

$$C_4 = C\left(\frac{1}{1-\delta_*\Gamma}\right)^{\frac{1}{m-\gamma}}, \ \ \zeta_4 = \delta_*\ell_1\Gamma, \ \ \Gamma = 1 - \left(\frac{C}{C_*}\right)^{m-\gamma} \tag{2.28}$$

$$\ell_1 = \left(\frac{b\gamma(m+1-2\gamma)\left(1-\left(\frac{1}{1-\delta_*\Gamma}\right)\left(\frac{C}{C_*}\right)^{m-\gamma}\right)}{\delta_*\Gamma(m-\gamma)C^{1-\gamma}}\right)^{\frac{m-\gamma}{m+1-2\gamma}} \tag{2.29}$$

$$g(\delta_*) = \max_{0\leq\delta\leq 1} g(\delta), \quad g(\delta) = \delta^{\frac{1-\gamma}{m-\gamma}}\left(1-\left(\frac{C}{C_*}\right)^{m-\gamma}\left(\frac{1}{1-\delta\Gamma}\right)\right) \tag{2.30}$$

$$\zeta_3 = -\zeta_* + \bar{\epsilon} - C_*^{\gamma-m} h^{m-\gamma}(\zeta_* - \bar{\epsilon}), \ \ \ell_0 = -\zeta_* + \bar{\epsilon} \tag{2.31}$$

where $0 < \bar{\epsilon} << 1$ is chosen sufficiently small to satisfy $\zeta_3 > 0$. In particular, (2.27) implies that if $C < C_*$ and $\gamma < 1$ then we have

$$\zeta_* \leq -\zeta_4 \tag{2.32}$$

Region (3)

Theorem 2.1.3. *Let $\gamma < 1$ and $\alpha > 1/(m-\gamma)$, then interface shrinks and*

$$\eta(t) \sim -\delta_* t^{1/(1+\alpha(1-\gamma))} \ \textit{as}\ t \to 0+ \tag{2.33}$$

where

$$\delta_* = C^{\frac{(\gamma-1)}{1+\alpha(1-\gamma)}}(b\gamma)^{\frac{1}{1+\alpha(1-\gamma)}}\left(\left[(1-\gamma)\alpha\right]^{\frac{1}{1+\alpha(1-\gamma)}} + \left[(1-\gamma)\alpha\right]^{\frac{\alpha(\gamma-1)}{1+\alpha(1-\gamma)}}\right) \tag{2.34}$$

For arbitrary $\delta \geq \delta_*$, *we have*

$$u\Big|_{x=\eta_\delta(t)} \sim C_\delta t^{\alpha/(1+\alpha(1-\gamma))}, \text{ as } t \to 0+ \tag{2.35}$$

where $\eta_\delta(t) = -\delta t^{1/(1+\alpha(1-\gamma))}$ *and*

$$C_{\delta_*} = [b\gamma(1-\gamma)\alpha C^{\frac{1}{\alpha}}]^{\frac{\alpha}{1+\alpha(1-\gamma)}} \tag{2.36}$$

Region (4)

Theorem 2.1.4. *If either* $1 \leq \gamma < \frac{m+1}{2}$ *and* $\alpha > \frac{1}{m-\gamma}$ *or* $\gamma \geq \frac{m+1}{2}$ *and* $\alpha \geq \frac{2}{m-1}$, *then interface has an initial waiting time.*

Further Details of Theorem 2.1.4: There are five different subcases in this case outlined as subregions (4a)-(4e) in Figure 2.1.

- Case (4a): $\alpha = 2/(m-1)$, $\gamma > (m+1)/2$. For $\forall\epsilon > 0$, $\exists\, x_\epsilon < 0, \delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^{\frac{2}{m-1}} \leq u(x,t) \leq (C+\epsilon)(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C+\epsilon}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}},$$
$$x_\epsilon < x < +\infty, 0 < t \leq \delta_\epsilon. \tag{2.37}$$

where

$$\bar{C} = \Big[\frac{(m-1)^2}{2m(m+1)}\Big]^{\frac{1}{m-1}} \tag{2.38}$$

- Case (4b): $\alpha = 2/(m-1)$, $\gamma = (m+1)/2$. If $C \leq C_*$ then the solution of the problem (1.8),(1.11) satisfies the following global estimation:

$$C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{2}{m-1}} \leq u(x,t) \leq C(-x)_+^{\frac{2}{m-1}},\ x \in \mathbb{R},\ 0 \leq t < +\infty \tag{2.39}$$

If $C > C_*$, then the solution of (1.8),(1.11) satisfies the estimate

$$C(-x)_+^{\frac{2}{m-1}} \le u(x,t) \le C(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}},\ x\in\mathbb{R}, 0\le t<T_c \quad (2.40)$$

$$T_c = \frac{(m-1)C^{1-m}}{2m(m+1)}$$

If u is a solution of the CP (1.8)-(1.10), then for $\forall\epsilon > 0$ $\exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that, if $C < C_*$, then

$$(C-\epsilon)\Big(\frac{-x}{1+(C-\epsilon)^{\gamma-1}b\gamma t}\Big)_+^{\frac{2}{m-1}} \le u(x,t) \le (C+\epsilon)(-x)_+^{\frac{2}{m-1}},$$
$$x_\epsilon \le x < +\infty,\ \ 0 \le t \le \delta_\epsilon, \quad (2.41)$$

while if $C \ge C_*$, then

$$(C-\epsilon)(-x)_+^{\frac{2}{m-1}} \le u(x,t) \le (C+\epsilon)(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}},$$
$$x_\epsilon \le x < +\infty,\ \ 0 \le t \le \delta_\epsilon, \quad (2.42)$$

- Case (4c): $\frac{1}{m-\gamma} < \alpha < \frac{1}{\gamma-1},\ 1<\gamma<\frac{m+1}{2}$. For $\forall\epsilon > 0,\ \exists\ x_\epsilon < 0, \delta_\epsilon > 0$ such that

$$C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{\gamma-1}} \le u(x,t) \le (C+\epsilon)\Big(\frac{-x}{1+(C+\epsilon)^{\gamma-1}b\gamma t}\Big)_+^{\alpha},$$
$$x_\epsilon \le x < +\infty,\ 0 \le t \le \delta_\epsilon. \quad (2.43)$$

Note that when $\alpha = \frac{1}{\gamma-1}$, upper bound and lower bounds in (2.43) match with the explicit solution of the convection equation (PDE (1.8) without diffusion term $(u^m)_{xx}$). In that case local estimation (2.43) holds for the solution of (1.8)-(1.10) with C replaced by $C-\epsilon$ on the left-hand side.

- Case (4d): either $\frac{m+1}{2} < \gamma < m,\ \frac{2}{m-1} < \alpha < \frac{1}{m-\gamma}$, or $\gamma \geq m,\ \alpha > \frac{2}{m-1}$. If u is a solution of the CP (1.8)-(1.10), then for $\forall \epsilon > 0\ \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^\alpha \leq u \leq (C+\epsilon)(-x)_+^\alpha (1-\epsilon t)^{\frac{1}{1-m}},\ x_\epsilon \leq x < +\infty, 0 \leq t \leq \delta_\epsilon. \tag{2.44}$$

The estimation (2.44) holds if $\frac{m+1}{2} < \gamma < m, \alpha = \frac{1}{m-\gamma}$ and $C > C_*$.

- Case (4e): $1 < \gamma < m$ and $\alpha > \max\left(\frac{1}{\gamma-1}; \frac{1}{m-\gamma}\right)$. If u is a solution of the CP (1.8)-(1.10), then for $\forall \epsilon > 0\ \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)\Big(\frac{-x}{1+(C-\epsilon)^{\gamma-1} b\gamma t}\Big)_+^\alpha \leq u(x,t) \leq (C+\epsilon)(-x)_+^\alpha,$$
$$x_\epsilon \leq x < +\infty, 0 \leq t \leq \delta_\epsilon. \tag{2.45}$$

The estimation (2.45) holds if $\frac{m+1}{2} < \gamma < m, \alpha = \frac{1}{m-\gamma}$ and $C < C_*$.

2.2 Preliminary results

Solution of the Cauchy Problem (1.8),(1.9) is understood in the following weak sense.

Definition 2.2.1 (Weak Solution)**.** ([3]) The function $u(x,t)$ is a local weak solution (respectively sub- or supersolution) of (1.12) in $Q_T = \mathbb{R} \times (0,T]$ if

- u is nonnegative and continuous in Q_T, satisfying initial-boundary conditions and $u \in L_\infty(Q_{T_1})$ for any finite $T_1 \in (0,T]$.

- For any finite t_0, t_1 such that $0 \leq t_0 < t_1 \leq T$ and for any C^∞ functions $\mu_i(t), t_0 \leq t \leq$

$t_1, i = 1,2$ for $t \in [t_0; t_1]$ the integral identity

$$\int_{t_0}^{t_1} \int_{\mu_1(t)}^{\mu_2(t)} (u\phi_t + u^m \phi_{xx} - bu^\gamma \phi_x - cu^\beta \phi) dxdt - \int_{\mu_1(t)}^{\mu_2(t)} u\phi \, dt \Big|_{t=t_0}^{t=t_1} - \int_{t_0}^{t_1} (u^m \phi_x) dt \Big|_{x=\mu_1(t)}^{x=\mu_2(t)} = (resp \geq or \leq)\ 0, \quad (2.46)$$

where $\phi \in C^{2,1}_{x,t}(\overline{D})$ is an arbitrary function (respectively nonnegative function) that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1, i = 1,2$, and

$$D = \{(x,t) : \mu_1(t) < x < \mu_2(t), t_0 < t < t_1\}$$

Despite implicit degeneration, nonlinear reaction-diffusion-convection equations inherited order preserving property of classical second order parabolic PDEs. The following is the standard comparison result [3, 59]:

Lemma 2.2.2. *Let g be a non-negative and continuous function in $\overline{Q}$, where:*

$$Q = \{(x,t) : \eta_0(t) < x < +\infty, 0 < t < T \leq +\infty\}$$

$g = g(x,t)$ is in $C^{2,1}_{x,t}$ in Q outside a finite number of curves: $x = \eta_j(t)$, which divide Q into a finite number of subdomains: Q^j, where $\eta_j \in C[0,T]$; for arbitrary $\delta > 0$ and finite $\Delta \in (\delta, T]$ the function η_j is absolutely continuous in $[\delta, \Delta]$. Let g satisfy the inequality:

$$Lg \equiv g_t - (g^m)_{xx} - b(g^\gamma)_x + cu^\beta \geq 0, (\leq 0),$$

at the points of Q where $g \in C^{2,1}_{x,t}$. Assume also that the function: $(g^m)_{xx}$ is continuous in

Q and $g \in L^{\infty}(Q \cap (t \le T_1))$ for any finite $T_1 \in (0, T]$. If in addition we have that:

$$g(\eta_0(t), t) \ge (\le)\, u(\eta_0(t), t),\ g(x, 0) \ge (\le)\, u(x, 0),$$

then

$$g \ge (\le)\, u,\ \textit{in}\ \overline{Q}$$

In general, weak solution is only Hölder continuous [59]. In fact, weak solution is classical smooth solution in the neighborhood of any point where solution is positive [109]. Hence, all the singularities of the weak solutions is concentrated on the boundary of the positivity set, or so called interfaces or free boundaries. Therefore the major open problem in the qualitative theory of nonlinear degenerate parabolic equations is the problem about the regularity and evolution of interfaces or free boundaries.

2.3 Asymptotic Properties of Solutions and Rescaling Principles

The proof of the main results formulated in Section 2.1 consists of two major steps. First step consists of the proof of the asymptotic formula for the solution of the diffusion-convection problem along some class of curves approaching the initial position of the interface origin from the support of the solution. The idea of the method is rescaling of the solution with subsequent iteration and use of compactness argument based on the interior Hölder regularity estimations. We formulate next lemmas outlining these results.

Lemma 2.3.1. *Let u be a solution to the CP* (1.8)*,* (1.9) *and u_0 satisfy* (1.10)*. Let either*

$b > 0$ or $b < 0$ and $\gamma \geq 1$. If $0 < \alpha < 1/(m - min(\gamma, (m+1)/2))$, then u satisfies (2.2).

Lemma 2.3.2. *Let u be a solution to the CP* (1.8), (1.11) *with $b > 0$, $\gamma < (m+1)/2$, $\alpha = 1/(m-\gamma)$, then*

$$u(x,t) = t^{1/(m+1-2\gamma)} h(\zeta), \tag{2.47}$$

$$\text{where } \zeta = x/t^{(m-\gamma)/(m+1-2\gamma)},\ h(\zeta) = u(\zeta, 1)$$

If $C > C_*$, then $h(0) = A_1$, where A_1 is a positive number depending on m, γ, C and b. If u_0 satisfies (1.10) *with $\alpha = 1/(m-\gamma)$, $C > C_*$, then u satisfies*

$$u(0,t) \sim A_1 t^{1/(m+1-2\gamma)} \text{ as } t \to 0+ \tag{2.48}$$

Lemma 2.3.3. *Let u be a solution to the CP*

$$\begin{cases} Lu := u_t - b(u^\gamma)_x = 0,\ x \in \mathbb{R},\ 0 < t \leq t_0 \\ u(x,0) = C(-x)_+^\alpha,\ x \in \mathbb{R} \end{cases} \tag{2.49}$$

with $b > 0$, $\gamma < 1$, and for any arbitrary $\alpha > 0$, then $\forall \delta \geq \delta_*(C, \gamma, \alpha)$ (see (2.33)*),there exist $C_\delta > 0$ and $\sigma > 0$ such that*

$$u(\eta_\delta(t), t) = C_\delta t^{\alpha/(1+\alpha(1-\gamma))}\ t \downarrow 0 \ \text{ along } \ \eta_\delta(t) = -\delta t^{1/(1+\alpha(1-\gamma))} \tag{2.50}$$

In particular, (2.36) *satisfies.*

Lemma 2.3.4. *Let u be a solution to the CP* (1.8), (1.10) *with $b > 0$, $\gamma < 1$, and $\alpha > 1/(m-\gamma)$. Then $\forall \delta \geq \delta_*(C, \gamma, \alpha)$ (see* (2.33)*) there exist $C_\delta > 0$ such that the asymptotic formula* (2.35) *is valid with $\eta_\delta(t) = -\delta t^{1/(1+\alpha(1-\gamma))}$.*

Lemma 2.3.5. *Let u be a solution to the CP* (2.49) *with $b > 0$, $\gamma > 1$ and $0 < \alpha < 1/(\gamma - 1)$, then $u(0,t) = 0$, and if $x < 0$ then $u(x,t) > 0$. For any $\delta > 0$, there exist $C_\delta \in (0, (\delta/b\gamma)^{1/(\gamma-1)})$ such that*

$$u(\xi_\delta(t), t) = C_\delta t^{(\alpha/(1-\alpha(\gamma-1))}, \ \ 0 \le t \le \delta \tag{2.51}$$

along $\xi_\delta(t) = -\delta t^{1/(1-\alpha(\gamma-1))}$

Lemma 2.3.6. *Let u be a solution to the CP* (1.8), (1.10) *with $b > 0$, $1 < \gamma < (m+1)/2$, and $1/(m-\gamma) < \alpha < 1/(\gamma - 1)$, then*

$$u(0,t) = 0$$

and for any $\delta > 0$, there exist $C_\delta \in (0, (\delta/b\gamma)^{1/(\gamma-1)})$ such that the asymptotic formula (2.51) *is valid with $\xi_\delta(t) = -\delta t^{1/(1-\alpha(\gamma-1))}$*

2.3.1 Proof of Lemma 2.3.1: Diffusion dominates over the convection

Suppose that u_0 satisfies (1.10). Then for arbitrary sufficiently small $\epsilon > 0$ there exists $x_\epsilon < 0$ such that

$$(C - \epsilon/2)(-x)_+^\alpha \le u_0(x) \le (C + \epsilon/2)(-x)_+^\alpha, \ : \ x_\epsilon \le x < +\infty \tag{2.52}$$

Let $u_\epsilon(x,t)$ (respectively, $u_{-\epsilon}(x,t)$) be a solution to the CP (1.8), (1.6), with initial data $(C+\epsilon)(-x)^\alpha$ (respectively, $(C-\epsilon)(-x)^\alpha$). Since the solution to the CP (1.8), (1.6) is continuous there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$u_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t), \quad u_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t) \quad \text{for} \quad 0 \leq t \leq \delta \tag{2.53}$$

From (2.52), (2.53), and a comparison principle it follows that

$$u_{-\epsilon}(x,t) \leq u(x,t) \leq u_\epsilon(x,t),\ 0 \leq t \leq \delta,\ x_\epsilon \leq x \tag{2.54}$$

Now, if we take

$$u_k(x,t) = ku(k^{-1/\alpha}x, k^{\alpha(m-1)-2/\alpha}t), \forall k \in R \tag{2.55}$$

Applying this to $u_{\mp\epsilon}$. Let us define

$$v_k^{\mp\epsilon} := ku_{\mp\epsilon}(k^{-1/\alpha}x, k^{\alpha(m-1)-2/\alpha}t) \tag{2.56}$$

then $v_k^{\mp\epsilon}$ satisfies the following problem:

$$u_t = (u^m)_{xx} + k^{\alpha(m-\gamma)-1/\alpha}b(u^\gamma)_x, x \in R, t > 0. \tag{2.57a}$$

$$u(x,0) = (C \mp \epsilon)(-x)_+^\alpha, \quad x \in R \tag{2.57b}$$

If $\gamma < (m+1)/2$, from [71], it follows that CP (2.57) has a unique solution. By using comparison principle and since $\alpha(m-\gamma) - 1 < 0$, we get

$$\lim_{k\to\infty} v_k^{\mp\epsilon}(x,t) = u_{\mp\epsilon}(x,t), x \in R, t > 0 \tag{2.58}$$

where $u_{\mp\epsilon}$ is a solution to the CP (1.8), (1.9) with $b = 0, u_0 = (C \mp \epsilon)(-x)_+^\alpha, T = +\infty$. Then $u_{\mp\epsilon}$ satisfies

$$u_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho; C \pm \epsilon)t^{\alpha/(2+\alpha(1-m))},\ t \geq 0, \tag{2.59}$$

Now, fixing $x = \xi_\rho(t)$, where $x = \xi_\rho(t)$ is the family of curves inside the positivity region of the interface and ρ is a an arbitrary number such that $\rho > \xi_*$, then from (2.58) it follows that

$$\lim_{k\to+\infty} ku_{\mp\epsilon}(k^{-1/\alpha}\xi_\rho(t), k^{(\alpha(m-1)-2)/\alpha}t) = f(\rho; C \mp \epsilon)t^{\alpha/(2-\alpha(m-1))},\ t > 0 \tag{2.60}$$

Taking $\tau = k^{(\alpha(m-1)-2)/\alpha}t$, then from (2.60) we get

$$u_{\mp\epsilon}(\xi_\rho(\tau), \tau) \sim f(\rho; C \mp \epsilon)\tau^{\alpha/(2-\alpha(m-1))} \ \text{ as }\ \tau \to 0^+. \tag{2.61}$$

As before, (2.2) follows from (2.54), (2.61).

For Case $b < 0$, consider two cases

(a) $0 < \gamma < (m+1)/2, 0 < \alpha < 1/(m-\gamma)$

(b) $(m+1)/2 \leq \gamma, 0 < \alpha < 2/(m-1)$

For case (a) when $b < 0$ and $\gamma < (m+1)/2$, from [71], it follows that CP (2.57) has a unique solution since $\alpha < 1/(\gamma - 1)$.

For case (b), let $u_{\mp\epsilon}$ is a solution of the Dirichlet problem

$$u_t - (u^m)_{xx} - b(u^\gamma)_x = 0, \quad |x| < |x_\epsilon|, \quad 0 < t \leq \delta \tag{2.62a}$$

$$u(x, 0) = (C \mp \epsilon)(-x)_+^\alpha \qquad |x| \leq |x_\epsilon| \tag{2.62b}$$

$$u(x_\epsilon, t) = (C \mp \epsilon)(-x)^\alpha \qquad u(-x_\epsilon, t) = 0, \quad 0 < t \leq \delta \tag{2.62c}$$

$v_k^{\mp\epsilon}$ satisfies the Dirichlet problem

$$u_t - (u^m)_{xx} - bk^{(\alpha(m-\gamma)-1)/\alpha}(u^\gamma)_x = 0, \quad in \quad D_\epsilon^k \tag{2.63a}$$

$$u(k^{\frac{1}{\alpha}} x_\epsilon, t) = k(C \mp \epsilon)(-x_\epsilon)^\alpha, \quad u(-k^{1/\alpha} x_\epsilon, t) = 0, \quad 0 \le t \le k^{(2-\alpha(m-1))/\alpha}\delta \tag{2.63b}$$

$$u(x,0) = (C \mp \epsilon)(-x)_+^\alpha \quad |x| < k^{1/\alpha}|x_\epsilon| \tag{2.63c}$$

Where

$$D_\epsilon^k = \{|x| < k^{\frac{1}{\alpha}}|x_\epsilon|, 0 < t \le k^{(2-\alpha(m-1))/\alpha}\delta\}$$

From [3], $\exists\delta > 0$ such that both (2.62a)- (2.62c) and (2.63a)-(2.63c) have a unique solution. For which there is a local solution and by comparison theorem implies $\delta = \delta(\epsilon) > 0$ may be chosen such that

$$u(-x_\epsilon, t) = 0, \quad 0 \le t \le \delta. \tag{2.64}$$

From [3] and comparison theorem we have , (2.52), (2.53) and (2.64), (2.54) follows for $|x| < |x_\epsilon|,\ 0 \le t \le \delta.$

To prove the convergence of the sequence $\{v_k^{\mp\epsilon}\}$ as $k \to +\infty$. We need to prove the uniform boundedness. Consider a function

$$g(x,t) = (C+1)(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{1}{1-m}}, \quad x \in R, \quad 0 \le t \le t_0 = \frac{\nu^{-1}}{2} \tag{2.65}$$

$$\nu = h_* + 1, \quad h_* = h(\alpha, m) = max_{x \in R} h(x), \tag{2.66}$$

where $\qquad h(x) = (C+1)^{m-1} m\alpha(m-1)(1+x^2)^{\frac{\alpha(m-1)-2}{2}}[x^2(m\alpha-2)(1+x^2)^{-1}+1]$

Then we get

$$L_k g = g_t - (g^m)_{xx} - bk^{\frac{\alpha(m-\gamma)-1}{\alpha}}(g^\gamma)_x = (C+1)(1+x^2)^{\frac{\alpha}{2}}(m-1)^{-1}(1-\nu t)^{\frac{m}{1-m}} S \quad in \quad D_\epsilon^k$$

$$S = \nu - h(x) - bk^{\frac{\alpha(m-\gamma)-1}{\alpha}}(C+1)^{\gamma-1}\gamma\alpha(1-\nu t)^{\frac{\gamma-m}{1-m}}(m-1)x(1+x^2)^{\frac{\alpha(\gamma-1)-2}{2}} \tag{2.67}$$

$$\text{and} \qquad S \leq 1+R \quad and \quad as \quad k\to\infty \quad in \quad D_0^k = D_\epsilon^k \cap \{0<t\leq t_0\} \tag{2.68}$$

where

$$R = O(k^{\frac{\alpha(m-\gamma)-1}{\alpha}}(k^{\frac{2}{\alpha}})^{\frac{\alpha(\gamma-1)-2}{2}}k^{\frac{1}{\alpha}}) = O(k^{\frac{\alpha(m-\gamma)-2}{\alpha}})\ uniformly\ for\ (x,t)\in D_0^k \quad as\ k\to+\infty$$

and we have for $0<\epsilon<<1$

$$g(x,0) \geq u_k^{\mp\epsilon}(x,0), \quad \text{for} \quad |x| < k^{\frac{1}{\alpha}}|x_\epsilon| \tag{2.69a}$$

$$g(\mp k^{\frac{1}{\alpha}}x_\epsilon, t) \geq u_k^{\mp\epsilon}(\mp k^{\frac{1}{\alpha}}x_\epsilon, t), \quad \text{for} \quad 0<t\leq t_0 \tag{2.69b}$$

Therefore, $\exists k_0 = k_0(\alpha; m)$ such that for $k \geq k_0$, by comparison theorem from [3] mplies

$$0 \leq u_k^{\mp\epsilon}(x,t) \leq g(x,t) \quad in \quad \bar{D}_{0_\epsilon}^k \tag{2.70}$$

For any compact $G \subset P = \{x\in R, 0<t\leq t_0\}$, and from (2.70), it follows that the sequences $u_k^{\mp\epsilon}$, $k\geq k_0$, are uniformly bounded in G and from [3] is uniformly Holder continuous which implies that for some subsequence k', $\exists v_{\mp\epsilon}$ such that

$$\lim_{k'\to\infty} u_{k'}^{\mp\epsilon} = v_{\mp\epsilon}(x,t), \quad (x,t)\in P \tag{2.71}$$

Since $\alpha(m-\gamma)-1<0$, passing to limit as $k'\to\infty$ from (2.46) for $u_k^{\mp\epsilon}$ it follows that $v_{\mp\epsilon}$ is a solution to the CP (1.8), (1.9) with $b=0, T=t_0, u_0=(C\pm\epsilon)(-x)_+^{\alpha}$.

As before, from (2.59), (2.60), (2.61), and (2.54), the required estimation (2.2) follows.

□

2.3.2 Proof of Lemma 2.3.2: Diffusion & convection are in balance

Consider a function

$$v_k(x,t)=ku(k^{\gamma-m}x, k^{2\gamma-m-1}t) \tag{2.72}$$

solves (1.8), (1.9). From [71] it follows that there exists a unique global solution to (1.8), (1.11). Therefore we have

$$u(x,t)=ku(k^{\gamma-m}x, k^{2\gamma-m-1}t), \quad k>0 \tag{2.73}$$

Using the scaled solution, we make the time variable equal to 1 by choosing k such that $k^{2\gamma-m-1}t=1$, then(2.73) implies (2.12) for u with $h(\zeta)=u(\zeta,1)$. h is nonlinear ordinary differential weak equation which is a unique solution to the problem

$$\begin{cases} \frac{d^2h^m}{d\zeta^2}+b\frac{dh^\gamma}{d\zeta}+\frac{m-\gamma}{m+1-2\gamma}\zeta\frac{dh}{d\zeta}-\frac{1}{m+1-2\gamma}h=0, \quad \zeta\in\mathbb{R} \\ h\sim C(-\zeta)^{\frac{1}{m-\gamma}} \;\; as\; \zeta\to-\infty,\; h(\zeta)=o(|\zeta|^{\frac{1}{m-\gamma}}) \;\; as \quad \zeta\downarrow+\infty, \end{cases}$$

and there exists a $\zeta_*>0$ such that h satisfies

$$(h(\zeta)^m)''+\frac{m-\gamma}{m+1-2\gamma}\zeta h'(\zeta)+b(h(\zeta)^\gamma)'-\frac{1}{m+1-2\gamma}h(\zeta)=0, \zeta\in\mathbb{R}$$

$$h(\zeta)\sim C(-\zeta)^{\alpha}\text{as } \zeta\downarrow-\infty, h(+\infty)=0$$

it is positive and smooth for $\zeta < \zeta_*$ and $h = 0$ for $\zeta \geq \zeta_*$([39]) , Thus (2.15) is valid.

If $u_0(x)$ satisfies (1.10) and $C > C_*$, $\alpha = \frac{1}{m-\gamma}$, then [96] implies that $h_1(0) = A_1 > 0$. Therefore we have $u(0,t) = A_1 t^{\frac{1}{m+1-2\gamma}}$, $0 \leq t \leq +\infty$.

Now if we define $u_{\pm\varepsilon}(x,t)$ to be solutions to CP (1.8), (1.9) with initial functions $u_0(x) = (C \pm \varepsilon)(-x)_+^{\alpha}$, then from (2.52), $\exists \delta_\varepsilon > 0$ s.t. for $0 \leq t \leq \delta_\varepsilon$, we have

$$u_{-\varepsilon}(0,t) \leq u(0,t) \leq u_\varepsilon(0,t) \tag{2.74}$$

then

$$u_{\pm\varepsilon}(0,t) = (A_1 \pm \varepsilon) A_1 t^{\frac{1}{m-2\gamma+1}}, t \geq 0 \tag{2.75}$$

Multiply (2.75) by $t^{\frac{-1}{m+1-2\gamma}}$ and taking the limit as $t \to 0^+$ and $\varepsilon \to 0^+$ gives us (2.48). □

2.3.3 Proof of Lemma 2.3.3: Asymptotic properties of the convection equation in strong regime

Consider CP (2.49). Consider a function v_k solves CP

$$\begin{cases} v_y - b\gamma v v_x = 0, & x \in R \\ v(x,0) = C^{\gamma-1}(-x)_+^{\alpha(\gamma-1)}, & x \in R \end{cases} \tag{2.76}$$

By applying a characteristic method we have

$$u = C(-x - b\gamma t u^{\gamma-1})_+^{\alpha} \tag{2.77}$$

its implicit equation which solves CP (2.49). Consider a family of curves $\eta(t) = -\delta t^{1/(1+\alpha(1-\gamma))}$, then the implicit equation (2.77), has a solution $u(\eta_\delta(t), t) = C_\delta t^{\alpha/(1+\alpha(1-\gamma))}$ such that

$$C_\delta = C(\delta - b\gamma C_\delta^{\gamma-1})_+^\alpha \tag{2.78}$$

$$\delta > b\gamma C_\delta^{\gamma-1} \tag{2.79}$$

Where C_δ is a positive constant. From (2.78), we have

$$F(C_\delta) = C^{\frac{-1}{\alpha}} C_\delta^{\gamma-1}\Big[C_\delta^{\frac{1+\alpha(1-\gamma)}{\alpha}} + b\gamma C^{\frac{1}{\alpha}}\Big] = \delta \tag{2.80}$$

Since $0 < \gamma < 1$, it is clear that $F(0) = +\infty$, $F(+\infty) = +\infty$, and $F(C_\delta) > 0$. By taking derivative to (2.80) we get $F'(C_\delta) = 0$ and $F'(C_\delta) \gtrless 0$ if $C_\delta \gtrless C_\delta^*$, where

$$C_{\delta_*} = \Big[b\gamma\alpha(1-\gamma)C^{\frac{1}{\alpha}}\Big]^{\frac{\alpha}{1+\alpha(1-\gamma)}} \tag{2.81}$$

Thus we have

$$F(C_{\delta_*}) = C^{\frac{(\gamma-1)}{1+\alpha(1-\gamma)}}(b\gamma)^{\frac{1}{1+\alpha(1-\gamma)}}\Big(\Big[(1-\gamma)\alpha\Big]^{\frac{1}{1+\alpha(1-\gamma)}} + \Big[(1-\gamma)\alpha\Big]^{\frac{\alpha(\gamma-1)}{1+\alpha(1-\gamma)}}\Big) \tag{2.82}$$

We need to find smallest $\delta > 0$ such that along $x = \eta_\delta(t) = -\delta t^{\frac{1}{1-\alpha(\gamma-1)}}$, we have

$$u(\eta_\delta(t), t) \sim C_\delta t^{\frac{\alpha}{1-\alpha(\gamma-1)}} \tag{2.83}$$

For that $\delta > 0$, we need $C_\delta < (\frac{\delta}{b\gamma})^{\frac{1}{\gamma-1}}$ which solve $F(C_\delta)$. Let

$$\hat{C} = C_{\delta_*} \text{ and } F(\hat{C}) = \delta_*(C, \gamma, \alpha) \text{ provided that } \hat{C}^{\gamma-1} < \frac{\delta_*}{b\gamma}$$

where δ_* is given in (2.34). It is easly to check that $\hat{C}^{\gamma-1} < \frac{\delta_*}{b\gamma}$. Hence, $\forall\delta \geq \delta_*$, (2.50) satisfies. □

2.3.4 Proof of Lemma 2.3.4: Convection strongly dominates over the diffusion

As before, (2.52), (2.53) follow from (1.11). Suppose $u_{\pm\epsilon}$ solves

$$\begin{cases} v_t - (v^m)_{xx} - b(v^\gamma)_x = 0,, & |x| < |x_\epsilon|, 0 < t \leq \delta \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,, & |x| \leq |x_\epsilon|, \\ v(x_\epsilon, t) = (C \pm \epsilon)(-x)_+^\alpha, \quad v(-x_\epsilon, t) = u(-x_\epsilon, t), & 0 \leq t \leq \delta \end{cases} \tag{2.84}$$

By continuity of $u_{\pm\epsilon}$ and comparsion theorem [3] for (2.52), (2.53) implies (2.54) for $|x| \leq |x_\epsilon|, 0 \leq t \leq \delta$. Scale $u_{\pm\epsilon}$ according to invariant scale of

$$\begin{cases} v_t - b(v^\gamma)_x = 0 \\ u(x,0) = C(-x)_+^\alpha \end{cases} \tag{2.85}$$

By uniqueness of (2.97), we get

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{\frac{-1}{\alpha}} x, k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t), \quad k > 0$$

then $u_k^{\pm\epsilon}$ satisfies the Dirichlet problem

$$\begin{cases} v_t - k^{\frac{1-\alpha(m-\gamma)}{\alpha}}(v^m)_{xx} - b(v^\gamma)_x = 0 & \text{in } E_\varepsilon^k = \left\{|x| \le k^{\frac{1}{\alpha}}|x_\varepsilon|, 0 \le t \le k^{\frac{1-\alpha(\gamma-1)}{\alpha}}\delta\right\} \\ v(x,0) = (C \pm \varepsilon)(-x)_+^\alpha & |x| \le k^{\frac{1}{\alpha}}|x_\varepsilon| \\ v\left(k^{\frac{1}{\alpha}}x_\varepsilon, t\right) = k(C \pm \varepsilon)(-x_\varepsilon)^\alpha & 0 \le t \le k^{\frac{1+\alpha(1-\gamma)}{\alpha}}\delta \\ v\left(-k^{\frac{1}{\alpha}}x_\varepsilon, t\right) = ku\left(-x_\varepsilon, k^{\frac{\alpha(\gamma-1)-1}{\alpha}}t\right) & 0 \le t \le k^{\frac{1-\alpha(\gamma-1)}{\alpha}}\delta \end{cases} \tag{2.86}$$

To prove $u_k^{\pm\epsilon}$ as $k \to +\infty$ is uniformly bounded sequence, we need to prove $u_k^{\pm\epsilon}$ as $k \to +\infty$ is a convergence sequence. Consider the function

$$g(x,t) = (C+1)(1+x^2)^{\alpha/2}e^t, x \in R \tag{2.87}$$

then we have

$$\begin{gathered} L_k g = g_t - k^{\frac{1-\alpha(m-\gamma)}{\alpha}}(g^m)_{xx} - b(g^\gamma)_x \ge \\ g\Big[1 - k^{\frac{1-\alpha(m-\gamma)}{\alpha}}2\alpha m(\frac{\alpha m}{2} - 1)(C+1)^{m-1}e^{(m-1)t}x^2(1+x^2)^{\frac{\alpha(m-1)}{2}-2} \\ -\alpha m k^{\frac{1-\alpha(m-\gamma)}{\alpha}}(C+1)^{(m-1)}e^{(m-1)t}(1+x^2)^{\frac{\alpha(m-1)}{2}-1} - \alpha\gamma b(C+1)^{\gamma-1}e^{(\gamma-1)t}x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1}\Big] \end{gathered} \tag{2.88}$$

in E_ϵ^k. By fixing $\sigma > 0$ and assume $E_{0\epsilon}^k = E_\epsilon^k \cap \{(x,t) : 0 < t \le \sigma\}$, then from (2.100), we have

$$L_k g = g(1 - h(x) + R)$$

where

$$h(x) = k^{\frac{\alpha(\gamma-m)+1}{\alpha}} \alpha m(C+1)^{m-1} e^{(m-1)t}(1+x^2)^{\frac{\alpha(m-1)}{2}-1}\Big[\frac{(\alpha m-1)x^2+1}{(1+x^2)}\Big] \tag{2.89}$$

$$R = -\alpha\gamma b(C+1)^{\gamma-1}e^{(\gamma-1)t}x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1} \tag{2.90}$$

We need to prove that $L_k g \geq 0$ in $E_\varepsilon^k = \Big\{|x| \leq k^{\frac{1}{\alpha}}|x_\varepsilon|, 0 \leq t \leq k^{\frac{1+\alpha(1-\gamma)}{\alpha}}\sigma\Big\}$. By proving $h(x)$ has a finite maximum and thus it is easily to check $h(x) \to 0$ as $x \to \infty$. Thus

$$h(x) = O(k^\theta) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

$$\theta = \begin{cases} \frac{\alpha(\gamma-m)+1}{\alpha} & \alpha \leq \frac{2}{m-1} \\ \frac{\alpha(\gamma-1)-1}{\alpha}, & \alpha > \frac{2}{m-1} \end{cases}$$

And we need to show that $\lim_{k\to\infty} R = 0$ uniformly in $|x| < k^{\frac{1}{\alpha}}|x_\varepsilon|$. Since R is continuous on $\mathbb{R}$, and therefore it is also continuous on the finite domain $|x| < k^{\frac{1}{\alpha}}|x_\epsilon|$. Thus

$$R = O(k^{\frac{\alpha(\gamma-1)-1}{\alpha}}) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

According to that, for $0 < \epsilon \ll 1$ we get

$$g(x,0) \geq u_k^{\pm\epsilon}(x,0) \quad \text{if } \; |x| < k^{\frac{1}{\alpha}}|x_\epsilon|$$

Since

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = o(k) \;\text{ if }\; 0 \leq t \leq \sigma \;\text{ as }\; k \to +\infty,$$

Therefore

$$g(\pm k^{\frac{1}{\alpha}}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}}x_\epsilon, t),\; 0 \leq t \leq \sigma$$

If $k \gg 1$ by comparison theorem [3], we have, as in the proof of lemma 2.3.1, if $k \gg 1$ implies (2.70) in $E^k_{0\epsilon}$. Hence, $\{u_k^{\pm\epsilon}\}$ is uniformly bounded in $\bar{E}^k_{o\epsilon}$ and accordingly [3] it is uniformly Hölder continuous on any compact $G \subset P = \{x \in \mathbb{R},\ 0 < t \leq \sigma\}$ so $\exists k'$ such that (2.71) solves

$$\begin{cases} v_t - b(v^\gamma)_x = 0,\ x \in \mathbb{R},\ 0 < t \leq t_0 \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,\ x \in \mathbb{R} \end{cases} \tag{2.91}$$

Now assume $k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t = \tau$ and $x = \eta_\delta(t) = -\delta t^{\frac{1}{1+\alpha(1-\gamma)}}$, from (2.71) it follows that

$$u_{\pm\epsilon}(\eta_\delta(\tau),\tau) \sim [-b\gamma(\gamma-1)\alpha(C \pm \epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1+\alpha(1-\gamma)}} \tau^{\frac{\alpha}{1+\alpha(1-\gamma)}},\ \text{as } \tau \downarrow 0 \tag{2.92}$$

For arbitrary $\epsilon > 0$, from (2.53) and (2.104), (2.35) follows. □

2.3.5 Proof of Lemma 2.3.5: Asymptotic properties of the convection equation in weak regime

From lemma 2.3.3, we have the implicit equation (2.77) of u.

$$x = 0 \Rightarrow u(0,t) = u = C(-b\gamma t u^{\gamma-1}(0,t))_+^\alpha = 0, \quad 0 < t \leq t_0$$

- If $\gamma > 1$ and $\alpha = \frac{1}{\gamma-1}$

We need to have an explicit solution of u. From (2.77), $\gamma > 1$ and $\alpha = \frac{1}{\gamma-1}$, consider $\bar{u}$ such that

$$\bar{u}^{\gamma-1} = C^{\gamma-1}(-x - b\gamma t \bar{u}^{\gamma-1})_+ \tag{2.93}$$

If $-x-b\gamma t\bar{u}^{\gamma-1}>0$, then $(1+b\gamma t)\bar{u}^{\gamma-1}=C^{\gamma-1}(-x)_+, x<0$. Hence

$$\bar{u}=C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{\gamma-1}} \tag{2.94}$$

which is the explicit solution of u.

- If $\gamma>1$ and $0<\alpha<\frac{1}{\gamma-1}$

Since $C(-x)_+^{\alpha}\geq C(-x)_+^{\frac{1}{\gamma-1}}, \quad x\in R$, then

$$u(x,t)\geq C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{\gamma-1}}, x\in R, 0<t\leq\delta$$

Hence

$$u(x,t)>0 \quad \text{if} \quad x<0$$

As before, along a family of curves $x=\xi_\delta(t)=-\delta t^{\frac{1}{1-\alpha(\gamma-1)}}$, the implicit equation (2.77) has a solution $u=C_\delta t^{\frac{\alpha}{1-\alpha(\gamma-1)}}$, such that

$$C_\delta=C(\delta-b\gamma C_\delta^{\gamma-1})_+^{\alpha} \tag{2.95}$$

From lemma 2.3.3, it is easily to solve (2.95). Assume that

$$F(C_\delta)=C^{\frac{-1}{\alpha}}C_\delta^{\frac{1}{\alpha}}+b\gamma C_\delta^{\gamma-1}=\delta$$

It is clear that F is increasing function on $(0,\infty)$. Therefore for any $\delta>0$, there is a unique $C_\delta>0$ such that $F(C_\delta)=\delta$. It follows that the condition $C_\delta<(\frac{\delta}{b\gamma})^{\frac{1}{\gamma-1}}$ satisfies. Therefore, $\forall\delta>0$, (2.51) hold. □

2.3.6 Proof of Lemma 2.3.6: Convection weakly dominates over the diffusion

As before, (2.52), (2.53) follow from (1.11). Suppose $u_{\pm\epsilon}$ solves

$$\begin{cases} v_t - (v^m)_{xx} - b(v^\gamma)_x = 0,, & |x| < |x_\epsilon|, 0 < t \le \delta \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha,, & |x| \le |x_\epsilon|, \\ v(x_\epsilon, t) = (C \pm \epsilon)(-x)_+^\alpha, \quad v(-x_\epsilon, t) = u(-x_\epsilon, t), & 0 \le t \le \delta \end{cases} \tag{2.96}$$

By continuity of $u_{\pm\epsilon}$ and comparsion theorem [3] for (2.52), (2.53) implies (2.54) for $|x| \le |x_\epsilon|, 0 \le t \le \delta$. Scale $u_{\pm\epsilon}$ according to invariant scale of

$$\begin{cases} v_t - b(v^\gamma)_x = 0 \\ u(x,0) = C(-x)_+^\alpha \end{cases} \tag{2.97}$$

By uniqueness of (2.97), we get

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{\frac{-1}{\alpha}} x, k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t), \quad k > 0$$

then $u_k^{\pm\epsilon}$ satisfies the Dirichlet problem

$$\begin{cases} v_t - k^{\frac{1-\alpha(m-\gamma)}{\alpha}} (v^m)_{xx} - b(v^\gamma)_x = 0 & \text{in } E_\varepsilon^k = \left\{ |x| \le k^{\frac{1}{\alpha}} |x_\varepsilon|, 0 \le t \le k^{\frac{1-\alpha(\gamma-1)}{\alpha}} \delta \right\} \\ v(x,0) = (C \pm \varepsilon)(-x)_+^\alpha & |x| \le k^{\frac{1}{\alpha}} |x_\varepsilon| \\ v\left(k^{\frac{1}{\alpha}} x_\varepsilon, t\right) = k(C \pm \varepsilon)(-x_\varepsilon)^\alpha & 0 \le t \le k^{\frac{1+\alpha(1-\gamma)}{\alpha}} \delta \\ v\left(-k^{\frac{1}{\alpha}} x_\varepsilon, t\right) = k u\left(-x_\varepsilon, k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t\right) & 0 \le t \le k^{\frac{1-\alpha(\gamma-1)}{\alpha}} \delta \end{cases} \tag{2.98}$$

To prove $u_k^{\pm\epsilon}$ as $k \to +\infty$ is uniformly bounded sequence, we need to prove $u_k^{\pm\epsilon}$ as $k \to +\infty$ is a convergence sequence. Consider the function

$$g(x,t) = (C+1)(1+x^2)^{\alpha/2}e^t, x \in R \tag{2.99}$$

then we have

$$L_k g = g_t - k^{\frac{1-\alpha(m-\gamma)}{\alpha}}(g^m)_{xx} - b(g^\gamma)_x \geq \tag{2.100}$$
$$g\Big[1 - k^{\frac{1-\alpha(m-\gamma)}{\alpha}} 2\alpha m(\frac{\alpha m}{2} - 1)(C+1)^{m-1}e^{(m-1)t}x^2(1+x^2)^{\frac{\alpha(m-1)}{2}-2}$$
$$-\alpha m k^{\frac{1-\alpha(m-\gamma)}{\alpha}}(C+1)^{(m-1)}e^{(m-1)t}(1+x^2)^{\frac{\alpha(m-1)}{2}-1} - \alpha\gamma b(C+1)^{\gamma-1}e^{(\gamma-1)t}x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1}\Big]$$

in E_ϵ^k. By fixing $\sigma > 0$ and assume $E_{0\epsilon}^k = E_\epsilon^k \cap \{(x,t) : 0 < t \leq \sigma\}$, then from (2.100), we have

$$L_k g = g(1 - h(x) + R)$$

where

$$h(x) = k^{\frac{\alpha(\gamma-m)+1}{\alpha}} \alpha m(C+1)^{m-1}e^{(m-1)t}(1+x^2)^{\frac{\alpha(m-1)}{2}-1}\Big[\frac{(\alpha m-1)x^2+1}{(1+x^2)}\Big] \tag{2.101}$$

$$R = -\alpha\gamma b(C+1)^{\gamma-1}e^{(\gamma-1)t}x(1+x^2)^{\frac{\alpha(\gamma-1)}{2}-1} \tag{2.102}$$

We need to prove that $L_k g \geq 0$ in $E_\varepsilon^k = \Big\{|x| \leq k^{\frac{1}{\alpha}}|x_\varepsilon|, 0 \leq t \leq k^{\frac{1+\alpha(1-\gamma)}{\alpha}}\sigma\Big\}$. By proving $h(x)$ has a finite maximum and thus it is easily to check $h(x) \to 0$ as $x \to \infty$. Thus

$$h(x) = O(k^\theta) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

$$\theta = \begin{cases} \frac{\alpha(\gamma-m)+1}{\alpha} & \alpha \leq \frac{2}{m-1} \\ \frac{\alpha(\gamma-1)-1}{\alpha}, & \alpha > \frac{2}{m-1} \end{cases}$$

And we need to show that $\lim_{k\to\infty} R = 0$ uniformly in $|x| < k^{\frac{1}{\alpha}}|x_\varepsilon|$. Since R is continuous on $\mathbb{R}$, and therefore it is also continuous on the finite domain $|x| < k^{\frac{1}{\alpha}}|x_\epsilon|$. Thus

$$R = O(k^{\frac{\alpha(\gamma-1)-1}{\alpha}}) \text{ uniformly for } (x,t) \in E^k_{o\epsilon} \text{ as } k \to \infty.$$

According to that, for $0 < \epsilon \ll 1$ we get

$$g(x,0) \geq u_k^{\pm\epsilon}(x,0) \quad \text{if} \ \ |x| < k^{\frac{1}{\alpha}}|x_\epsilon|$$

Since

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = o(k) \ \text{ if } \ 0 \leq t \leq \sigma \ \text{ as } \ k \to +\infty,$$

Therefore

$$g(\pm k^{\frac{1}{\alpha}}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}}x_\epsilon, t), \ 0 \leq t \leq \sigma$$

If $k \gg 1$ by comparison theorem [3], we have, as in the proof of lemma 2.3.1, if $k \gg 1$ implies (2.70) in $E^k_{0\epsilon}$. Hence, $\{u_k^{\pm\epsilon}\}$ is uniformly bounded in $\bar{E}^k_{o\epsilon}$ and accordingly [3] it is uniformly Hölder continuous on any compact $G \subset P = \{x \in \mathbb{R}, \ 0 < t \leq \sigma\}$ so $\exists k'$ such that (2.71) solves

$$\begin{cases} v_t - b(v^\gamma)_x = 0, \ x \in \mathbb{R}, \ 0 < t \leq t_0 \\ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \ x \in \mathbb{R} \end{cases} \tag{2.103}$$

Now assume $k^{\frac{\alpha(\gamma-1)-1}{\alpha}}t = \tau$ and $x = \eta_\delta(t) = -\delta t^{\frac{1}{1+\alpha(1-\gamma)}}$, from (2.71) it follows that

$$u_{\pm\epsilon}(\eta_\delta(\tau),\tau) \sim [-b\gamma(\gamma-1)\alpha(C\pm\epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1+\alpha(1-\gamma)}}\tau^{\frac{\alpha}{1+\alpha(1-\gamma)}},\ \text{as } \tau\downarrow 0 \tag{2.104}$$

For arbitrary $\epsilon > 0$, from (2.53) and (2.104), (2.35) follows. □

2.4 Proofs of the main results

In this section, we prove the main results in case when $b > 0$ and $m > 1$.

2.4.1 Diffusion dominates and interface expands

Region (1)

Proof of Theorem 2.1.1. Consider $\alpha < \frac{1}{m-\min(\gamma,\frac{m+1}{2})}$. The formula (2.2) follows from lemma 2.3.1. We will prove $\eta(t) \sim \xi_* t^{\frac{1}{2+\alpha(1-m)}}$ as $t \to 0^+$. Let's first express the result from lemma 2.3.1 using limits. Since $\rho > \xi_*$, we can say $\rho = \xi_* - \varepsilon$ for some $\varepsilon > 0$,

$$\lim_{t\to 0^+} u(\xi_\rho(t),t)t^{\frac{\alpha}{\alpha(m-1)-2}} \geq \frac{f(\rho)}{2},\ 0<t<\delta_\varepsilon$$

$\exists\delta > 0$ for $0 < t < \delta_\varepsilon$ such that

$$\liminf_{t\to 0^+} \eta(t)\, t^{\frac{1}{\alpha(m-1)-2}} \geq \xi_* - \varepsilon$$

and then passing $\varepsilon \to 0$ yields

$$\liminf_{t\to 0^+} \eta(t)\, t^{\frac{1}{\alpha(m-1)-2}} \geq \xi_* \tag{2.105}$$

Since u_ϵ is a supersolution of CP (1.8) with initial function

$$u_\epsilon(x,0) = (C+\varepsilon)(-x)_+^\alpha \geq u(x,0),$$

and from (2.52), (2.53) and a comparison theorem, (2.54) follows. From lemma in [13] we have

$$\eta(t) \leq (C+\varepsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi_*^{'} t^{\frac{1}{2-\alpha(m-1)}}$$

Using the definition of $\xi_*^{'}$ and now passing the limit, $\varepsilon \to 0$, results in

$$\limsup_{t\to 0} \eta(t) t^{\frac{1}{\alpha(m-1)-2}} \leq \xi_*^{'}$$

□

2.4.2 Borderline case: diffusion & convection are in balance

Region (2)

Proof of Theorem 2.1.2. First, assume that the global case of (1.11). The problem (1.8), (1.11) has a unique global solution and the comparison principle is valid.

If $\gamma = 1$, it is easy to check that the problem (1.8), (1.11) has an explicit solution is given by (2.16).

Let $\gamma \neq 1$. The self-similar form (2.12) follows from Lemma 2.3.2. Let $C > C_*$.
Consider a function

$$g(x,t) = t^{\frac{1}{m+1-2\gamma}} h(\zeta), \quad \zeta := xt^{\frac{\gamma-m}{m+1-2\gamma}} \tag{2.106}$$

then

$$Lg = t^{\frac{2\gamma-m}{m+1-2\gamma}} L^0 h_1 \tag{2.107}$$

$$L^0 h_1 = \frac{1}{m+1-2\gamma} h + \frac{\gamma-m}{m+1-2\gamma} \zeta h_1' - (h^m)'' - b(h^\gamma)' \tag{2.108}$$

Choose function h_1 as

$$h_1(\zeta) = C_0(\lambda_0-\zeta)_+^{\frac{1}{m-\gamma}}, 0 \le \zeta \le \lambda_0$$

where C_0 and λ_0 are some positive constants. From (2.108) we have

$$L^0 h_1 = bC_0^\gamma \frac{\gamma}{m-\gamma} (\lambda_0-\zeta)_+^{\frac{2\gamma-m}{m-\gamma}} \left[1 - \left(\frac{C_0}{C_*}\right)^{m-\gamma} + \frac{C_0^{1-\gamma}(m-\gamma)}{b\gamma(m+1-2\gamma)} \lambda_0 (\lambda_0-\zeta)^{\frac{1-\gamma}{m-\gamma}}\right] \tag{2.109}$$

To estimate upper bound, we take $C_0 = C_2$ and $\lambda_0 = \lambda_2$ and $\alpha = \frac{1}{m-\gamma}$.

If $\gamma < 1$, then we have

$$L^0 h_1 \ge bC_2^\gamma \frac{\gamma}{m-\gamma} (\lambda_2-\zeta)_+^{\frac{2\gamma-m}{m-\gamma}} \left[1 - \left(\frac{C_2}{C_*}\right)^{m-\gamma}\right] \ge 0, \text{ for } 0 \le \lambda_2 - \zeta \le \lambda_2$$

From (2.107), it follows that

$$Lg \ge 0 \quad \text{for} \quad 0 < x < \lambda_2 t^{\frac{m-\gamma}{m+1-2\gamma}}, \ 0 < t < +\infty \tag{2.110}$$

$$Lg = 0 \quad \text{for} \quad x > \lambda_2 t^{\frac{m-\gamma}{m+1-2\gamma}}, \ 0 < t < +\infty \tag{2.111}$$

From lemma 2.2.2 we have g is a super solution of CP (1.8) in $\{(x,t) : x > 0, t > 0\}$. Since

$$g(x,0) = u(x,0) = 0 \quad \text{for} \quad 0 \le x < +\infty, \tag{2.112}$$

$$g(0,t) = u(0,t) \quad \text{for} \quad 0 \le t < +\infty, \tag{2.113}$$

then we have

$$u(x,t) \le g(x,t) = C_*(\lambda_2 - \zeta)_+^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \le x, t \le +\infty \tag{2.114}$$

$$\eta(t) \le \lambda_2 t^{\frac{m-\gamma}{m+1-2\gamma}}, \qquad 0 \le t < +\infty \tag{2.115}$$

To estimate lower bound when $\gamma < 1$, we take $C_0 = C_1$ and $\lambda_0 = \lambda_1$ and $\alpha = \frac{1}{m-\gamma}$. From (2.109) it follows

$$L^0 h_1 \le bC_1^\gamma \frac{\gamma}{m-\gamma} \lambda_1^{\frac{2\gamma-m}{m-\gamma}} \left[1 - \left(\frac{C_1}{C_*} \right)^{m-\gamma} + \frac{C_1^{1-\gamma}(m-\gamma)}{b\gamma(m+1-2\gamma)} \lambda_1^{\frac{1+m-2\gamma}{m-\gamma}} \right] \le 0, \text{ for } \zeta = 0 \tag{2.116}$$

From (2.107), it follows that

$$Lg \le 0 \quad \text{for} \quad 0 < x < \lambda_1 t^{\frac{m-\gamma}{m+1-2\gamma}},\ 0 < t < +\infty \tag{2.117}$$

$$Lg = 0 \quad \text{for} \quad x > \lambda_1 t^{\frac{m-\gamma}{m+1-2\gamma}},\ 0 < t < +\infty \tag{2.118}$$

From lemma 2.2.2 and (2.112), (2.113), we have

$$u(x,t) \ge g(x,t) = C_1(\lambda_1 - \zeta)_+^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \le x, t \le +\infty \tag{2.119}$$

$$\lambda_1 t^{\frac{m-\gamma}{m+1-2\gamma}} \le \eta(t), \quad 0 \le t < +\infty \tag{2.120}$$

If $\gamma > 1$, we need to bound $u(x,t)$, so we have the following upper and lower estimation of $u(x,t)$ when $\gamma > 1$, $C > C_*$, $0 \le x < \infty$ and $0 \le t < \infty$.

To estimate upper bound, we take $C_0 = C_2$ and $\lambda_0 = \lambda_2$ and $\alpha = \frac{1}{m-\gamma}$. From (2.108) we have

$$L^0 h_1 = \frac{C_2(\lambda_2-\zeta)^{\frac{2-m}{m-1}}}{m+1-2\gamma}\left[\lambda_2+\frac{1-\gamma}{m-1}\zeta-\frac{C_2^{m-1}m(m+1-2\gamma)}{(m-1)^2}+\frac{bC_2^{\gamma-1}\gamma(m+1-2\gamma)(\lambda_2-\zeta)^{\frac{\gamma-1}{m-1}}}{m-1}\right] \tag{2.121}$$

then taking $\zeta = \lambda_2$ to achieve the sharpest upper bound, we have,

$$L^0 h_1 \geq \frac{C_2(\lambda_2-\zeta)^{\frac{2-m}{m-1}}}{m+1-2\gamma}\left[\lambda_2+\frac{1-\gamma}{m-1}\lambda_2-\frac{C_2^{m-1}m(m+1-2\gamma)}{(m-1)^2}\right] \geq 0$$

which implies (2.110) and (2.111). Again from lemma 2.2.2 and comparsion theorem [3] , (2.114) follows. Thus g is a supersolution of u and as a result we have found a sharper upper bound.

It is easy to check the lower bound by following the same analysis if $C_0 = C_1$ and $\lambda_0 = \lambda_1$ and $\alpha = \frac{1}{m-\gamma}$.

Assume $\gamma < 1$, $C < C_*$. We are going to establish the rough estimation. Consider a function

$$u_0(x) = C(-x)^{\alpha}, \quad x \in R \tag{2.122}$$

with $\alpha = \frac{1}{m-\gamma}$, then

$$Lu_0 = b(u_0^{\gamma})_x\Big[1-\Big(\frac{C}{C_*}\Big)^{m-\gamma}\Big] \geq 0 \quad \text{for} \quad x \in R, \quad t \geq 0 \tag{2.123}$$

To establish a more accurate estimation, from (2.123), consider a function

$$g(x,t) = C_0\Big(-\zeta_0 t^{\frac{m-\gamma}{m+1-2\gamma}} - x\Big)_+^{\frac{1}{m-\gamma}} \quad \text{in} \quad G_\ell \tag{2.124}$$

$$G_\ell = \{(x,t) : \zeta_\ell(t) = -\ell t^{\frac{m-\gamma}{m+1-2\gamma}} < x < +\infty, \quad 0 < t < +\infty\}$$

where $C_0 > 0$, $\zeta_0 > 0$ and $\ell > \zeta_0$ are some constants.

To estimate upper bound, we will approximate this curve using $x = \zeta_\ell(t) = -\ell t^{\frac{m-\gamma}{m+1-2\gamma}}$.

Calculating Lg in

$$G_\ell^+ = \{(x,t) : \zeta_\ell(t) = -\ell t^{\frac{m-\gamma}{m+1-2\gamma}} < x < -\zeta_0 t^{\frac{m-\gamma}{m+1-2\gamma}}, \quad 0 < t < \delta_\epsilon\}$$

thus

$$Lg = -b(g^\gamma)_x S, \tag{2.125}$$

$$S = 1 - \left(\frac{C_0}{C_*}\right)^{m-\gamma} - \frac{\zeta_0 C_0^{1-\gamma}(m-\gamma)}{b\gamma(m+1-2\gamma)} t^{\frac{\gamma-1}{m+1-2\gamma}} \left(-\zeta_0 t^{\frac{m-\gamma}{m+1-2\gamma}} - x\right)_+^{\frac{1-\gamma}{m-\gamma}}$$

then taking $x > \zeta_\ell(t)$ yields

$$S \geq 1 - \left(\frac{C_0}{C_*}\right)^{m-\gamma} - \frac{\zeta_0(m-\gamma)}{b\gamma(m+1-2\gamma)} C_0^{1-\gamma}(\ell - \zeta_0)^{\frac{1-\gamma}{m-\gamma}} \geq 0 \tag{2.126}$$

Then we can find the optimal values of C_0, ℓ and ζ_0, corresponding to C_4, ℓ_1 and ζ_4 such that

$$Lg \geq 0 \quad \text{in} \quad G_{\ell_1}^+, \; Lg = 0 \quad \text{in} \quad G_{\ell_1} \setminus \bar{G}_{\ell_1}^+$$

$$u(\zeta_\ell(t), t) \leq C \ell_1^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}} = C_3(\ell_1 - \zeta_4)_+^{\frac{1}{m-\gamma}} t^{\frac{1}{m+1-2\gamma}} = g(\zeta_\ell(t), t), \quad t \geq 0$$

$$g(x,0) = u(x,0) = 0, \quad 0 \leq x \leq x_0 \tag{2.127}$$

$$g(x_0,0) = u(x_0,0) = 0, \quad t \geq 0, \tag{2.128}$$

By comparison theorem upper estimation is valid. To estimate lower bound when $\gamma < 1$

and $C < C_*$, define a function (2.124) in

$$G_\ell = \{(x,t) : x > \zeta_\ell(t) = -\ell t^{\frac{m-\gamma}{m+1-2\gamma}}, \quad 0 < t < +\infty\}$$

where $C_0 > 0$, $\zeta_0 > 0$ and $\ell > \zeta_0$ are some constants. From (2.125), we apply the same analysis and take $x < \zeta_\ell(t)$ yields

$$S \leq 1 - \left(\frac{C_0}{C_*}\right)^{m-\gamma}$$

Choose $C_0 = C_*$, $\zeta_0 = \zeta_3$ then $S \leq 0$, thus

$$Lg \leq 0 \quad in \quad G_\ell^+; \quad Lg = 0 \quad in \quad G_\ell \setminus G_\ell^+ \tag{2.129}$$

$$g(\zeta_\ell(t),t) = C_3\,(\ell_0 - \zeta_3)_+^{\frac{1}{m-\gamma}}\, t^{\frac{1}{m+1-2\gamma}} \leq h(-\ell) t^{\frac{1}{m+1-2\gamma}} \leq u(\zeta_\ell(t),t), \quad 0 \leq t < +\infty$$

$\forall\ 0 < \epsilon << 1$, then (2.127), (2.128) are valid where $x_0 > 0$ is arbitrary fixed number. Since $x_0 > 0$ is arbitrary number then lower estimation from (2.127), (2.128) follows. and

$$-\zeta_1 t^{\frac{m-\gamma}{m+1-2\gamma}} \leq \eta(t) \leq -\zeta_4 t^{\frac{m-\gamma}{m+1-2\gamma}}, \quad 0 \leq t < +\infty$$

Let $\gamma > 1$, $0 < C < C_*$. To estimate lower bound, from lemma 2.3.3, we have Convection equation (2.49) is equal zero and from lemma 2.3.5 we have (2.94) which is explicit solution. Now, we calculate $L\bar{u}$, then

$$L\bar{u} = -\frac{m(m-\gamma+1)C_0^m}{(\gamma-1)^2(1+b\gamma t)^2}\left(\frac{-x}{1+b\gamma t}\right)_+^{\frac{m-2\gamma+2}{\gamma-1}} \tag{2.130}$$

Since $\gamma < \frac{m+1}{2}$, therefore

$$L\bar{u} = -(\bar{u}^m)_{xx} \leq 0$$

So $\bar{u}(x,t)$ is a lower bound for $u(x,t)$ which solves CP (1.8), (1.11). Since $\frac{1}{m-\gamma} < \frac{1}{\gamma-1}$, then

$$u(x_\epsilon, 0) = C(-x_\epsilon)_+^{\frac{1}{m-\gamma}} \geq C_*(-x_\epsilon)_+^{\frac{1}{\gamma-1}} = \bar{u}(x_\epsilon, 0) \quad \text{for} \quad 0 < |x_\epsilon| << 1 \tag{2.131}$$

moreover, $\forall \epsilon > 0$, $\exists \delta_\epsilon > 0$ such that

$$u(x_\epsilon, t) \geq \bar{u}(x_\epsilon, t), \quad 0 \leq t \leq \delta_\epsilon$$

From (2.131), we have

$$u(x,0) \geq \bar{u}(x,0), \quad x \in R, \quad x_\epsilon < x < +\infty$$

$$L\bar{u} \leq 0, \quad x_\epsilon < x < +\infty, \quad t \in (0, \delta_\epsilon)$$

By comparison theorem, upper estimation follows.

For upper bound when $C < C_*$ and $\gamma > 1$, we need upper bound with stationary interface. We can easily deduce that

$$u(x,t) \leq u_*(x) = C_*(-x)_+^{\frac{1}{m-\gamma}}, \quad x \in R$$

Since u_* solves Diffusion Convection equation $(u_*^m)_{xx} + b(u_*^\gamma)_x = 0$ then $Lu_* = 0$. By comparison theorem

$$u(x,0) = C(-x)_+^{\frac{1}{m-\gamma}} \leq C_*(-x)_+^{\frac{1}{m-\gamma}} = u_*(x,0)$$

Consider a function (2.122). Since $0 < C < C_* \Rightarrow S > 0$, then

$$u(x,t) \le C(-x)_+^{\frac{1}{m-\gamma}}, \quad x \in R, t > 0$$

For all $\epsilon > 0$, we have

$$(C-\epsilon)\Big(\frac{-x}{1+(C-\epsilon)^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{\gamma-1}} \le u(x,t) \le C(-x)_+^{\frac{1}{m-\gamma}}, \quad x_\epsilon \le x < +\infty, \quad t \ge 0 \quad (2.132)$$

Now we need to have sharp lower bound for u, for that if $\alpha = \frac{1}{m-\gamma}$ consider a function

$$g(x,t) = C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1}{m-\gamma}} \quad (2.133)$$

As before, apply the same calculation for the operator Lg, and thus

$$Lg = -b(g^\gamma)_x S, \quad (2.134)$$
$$S = 1 - \Big(\frac{C}{C_*}\Big)^{m-\gamma}\frac{1}{(1+C^{\gamma-1}b\gamma t)} - C^{1-\gamma}\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{1+m-2\gamma}{m-\gamma}}$$

Since $0 < C < C_* \Rightarrow 1-\Big(\frac{C}{C_*}\Big)^{m-\gamma} > 0$, and since $\gamma < \frac{m+1}{2} \Rightarrow \exists\, \bar{x} < 0,$ such that $S \ge 0,\ \bar{x} \le x < 0,\ t > 0$. thus we have

$$g(x,0) = C(-x)_+^{\frac{1}{m-\gamma}}, \text{ in } \{\bar{x} < x < +\infty, 0 < t \le \delta\}$$

For sufficiently smal $\epsilon > 0$, consider a function (2.133) with replacing C by $C+\epsilon$, it follows

$$1 - \Big(\frac{C+\epsilon}{C_*}\Big)^{m-\gamma} > 0$$

thus

$$\exists x_\epsilon < 0 \quad \text{such that} \quad \delta_\epsilon \geq 0, x_\epsilon \leq x < +\infty$$

Since

$$g_\epsilon(x_\epsilon, 0) = (C + \epsilon)(-x_\epsilon)^{\frac{1}{m-\gamma}} > C(-x_\epsilon)^{\frac{1}{m-\gamma}} = u(x_\epsilon, 0)$$

then

$$g_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t), \quad 0 \leq t < \delta_\epsilon$$

By comparison theorem lower estimation follows. □

2.4.3 Convection dominates and interface shrinks

Region (3)

Proof of Theorem 2.1.3. Take an arbitrary sufficiently small $\epsilon > 0$. From (1.10), (2.52) follows. Then consider a function

$$g(x,t) = C_0(-\zeta_0 t^{\frac{1}{1+\alpha(1-\gamma)}} - x)_+^{\alpha}, \quad \text{in} \quad G_\delta \tag{2.135}$$

$$G_\delta = \{(x,t) : \eta_{\delta_*}(t) < x < +\infty, \quad 0 < t < \delta\}$$

From (2.35), it follows that for $\ell > \delta_*$ and $\epsilon > 0$ there exists a $\delta(\epsilon, \ell) > 0$ such that

$$u(\eta_{\delta_*}(t), t) \leq C\delta_*^{\alpha} t^{\frac{\alpha}{1+\alpha(1-\gamma)}}, \quad 0 \leq t \leq \delta \tag{2.136}$$

We need to estimate Lg in

$$G_\delta^+ = \{(x,t) : -\ell t^{\frac{1}{1+\alpha(1-\gamma)}} < x < -\zeta_0 t^{\frac{1}{1+\alpha(1-\gamma)}}, 0 < t < \delta\}$$

$$Lg = -b(g^{\gamma})_x S,$$

$$S = 1 - \frac{\zeta_0 C_0^{1-\gamma}}{b\gamma(1+\alpha(1-\gamma))}(-\zeta_0 t^{\frac{1}{1+\alpha(1-\gamma)}} - x)_+^{\alpha(1-\gamma)} t^{\frac{\alpha(\gamma-1)}{1+\alpha(1-\gamma)}} - \frac{m(\alpha m-1)C_0^{m-\gamma}}{b\gamma}\times$$
$$(-\zeta_0 t^{\frac{1}{1+\alpha(1-\gamma)}} - x)_+^{\alpha(m-\gamma)-1}$$

Take

$$S_x = \frac{\zeta_0\alpha(1-\gamma)C_0^{1-\gamma}}{b\gamma(1+\alpha(1-\gamma))}\times$$
$$(-\zeta_0 t^{\frac{1}{1+\alpha(1-\gamma)}} - x)_+^{\alpha(1-\gamma)-1} t^{\frac{\alpha(\gamma-1)}{1+\alpha(1-\gamma)}} + \frac{m(\alpha m-1)(\alpha(m-\gamma)-1)C_0^{m-\gamma}}{b\gamma}(-\zeta_0 t^{\frac{\alpha}{1+\alpha(1-\gamma)}} - x)_+^{\alpha(m-\gamma)-2}$$

Since $S_x \geq 0$ in G_δ^+, we have

$$S \geq S|_{x=\eta_{\delta_*}(t)} = 1 - \frac{\zeta_0}{b\gamma(1+\alpha(1-\gamma))}\times$$
$$C_0^{1-\gamma}(\delta_* - \zeta_0)^{\alpha(1-\gamma)} - \frac{m(\alpha m-1)}{b\gamma}C_0^{m-\gamma}(\delta_* - \zeta_0)_+^{\alpha(m-\gamma)-1} t^{\frac{\alpha(m-\gamma)-1}{1+\alpha(1-\gamma)}}$$

Then

$$S \geq \epsilon - \frac{m(\alpha m-1)}{b\gamma}C_0^{m-\gamma}(\delta_* - \zeta_0)_+^{\alpha(m-\gamma)-1} t^{\frac{\alpha(m-\gamma)-1}{1+\alpha(1-\gamma)}} \geq 0, \quad 0 \leq t \leq \delta_\epsilon$$

Choose $\delta_\epsilon > 0$ such that

$$Lg \geq -b(g^{\gamma})_x \text{ in } G_{\ell,\delta}^+ \tag{2.137}$$

By using (2.136), we can apply comparison theorem in $G'_\delta = G_\delta \cap \{x < x_0\}, x_0 > 0$, thus we have

$$Lg = 0 \quad \text{in} \quad G'_{\ell,\delta} \setminus \bar{G}^+_{\ell,\delta} \tag{2.138}$$

Choose $C_0 = C_5$ and $\zeta_0 = \zeta_5$, therefore

$$u(\eta_{\delta_*}(t),t) = C\delta_*^{\alpha} \le C_5(\delta_* - \zeta_5)^{\alpha} = g(\eta_{\delta_*}(t),t), \quad 0 \le t \le \delta \tag{2.139}$$

and

$$u(x_0,t) = g(x_0,t) = 0, \quad 0 \le t \le \delta \tag{2.140}$$

$$u(x,0) = g(x,0) = 0, \quad 0 \le x \le x_0 \tag{2.141}$$

For $x_0 > 0$, from (2.137)-(2.141) and by comparison principle we have $\forall \epsilon > 0, \exists \delta_\epsilon > 0$ such that

$$u(x,t) \le C_5(-\zeta_5 t^{\frac{1}{1+\alpha(1-\gamma)}} - x)_+^{\alpha} = g(x,t), \quad \text{in} \quad \bar{G}_\delta^+$$

Since (2.35) is valid along a family of curves $x = \eta_\ell(t)$, choose δ as small as possible such that

$$-\ell t^{\frac{1}{1+\alpha(1-\gamma)}} \le \eta(t) \le -\zeta_5 t^{\frac{1}{1+\alpha(1-\gamma)}}, \quad 0 \le t \le \delta \tag{2.142}$$

$\forall \ell > \delta_*, \exists \epsilon > 0$ such that (2.33) follows from (2.142). □

2.4.4 Interface has initial waiting time

Region (4)

Proof of Theorem 2.1.4. • Case (4a).

Suppose $m > 1, \alpha = \frac{2}{m-1}$ and $\gamma > \frac{m+1}{2}$.

Since the explicit solution

$$u_c(x,t) = C(-x)_+^{\frac{2}{m-1}}\left[1 - \left(\frac{C}{\bar{C}}\right)^{m-1}(m-1)t\right]^{\frac{1}{1-m}}, \quad x \in R, 0 < t \le T_c \tag{2.143}$$

solves $u_{ct} = (u_c^m)_{xx}$, then upper bound estimation is immediate.

To prove a lower estimation, we assume a function

$$u_0(x) = C(-x)^{\frac{2}{m-1}} \tag{2.144}$$

Calculate the operator Lu_0, hence

$$Lu_0 = C(-x)_+^{\frac{2}{m-1}} S,\ S = -\Big(\frac{C}{\bar{C}}\Big)^{m-1} + bC^{\gamma-1}\frac{2\gamma}{m-1}(-x)_+^{\frac{2\gamma-m-1}{m-1}} \tag{2.145}$$

When $\gamma > \frac{m+1}{2}$, then $Lu_0 \le 0$ if $x_1 \le x < +\infty, x_1 < 0$. From (2.145), we have

$$x \ge -\Big[b^{-1}C^{1-\gamma}\frac{m-1}{2\gamma}\Big(\frac{C}{\bar{C}}\Big)^{m-1}\Big]^{\frac{m-1}{2\gamma-m-1}} =: x_1$$

Since $u_{0\epsilon}(x) = (C-\epsilon)(-x)_+^{\frac{2}{m-1}}$, then $\exists\ \bar{x} < 0$ and $\delta_\epsilon > 0$ *s.t.*

$$Lu_{0\epsilon} \le 0,\ \bar{x} < x < +\infty, 0 < t \le \delta_\epsilon$$

it follows that

$$u_{0\epsilon}(\bar{x}) \le u(\bar{x}, t),\ \ 0 \le t \le \delta_\epsilon$$

$$u_{0\epsilon}(x) \le u(x, 0),\ \ \bar{x} \le x \le +\infty$$

By Comparison Theorem

$$(C-\epsilon)(-x)_+^{\frac{2}{m-1}} \le u(x,t),\ x \ge \bar{x}, 0 < t \le \delta_\epsilon \tag{2.146}$$

Hence, lower estimation is proved.

Since u_0 solves (1.8), (1.10), then (2.52) satisfies with $\alpha = \frac{2}{m-1}$ and by continuity of u_0, (2.54) follows. Then (2.37) hold.

- Case (4b).

Suppose $m>1, \alpha=\frac{2}{m-1}$ and $\gamma=\frac{m+1}{2}$. Consider a function (2.144) solves CP (1.8), (1.11), as before applying the same analysis to (2.144) and from (2.145)

$S=0$ if $C=C_*$, then $Lu_0=0$ and $u(x,t)=C_*(-x)_+^{\frac{2}{m-1}}$ is stationary global solution.

To get lower bound, $S<0$ if $C>C_*$, then $Lu_0\leq 0$, thus

$$C(-x)_+^{\frac{2}{m-1}} \leq u(x,t),\ x\in R^1,\ 0\leq t<+\infty$$

by (2.143), therefore (2.40) follows.

To get upper bound, $S>0$ if $C<C_*$, then $Lu_0\geq 0$, thus

$$u(x,t)\leq C(-x)_+^{\frac{2}{m-1}},\ x\in R^1,\ 0\leq t<+\infty$$

where

$$C\geq \Big(\frac{b(m-1)}{2m}\Big)^{\frac{2}{m-1}} := C_* \tag{2.147}$$

To get sharp lower bound,consider

$$g_\epsilon(x,t)=C\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{2}{m-1}}$$

calculate Lg_ϵ, we get

$$Lg_\epsilon=-b(g_\epsilon^\gamma)_x S_\epsilon,$$

$$S_\epsilon=1-C^{1-\gamma}\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{m-2\gamma+1}{m-1}}-\frac{C^{m-\gamma}m(m+1)}{b\gamma(m-1)(1+C^{\gamma-1}b\gamma t)}\Big(\frac{-x}{1+C^{\gamma-1}b\gamma t}\Big)_+^{\frac{m-2\gamma+1}{m-1}}$$

For sufficiently small ϵ , $\exists x_\epsilon < 0, \delta_\epsilon > 0, x_\epsilon \leq x < +\infty$ such that

$$S_\epsilon \leq 0,$$

$$Lg_\epsilon \leq 0, \text{ for } x_\epsilon \leq x < +\infty$$

$$g_\epsilon(x_\epsilon, t) \leq u(x_\epsilon, t), \quad 0 \leq t < \delta_\epsilon$$

By comparison theorem

$$C\Big(\frac{-x}{1 + C^{\gamma-1}b\gamma t}\Big)_+^{\frac{2}{m-1}} \leq u(x,t), \quad x_\epsilon \leq x < +\infty, \quad 0 \leq t \leq \delta$$

If $C < C_*^1$, then (2.39) holds. Consider a function (2.144) solves CP (1.8), (1.10) with $\alpha = \frac{2}{m-1}$, the (2.41), (2.42) follows.

- Case (4c).

Suppose $m > 1, \frac{1}{m-\gamma} < \alpha < \frac{1}{\gamma-1}$ and $1 < \gamma < \frac{m+1}{2}$. To prove lower and upper estimation, By using an explicit solution (2.94) for convection equation (2.49) when $\alpha = \frac{1}{\gamma-1}$, and by applying the same analysis in the case when $\gamma > 1$ and $0 < C < C_*$ in the region (3), applying comparison principle, then (2.43) hold with $\frac{1}{m-\gamma} < \alpha < \frac{1}{\gamma-1}$.

- Case (4d).

Either $\frac{m+1}{2} < \gamma < m, \ \frac{2}{m-1} < \alpha \leq \frac{1}{m-\gamma}$, or $\gamma \geq m, \ \alpha > \frac{2}{m-1}$. To prove lower estimation, for arbitrary $\alpha > 0$ and $\forall \epsilon > 0$ take a function (2.122) with $C = C - \epsilon$. As before, calculate Lu_0. Hence

$$Lu_0 = (u_0^m)_{xx} S, \tag{2.148}$$

$$S = -1 + \frac{b\gamma(C-\epsilon)^{\gamma-m}}{m(\alpha m - 1)}(-x)_+^{1-\alpha(m-\gamma)}, -1 << x < 0$$

If $\alpha < \frac{1}{m-\gamma}$, then $Lu_0 \leq 0$. Therefore,

$$\forall \epsilon > 0 \; \exists x_\epsilon < 0, \; \delta_\epsilon > 0 \;\; s.t.$$
$$(C-\epsilon)(-x)_+^\alpha \leq u(x,t), \; x \in R^1, \; 0 \leq t < +\infty \tag{2.149}$$

To estimate upper bound, $\forall \epsilon > 0$ consider a function

$$g_C(x,t) = (C+\epsilon)(-x)_+^\alpha (1-Lt)^{\frac{1}{1-m}} \tag{2.150}$$

Calculating Lg_C, then

$$Lg_C = \frac{C+\epsilon}{m-1}(-x)_+^\alpha (1-Lt)^{\frac{m}{1-m}} S,$$
$$S = L - \alpha m(\alpha m-1)(m-1)(C+\epsilon)^{m-1}(-x)^{\alpha(m-1)-2} + b\alpha\gamma(m-1)(C+\epsilon)^{\gamma-1} \times$$
$$(-x)^{\alpha(\gamma-1)-1}(1-Lt)^{\frac{\gamma-m}{1-m}}$$

$$Lg_C > 0 \text{ for small } 0 < t << 1 \text{ if}$$
$$L - \alpha m(\alpha m-1)(m-1)(C+\epsilon)^{m-1}(-x)^{\alpha(m-1)-2} + b\alpha\gamma(m-1)(C+\epsilon)^{\gamma-1} \times$$
$$(-x)^{\alpha(\gamma-1)-1} > 0$$

For sufficiently small $\epsilon > 0$, we have

$$g_C(x,t) = (C+\epsilon)(-x)^\alpha (1-\epsilon t)^{\frac{1}{1-m}}$$
$$\epsilon = \alpha m(\alpha m-1)(m-1)(C+\epsilon)^{m-1}(-x)^{\alpha(m-1)-2} - b\alpha\gamma(m-1)(C+\epsilon)^{\gamma-1} \times$$
$$(-x)^{\alpha(\gamma-1)-1} \tag{2.151}$$

Hence, Lg_C is superslution of u, such that

$$\forall\epsilon > 0, \exists x_\epsilon < 0, \delta_\epsilon > 0 \quad \text{then}$$
$$u(x,t) \le (C+\epsilon)(-x)^\alpha (1-\epsilon t)^{\frac{1}{1-m}}, \quad x_\epsilon \le x < +\infty, 0 \le t \le \delta_\epsilon$$

If $\alpha > \frac{2}{m-1}$, then (2.44) satisfies.

- Case (4e).

Suppose $1 < \gamma < m$ and $\alpha > \max\left(\frac{1}{\gamma-1}; \frac{1}{m-\gamma}\right)$.
For arbitrary $\alpha > 0$ and by applying comparsion theorem, it may easily to check lower bound by following the same analysis in the case (c). To prove upper estimation, for arbitrary $\alpha > 0$ and $\forall\epsilon > 0$ take a function (2.122) with $C = C + \epsilon$. As before, calculate Lu_0. Hence

$$Lu_0 = -b(u_0^\gamma)_x S, \tag{2.152}$$
$$S = 1 - \frac{m(\alpha m - 1)(C+\epsilon)^{m-\gamma}}{b\gamma}(-x)_+^{\alpha(m-\gamma)-1}$$

If $\alpha > \frac{1}{\gamma-1}$, then $Lu_0 \ge 0$, thus

$$u(x,t) \le (C+\epsilon)(-x)_+^\alpha, \quad x_\epsilon \le x < +\infty, \quad t \ge 0$$

Hence $\forall\epsilon > 0, \exists x_\epsilon < 0, \delta_\epsilon > 0$, (2.45) satisfies.

□

Chapter 3

Classification of Interfaces and Asymptotic Properties of Solutions for the Nonlinear Degenerate Diffusion-Convection Equation: Convection alongside Diffusion

In this chapter, we present a full classification of the short-time behavior of the interfaces and local solutions near the interfaces in Cauchy Problem (1.8)-(1.10) with $b < 0$. It will be specifically mentioned when u is a global solution to the problem (1.8), (1.11).

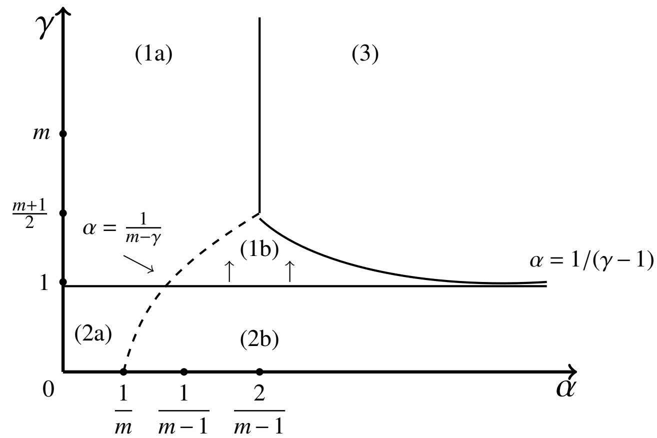

Figure 3.1: Classification of different cases in the (α,γ) plane for interface development in problem (1.8)-(1.11) when $b < 0$

3.1 Description of Main Results

In the Figure 3.1 we present a full classification of different cases in the (α,γ) plane for interface development in problem (1.8), (1.10) if $b < 0$.

- **Region (1):** $\gamma \geq 1,\ 0 \leq \alpha < 1/(m - \min\{\gamma, (m+1)/2\})$; Diffusion strongly dominates over the convection and interface expands.

- **Region (2):** $1 \leq \gamma < (m+1)/2,\ 1/(m-\gamma) < \alpha < 1/(\gamma - 1)$; Convection dominates over the diffusion and interface expands.

- **Region (3):** $1 \leq \gamma < (m+1)/2,\ \alpha = 1/(m-\gamma)$; Diffusion and convection are in balance and interface initially expands.

- **Region (4):** $\gamma < 1$; Convection strongly dominates over the diffusion and causes infinite speed of propagation. Interface is absent.

- **Region (5):** $\gamma > 1,\ \alpha \geq max(2/(m-1), 1/(\gamma-1))$; Both diffusion and convection are weak, and interface has initial ”waiting time”.

The following are the main results on the asymptotic properties of the interface curve $\eta(t)$, and local solution near the interface in respective regions (1)-(4).

Region (1)

Theorem 3.1.1. *If* $\gamma \geq 1$ *and* $0 \leq \alpha < 1/(m - min\{\gamma, (m+1)/2\})$*, then solution u satisfies asymptotic property* (2.2)*, and its interface initially expands with*

$$\eta(t) \sim \bar{\xi} t^{1/(2-\alpha(m-1))} \;\; as\; t \to 0+, \tag{3.1}$$

where $\bar{\xi} = \xi_*$*, if* $\alpha < 1/(m-1)$*, and* $\bar{\xi} \in [\xi_*, \xi_1]$*, if* $\alpha \geq 1/(m-1)$ *with*

$$\xi_1 = \Big(m(m-1)^{-1}(2+\alpha(1-m))f^{m-1}(0)\Big)^{\frac{1}{2}}. \tag{3.2}$$

Further Details of Theorem 3.1.1: We refer to relevant further details of the Theorem 2.1.1. In the proof of Theorem 3.1.1 we establish the following estimation: for arbitrary $\forall \epsilon \; \exists \; \delta_\epsilon > 0$ such that

$$t^{\frac{\alpha}{2+\alpha(1-m)}} C_2(\xi_2^\epsilon - \xi)_+^{\frac{1}{m-1}} \leq u(x,t) \leq t^{\frac{\alpha}{2+\alpha(1-m)}} C_1(\xi_1^\epsilon - \xi)_+^{\frac{1}{m-1}}, \; 0 \leq x \leq +\infty, 0 \leq t \leq \delta_\epsilon \tag{3.3}$$

$$\xi_1^\epsilon = \Big[\frac{(m+\epsilon)(2+\alpha(1-m))(f(0)+\epsilon)^{m-1}}{\min(\alpha; 1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}}, \; C_1 = (\xi_1^\epsilon)^{\frac{1}{1-m}}(f(0)+\epsilon)$$

$$\xi_2^\epsilon = \Big[\frac{m(2+\alpha(1-m))(f(0)-\epsilon)^{m-1}}{\max(\alpha; 1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}}, \; C_2 = (\xi_2^\epsilon)^{\frac{1}{1-m}}(f(0)+\epsilon)$$

Estimation (3.3) is just ϵ-modification of the similar estimation for the nonlinear diffusion equation (PDE (1.8) with $b = 0$)[13]. In particular, it implies estimation for the interface constant ξ_*:

$$\xi_2 \leq \xi_* \leq \xi_1, \tag{3.4}$$

where

$$\xi_1 = \Big[\frac{m(2+\alpha(1-m))f^{m-1}(0)}{\min(\alpha;1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}},\ \xi_2 = \Big[\frac{m(2+\alpha(1-m))f^{m-1}(0)}{\max(\alpha;1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}}$$

Note that ξ_1 coincides with (3.2) due to condition $\alpha \geq 1/(m-1)$.

Region (2)

Theorem 3.1.2. *Let* $1 \leq \gamma < (m+1)/2$ *and* $1/(m-\gamma) < \alpha < 1/(\gamma-1)$*, then interface initially expands and*

$$\eta(t) \sim \tilde{\xi} t^{1/(1-\alpha(\gamma-1))} \text{ as } t \to 0+ \tag{3.5}$$

where

$$\tilde{\xi} = (1-\alpha(\gamma-1))[-b\gamma C^{(\gamma-1)}]^{\frac{1}{1-\alpha(\gamma-1)}}[(\gamma-1)\alpha]^{\frac{\alpha(\gamma-1)}{1-\alpha(\gamma-1)}} \tag{3.6}$$

if $\gamma > 1$*, and* $\tilde{\xi} = -b$*, if* $\gamma = 1$*. For any* $\delta \in (0, \tilde{\xi}]$*, we have*

$$u(\xi_\delta(t), t) \sim C_\delta t^{\alpha/(1-\alpha(\gamma-1))}, \text{ as } t \to 0+ \tag{3.7}$$

along the curve $\xi_\delta(t) = \delta t^{1/(1-\alpha(\gamma-1))}$ *and*

$$C_{\tilde{\xi}} = [-b\gamma(\gamma-1)\alpha C^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} \tag{3.8}$$

Region (3)

Theorem 3.1.3. *Let* $1 \leq \gamma < (m+1)/2,\ \alpha = 1/(m-\gamma)$*. Then interface initially expands, and asymptotic formulae* (2.10) *and* (2.11) *are satisfied.*

Further Details of Theorem 3.1.3: We refer to the details of the Theorem 2.1.2 for relevance. Precise values of the constant ζ_* and the function h are associated with the one-dimensional Cauchy Problem (1.8),(1.11), which has a unique solution of self-similar form (2.12), with the shape function h solving nonlinear ODE problem (2.13), and having a finite interface $\zeta_* > 0$ as in (2.14),(2.15). In the proof of Theorem 3.1.3 we establish the following global estimation for the solution of the Cauchy Problem (1.8),(1.11) if $\gamma > 1$:

$$C_1(\zeta_1 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}} \le u(x,t) \le C_2(\zeta_2 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}},\ 0 \le x, t \le +\infty, \tag{3.9}$$

$$\zeta_1 = \frac{[A_1^{m-1} m(m+1-2\gamma)]^{\frac{1}{2}}}{(m-1)},\ C_1 = \zeta_1^{\frac{1}{1-m}} A_1,\ C_2 = \zeta_2^{\frac{1}{1-m}} A_1,\ A_1 = h(0) \tag{3.10}$$

ζ_2 is a positive solution of the quadratic equation

$$z^2 + \frac{b\gamma(m+1-2\gamma)A_1^{\gamma-1}}{m-\gamma} z - \frac{m(m+1-2\gamma)A_1^{m-1}}{(m-1)(m-\gamma)} = 0$$

(3.9) implies the estimation for ζ_*:

$$\zeta_1 \le \zeta_* \le \zeta_2. \tag{3.11}$$

In the particular case of $\gamma = 1$, the problem (1.8),(1.11) has an explicit solution (2.16),(2.17) with

$$\zeta_* = \frac{C^{m-1} m}{m-1} - b$$

Region (4)

Theorem 3.1.4. *If $\gamma < 1$, then there is an infinite speed of propagation ($\eta(t) = +\infty$) and for some $\delta > 0$ we have*

$$u(x,t) \ge [(-b\gamma)t]^{\frac{1}{1-\gamma}} (1+x)^{\frac{1}{\gamma-1}},\ 0 \le x \le +\infty,\ 0 \le t \le \delta. \tag{3.12}$$

Region (5)

Theorem 3.1.5. *If* $\gamma > 1$ *and* $\alpha \geq max(2/(m-1), 1/(\gamma-1))$*, then the interface has an initial waiting time.*

3.2 Asymptotic Properties of Solutions and Rescaling Principles

The proof of the main results formulated in Section 3.1 consists of two major steps. First step consists of the proof of the asymptotic formula for the solution of the diffusion-convection problem along some class of curves approaching the initial position of the interface origin from the support of the solution. The idea of the method is rescaling of the solution with subsequent iteration and use of compactness argument based on the interior Hölder regularity estimations. We formulate next lemmas outlining these results.

Lemma 3.2.1. *Let* u *be a solution to the CP* (2.49) *with* $b < 0$*,* $1 < \gamma$ *and* $0 < \alpha < 1/(\gamma-1)$*, then*

$$u(0,t) = C^{\frac{1}{1-\alpha(\gamma-1)}}[-b\gamma t]^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \quad 0 \leq t \leq \sigma$$

and for any $\delta \in (0, \xi_1]$*(see* (3.5) *),there exist* $C_\delta > 0$ *such that*

$$u(\xi_\delta(t), t) = C_\delta t^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \ t \downarrow 0 \quad \textit{along} \tag{3.13}$$

$$\xi_\delta(t) = \delta t^{\frac{1}{1-\alpha(\gamma-1)}} \tag{3.14}$$

In particular, (3.8) *satisfies.*

Lemma 3.2.2. *Let u be a solution to the CP* (1.8), (1.10) *with* $b < 0,\ 1 < \gamma < (m+1)/2,\ 1/(m-\gamma) < \alpha < 1/(\gamma-1)$*.Then*

$$u(0,t) \sim [-b\gamma C^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} t^{\frac{\alpha}{1-\alpha(\gamma-1)}},\ \ 0 < t \le \delta$$

For any $l \in (0,\xi_1]$ *(see* (3.5)*) there exist* $C_l > 0$*, such that the asymptotic formula* (3.7) *is valid with* $\xi_l(t) = C_l t^{\frac{1}{1-\alpha(\gamma-1)}}$.

Lemma 3.2.3. *Let u be a solution to the CP* (2.49) *with* $b < 0,\ \gamma < 1$*, then for any arbitrary* $\delta \ge 0$*,there exist* $C_\delta \in (0, (\frac{\delta}{-b\gamma})^{\frac{1}{\gamma-1}})$ *and* $\sigma_\delta > 0$ *such that*

$$u(\zeta_\delta(t),t) = C_\delta t^{\frac{\alpha}{1+\alpha(1-\gamma)}},\ \ t \downarrow 0 \ along\ \ \zeta_\delta(t) = \delta t^{\frac{1}{1+\alpha(1-\gamma)}} \tag{3.15}$$

In particular,

$$C_0 = C^{\frac{1}{1+\alpha(1-\gamma)}}(-b\gamma)^{\frac{\alpha}{1+\alpha(1-\gamma)}} > 0 \tag{3.16}$$

satisfies.

Lemma 3.2.4. *Let u be a solution to the CP* (1.8), (1.10) *with* $b < 0,\ \gamma < 1$*, then for any arbitrary* $\delta \ge 0$*,there exist* $C_\delta \in (0, (\frac{\delta}{-b\gamma})^{\frac{1}{\gamma-1}})$ *such that the asymptotic formula*

$$u(\zeta_\delta(t),t) \sim C_\delta t^{\frac{\alpha}{1+\alpha(1-\gamma)}},\ as\ t \to 0+ \tag{3.17}$$

is valid with $\zeta_\delta(t) = \delta t^{\frac{1}{1+\alpha(1-\gamma)}}$.

3.2.1 Proof of Lemma 3.2.1: Asymptotic properties of the convection equation

$$\text{If } x = 0 \text{ then } u(0,t) = C^{\frac{1}{1-\alpha(\gamma-1)}}(-b\gamma t)^{\frac{\alpha}{1-\alpha(\gamma-1)}},\ \ 0 < t \le t_0$$

As before, we follow the same analysis in lemma 2.3.3 to solve (2.77) along the family of curves (3.14), then

$$u(\xi_\delta(t),t) = C(-\xi_\delta(t) - b\gamma u(\xi_\delta(t),t)^{\gamma-1}t)_+^{\alpha} \tag{3.18}$$

To solve (3.18), assume that

$$u(\xi_\delta(t),t) \sim \hat{C} t^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

therefore, the implicit equation (2.77) has a solution $u = C_\delta t^{\frac{\alpha}{1-\alpha(\gamma-1)}}$,if we plug it in (3.18), we have

$$C_\delta = C(-\delta - b\gamma C_\delta^{\gamma-1})_+^{\alpha} \tag{3.19}$$

$$C_\delta > 0 \text{ if } C_\delta > [\frac{\delta}{-b\gamma}]^{\frac{1}{\gamma-1}} \tag{3.20}$$

From (3.19) and (3.20), we get

$$F(C_\delta) = C^{\frac{-1}{\alpha}} C_\delta^{\frac{1}{\alpha}} + b\gamma C_\delta^{\gamma-1} = -\delta \tag{3.21}$$

It is clear that

$$F(0) = 0,\ F(+\infty) = +\infty \text{ and } F(C_\delta) = 0 \text{ iff } \hat{C}_\delta = \Big[-b\gamma C^{\frac{1}{\alpha}}\Big]^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

By taking derivative to (3.21) and since $\alpha(\gamma-1) < 1$, we have $0 < C_\delta^* < C_\delta$, where C_δ^* is given in (3.8). Thus, $F'(C_\delta^*) < 0$. Now, we need to find largest $\delta > 0$ along $\xi_\delta(t)$ such that

$$C_\delta > [\frac{\delta}{-b\gamma}]^{\frac{1}{\gamma-1}} \tag{3.22}$$

which solves (3.21). From (3.8) and (3.21), it follows that , there exists $\xi_1 > 0$ such that (3.22) satisfies. Hence (3.13) holds. □

3.2.2 Proof of Lemma 3.2.2: Convection dominates over the diffusion

As before, from lemma 2.3.4 we can apply the same analysis to prove the existence and uniqueness of CP (1.8), (1.11) with $b < 0$, and by using comparison theorem to CP (1.8) and (1.11) with $u_0 = (C \pm \epsilon)(-x)_+^{\alpha}$, $T = +\infty$, then the results hold. Again, by using rescaling laws, we get that the sequence $u_k^{\pm\epsilon}$ satisfies Dirichlet problem (2.98). Moreover, we need to prove $u_k^{\pm\epsilon}$ is uniformly bounded sequence by considering the function (2.99). Calculating Lg and by using comparison theorem [3], we have, as in the proof of lemma 2.3.1, if $k \gg 1$ implies (2.70) in $E_{0\epsilon}^k$. Hence, $\{u_k^{\pm\epsilon}\}$ is uniformly bounded in $\bar{E}_{o\epsilon}^k$ and accordingly [3], it is uniformly Hölder continuous on any compact $G \subset P = \{x \in \mathbb{R},\ 0 < t \leq \sigma\}$ so $\exists k'$ such that (2.71) solves (2.103).
Now, Take $\tau = k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t$, we obtain

$$\lim_{\tau \downarrow 0} \frac{k u_{\pm\epsilon}(\xi_\rho(\tau), \tau)}{t^{\frac{\alpha}{1-\alpha(\gamma-1)}}} = \lim_{\tau \downarrow 0} \frac{u_{\pm\epsilon}(\xi_\delta(\tau), \tau)}{\tau^{\frac{\alpha}{1-\alpha(\gamma-1)}}} = [b\gamma(\gamma-1)\alpha(C \pm \epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

$$u_{\pm\epsilon}(\xi_\delta(\tau), \tau) \sim [b\gamma(\gamma-1)\alpha(C \pm \epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} \tau^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \text{ as } \tau \downarrow 0.$$

by comparison principle results, we get

$$[b\gamma(\gamma-1)\alpha(C-\epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} \leq t^{-\frac{\alpha}{1-\alpha(\gamma-1)}} u(\xi_\delta(t), t) \leq [b\gamma(\gamma-1)\alpha(C+\epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}},$$

Now, recall that the global solutions $u_{\pm\epsilon}$ have the asymptotic equivalence along $x = \xi_\rho(t)$, we obtain as $t \downarrow 0$ and $\epsilon \downarrow 0$

$$u(x,t) \sim [b\gamma(\gamma-1)\alpha C^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} t^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \text{ where } x = \xi_\delta(t) = \delta t^{\frac{1}{1-\alpha(\gamma-1)}}$$

Hence

$$u_{\pm\epsilon}(\xi_\delta(\tau),\tau) \sim [b\gamma(\gamma-1)\alpha(C\pm\epsilon)^{\frac{1}{\alpha}}]^{\frac{\alpha}{1-\alpha(\gamma-1)}} \tau^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \text{ as } \tau \downarrow 0 \tag{3.23}$$

For arbitrary $\epsilon > 0$, from (2.53) and (3.23), (3.7) follows. □

3.2.3 Proof of Lemma 3.2.3: Asymptotics of the convection equation in a strong regime

Lemma 2.3.3, we have (2.77) is the implicit equation of u. If $x = 0$, $\exists\, C_0 > 0$, such that

$$[u(0,t)]^{1+\alpha(1-\gamma)} = C^{\frac{1}{1+\alpha(1-\gamma)}}(-b\gamma t)^{\frac{\alpha}{1+\alpha(1-\gamma)}} = C_0$$

Along $\zeta_\delta(t)$, we have

$$C_\delta = C(-\delta - b\gamma C_\delta^{\gamma-1})_+^\alpha \text{ if } C_\delta < (\frac{\delta}{-b\gamma})^{\frac{1}{\gamma-1}}$$

$\delta \geq 0$ and $\alpha > 0$ is arbitrary if $\gamma < 1$. For sufficiently small $\delta > 0$, we need to prove

$$C_\delta < (\frac{\delta}{-b\gamma})^{\frac{1}{\gamma-1}} \tag{3.24}$$

By (3.21), since $0 < \gamma < 1$ then $F(0) = -\infty$, $F(+\infty) = +\infty$, and $F(C_\delta) > 0$.

$$F(C_\delta) = 0 \iff C_\delta = C^{\frac{1}{1+\alpha(1-\gamma)}}\Big(-b\gamma\Big)^{\frac{\alpha}{1+\alpha(1-\gamma)}} > 0$$

Therefore, $\exists\, C_0 > 0$ such that $F(C_0) = 0 \iff C_0 = C^{\frac{1}{1+\alpha(1-\gamma)}}\Big(-b\gamma\Big)^{\frac{\alpha}{1+\alpha(1-\gamma)}} > 0$. By taking derivative of F,

$$F'(C_\delta) = \frac{1}{\alpha} C^{\frac{-1}{\alpha}} C_\delta^{\gamma-2}\Big[C_\delta^{\frac{1+\alpha(1-\gamma)}{\alpha}} + b\gamma(\gamma-1)\alpha C^{\frac{1}{\alpha}}\Big] > 0$$

therefore we get F is monotonically increasing. Then $\forall\delta > 0, \exists!\ C_\delta$ such that (3.16) satisfies. From (3.21), it may easily to check (3.24). Hence (3.15) holds. □

3.2.4 Proof of Lemma 3.2.4: Convection strongly dominates over the diffusion

As before, from lemma 2.3.1 we can apply the same analysis to prove the existence and uniqueness of CP (1.8), (1.11) with $b < 0$, and by using comparison theorem to CP (1.8) and (1.11) with $u_0 = (C \pm \epsilon)(-x)_+^\alpha$, $T = +\infty$, then the results hold. Again, by using rescaling laws, the sequence $u_k^{\pm\epsilon}$ satisfies Dirichlet problem (2.63a), (2.63b) and (2.63c). Moreover, we need to prove $u_k^{\pm\epsilon}$ is uniformly bounded sequence by considering the function (2.65). Calculating Lg , it follows

$$L_k g = (C+1)(1+x^2)^{\frac{\alpha}{2}}(m-1)^{-1}(1-vt)^{\frac{m}{1-m}} S \quad in \quad D_\epsilon^k$$

where S defines as (2.67). We have, $h_* = h(\alpha, m) = max_R h(x)$ is finite, and

$$R(x) = O(k^{\frac{\alpha(\gamma-1)-1}{\alpha}}) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

Therefore, $S = 1 + h(x) - R \to 1 + R$, as $k \to \infty$, $\Rightarrow 1 + R \leq S > 0 \Rightarrow S > 0 \; as \; k \to \infty$

So

$$L_k g \geq 0 \text{ in } E^k_{0\epsilon} = E^k_\epsilon \cap \{0 < t \leq \sigma\}$$

Moreover, for $0 < \epsilon \ll 1 \Rightarrow$ then (2.69a) and (2.69b) satisfy. As before, from comparison theorem, (2.70) satisfies. Hence, $\{u_k^{\pm\epsilon}\}$ is uniformly bounded in $\bar{E}^k_{o\epsilon}$ and accordingly [3], it is uniformly Hölder continuous on any compact $G \subset P = \{x \in \mathbb{R},\ 0 < t \leq \sigma\}$ so $\exists k'$ such that (2.71) solves (2.103).

Let $\tau = k^{\frac{\alpha(\gamma-1)-1}{\alpha}} t$, it is easy to satisfies (3.12) as $\tau \to 0$, $k \to +\infty$. For arbitrary $\epsilon > 0$, from (2.53) and (3.12), (3.17) follows. □

3.3 Proofs of the main results

In this section, we prove the main results in the case when $b < 0$ and $m > 1$.

3.3.1 Diffusion dominates and interface expands

Region (1)

Proof of Theorem 3.1.1. • **First:** Linear Convection: Consider the CP (1.8), (1.11) with linear convection, $\gamma = 1$:

Define a new function $v(y,t)$ where $y = x + bt$ and v satisfies

$$v(y,t) = u(y - bt, t),\ u(x,t) = v(x + bt, t) \tag{3.25}$$

By expressing CP (1.8), (1.11) in terms of v and y, we have v solves the following problem:

$$\begin{cases} v_t = (v^m)_{yy} & y \in \mathbb{R}, 0 < t \le T, m > 1 \\ v(y,0) = C(-y)_+^{\alpha} & y \in \mathbb{R} \end{cases} \tag{3.26}$$

By applying lemma 2.3.1, v has the following self-similar form:

$$v(y,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi),\ \xi := yt^{-\frac{1}{2+\alpha(1-m)}},\ f(\xi) := v(\xi,1) \tag{3.27}$$

By (3.27), we have

$$u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f((x+bt)t^{-\frac{1}{2+\alpha(1-m)}} \tag{3.28}$$

$$\eta_v(t) \sim \xi_* t^{1/2+\alpha(1-m)},\ t \to 0^+ \tag{3.29}$$

To investigate interface function for $u(x,t)$, from (3.25) and (3.29) it follows

$$\eta(t) \sim \xi_* t^{1/2-\alpha(m-1)} - bt,\ t \to 0^+ \tag{3.30}$$

We will simply consider the three cases of $\alpha < 1/(m-1)$, $\alpha = 1/(m-1)$, and $1/(m-1) < \alpha < 2/(m-1)$. We will do this analysis for both global and local cases. As for the local behavior of $\eta(t)$,

$$\frac{\eta(t)}{\xi_* t^{\frac{1}{2+\alpha(1-m)}} - bt} \to 1, \quad t \to 0^+$$

$$\forall \epsilon > 0, \exists \delta_\epsilon > 0\ s.t.$$

$$(\xi_* - \epsilon)t^{\frac{1}{2+\alpha(1-m)}} - bt \le \eta(t) \le (\xi_* + \epsilon)t^{\frac{1}{2+\alpha(1-m)}} - bt \quad 0 \le t \le \delta_\epsilon$$

Case I: Consider the case where $\gamma = 1$, $0 < \alpha < \frac{1}{m-1}$. Since $\frac{1}{2+\alpha(1-m)} < 1$, this implies

$|\eta'(t)|$ increases without bound as $t \to 0^+$. On the other hand, for large t,

$$(\xi_* - \epsilon) - bt^{\frac{1+\alpha(1-m)}{2+\alpha(1-m)}} \le \eta(t)t^{-\frac{1}{2+\alpha(1-m)}} \le (\xi_* + \epsilon) - bt^{\frac{1+\alpha(1-m)}{2+\alpha(1-m)}}, \quad 0 \le t \le \delta_\epsilon$$

Passing to the limit as $t \to 0^+$ and $\epsilon \to 0^+$, the limit must exist and there exists $\bar{\xi} = \xi_*$ such that

$$\eta(t) = \bar{\xi}t^{\frac{1}{2-\alpha(m-1)}} - bt \sim \bar{\xi}t^{\frac{1}{2-\alpha(m-1)}} \tag{3.31}$$

and so diffusion dominates for large t.

Case II: Consider the case where $\gamma = 1$, $\frac{1}{m-1} < \alpha < \frac{2}{m-1}$. Since $\frac{1}{2+\alpha(1-m)} > 1$, the derivative of (3.30) at zero is $-b$ and convection dominates near $t = 0$. However, passing to the limit as $t \to 0^+$ and $\epsilon \to 0^+$, so the limit exist such that

$$\eta(t) \sim -bt \tag{3.32}$$

Therefore, diffusion dominates for small t but convection dominates for large t.

Case III: Consider the borderline case where $\gamma = 1$, $\alpha = \frac{1}{m-1}$. As before, passing to the limit as $t \to 0^+$ and $\epsilon \to 0^+$, then

$$\eta(t) \sim (\xi_* - b)t \quad \text{as} \quad t \to 0^+ \tag{3.33}$$

- **Second:** Consider the case when $b < 0$ and $\gamma > 1$.

Let u_ε be a solution to CP (1.8), (1.11) with $C = C + \epsilon$. We want to prove that u_ε is a supersolution. We have

$$Lu_\varepsilon = b(u_\varepsilon)_x\left[1 - \gamma(u_\varepsilon)^{\gamma-1}\right] \ge 0 \text{ if } |u_\varepsilon| \ll 1.$$

Case I: Consider $\gamma > 1$, $0 < \alpha < \frac{1}{m-1}$

From interface function (3.30), we have

$$\limsup_{t\to 0} \eta(t) \le \bar{\xi} t^{\frac{1}{2+\alpha(1-m)}} \tag{3.34}$$

Case II: Consider $\gamma > 1, \alpha = \frac{1}{m-1}$. Lemma (2.3.1), we have

$$\xi_* \le \liminf_{t\to 0^+} \eta(t)/t \le \limsup_{t\to 0^+} \eta(t)/t \le (\xi_* - b)$$

Case III: Consider $\gamma > 1$, $\frac{1}{m-1} \le \alpha < \frac{2}{m-1}$

Let $\frac{1}{m-1} \le \alpha < \frac{1}{m-\min(\gamma,\frac{m+1}{2})}$ and $\gamma \ge 1$. From asymptotic estimation (2.2) and lemma 2.3.1, $\forall \epsilon > 0$, $\exists \delta_1 = \delta_1(\epsilon) > 0$ such that

$$(f(0)-\epsilon) t^{\frac{\alpha}{2-\alpha(m-1)}} \le u(0,t) \le (f(0)+\epsilon) t^{\frac{\alpha}{2-\alpha(m-1)}}, \quad 0 \le t \le \delta_1 \tag{3.35}$$

is valid. Let $\gamma \ge 1$, consider a function

$$g(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi)\ , \ \xi = x t^{-\frac{1}{2+\alpha(1-m)}} \tag{3.36}$$

$$g(0,t) = C_0 \xi^{\mu} t^{\frac{\alpha}{2+\alpha(1-m)}}$$

Calculating Lg, then

$$Lg = t^{\frac{\alpha m-2}{2+\alpha(1-m)}} \left(\frac{\alpha}{2+\alpha(1-m)} f - \frac{\xi}{(2+\alpha(1-m))} f' - (f^m)'' - b t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}} (f^{\gamma})' \right) \tag{3.37}$$

As a function f we take

$$f(\xi) = C_0 \Big(\xi_0 - \xi\Big)_+^{\mu} \quad 0 \le \xi < +\infty \tag{3.38}$$

where C_0, ξ_0, μ are positive constants and choose $\mu = \frac{1}{m-1}$. We compute (3.38), and by (3.37), we have

$$Lg = t^{\frac{\alpha m-2}{2+\alpha(1-m)}} L_t f,$$

$$L_t f = \frac{\alpha}{2+\alpha(1-m)} C_0\Big(\xi_0 - \xi\Big)_+^{\frac{1}{m-1}} +$$
$$\frac{\xi}{(m-1)(2+\alpha(1-m))} C_0\Big(\xi_0-\xi\Big)_+^{\frac{2-m}{m-1}} - \frac{m}{(m-1)^2} C_0^m\Big(\xi_0-\xi\Big)_+^{\frac{2-m}{m-1}} + \frac{b\gamma}{m-1} t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}} C_0^{\gamma}\Big(\xi_0-\xi\Big)_+^{\frac{\gamma-m+1}{m-1}}$$

where

$$L_t f = (2+\alpha(1-m))^{-1} C_0\Big(\xi_0-\xi\Big)_+^{\frac{2-m}{m-1}} \times \tag{3.39}$$
$$\Big[\alpha\Big(\xi_0-\xi\Big)_+ + \xi(m-1)^{-1} - m(2+\alpha(1-m))(m-1)^{-2} C_0^{m-1} + b\gamma(2+\alpha(1-m))(m-1)^{-1} \times$$
$$t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}} C_0^{\gamma-1}\Big(\xi_0-\xi\Big)_+^{\frac{\gamma-1}{m-1}}\Big]$$

Simplifying (3.39), we get

$$L_t f = \alpha\xi_0 + \Big(\frac{1}{m-1} - \alpha\Big)\xi -$$
$$m(2+\alpha(1-m))(m-1)^{-2} C_0^{m-1} + b\gamma(2+\alpha(1-m))(m-1)^{-1} t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}} C_0^{\gamma-1}\Big(\xi_0-\xi\Big)_+^{\frac{\gamma-1}{m-1}} \tag{3.40}$$

To estimate upper bound, since $\frac{1}{m-1} - \alpha < 0$, we have $\xi = \xi_0$

$$L_t f \geq 0 \quad \text{if} \quad \xi = \xi_0$$

For any $\epsilon > 0$,

$$L_t f \geq \frac{1}{m-1}\xi_0 - m(2+\alpha(1-m))(m-1)^{-2} C_0^{m-1} + \epsilon \geq 0 \tag{3.41}$$

Choose $\xi_0 = \xi_1^\epsilon$ and $C_0 = C_1$ such that

$$L_t f \geq \epsilon(2+\alpha(1-m))(m-1)^{-2}C_1^{m-1} + b\gamma(2+\alpha(1-m))(m-1)^{-1}t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}}C_1^{\gamma-1}\xi_1^{\epsilon\,\frac{\gamma-1}{m-1}} \geq 0$$

$$-b\gamma(2+\alpha(1-m))(m-1)^{-1}t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}}C_1^{\gamma-1}\xi_1^{\epsilon\,\frac{\gamma-1}{m-1}} \leq \epsilon(2+\alpha(1-m))(m-1)^{-2}C_1^{m-1}, \quad 0 \leq t \leq \delta_\epsilon$$

By comparison theorem we have

$$Lg \geq 0 \quad \text{for} \quad 0 \leq x < +\infty, \quad 0 \leq t \leq \delta_\epsilon \tag{3.42}$$

$$u(x,t) \leq g(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}}C_1(\xi_1^\epsilon - \xi)_+^{\frac{1}{m-1}}, \quad 0 \leq x \leq +\infty, \quad 0 \leq t \leq \delta_\epsilon$$

$$C_1 = (\xi_1^\epsilon)^{\frac{1}{1-m}}(f(0)+\epsilon)$$

$$\xi_1^\epsilon = \Big[\frac{(m+\epsilon)(2+\alpha(1-m))(f(0)+\epsilon)^{m-1}}{\min(\alpha; 1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}}$$

Since $\xi = \xi_1^\epsilon$, we have

$$\eta(t) \leq \xi_1^\epsilon t^{\frac{1}{2+\alpha(1-m)}}, \quad 0 \leq t \leq \delta_\epsilon \tag{3.43}$$

To estimate lower bound, we have $0 \leq \xi \leq \xi_0$

$$L_t f \leq 0 \quad \text{if} \quad \xi = 0$$

$$L_t f \leq \alpha\xi_0 -$$

$$m(2+\alpha(1-m))(m-1)^{-2}C_0^{m-1} + b\gamma(2+\alpha(1-m))(m-1)^{-1}t^{\frac{\alpha(\gamma-m)+1}{2+\alpha(1-m)}}C_0^{\gamma-1}\xi_0^{\frac{\gamma-1}{m-1}}$$

Choose $\xi_0 = \xi_2^\epsilon$ and $C_0 = C_2$ such that

$$L_t f \leq \alpha\xi_2^\epsilon - m(2+\alpha(1-m))(m-1)^{-2}C_2^{m-1} \leq 0$$

By comparison theorem we have

$$Lg \le 0 \quad \text{for} \quad 0 \le x < +\infty, \quad 0 \le t \le \delta_\epsilon \tag{3.44}$$

$$u(x,t) \ge g(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} C_2(\xi_2^\epsilon - \xi)_+^{\frac{1}{m-1}}, \quad 0 \le x \le +\infty, \quad 0 \le t \le \delta_\epsilon$$

$$C_2 = (\xi_2^\epsilon)^{\frac{1}{1-m}} (f(0)+\epsilon)$$

$$\xi_2^\epsilon = \Big[\frac{m(2+\alpha(1-m))(f(0)-\epsilon)^{m-1}}{\max(\alpha; 1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}}$$

Since $0 \le \xi \le \xi_2^\epsilon$, we have

$$\eta(t) \ge \xi_2^\epsilon t^{\frac{1}{2+\alpha(1-m)}}, \quad 0 \le t \le \delta_\epsilon \tag{3.45}$$

Therefore, (3.3) satisfies. □

3.3.2 Convection dominates and interface expands

Region (2)

Proof of Theorem 3.1.2. Let u solves Cauchy problem (1.8). Consider a function

$$g(x,t) = t^{\frac{\alpha}{1-\alpha(\gamma-1)}} f(\xi), \; \xi = x t^{-\frac{1}{1-\alpha(\gamma-1)}} \tag{3.46}$$

Now we calculate Lg, then

$$Lg = t^{\frac{\alpha\gamma-1}{1-\alpha(\gamma-1)}} S, \tag{3.47}$$

$$S = \frac{\alpha}{1-\alpha(\gamma-1)} f(\xi) - \frac{1}{1-\alpha(\gamma-1)} \xi (f(\xi))' - t^{\frac{\alpha(m-\gamma)-1}{1-\alpha(\gamma-1)}} \Big(f^m(\xi)\Big)'' - b(f^\gamma(\xi))'$$

To estimate upper bound, from (3.38) when $\mu = \frac{1}{m-1}$ and (3.47), it follows

$$S = \Big(\frac{[\alpha(\xi_0-\xi)+\mu\xi]}{1-\alpha(\gamma-1)} - t^{\frac{\alpha(m-\gamma)-1}{1-\alpha(\gamma-1)}} C_0^{m-1}\frac{m}{(m-1)^2} + bC_0^{\gamma-1}\frac{\gamma}{m-1}(\xi_0-\xi)^{\frac{\gamma-1}{m-1}}\Big)\Big(\xi_0-\xi\Big)_+^{\frac{2-m}{m-1}}$$

$$= \Big(\xi_0-\xi\Big)_+^{\frac{2-m}{m-1}} S_1,$$

$$S_1 = \frac{[\alpha\xi_0 + (\frac{1}{m-1}-\alpha)\xi]}{1-\alpha(\gamma-1)} - t^{\frac{\alpha(m-\gamma)-1}{1-\alpha(\gamma-1)}} C_0^{m-1}\frac{m}{(m-1)^2} + bC_0^{\gamma-1}\frac{\gamma}{m-1}(\xi_0-\xi)^{\frac{\gamma-1}{m-1}}$$

Since $\frac{1}{m-1} < \alpha < \frac{1}{\gamma-1}$, then

$$S_1 \geq \frac{\xi_0}{(m-1)(1-\alpha(\gamma-1))} - t^{\frac{\alpha(m-\gamma)-1}{1-\alpha(\gamma-1)}} C_0^{m-1}\Big(\frac{m}{m-1}\Big)^2 + bC_0^{\gamma-1}\frac{\gamma}{m-1}\xi_0^{\frac{\gamma-1}{m-1}}$$

$$\frac{\xi_0}{(m-1)(1-\alpha(\gamma-1))} + bC_0^{\gamma-1}\frac{\gamma}{m-1}\xi_0^{\frac{\gamma-1}{m-1}} \geq \epsilon \tag{3.48}$$

Thus

$$g(0,t) \geq u(0,t), \quad \text{if}$$

$$C_0\xi_0^{\frac{1}{m-1}} \geq C^{\frac{1}{1-\alpha(\gamma-1)}}\Big[-b\gamma\Big]^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

Choose $C_0 = C_2$ and $\xi_0 = \xi_2$ such that (3.48) becomes

$$\xi_2 \geq (1-\alpha(\gamma-1))C^{\frac{\gamma-1}{1-\alpha(\gamma-1)}}\Big[-b\gamma\Big]^{\frac{1}{1-\alpha(\gamma-1)}} + \epsilon(m-1)(1-\alpha(\gamma-1))$$

Hence we can set upper bound for any

$$\xi_2 > (1-\alpha(\gamma-1))\Big[-b\gamma C^{\gamma-1}\Big]^{\frac{1}{1-\alpha(\gamma-1)}}$$

$$C_2 = \xi_2^{-\frac{1}{m-1}} C^{\frac{1}{1-\alpha(\gamma-1)}}\Big[-b\gamma\Big]^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

Hence

$$u(x,t) \le g(x,t) = C_2 t^{\frac{\alpha}{1-\alpha(\gamma-1)}} \Big(\xi_2 - \frac{x}{t^{\frac{1}{1-\alpha(\gamma-1)}}}\Big)^{\frac{1}{m-1}}$$

$$\eta(t) \le \bar{\xi} t^{\frac{1}{1-\alpha(\gamma-1)}}$$

As the same analysis, we estimate lower estimation: from (3.38) when $\mu = \frac{1}{\gamma-1}$ and (3.47), it follows

$$S = \frac{C_0}{1-\alpha(\gamma-1)} \Big(\xi_0 - \xi\Big)_+^{\mu-1} \times$$

$$\Big[\alpha(\xi_0-\xi) + \mu\xi\Big] - t^{\frac{\alpha(m-\gamma)-1}{1-\alpha(\gamma-1)}} C_0^m \mu(\mu m - 1)\Big(\xi_0 - \xi\Big)_+^{\mu m-2} + bC_0^\gamma \gamma\mu(\xi_0-\xi)^{\gamma\mu-1}$$

Since

$$\eta(t) \le \bar{\xi} t^{\frac{1}{1-\alpha(\gamma-1)}} \quad \text{and} \quad \xi_\delta(t) = \delta t^{\frac{1}{1-\alpha(\gamma-1)}}$$

$$\begin{aligned} S &\le \frac{C_0}{1-\alpha(\gamma-1)} \Big(\xi_0-\xi\Big)_+^{\frac{2-\gamma}{\gamma-1}} \Big[\alpha(\xi_0-\xi) + \frac{1}{\gamma-1}\xi\Big] + bC_0^\gamma \frac{\gamma}{\gamma-1}(\xi_0-\xi)^{\frac{1}{\gamma-1}} \\ &= \Big(\xi_0-\xi\Big)_+^{\frac{1}{\gamma-1}} \Big[\frac{bC_0^\gamma \gamma}{\gamma-1} + (\xi_0-\xi)^{-1}\Big(\alpha(\xi_0-\xi) + \frac{\xi}{\gamma-1}\Big)\Big] \end{aligned}$$

Now, consider a function

$$g(x,t) = C_0\Big(\xi_0 t^{\frac{1}{1-\alpha(\gamma-1)}} - x\Big)_+^\alpha \tag{3.49}$$

defined in the region

$$G^+ = \{\delta t^{\frac{1}{1-\alpha(\gamma-1)}} < x < \xi_0 t^{\frac{1}{1-\alpha(\gamma-1)}}\}$$

As before, calculating Lg, then

$$\begin{aligned} Lg &= \left(\xi_0 t^{\frac{1}{1-\alpha(\gamma-1)}} - x\right)_+^{\alpha\gamma-1} S, \\ S &= \frac{\alpha C_0 \xi_0}{1-\alpha(\gamma-1)} t^{\frac{\alpha(\gamma-1)}{1-\alpha(\gamma-1)}} \times \\ &\left(\xi_0 t^{\frac{1}{1-\alpha(\gamma-1)}} - x\right)_+^{\alpha(1-\gamma)} - C_0^m \alpha m(\alpha m-1)\left(\xi_0 t^{\frac{1}{1-\alpha(\gamma-1)}} - x\right)_+^{\alpha(m-\gamma)-1} + bC_0^\gamma \end{aligned} \tag{3.50}$$

If $G^+ = \{0 < x < \delta_0 t^{\frac{1}{1-\alpha(\gamma-1)}}\}$, then

$$\begin{aligned} S &\leq \frac{\alpha C_0 \xi_0}{1-\alpha(\gamma-1)} t^{\frac{\alpha(\gamma-1)}{1-\alpha(\gamma-1)}} \times \\ &\left(\xi_0 - \delta\right)_+^{\alpha(1-\gamma)} t^{\frac{\alpha(1-\gamma)}{1-\alpha(\gamma-1)}} - C_0^m \alpha m(\alpha m-1)\left(\xi_0 - \delta\right)_+^{\alpha(m-\gamma)-1} t^{\frac{\alpha(m-\gamma)}{1-\alpha(\gamma-1)}} + b\alpha C_0^\gamma \end{aligned}$$

Therefore

$$S \leq \frac{\alpha C_0 \xi_0}{1-\alpha(\gamma-1)}\left(\xi_0 - \delta\right)_+^{\alpha(1-\gamma)} + b\alpha C_0^\gamma$$

Choose $C_0 = C_1$ and $\xi_0 = \xi_1$, then

$$\alpha\xi_1(\xi_1 - \delta)_+^{\alpha(1-\gamma)} \leq -b\alpha C_1^{\gamma-1}(1-\alpha(\gamma-1)) \tag{3.51}$$

$$g(0,t) = C_1 \xi_1^\alpha t^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

Along $\xi_\delta(t)$, we get

$$g(\xi_\delta(t),t) = C_1\left(\xi_1 - \delta\right)^\alpha t^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

$$u(0,t) \geq (C-\epsilon)^{\frac{1}{1-\alpha(\gamma-1)}}\left[-b\gamma t\right]^{\frac{\alpha}{1-\alpha(\gamma-1)}}$$

Hence, $\forall\ 0 < \delta << 1,\ \exists\ C_\delta$ such that

$$u(\xi_\delta(t),t) \geq C_\delta t^{\frac{\alpha}{1-\alpha(\gamma-1)}}, \quad 0 < t < \delta$$
$$\eta(t) \geq \bar{\xi} t^{\frac{1}{1-\alpha(\gamma-1)}}$$

□

3.3.3 Diffusion and convection are in balance

Region (3)

Proof of Theorem 3.1.3. We need to bound $u(x,t)$, so we have the following lower and upper estimation of $u(x,t)$ when $\gamma > 1$, $0 \leq x < \infty$ and $0 \leq t < \infty$.

$h_1(0) = A_1 > 0$, then

$$u(x,t) = t^{\frac{1}{m+1-2\gamma}} h_1\left(\frac{x}{t^{\frac{m-\gamma}{m+1-2\gamma}}}\right)$$
$$u(0,t) = A_1 t^{\frac{1}{m+1-2\gamma}}$$

Consider

$$g(x,t) = t^{\frac{1}{m+1-2\gamma}} h_1(\zeta), \quad h_1(\zeta) = C_0(\zeta_0 - \zeta)_+^{\frac{1}{m-1}},\ 0 \leq \zeta \leq \zeta_0$$

Calculating Lg, then

$$Lg = t^{\frac{2\gamma-m}{m+1-2\gamma}} L^0 h_1$$
$$L^0 h_1 = \frac{C_0(\zeta_0-\zeta)^{\frac{2-m}{m-1}}}{m+1-2\gamma} \times \tag{3.52}$$
$$\left[\zeta_0 + \frac{1-\gamma}{m-1}\zeta_0 - \frac{C_0^{m-1} m(m+1-2\gamma)}{(m-1)^2} + \frac{C_0^{\gamma-1} b\gamma(m+1-2\gamma)(\zeta_0-\zeta)^{\frac{\gamma-1}{m-1}}}{m-1}\right]$$

For lower bound, taking $C_0 = C_1$, $\zeta_0 = \zeta_1$ to achieve the sharpest lower bound, we have,

$$L^0 h_1 \leq 0 \quad \text{if} \quad \zeta_1 - \frac{C_1^{m-1} m(m+1-2\gamma)}{(m-1)^2} \leq 0$$

$$g(0,t) = h_1(0) t^{\frac{1}{m+1-2\gamma}} = C_1 \zeta_1^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}}, \quad \text{then} \quad C_1 = \zeta_1^{\frac{1}{1-m}} A_1$$

$$g(0,t) \leq u(0,t)$$

Since

$$C_1 = \zeta_1^{\frac{1}{1-m}} A_1 \quad \text{and} \quad \zeta_1 - \frac{C_1^{m-1} m(m+1-2\gamma)}{(m-1)^2} \leq 0$$

then (3.10) hold. Hence

$$u(x,t) \geq g(x,t) = C_1(\zeta_1 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \leq x, t < +\infty$$

$$\eta(t) \geq \zeta_1 t^{\frac{m-\gamma}{m+1-2\gamma}}, \quad 0 \leq x, t < +\infty$$

To estimate upper bound, by applying same analysis in lower bound case. Choose $C_0 = C_2$, $\zeta_0 = \zeta_2$, we consider (3.52), then

$$g(0,t) = h_1(0) t^{\frac{1}{m+1-2\gamma}} = C_2 \zeta_2^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}}, \quad \text{then} \quad C_2 = \zeta_2^{\frac{1}{1-m}} A_1$$

$$g(0,t) \geq u(0,t)$$

We have

$$L^0 h_1 \geq 0 \quad \text{if} \tag{3.53}$$

$$\zeta_2 + \frac{1-\gamma}{m-1}\zeta_2 - \frac{C_2^{m-1} m(m+1-2\gamma)}{(m-1)^2} + \frac{b C_2^{\gamma-1} \gamma(m+1-2\gamma) \zeta_2^{\frac{\gamma-1}{m-1}}}{m-1} \geq 0$$

Since $C_2 = \zeta_2^{\frac{1}{1-m}} A_1$, then (3.53) becomes

$$\zeta_2^2 + \frac{b\gamma(m+1-2\gamma)A_1^{\gamma-1}}{m-\gamma}\zeta_2 - \frac{m(m+1-2\gamma)A_1^{m-1}}{m-1} = 0 \tag{3.54}$$

After solving quadratic equation (3.54), we estimate upper bound when $\gamma > 1$. Hence

$$u(x,t) \le g(x,t) = C_2(\zeta_2 - \zeta)_+^{\frac{1}{m-1}} t^{\frac{1}{m+1-2\gamma}}, \quad 0 \le x, t < +\infty$$
$$\eta(t) \le \zeta_2 t^{\frac{m-\gamma}{m+1-2\gamma}}, \quad 0 \le x, t < +\infty$$

□

3.3.4 Strong convection causes infinite speed of propagation

Region (4)

Proof of Theorem 3.1.4. We want to estimate lower bound, in the case when $\gamma < 1$. Consider a function

$$g(x,t) = \Big[\big(-b\gamma\big)t\Big]^{\frac{1}{1-\gamma}}(1+x)^{\frac{1}{\gamma-1}} \tag{3.55}$$

Calculating Lg, then

$$(g_\epsilon)_t - b(g_\epsilon^\gamma)_x = -\frac{b\gamma}{1-\gamma}\Big[\big(-b\gamma(1+\epsilon)\big)t\Big]^{\frac{\gamma}{1-\gamma}}(1+x)^{\frac{1}{\gamma-1}}\epsilon$$

If $\epsilon = 0$, then $g_0 = \Big[(-b\gamma)t\Big]^{\frac{1}{1-\gamma}}(1+x)^{\frac{1}{\gamma-1}}$ solves $g_t - b(g^\gamma)_x = 0$

therefore

$$g_\epsilon(0,t) = \Big[\big(-b\gamma(1+\epsilon)\big)t\Big]^{\frac{1}{1-\gamma}}$$

$$u(0,t) \sim C_0 t^{\frac{\alpha}{1+\alpha(1-\gamma)}}$$

Since $\frac{1}{1-\gamma} > 0$, we get

$$g_\epsilon(0,t) < u(0,t), \quad 0 \le t \le \delta_\epsilon$$

Therefore, $\forall \epsilon > 0$, $\exists \delta_\epsilon$ s.t.

$$\Big[(-b\gamma)t\Big]^{\frac{1}{1-\gamma}}(1+x)^{\frac{1}{\gamma-1}} \le u(x,t), \quad 0 \le x < +\infty, \quad 0 < t < \delta_\epsilon \tag{3.56}$$

□

3.3.5 Waiting time phenomena

Region (5)

Proof of Theorem 3.1.5. • Case (4a).

Suppose $m > 1, \alpha = \frac{2}{m-1}$ and $\gamma > \frac{m+1}{2}$. Then

$$\forall \epsilon > 0, \ \exists\, x_\epsilon < 0, \delta_\epsilon > 0 \ s.t.$$

$$(C-\epsilon)(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C-\epsilon}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}} \le u(x,t)$$

$$\le (C+\epsilon)(-x)_+^{\frac{2}{m-1}}\left[1-\left(\frac{C+2\epsilon}{\bar{C}}\right)^{m-1}(m-1)t\right]^{\frac{1}{1-m}},$$

$$x_\epsilon < x < +\infty, 0 < t \le \delta_\epsilon$$

To estimate lower bound, since the explicit solution (2.143) solves $u_{ct} = (u_c^m)_{xx}$, so we need to prove that $u_c(x,t)$ is an lower bound for (1.8),(1.11)

$$Lu_c = -b(u_c^\gamma)_x < 0, \ x < 0$$

Consider

$$u_c(x,0) = C(-x)_+^{\frac{2}{m-1}}$$

such that $u_c(x,t)$ is a global lower bound for (1.8),(1.11) with $u_0(x) = C(-x)_+^{\frac{2}{m-1}}$

$$u_c(x,t) \leq u(x,t)$$

$$C(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}} \leq u(x,t), \quad x \in R^1, 0 < t < T_c \tag{3.57}$$

To have upper bound estimation, consider Consider

$$g_{c+\epsilon}(x,t) = (C+\epsilon)(-x)_+^{\alpha}(1-Lt)^{\frac{1}{1-m}} \tag{3.58}$$

Applying the same analysis in 2.1.4, case (b) and calculating Lg, to get sharp upper bound, consider $g_{C+\epsilon}$ with $\epsilon > 0$, such that

$$Lg_{C+\epsilon} = \frac{(C+\epsilon)}{m-1}(-x)_+^{\frac{2}{m-1}}(1-Lt)^{\frac{m}{1-m}}S \tag{3.59}$$

$$S = L - \frac{2m(m+1)}{m-1}(C+\epsilon)^{m-1} + 2b\gamma(C+\epsilon)^{\gamma-1}(-x)_+^{\frac{2\gamma-m-1}{m-1}}(1-Lt)^{\frac{\gamma-m}{1-m}} \tag{3.60}$$

$$L - \frac{2m(m+1)}{m-1}(C+\epsilon)^{m-1} \geq 0$$

For all $\epsilon > 0$ and by using (2.38) we get

$$L-(m-1)\Big(\frac{C+\epsilon}{\bar{C}}\Big)^{m-1} = \epsilon$$

Choose

$$L = (m-1)\Big(\frac{C+2\epsilon}{\bar{C}}\Big)^{m-1} \geq 0$$

Therefore

$$Lg_{C+\epsilon} = \frac{(C+\epsilon)}{m-1}(-x)_+^{\frac{2}{m-1}}(1-Lt)^{\frac{m}{1-m}}\times$$
$$\Big(\frac{m-1}{\bar{C}^{m-1}}\Big((C+2\epsilon)^{m-1}-(C+\epsilon)^{m-1}\Big)+2b\gamma(C+\epsilon)^{\gamma-1}(-x)_+^{\frac{2\gamma-m-1}{m-1}}(1-Lt)^{\frac{\gamma-m}{1-m}}\Big)\geq 0$$
$$x_\epsilon \leq x < \infty,\ 0\leq t\leq \delta_\epsilon$$

$\forall\epsilon > 0, \exists x_\epsilon < 0, \delta_\epsilon > 0$, sharp lower estimation when $\alpha = \frac{2}{m-1}, \gamma > \frac{m+1}{2}$ is

$$u(x,t)\leq (C+\epsilon)(-x)_+^{\frac{2}{m-1}}\left[1-\left(\frac{C+2\epsilon}{\bar{C}}\right)^{m-1}(m-1)t\right]^{\frac{1}{1-m}},\quad x_\epsilon \leq x < +\infty, 0\leq t\leq\delta_\epsilon$$

Hence

$$\forall\epsilon > 0,\ \exists\ x_\epsilon < 0, \delta_\epsilon > 0\ s.t.$$
$$(C-\epsilon)(-x)_+^{\frac{2}{m-1}}\Big[1-\Big(\frac{C-\epsilon}{\bar{C}}\Big)^{m-1}(m-1)t\Big]^{\frac{1}{1-m}}\leq u(x,t)\leq$$
$$(C+\epsilon)(-x)_+^{\frac{2}{m-1}}\left[1-\left(\frac{C+2\epsilon}{\bar{C}}\right)^{m-1}(m-1)t\right]^{\frac{1}{1-m}},$$
$$x_\epsilon < x < +\infty, 0 < t\leq\delta_\epsilon$$

□

Chapter 4

Conclusions and Future Research

4.1 Conclusions

This dissertation analyzes the short-time evolution of interfaces and local solutions near interfaces in the Cauchy Problem for the nonlinear diffusion-convection equation

$$u_t = (u^m)_{xx} + b(u^\gamma)_x,\ x \in \mathbb{R},\ t > 0$$

$$u(x,0) = u_0(x) \sim C(-x)^\alpha,\ \text{as } x \to 0+;\ u_0(x) \equiv 0,\ x \geq 0,$$

where $m > 1, \gamma > 0, b \in \mathbb{R}, C, \alpha > 0$. The direction of the movement of interface curve

$$\eta(t) := \sup\{x : u(x,t) > 0\}$$

and its asymptotic properties depends on the relative strength of the diffusion and convection terms (m vs. γ), direction of the movement of the convection (*sign b*), asymptotics of the initial function near its support, and whether left or right interface curve is under consideration. The main results on the short-time asymptotic properties of the interface $\eta(t)$ are the following:

- If $b > 0, \alpha < 1/(m - \min\{\gamma, (m+1)/2\})$, then interface expands under the strong domination of the diffusion over the convection and

$$\eta(t) \sim \xi_* t^{1/(2-\alpha(m-1))} \text{ as } t \to 0+;\ \xi_* = \xi_*(m, \alpha, C) > 0.$$

- If $b > 0, \alpha = 1/(m-\gamma), \gamma < (m+1)/2$, then diffusion and convection are in balance in this borderline case. There is a critical constant C_* such that the interface expands for $C > C_*$, shrinks for $C < C_*, \gamma \leq 1$, and has a waiting time for $C < C_*, \gamma > 1$. In the first two cases we have

$$\eta(t) \sim \zeta_* t^{\frac{m-\gamma}{m+1-2\gamma}} \quad \text{as } t \to 0+;\ \zeta_* = \zeta_*(m, \gamma, b, C), \tag{4.1}$$

  where $\zeta_* \lesseqgtr 0$ if $C \lesseqgtr C_*$.

- If $b > 0, \gamma < 1$ and $\alpha > 1/(m-\gamma)$, then interface shrinks under the strong domination of convection over the diffusion and

$$\eta(t) \sim -\delta_* t^{1/(1+\alpha(1-\gamma))} \text{ as } t \to 0+;\ \delta_* = \delta_*(\gamma, \alpha, b, C) > 0.$$

- If $b > 0$, and either $1 \leq \gamma < \frac{m+1}{2}$ and $\alpha > \frac{1}{m-\gamma}$ or $\gamma \geq \frac{m+1}{2}$ and $\alpha \geq \frac{2}{m-1}$, then both diffusion and convection are weak, and interface has an initial waiting time.

- If $b < 0, \gamma \geq 1$ and $0 \leq \alpha < \frac{1}{m-\min(\gamma, \frac{m+1}{2})}$, then interface expands under the domination of the diffusion over the convection and

$$\eta(t) \sim \bar{\xi} t^{1/(2-\alpha(m-1))} \text{ as } t \to 0+; \ \bar{\xi} = \bar{\xi}(m,\alpha,C) > 0.$$

- If $b < 0, 1 \le \gamma < \frac{m+1}{2}$ and $\frac{1}{m-\gamma} < \alpha < \frac{1}{\gamma-1}$, then interface expands under the domination of convection over the diffusion and

$$\eta(t) \sim \tilde{\xi} t^{\frac{1}{1-\alpha(\gamma-1)}} \text{ as } t \to 0+; \ \tilde{\xi} = \tilde{\xi}(\gamma,b,\alpha) > 0.$$

- If $b < 0, 1 \le \gamma < (m+1)/2, \ \alpha = 1/(m-\gamma)$ then diffusion and convection are in balance, and the interface expands under the collective efforts of both with asymptotics (4.1).
- If $b < 0, \gamma < 1$, then there is an infinite speed of propagation and $\eta(t) = +\infty$.
- $b < 0, \gamma > 1$ and $\alpha \ge max(\frac{2}{m-1}, \frac{1}{\gamma-1})$, then then both diffusion and convection are weak, and interface has an initial waiting time.

4.2 Conference Presentations

The results of the dissertation are presented in the following conferences:

- U.G. Abdulla & L. Alzaki, Analysis of Interfaces for the Nonlinear Degenerate Diffusion Equation with Convection, AMS Contributed Session on Partial Differential Equations, Thursday, 10:45-11am, January 17, 2019; Joint Mathematics Meeting of the American Mathematical Society and Mathematical Association of America, Baltimore, Maryland.

- U.G. Abdulla & L. Alzaki, Analysis of Interfaces for the Nonlinear Degenerate Diffusion Equation with Convection, Sunday, 10:30-10:50 am, October 7, 2018; 38th Southeastern -Atlantic Regional Conference on Differential Equations (SEARCDE 2018), University of North Georgia, Atlanta, Georgia.

- U.G. Abdulla & L. Alzaki, Analysis of Interfaces for the Nonlinear Degenerate Second Order Parabolic Equations Modeling Diffusion-Convection Processes, AMS Contributed Session on Partial Differential Equations, Saturday, 10:30-10:45 am, January 13, 2018; Joint Mathematics Meeting of the American Mathematical Society and Mathematical Association of America, San Diego, California.

- U.G. Abdulla & L. Alzaki, Analysis of Interfaces for the Nonlinear Degenerate Second Order Parabolic Equations Modeling Diffusion-Convection Processes, AMS Session for Contributed Papers, Saturday, 3:30-3:45 pm, September 23, 2017; American Mathematical Society (AMS) Sectional Meetings, University of Central Florida, Orlando, Florida.

4.3 Future Research

The results of the dissertation motivate the development of the implemented methods to different open problems in the field, such as

- To present full classification of the development of interfaces and local solutions near the interfaces for the nonlinear reaction-diffusion-convection equation

$$u_t - (u)^m_{xx} + b(u^\gamma)_x + cu^\beta = 0.$$

- Solving interface problem for the nonlinear non-homogeneous reaction-diffusion-convection equation of the type

$$u_t = (a(x)u^m)_{xx} + b(x)\left(u^\gamma\right)_x + c(x)u^\beta = 0.$$

- Solving interface problem in multi-dimensional diffusion-convection equations of the type

  $$u_t - \Delta u^m + \mathbf{b} \cdot \nabla(u^\gamma),$$

  where the components of the vector **b**, in general, do not preserve the uniform sign, and thus the direction of the convection varies for different space variables.

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Appendix

Part A: We give here explicit values of the constants used in section 2.1 in the outline of the results for regions (2) and (3) and later in section 2.4 during the proof of these results.

$\zeta_* = b\gamma C^{\gamma-1}\left[1-\left(\frac{C}{C_*}\right)^{m-\gamma}\right] \quad \text{if} \quad \gamma = 1,$

$\lambda_2 = \left(\frac{A_1}{C_*}\right)^{m-\gamma},\ C_* \equiv \left[\frac{b(m-\gamma)}{m}\right]^{\frac{1}{m-\gamma}} \quad \text{if} \quad \gamma < 1,$

$\lambda_2 = \left(\frac{A_1}{C_2}\right)^{m-1},\ C_2 = \left(\frac{(m-\gamma)(m-1)}{m(m+1-2\gamma)}\right)^{\frac{1}{2(m-1)}} A_1^{\frac{1}{2}} \quad \text{if} \quad \gamma > 1,$

$\lambda_1 = \bar{z}^{-1}A_1^{m-\gamma},\ C_1 = \bar{z}^{\frac{1}{m-\gamma}} \quad \text{if} \quad \gamma < 1 \text{ and } z^2 - C_*^{m-\gamma}z - \frac{(m-\gamma)^2 A_1^{1+m-2\gamma}}{m\gamma(m+1-2\gamma)} = 0,$

$\lambda_1 = \bar{z}^{-1}A_1^{m-1},\ C_1 = \bar{z}^{\frac{1}{m-1}}, \quad \text{if} \quad \gamma > 1 \text{ and } z^2 - b\gamma m^{-1}(m-1)A_1^{\gamma-1}z - \frac{(m-1)^2 A_1^{m-1}}{m(m+1-2\gamma)} = 0,$

$\ell_0 = -\zeta_* + \bar{\epsilon},$

$\zeta_3 = -\zeta_* + \bar{\epsilon} - C_*^{\gamma-m}h^{m-\gamma}(\zeta_* - \bar{\epsilon}),$

$\ell_1 = \left(\frac{b\gamma(m+1-2\gamma)\left(1-\left(\frac{1}{1-\delta_*\Gamma}\right)\left(\frac{C}{C_*}\right)^{m-\gamma}\right)}{\delta_*\Gamma(m-\gamma)C^{1-\gamma}}\right)^{\frac{m-\gamma}{m+1-2\gamma}}$, where $\delta_* \in (0,1)$ satisfies

$g(\delta_*) = \max_{0\le\delta\le1} g(\delta), \quad (\delta) = \delta^{\frac{1-\gamma}{m-\gamma}}\left(1-\left(\frac{C}{C_*}\right)^{m-\gamma}\left(\frac{1}{1-\delta\Gamma}\right)\right),$

$\zeta_4 = \delta_*\ell_1\Gamma,\ \Gamma = 1-\left(\frac{C}{C_*}\right)^{m-\gamma},\ C_4 = C\left(\frac{1}{1-\delta_*\Gamma}\right)^{\frac{1}{m-\gamma}},$

$\zeta_5 = \delta_*(1+\epsilon)^{\gamma-1}(1-\epsilon),$

$C_5 = C_{\delta_*}(1+\epsilon)\delta_*^{-\alpha}\left(1-(1-\epsilon)(1+\epsilon)^{\gamma-1}\right)^{\alpha}$

Part B: We given here explicit values of the constants used in Section 3.1 in the outline of the results for regions (1) and (3) and later in section 3.3 during the proof of these results.

$$\xi_1 = \Big(m(m-1)^{-1}(2+\alpha(1-m))f^{m-1}(0)\Big)^{\frac{1}{2}},$$

$$\xi_1^{\epsilon} = \Big[\frac{(m+\epsilon)(2+\alpha(1-m))(f(0)+\epsilon)^{m-1}}{\min(\alpha;1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}},$$

$$C_1 = \Big(\frac{(m+\epsilon)(2+\alpha(1-m))(f(0)+\epsilon)^{m-1}}{\min(\alpha;1/(m-1))(m-1)^2}\Big)^{\frac{1}{2(1-m)}}(f(0)+\epsilon) \quad \text{if} \quad \gamma \geq 1$$

$$\xi_2^{\epsilon} = \Big[\frac{m(2+\alpha(1-m))(f(0)-\epsilon)^{m-1}}{\max(\alpha;1/(m-1))(m-1)^2}\Big]^{\frac{1}{2}},$$

$$C_2 = \Big(\frac{m(2+\alpha(1-m))(f(0)-\epsilon)^{m-1}}{\max(\alpha;1/(m-1))(m-1)^2}\Big)^{\frac{1}{2(1-m)}}(f(0)+\epsilon) \quad \text{if} \quad \gamma \geq 1,$$

$$\zeta_* = \frac{C^{m-1}m}{m-1} - b \quad \text{if} \quad \gamma = 1,$$

$$\zeta_1 = \frac{[A_1^{m-1}m(m+1-2\gamma)]^{\frac{1}{2}}}{(m-1)}, \quad C_1 = \zeta_1^{\frac{1}{1-m}}A_1 \quad \text{if} \quad \gamma > 1,$$

$$C_2 = \zeta_2^{\frac{1}{1-m}}A_1,\ A_1 = h(0) \quad \text{if} \quad \gamma > 1,$$

ζ_2 is a positive solution of the quadratic equation

$$z^2 + \frac{b\gamma(m+1-2\gamma)A_1^{\gamma-1}}{m-\gamma}z - \frac{m(m+1-2\gamma)A_1^{m-1}}{(m-1)(m-\gamma)} = 0$$