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Qualitative Analysis of the Nonlinear Double Degenerate Parabolic Equation of Turbulent Filtration with Absorption

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by

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We the undersigned committee
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Qualitative Analysis of the Nonlinear Double Degenerate Parabolic Equation of Turbulent Filtration with Absorption by

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ABSTRACT

Title:

Qualitative Analysis of the Nonlinear Double Degenerate Parabolic Equation of Turbulent Filtration with Absorption

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The goal of the dissertation is to pursue qualitative analysis of the mathematical model of turbulent polytropic filtration of a gas in a porous media with reaction or absorption described by the second order nonlinear double degenerate parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} F\left[\frac{\partial u^m}{\partial x}\right] + Q(u) = 0, \tag{1}$$

where

$$F(y) = |y|^{p-1}y,\ Q(u) = bu^\beta,\ m, p, \beta > 0,\ b \in \mathbb{R}.$$

In the absence of the reaction term there is a finite speed of propagation with an expanding interface in the case of slow diffusion ($mp > 1$), and infinite speed of propagation in the case of fast diffusion ($0 < mp < 1$). In general, qualitative properties of the turbulent filtration is an outcome of the competition between the diffusion and reaction or absorption forces. In the slow diffusion case, the strong domination of the diffusion causes an expanding interface. In the fast diffusion case, the strong domination of the diffusion causes infinite speed of propagation and absence of interfaces, while weak domination of the diffusion causes an expanding interface. Domination of the absorption causes a shrinking interface. If diffusion and absorption are in balance then initial density profile dictates the direction of the interface movement. When the interface exists, explicit asymptotic formulas and estimates for the interface function and the local solution near the interfaces with accuracy up to precise constants are proved. Explicit asymptotic formulas for the local solution at infinity are proved in all cases where interface does not exist. The results of the dissertation can be applied to problems in the oil and gas industry to pursue the estimation and control of the time evolution of the size of oil and gas resources.

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- $(\cdot)_+ := \max\{\cdot, 0\}$.
- $C^k(\Omega)$ is the space of k-times differentiable functions in Ω with finite norm

$$\|f\|_{C^k(\Omega)} := \sum_{n=0}^{k} \sup_{x\in\Omega} \left| f^{(n)}(x) \right|.$$

  If $k = 0$, $C^0(\Omega) \equiv C(\Omega)$ denotes the space of continuous functions on Ω.

- $C^k_0(\Omega)$ is the space of k-times differentiable functions in Ω with compact support: the space of k-times differentiable functions in Ω that are identically zero outside of a compact (bounded and closed) set.
- $L^p(\Omega)$ $(1 \le p < +\infty)$ is the space of p-integrable functions in Ω with finite norm

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

- $L^p_{\text{loc}}(\Omega)$ is the space of locally p-integrable functions in Ω: the space of p-integrable functions on compact subsets of Ω.
- $L^\infty(\Omega)$ is the space of all essentially bounded and measurable functions on Ω with

finite norm

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{C \geq 0 : |f(x)| \leq C, \text{ for almost every } x \in \Omega\}.$$

- Let $I \subset \mathbb{R}$ and $u \in L^p(I)$. The function $v \in L^p(I)$ is called the weak derivative of u if the following integral identity holds for all test functions $\varpi \in C_0^\infty(I)$

$$\int_I u(z)\varpi'(z)dz = -\int_I v(z)\varpi(z)dz.$$

- $W^{k,p}(\Omega)$ $(1 \leq p \leq +\infty)$ is the time-independent Sobolev space of functions in $L^p(\Omega)$ such that the functions themselves and their weak derivatives up to order k have a finite L^p norm, with finite Sobolev norm

$$\|f\|_{W^{k,p}(\Omega)} := \Big(\sum_{i=0}^{k} \|f^{(i)}\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

- Time-dependent Sobolev spaces. The following spaces are of importance to the study of time-dependent problems, specifically parabolic problems

$$C(0,T;L^1_{\text{loc}}(\Omega)) := \Big\{u = u(t) : [0,T] \to L^1_{\text{loc}}(\Omega)\Big\},$$

let $K \subset \Omega$ be compact,

$$\|u\|_{C(0,T;L^1(K))} := \max_{0\leq t\leq T} \|u\|_{L^1(K)}.$$

We define the time-dependent Sobolev space $L^p\Big(0,\mathrm{T};W^{1,p}_{\mathrm{loc}}(K)\Big)$ as

$$L^p\Big(0,\mathrm{T};W^{1,p}_{\mathrm{loc}}(K)\Big) := \Big\{u = u(t) : [0,T] \to W^{1,p}_{\mathrm{loc}}(K)\Big\},$$

with finite norm

$$||u||_{L^p\left(0,\mathrm{T};W^{1,p}(K)\right)} := \left(\int_0^T ||u||^p_{W^{1,p}(K)} dt\right)^{1/p} = \left(\int_0^T \int_K (|u(x,t)|^p + |Du(x,t)|^p) dxdt\right)^{1/p}.$$

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I would like to thank my advisor, Professor Ugur G. Abdulla, for sharing his deep expertise in the field nonlinear partial differential equations, in general, and for his constant patience with me through many meetings over several years. Without his help, this work would not have been possible. I would also like to acknowledge my wife, Megan, and son, Benjamin, for their love, encouragement, and support, throughout the completion of this work. Finally, I would like to thank my mother, Michele, without whom I would not have been able to complete my undergraduate education, and so, would not have been able to pursue my doctorate at all. I'm at this point because of all of you, and I cannot thank you enough for helping me through this challenging journey.

Dedication

This dissertation is dedicated to my wife, Megan, my son, Benjamin, my mother, Michele, and my mom-mom, Betsy. I love you all very much.

Chapter 1

Introduction

1.1 Physical Derivation of the Equation

Consider the physical problem of turbulent filtration of a fluid, specifically a gas, in a porous media. This problem is governed by the following physical laws:

- Polytropic equation of state, relating the fluid pressure to its density

$$\mathcal{P} = c\vartheta^n, \tag{1.1}$$

  where c is a positive constant, $\vartheta = \vartheta(x,t)$ is a density of the fluid, and $\mathcal{P}$ is the pressure of the fluid.

- Continuity equation, relating the fluid density to its velocity

$$k\vartheta_t + (\vartheta\mathcal{V})_x = 0, \tag{1.2}$$

  where $\mathcal{V}$ is the velocity of the fluid at the point x and moment t and k is a positive constant.

- Flux under turbulence

$$\vartheta\mathcal{V} = -Q|\phi_x|^{p-1}\phi_x, \tag{1.3}$$

where $\phi = \mathcal{P}^{\frac{n+1}{n}}$, Q is a positive physical constant, with $n \geq 1$ and $p \geq 1$.
Combining (1.1)-(1.3), we have

$$k\vartheta_t = Qc^{\frac{p(n+1)}{n}}\left(\left|(\vartheta^{n+1})_x\right|^{p-1}(\vartheta^{n+1})_x\right)_x. \tag{1.4}$$

By rescaling t in (1.4) and setting $\vartheta = u = u(x,t)$ we obtain the nonlinear double degenerate diffusion equation

$$u_t = \left(\left|(u^m)_x\right|^{p-1}(u^m)_x\right)_x,$$

where $m = n+1 \geq 2$ under these particular conditions, however, this is relaxed throughout the dissertation: we only assume m and p are positive constants.

The case where $mp > 1$ is called the slow diffusion case and the case $0 < mp < 1$ is called the fast diffusion case [74]. If $m = 1$, then we have the non-Newtonian elastic filtration equation or the parabolic p-Laplacian equation, while if $m = p = 1$ we have the linear diffusion equation or heat equation.

1.1.1 The Barenblatt Solution

Consider the Barenblatt problem for the nonlinear double degenerate parabolic equation, a standard model problem for nonlinear second order degenerate equations

$$\begin{cases} u_t = (|(u^m)_x|^{p-1}(u^m)_x)_x,\ x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x),\ \forall\, x \neq 0, \end{cases}$$

where $\delta(\cdot)$ is the Dirac measure on $\mathbb{R}$.

The solution, called the Barenblatt solution, is given by the formula

$$u_*(x,t) = t^{\frac{-1}{p(m+1)}} \left[\Gamma - k(m,p) \left(|x| t^{\frac{-1}{p(m+1)}} \right)^{\frac{1+p}{p}} \right]_+^{\frac{p}{mp-1}}, \tag{1.5}$$

where $k(m,p) = \frac{mp-1}{m(1+p)} \left(\frac{1}{p(m+1)} \right)^{\frac{1}{p}}$ and Γ is an integration constant defined by the conservation of energy condition. This solution, (1.5), is called the Barenblatt solution. In the slow diffusion case ($mp > 1$) [42, 43], the solution travels with a finite speed of propagation (the interface function exists). In the fast diffusion case ($0 < mp < 1$), solution of the above problem exhibits an infinite speed of propagation. With $mp > 1$, the case of slow diffusion, the solution exhibits a finite speed of propagation, and so, is not a classical.

1.2 Historical Review and the Open Problem

The solutions to equation (1) are not classical and are understood in the following, *weak*, sense

Definition 1.2.1. For the equation (1), we introduce the notion of the weak solution. A continuous nonnegative function $u(x,t)$ defined in $\mathbb{R} \times [0,T)$ is a weak solution of

(1.7), (1.8) if for any $T_1 \in (0,T)$ and any bounded interval $(a,b) \in \mathbb{R}$, $(u^m)_x \in L^{p+1}((a,b)\times (0,T_1))$ and

$$\int_0^{T_1}\int_a^b \Big(-u\phi_t + |(u^m)_x|^{p-1}(u^m)_x\phi_x + bu^\beta\phi\Big)dxdt = \int_a^b u\phi\Big|_{t=T_1}^{t=0}dx \qquad (1.6)$$

for arbitrary $\phi \in C^1([a,b]\times[0,T_1])$ such that $\phi\Big|_{x=a} = \phi\Big|_{x=b} = 0$.

The Barenblatt solution above, u_*, when $mp > 1$, is a weak solution in the sense of the Definition 1.2.1 due to having a finite speed of propagation.

The general theory of nonlinear degenerate parabolic equations is well studied (refer to [6, 11, 2, 5, 3, 78, 97, 85, 40, 73, 72, 87, 94, 52, 61, 55, 51, 93]). For existence of solutions to general doubly nonlinear parabolic problems we refer to [45, 47, 83] For advances such as the analysis of fine properties of solutions and the development of Harnack-type inequalities for general doubly nonlinear parabolic problems, refer to [70, 95, 89, 50, 82, 101, 58, 49, 76, 79, 99, 96, 69, 91, 77, 59, 98, 60, 48, 39, 88, 56, 46, 84, 37, 90, 36, 86, 81, 102, 38, 41, 100, 83].

Problems of existence, uniqueness of solutions of general parabolic initial-boundary value problems, the formulation of comparison results, analysis of the regularity properties of weak solutions, and the optimal control of free boundary problems are considered in [30], [18], [19], [42].

1.2.1 The Open Problem

We consider the Cauchy problem (CP) for the nonlinear degenerate parabolic equation

$$Lu \equiv u_t - (|(u^m)_x|^{p-1}(u^m)_x)_x + bu^\beta = 0,\ x \in \mathbb{R},\ 0 < t < T, \qquad (1.7)$$

with nonnegative and continuous initial function

$$u(x,0) = u_0(x), \ x \in \mathbb{R}, \tag{1.8}$$

where $u = u(x,t)$; m, $p, \beta > 0$, $b \in \mathbb{R}$, and $T \leq +\infty$, with the conditions that if $b > 0$ then $\beta < 1$ while if $b \leq 0$ then $\beta \geq 1$. Notably, equation (1.7) models turbulent polytropic filtration of a gas (or general fluid) in a porous medium [42, 57, 78]. We define the interface function as

$$\eta(t) := \sup\{x : u(x,t) > 0\},$$

with $\eta(0) = 0$. We want to analyze the short-time behavior of the interface function $\eta(t)$, and local solution near the interface function. In general, we shall assume

$$u_0(x) \sim C(-x)_+^{\alpha}, \text{ as } \ x \to 0^-, \text{ for some } C > 0, \alpha > 0. \tag{1.9}$$

In some special cases, in global cases in particular, we will consider

$$u_0(x) = C(-x)_+^{\alpha}, x \in \mathbb{R}. \tag{1.10}$$

This problem is well understood in the case of (1.7) with $p = 1$

$$u_t - (u^m)_{xx} + bu^{\beta} = 0 \ x \in \mathbb{R}, 0 < t < T. \tag{1.11}$$

Solutions and regularity analysis for the Cauchy problem (1.11), (1.8) is well studied [1, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 25, 27, 28, 29, 31, 32, 34, 35, 62, 63]. If $u_0 \in C(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ and nonnegative, then the existence, uniqueness and comparison theorems for the weak solution of the CP (1.7), (1.8) was proved in [57] for the case

$b = 0$, and in [92] for $b > 0$. In [57] it is proved that the weak solution of (1.7), $b = 0$, is locally Hölder continuous. Local Hölder continuity of the locally bounded weak solutions of the general second order multidimensional nonlinear degenerate parabolic equations with double degenerate diffusion term is proved in [68, 67].

The interface problem for the CP (1.11), (1.9) was presented in [23] for the slow diffusion case ($m > 1$) and in [33] for the fast diffusion case ($0 < m < 1$). A similar classification for the reaction-diffusion equation with p-Laplacian type diffusion ($m = 1$ in (1.7)) is presented in the recent paper [22] for the case of slow diffusion case ($p > 1$) and in [21] for the fast diffusion case ($0 < p < 1$).

The overall goal of this dissertation is to apply and extend the analytical methods developed in [23, 33, 6] to prove accurate asymptotic estimations for the interface function and local solution near the interface function for the Cauchy problem (1.7), (1.8) in the case of both slow diffusion ($mp > 1$) and fast diffusion ($0 < mp < 1$).

Chapter 2

Evolution of Interfaces for the Nonlinear Double Degenerate Parabolic Equation with Slow Diffusion

In this chapter we present full classification of the evolution of interfaces and local structure of solution near the interfaces of the problem (1.7), (1.9) in the slow diffusion case ($mp > 1$). The results of Chapter 2 are published in [20].

2.1 Description of the Main Results

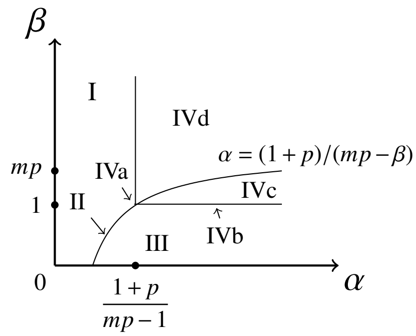


Figure 2.1: (α,β) parameter space diagram for the interface development for the CP (1.7), (1.9) with $mp > 1$, which is presented in [20].

Throughout this section we assume that u is a unique weak solution of the CP (1.7)-(1.9). There are four different subcases; the main results are outlined below in Theorems 2.1.1, 2.1.2, 2.1.3 and 2.1.4 corresponding directly to the cases I, II, III and IVa-IVd in Figure 2.1.

Theorem 2.1.1. *If $b > 0$ and $0 < \alpha < \frac{1+p}{mp-min\{1,\beta\}}$, then the interface initially expands and*

$$\eta(t) \sim \xi_* t^{1/(1+p-\alpha(mp-1))}, \text{ as } t \to 0^+, \tag{2.1}$$

where

$$\xi_* = C^{\frac{mp-1}{1+p-\alpha(mp-1)}} \xi'_*, \tag{2.2}$$

and ξ'_*, is a positive number depending only on m, p, and α. For arbitrary $\rho < \xi_*$, there exists a positive number $f(\rho)$ depending on C, m, p, and α such that

$$u(x,t) \sim t^{\alpha/(1+p-\alpha(mp-1))} f(\rho), \text{ as } t \to 0^+, \tag{2.3}$$

along the curve $x = \xi_\rho(t) = \rho t^{1/(1+p-\alpha(mp-1))}$.

Theorem 2.1.2. *Let* $b > 0$, $0 < \beta < 1$, $\alpha = (1+p)/(mp-\beta)$, *and*

$$C_* = \left[\frac{b(mp-\beta)^{1+p}}{(m(1+p))^p p(m+\beta)}\right]^{\frac{1}{mp-\beta}}.$$

Then interface expands or shrinks according as $C > C_*$ *or* $C < C_*$ *and*

$$\eta(t) \sim \zeta_* t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \text{ as } t \to 0^+, \tag{2.4}$$

where $\zeta_* \lesseqgtr 0$ *if* $C \lesseqgtr C_*$, *and for arbitrary* $\rho < \zeta_*$ *there exists* $f_1(\rho) > 0$ *such that*

$$u(x,t) \sim t^{1/(1-\beta)} f_1(\rho), \text{ for } x = \rho t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \text{ as } t \to 0^+. \tag{2.5}$$

Theorem 2.1.3. *If* $b > 0$, $0 < \beta < 1$, *and* $\alpha > (1+p)/(mp-\beta)$, *then the interface shrinks and*

$$\eta(t) \sim -\ell_* t^{1/\alpha(1-\beta)}, \text{ as } t \to 0^+, \tag{2.6}$$

where $\ell_* = C^{-1/\alpha}(b(1-\beta))^{1/\alpha(1-\beta)}$.*For arbitrary* $\ell > \ell_*$ *we have*

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{1/(1-\beta)}, \text{ as } t \to 0^+, \tag{2.7}$$

along the curve $x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}$.

Theorem 2.1.4. *If* $b \in \mathbb{R}$, $\beta \geq 1$, *and* $\alpha \geq (1+p)/(mp-1)$, *then interface initially remains stationary.*

2.2 Further Details of the Main Results

In this section we outline some essential details of the main results described in Theorems 2.1.1-2.1.4 of Section 2.1. In particular, we describe global estimations for minimal solutions of the CP (1.7), (1.10).

Further details of Theorem 2.1.1. f is a shape function of the self-similar solution to the problem (1.7), (1.10) with $b = 0$

$$u(x,t) = t^{\frac{\alpha}{1+p-\alpha(mp-1)}} f(\xi), \xi = xt^{-\frac{1}{1+p-\alpha(mp-1)}}. \tag{2.8}$$

In fact, f is a solution of the following nonlinear ODE problem in $\mathbb{R}$

$$(|(f^m(\xi))'|^{p-1}(f^m(\xi))')' + (1 + p - \alpha(mp-1))^{-1}(\xi f'(\xi) - \alpha f(\xi)) = 0, \tag{2.9}$$

$$f(\xi) \sim C(-\xi)^{\alpha}, \text{ as } \xi \downarrow -\infty, \ f(+\infty) = 0. \tag{2.10}$$

Moreover, there exists $\xi_* > 0$ such that: $f(\xi) \equiv 0$ for $\xi \geq \xi_*$; $f(\xi) > 0$ for $\xi < \xi_*$. Dependence on C is given by the following relation

$$f(\rho) = C^{1+p/(1+p-\alpha(mp-1))} f_0\Big(C^{(mp-1)/(\alpha(mp-1)-(1+p))}\rho\Big), \tag{2.11a}$$

$$f_0(\rho) = w(\rho, 1), \quad \xi'_* = \sup\{\rho : f_0(\rho) > 0\} > 0, \tag{2.11b}$$

where w is a minimal solution of the CP (1.7), (1.10) with $b = 0$, $C = 1$. Lower and upper estimations for f are given in (2.30). We also have that

$$\xi'_* = A_0^{\frac{mp-1}{1+p}} \left[\frac{(mp)^p(1 + p - \alpha(mp-1))}{(mp-1)^p}\right]^{\frac{1}{1+p}} \xi''_*, \tag{2.12}$$

where $A_0 = w(0,1)$ and ξ''_* is some number belonging to the segment $[\xi_1, \xi_2]$ (see 6.1). In the particular case $\alpha = p(mp-1)^{-1}$ and $mp > 1 + p - p(\min\{1,\beta\})$, the explicit solution of (1.7), (1.10) with $b = 0$ is given by (2.28) and

$$\xi_1 = \xi_2 = 1,\ \xi'_* = (mp)^p(mp-1)^{-p},\ f_0(x) = (\xi'_* - x)_+^{p/(mp-1)} \tag{2.13}$$

Further details of Theorem 2.1.2. If $p(m+\beta) = 1 + p$, the solution to (1.7), (1.10) is

$$u(x,t) = C(\zeta_* t - x)_+^{\frac{1+p}{mp-\beta}},\ \zeta_* = b(1-\beta)C^{\beta-1}((C/C_*)^{mp-\beta} - 1). \tag{2.14}$$

Let $p(m+\beta) \neq 1 + p$. If $C = C_*$, then u_0 is a stationary solution to (1.7), (1.10). If $C \neq C_*$, then the minimal solution to (1.7), (1.10) is of the self-similar form

$$u(x,t) = t^{1/(1-\beta)} f_1(\zeta),\ \zeta = xt^{-\frac{mp-\beta}{(1+p)(1-\beta)}}, \tag{2.15}$$

$$\eta(t) = \zeta_* t^{\frac{mp-\beta}{(1+p)(1-\beta)}},\ 0 \le t < +\infty, \tag{2.16}$$

where $f_1(\zeta)$ solves the following nonlinear ODE problem

$$(|(f_1^m)'|^{p-1}(f_1^m)')' + \frac{mp-\beta}{(1+p)(1-\beta)}\zeta f_1' - \frac{1}{1-\beta} f_1 - b f_1^\beta = 0,\ \zeta \in \mathbb{R}, \tag{2.17}$$

$$f_1(\zeta) \sim C(-\zeta)^{(1+p)/(mp-\beta)},\ \text{as } \zeta \downarrow -\infty,\ \text{and } f_1(\zeta) \to 0,\ \text{as } \zeta \uparrow +\infty. \tag{2.18}$$

Moreover, there exists ζ_* such that $f(\zeta) \equiv 0$ for $\zeta \ge \zeta_*$; $f(\zeta) > 0$ for $\zeta < \zeta_*$. If $C > C_*$

then the interface expands, $f_1(0) = A_1 > 0$ (see Lemma 2.3.7), and

$$C_1 t^{\frac{1}{1-\beta}}\Big(\zeta_1 - \zeta\Big)_+^{\mu} \le u \le C_2 t^{\frac{1}{1-\beta}}\Big(\zeta_2 - \zeta\Big)_+^{\mu}, \ 0 \le x < +\infty, \ 0 < t < +\infty, \tag{2.19}$$

where

$$\begin{cases} \mu = p(mp-1)^{-1}, & \text{if } p(m+\beta) > 1+p, \\ \mu = (1+p)(mp-\beta)^{-1}, & \text{if } p(m+\beta) < 1+p, \end{cases}$$

which implies

$$\zeta_1 \le \zeta_* \le \zeta_2. \tag{2.20}$$

If $0 < C < C_*$, then the interface shrinks. If $p(m+\beta) > 1+p$ then

$$\begin{aligned} [C^{1-\beta}(-x)_+^{\frac{(1+p)(1-\beta)}{mp-\beta}} - b(1-\beta)t]_+^{\frac{1}{1-\beta}} \le u \le \\ [C^{1-\beta}(-x)_+^{\frac{(1+p)(1-\beta)}{mp-\beta}} - b(1-\beta)(1-(C/C_*)^{mp-\beta})t]_+^{\frac{1}{1-\beta}}, \ x \in \mathbb{R}, \ t \ge 0, \end{aligned} \tag{2.21}$$

which also implies (2.20), where we replace ζ_1 (respectively, ζ_2) with respective negative values given in Appendix 6.1. However, if $p(m+\beta) < 1+p$, then

$$C_*\Big(-\zeta_3 t^{\frac{mp-\beta}{(1+p)(1-\beta)}} - x\Big)_+^{\frac{1+p}{mp-\beta}} \le u \le C_3\Big(-\zeta_4 t^{\frac{mp-\beta}{(1+p)(1-\beta)}} - x\Big)_+^{\frac{1+p}{mp-\beta}}, \ x \in \mathbb{R}, \ t > 0, \tag{2.22}$$

where the left-hand side is valid for $x \ge -\ell_0 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}$, while the right-hand side is valid for $x \ge -\ell_1 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}$. From (2.22), (2.20) follows if we replace ζ_1 and ζ_2 with $-\zeta_3$ and $-\zeta_4$, respectively.

Further Details of Theorem 2.1.4. There are four different subcases (see Figure 2.1).

(4a) If $\beta = 1, \alpha = (1+p)/(mp-1)$, the unique minimal solution to (1.7), (1.10) is

$$u_C = C(-x)_+^{(1+p)/(mp-1)} e^{-bt}[1-(C/\bar{C})^{mp-1} b^{-1}(1-e^{-b(mp-1)t})]^{1/(1-mp)} \tag{2.23}$$

where

$$T = +\infty, \quad \text{if } \ b \geq (C/\bar{C})^{mp-1},$$
$$T = (b(1-mp))^{-1}\ln[1-b(\bar{C}/C)^{mp-1}], \quad \text{if } -\infty < b < (C/\bar{C})^{mp-1},$$

(4b) Let $\beta = 1$ and $\alpha > (1+p)/(mp-1)$. Then for any $\varepsilon > 0$ there exists $x_\varepsilon < 0$ and $\delta_\varepsilon > 0$ such that

$$(C-\varepsilon)(-x)_+^\alpha e^{-bt} \leq u \leq (C+\varepsilon)(-x)_+^\alpha e^{-bt}$$
$$[1-\varepsilon(b(mp-1))^{-1}(1-e^{-b(mp-1)t})]^{1/(1-mp)}, \ x > x_\varepsilon, \ \ 0 \leq t \leq \delta_\varepsilon. \tag{2.24}$$

(4c) Let $1 < \beta < mp$ and $\alpha \geq (1+p)/(mp-\beta)$. Then for any$\varepsilon > 0$ there exists $x_\varepsilon < 0$ and $\delta_\varepsilon > 0$ such that

$$g_{-\varepsilon}(x,t) \leq u(x,t) \leq g_\varepsilon(x,t), \ x \geq x_\varepsilon, \ 0 \leq t \leq \delta_\varepsilon, \tag{2.25}$$

where

$$g_\varepsilon(x,t) = \begin{cases} [(C+\varepsilon)^{1-\beta}|x|^{\alpha(1-\beta)} + b(\beta-1)(1-\varepsilon-\kappa_\varepsilon)t]^{1/(1-\beta)}, & \text{if } x_\varepsilon \leq x < 0, \\ 0, & \text{if } x \geq 0, \end{cases}$$

where $\kappa_\varepsilon = 0$, if $\alpha > (1+p)/(mp-\beta)$; $\kappa_\varepsilon = ((C+\varepsilon)/C_*)^{mp-\beta}$, if $\alpha = (1+p)/(mp-\beta)$.

(4d) Let either $1 < \beta < mp, \ (1+p)/(mp-1) \leq \alpha < (1+p)/(mp-\beta)$, or $\beta \geq mp, \ \alpha \geq$

$(1+p)/(mp-1)$. If $\alpha=(1+p)/(mp-1)$ then for any $\varepsilon>0$ there exists $x_\varepsilon<0$ and $\delta_\varepsilon>0$ such that

$$(C-\varepsilon)(-x)_+^{\frac{1+p}{mp-1}}(1-\gamma_{-\varepsilon}t)^{\frac{1}{1-mp}}\le u\le(C+\varepsilon)(-x)_+^{\frac{1+p}{mp-1}}(1-\gamma_\varepsilon t)^{\frac{1}{1-mp}} \tag{2.26}$$

for $x>x_\varepsilon,\ 0\le t\le\delta_\varepsilon$ (see 6.1 for γ_ε). However, if $\alpha>(1+p)/(mp-1)$, then for arbitrary sufficiently small $\varepsilon>0$, there exist $x_\varepsilon<0$ and $\delta_\varepsilon>0$ such that

$$(C-\varepsilon)(-x)_+^\alpha\le u\le(C+\varepsilon)(-x)_+^\alpha(1-\varepsilon t)^{1/(1-mp)},\ x>x_\varepsilon,\ 0\le t\le\delta_\varepsilon. \tag{2.27}$$

Results for the case $b=0$.

(1) If $\alpha=p/(mp-1)$ the minimal solution to the problem (1.7), (1.10) is

$$u(x,t)=C(\xi_*t-x)_+^{p/(mp-1)},\xi_*=C^{mp-1}\Big(\frac{mp}{mp-1}\Big)^p. \tag{2.28}$$

If $0<\alpha<(1+p)/(mp-1)$, then the minimal solution to (1.7), (1.10) has the self-similar form (2.8) and

$$\eta(t)=\xi_*t^{\frac{1}{1+p-\alpha(mp-1)}},0\le t<+\infty, \tag{2.29}$$

where ξ_* and f solve (2.9)-(2.10). We have the estimation

$$C_4t^{\frac{\alpha}{1+p-\alpha(mp-1)}}(\xi_3-\xi)_+^{\frac{p}{mp-1}}\le u\le C_5t^{\frac{\alpha}{1+p-\alpha(mp-1)}}(\xi_4-\xi)_+^{\frac{p}{mp-1}},\ x\ge0,\ t\ge0, \tag{2.30}$$

(see Appendix 6.1). If $\alpha=p/(mp-1)$, then $\xi_3=\xi_4=\xi_*$ and both lower and upper estimations in (2.30) coincide with the solution (2.28).

(2) If $\alpha=(1+p)/(mp-1)$, then interface initially remains stationary. Explicit solu-

tion to (1.7), (1.10) is

$$u_C(x,t) = C(-x)_+^{(1+p)/(mp-1)}[\lambda(t_* - t)(1-mp)]^{1/(1-mp)},\ x \in \mathbb{R},\ 0 \le t < t_*, \tag{2.31}$$

where

$$t_* = 1/\lambda(1-mp), \text{with } \lambda = -C^{mp-1}\frac{p(m+1)(m(1+p))^p}{(mp-1)^{1+p}}.$$

(3) If $\alpha > (1+p)/(mp-1)$, then interfaces again remain stationary, and for any $\varepsilon > 0$, there exists a number $x_\varepsilon < 0$ and $\delta_\varepsilon > 0$, such that

$$(C-\varepsilon)(-x)_+^\alpha \le u \le (C+\varepsilon)(-x)_+^\alpha(1-\varepsilon t)^{1/(1-mp)},\ x_\varepsilon \le x,\ 0 \le t \le \delta_\varepsilon. \tag{2.32}$$

2.3 Preliminary Results

The following is a standard comparison result (comparison theorem) to be used throughout the dissertation.

Lemma 2.3.1. *Let g be a non-negative and continuous function in $\overline{Q}$, where:*

$$Q = \{(x,t) : \eta_0(t) < x < +\infty,\ 0 < t < T \le +\infty\}$$

$g = g(x,t)$ is in $C^{2,1}_{x,t}$ in Q outside a finite number of curves: $x = \eta_j(t)$, which divide Q into a finite number of subdomains: Q^j, where $\eta_j \in C[0,T]$; for arbitrary $\delta > 0$ and finite $\Delta \in (\delta, T]$ the function η_j is absolutely continuous in $[\delta,\Delta]$. Let g satisfy the inequality:

$$Lg \equiv g_t - \left(|(g^m)_x|^{p-1}(g^m)_x\right)_x + bg^\beta \ge 0, (\le 0),$$

at the points of Q where $g \in C^{2,1}_{x,t}$. Assume also that the function: $|(g^m)_x|^{p-1}(g^m)_x$ is *continuous in Q and $g \in L^\infty(Q \cap (t \le T_1))$ for any finite $T_1 \in (0,T)$. If u is a weak solution of (1.7) in Q, $u \in C(\overline{Q})$, and*

$$g(\eta_0(t),t) \ge (\le)\, u(\eta_0(t),t),\; g(x,0) \ge (\le)\, u(x,0).$$

Then

$$g \ge (\le)\, u, \text{ in } \overline{Q}$$

Suppose that $u_0 \in C(\mathbb{R})$, and it has unbounded growth as $|x| \to +\infty$. It is well known that in this case some restriction must be imposed on the growth rate for existence, uniqueness of the solution to the CP (1.7), (1.8). For the particular cases of the equation (1.7) with $b = 0$ this question was settled down in [44, 64] for the porous medium equation ($p = 1$) with slow ($m > 1$) and fast ($0 < m < 1$) diffusion; and in [53, 54] for the p-Laplacian equation ($m = 1$) with slow ($p > 1$) and fast ($0 < p < 1$) diffusion; The case of reaction-diffusion equation $m > 1, p = 1, b > 0$ is analyzed in [71, 75, 26]. Surprisingly, only a partial result is available for the double-degenerate PDE (1.7). The sharp sufficient condition for the existence of the solution to the CP for (1.7), $b = 0$ is established in [66]. In particular, it follows from [66] that the CP (1.7),(1.10) has a solution if and only if $\alpha \le (1+p)/(mp-1)$. Moreover, solution is global ($T = +\infty$) if $\alpha < (1+p)/(mp-1)$ and only local in time if $\alpha = (1+p)/(mp-1)$. Uniqueness of the solution is an open problem. For our purposes it is satisfactory to employ the notion of the minimal solution.

Definition 2.3.2 (Minimal Solution)**.** Weak solution of the CP (1.7), (1.8) is called a *minimal solution* if

$$0 \le u(x,t) \le v(x,t), \tag{2.33}$$

for any weak solution v of the same problem (1.7), (1.8).

Note that the minimal solution is unique by definition. The following standard comparison result is true in the class of minimal solutions.

Lemma 2.3.3. *Let u and v be minimal solutions of the CP (1.7), (1.8). If*

$$u(x,0) \geq (\leq)\ v(x,0),\ x \in \mathbb{R},$$

then

$$u(x,t) \geq (\leq)\ v(x,t),\ (x,t) \in \mathbb{R} \times (0,T).$$

If the function $u(x,t)$ is a minimal solution to CP (1.7), (1.10) with $b = 0$, then the function

$$\bar{u}(x,t) = \exp(-bt)u(x,(b(1-mp))^{-1}(\exp(-b(mp-1))t-1)),$$

is a minimal solution to (1.7), (1.10) with $b \neq 0$ and $\beta = 1$. Hence, from the above mentioned result it follows that the unique minimal solution to CP (1.7), (1.10) with $mp > 1, b > 0$, $\beta = 1$, and $\alpha = (1+p)/(mp-1)$, is the function $\bar{u}_C(x,t)$ from (2.23).

2.3.1 Asymptotic Properties of Solutions With $mp > 1$

In the following lemmas we establish some preliminary estimations of the solution to the CP.

2.3.1.1 Diffusion Dominates Reaction

Lemma 2.3.4. *If $b = 0, 0 < \alpha < (1+p)/(mp-1)$, then the minimal solution u of the CP (1.7), (1.10) has a self-similar form (2.8), where the self-similarity function f satisfies*

(2.11). If u_0 satisfies (1.9), and u be a unique weak solution to CP (1.7), (1.8), then u satisfies (2.1)-(2.3).

Proof of Lemma 2.3.4. Let u be a unique minimal solution of the problem (1.7), (1.10). If we consider a function

$$u_k(x,t) = ku(k^{-1/\alpha}x, k^{(\alpha(mp-1)-(1+p))/\alpha}t),\ k > 0, \tag{2.34}$$

it may easily be checked that this satisfies (1.7), (1.10). Since u is a minimal solution we have

$$u(x,t) \le ku(k^{-1/\alpha}x, k^{(\alpha(mp-1)-(1+p))/\alpha}t),\ k > 0. \tag{2.35}$$

By changing the variable in (4.38) as

$$y = k^{-1/\alpha}x,\ \tau = k^{(\alpha(mp-1)-(1+p))/\alpha}t, \tag{2.36}$$

we derive (4.38) with k replaced with k^{-1}. Since $k > 0$ is arbitrary, (4.38) follows with "=". If we choose $k = t^{\alpha/(1+p-\alpha(mp-1))}$, the latter implies (2.8) with $f(\xi) = u(\xi, 1)$, where f is a nonnegative and continuous solution of (2.9), (2.10). By [42], PDE (1.7) has a finite speed of propagation property, and minimal solution of (1.7), (1.10) has an expanding interface. Therefore, the upper bound ξ_* of the support of f is positive and finite; f is positive and smooth for $\xi < \xi_*$ and $f = 0$ for $\xi \ge \xi_*$. Thus, (2.29) is valid. Proof of (2.2) and (2.11) coincide with the proof given in Lemma 3.2 of [23].

Now suppose that u_0 satisfies (1.9). Then for arbitrary sufficiently small $\varepsilon > 0$, there exists an $x_\varepsilon < 0$ such that

$$(C-\varepsilon)(-x)_+^\alpha \le u_0(x) \le (C+\varepsilon)(-x)_+^\alpha,\ x \ge x_\varepsilon. \tag{2.37}$$

Let $u_\varepsilon(x,t)$ (respectively, $u_{-\varepsilon}(x,t)$) be a minimal solution to the CP (1.7), (1.8) with initial data $(C+\varepsilon)(-x)_+^\alpha$ (respectively, $(C-\varepsilon)(-x)_+^\alpha$). Since the solution to the CP (1.7), (1.8) is continuous, there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$u_\varepsilon(x_\varepsilon,t) \geq u(x_\varepsilon,t),\ u_{-\varepsilon}(x_\varepsilon,t) \leq u(x_\varepsilon,t),\ \text{for } 0 \leq t \leq \delta. \tag{2.38}$$

From (2.37), (2.38), and by applying the comparison result, lemma 2.3.1, it follows that:

$$u_{-\varepsilon} \leq u \leq u_\varepsilon,\ \text{for } x \geq x_\varepsilon,\ 0 \leq t \leq \delta. \tag{2.39}$$

We have

$$u_{\pm\varepsilon}(\xi_\rho(t),t) = t^{\alpha/(1+p-\alpha(mp-1))} f(\rho; C \pm \varepsilon), \rho < \xi_*, t \geq 0. \tag{2.40}$$

(Furthermore, we denote the right-hand side of (2.11a) by $f(\rho, C)$). Now taking $x = \xi_\rho(t)$ in (2.39), after multiplying by $t^{-\alpha/(1+p-\alpha(mp-1))}$ and passing to the limit, first as $t \to 0^+$ and then as $\varepsilon \to 0^+$, we can easily derive (2.3). Similarly, from (2.39), (2.40) and (2.29), (2.1) easily follows. The lemma is proved. □

Lemma 2.3.5. *If either $b > 0$ and $0 < \alpha < \frac{1+p}{mp-\min\{1,\beta\}}$ or $b < 0$, $\beta \geq 1$ and $0 < \alpha < \frac{1+p}{mp-1}$ and u is the unique weak solution to the CP (1.7)-(1.9), then u satisfies (2.3).*

Proof of Lemma 2.3.5. Let $b > 0$. Then, as in the proof of Lemma 2.3.4, (2.37) and (2.38) follow from (1.9). From [66] and Lemma 2.3.3 it follows that the existence, uniqueness, and comparison result for the minimal solution of the CP (1.7), (1.8) with $u_0 = (C \pm \varepsilon)(-x)_+^\alpha$ and $T = +\infty$ hold. As before, from (2.37) and (2.38), (2.39) follows. For arbitrary $k > 0$, the function

$$u_k^{\pm\varepsilon}(x,t) = k u_{\pm\varepsilon}\Big(k^{-1/\alpha} x, k^{(\alpha(mp-1)-(1+p))/\alpha} t\Big), k > 0, \tag{2.41}$$

is a minimal solution of the following problem

$$u_t - \Big(|(u^m)_x|^{p-1}(u^m)_x\Big)_x + bk^{(\alpha(mp-\beta)-(1+p))/\alpha}u^\beta = 0,\ x\in\mathbb{R},\ t>0, \tag{2.42a}$$

$$u(x,0) = (C\pm\varepsilon)(-x)_+^\alpha,\ x\in\mathbb{R}. \tag{2.42b}$$

Since $\alpha(mp-\beta)-(1+p)<0$, it follows that

$$\lim_{k\to+\infty} u_k^{\pm\varepsilon}(x,t) = v_{\pm\varepsilon}(x,t),\ \ x\in\mathbb{R},\ \ t\geq 0, \tag{2.43}$$

where $v_{\pm\varepsilon}$ is a minimal solution to CP (1.7), (1.8) with $b=0,\ u_0=(C\pm\varepsilon)(-x)_+^\alpha$, and $T=+\infty$. Hence, $v_{\pm\varepsilon}$ satisfies (2.40). Taking $x=\xi_\rho(t)$, where $\rho<\xi_*$ is fixed, from (2.43) it follows that for arbitrary $t>0$

$$\lim_{k\to+\infty} ku_{\pm\varepsilon}\Big(k^{-1/\alpha}\xi_\rho(t), k^{(\alpha(mp-1)-(1+p))/\alpha}t\Big) = t^{\alpha(1+p-\alpha(mp-1))}f(\rho;C\pm\varepsilon). \tag{2.44}$$

Letting $\tau = k^{(\alpha(mp-1)-(1+p))/\alpha}t$, then (2.44) implies

$$u_{\pm\varepsilon}(\xi_\rho(\tau),\tau) \sim \tau^{\alpha/(1+p-\alpha(mp-1))}f(\rho;C\pm\varepsilon),\ \text{as } \tau\to 0^+. \tag{2.45}$$

As before, (2.3) easily follows from (2.39) and (2.45).

Let $b<0$. From (1.9) it follows that for all $\varepsilon>0$ there exists a number $x_\varepsilon<0$ such that

$$(C-\varepsilon/2)(-x)_+^\alpha \leq u_0(x) \leq (C-\varepsilon/2)(-x)_+^\alpha,\ \ x\geq x_\varepsilon. \tag{2.46}$$

Assume that $u_{\pm\varepsilon}$ is a solution of the boundary value problem

$$Lu_{\pm\varepsilon} = 0,\ |x|<|x_\varepsilon|, 0<t\leq\delta, \tag{2.47}$$

$$u_{\pm\varepsilon}(x,0) = (C \pm \varepsilon)(-x)_+^{\alpha},\ |x| \le |x_\varepsilon|,$$

$$u_{\pm\varepsilon}(x_\varepsilon,t) = (C \pm \varepsilon)(-x_\varepsilon)^{\alpha},\ u_{\pm\varepsilon}(-x_\varepsilon,t) = u(-x_\varepsilon,t) = 0,\ 0 \le t \le \delta, \tag{2.48}$$

where $\delta = \delta(\varepsilon) > 0$ is chosen such that

$$u_\varepsilon(x_\varepsilon,t) \ge u(x_\varepsilon,t),\ u_{-\varepsilon}(x_\varepsilon,t) \le u(x_\varepsilon,t),\ 0 \le t \le \delta. \tag{2.49}$$

From the comparison theorem it follows that

$$u_{-\varepsilon} \le u \le u_\varepsilon,\ |x| \le |x_\varepsilon|, 0 \le t \le \delta. \tag{2.50}$$

Now if we rescale

$$u_k^{\pm\varepsilon}(x,t) = k u_{\pm\varepsilon}(k^{-1/\alpha}x, k^{(\alpha(mp-1)-(1+p))/\alpha}t),\ k > 0, \tag{2.51}$$

then $u_k^{\pm\varepsilon}$ satisfies the following problem

$$L_k u_k^{\pm\varepsilon} \equiv (u_k^{\pm\varepsilon})_t - \Big(|((u_k^{\pm\varepsilon})^m)_x|^{p-1}((u_k^{\pm\varepsilon})^m)_x\Big)_x + bk^{(\alpha(mp-\beta)-(1+p))/\alpha}(u_k^{\pm\varepsilon})^{\beta} = 0 \text{ in } D_\varepsilon^k, \tag{2.52}$$

$$u_k^{\pm\varepsilon}(x,0) = (C \pm \varepsilon)(-x)_+^{\alpha},\ |x| \le k^{1/\alpha}|x_\varepsilon|, \tag{2.53}$$

$$u_k^{\pm\varepsilon}(k^{1/\alpha}x_\varepsilon,t) = k(C \pm \varepsilon)(-x_\varepsilon)^{\alpha},\ u_k^{\pm\varepsilon}(-k^{1/\alpha}x_\varepsilon,t) = 0, 0 \le t \le k^{\frac{p+1+\alpha(1-mp)}{\alpha}}\delta, \tag{2.54}$$

where

$$D_\varepsilon^k = \{(x,t) : |x| < k^{1/\alpha}|x_\varepsilon|, 0 < t \le k^{(\alpha(1-mp)+(1+p))/\alpha}\delta\}.$$

The next step is to prove the convergence of the sequence $\{u_k^{\pm\varepsilon}\}$ as $k \to +\infty$. If $b > 0$ the proof coincides with that given for Lemma 3.2 of [23]. If $b < 0$ we consider a function

$$g(x,t) = (C+1)(1+|x|^\mu)^{\frac{\alpha}{\mu}}(1-\nu t)^\gamma,\ x \in \mathbb{R}, 0 \le t \le t_0 = \nu^{-1}/2,$$

where

$$\gamma < 0,\ \mu > \frac{p+1}{p},\ \nu = -h_* + 1,\ h_* = \min_{\mathbb{R}} h(x) > -\infty$$
$$h(x) = p(\alpha m)^p (C+1)^{p-1}\gamma^{-1}(1-\nu t)^{\gamma(mp-1)+1}(1+|x|^\mu)^{\frac{\alpha(mp-1)-\mu(p+1)}{\mu}}|x|^{(\mu-1)p-1}$$
$$\times[(\mu-1)(1+|x|^\mu) + (\alpha m - \mu)\mu|x|^\mu].$$

Then we have

$$L_k g = -\gamma(C+1)(1+|x|^\mu)^{\frac{\alpha}{\mu}}(1-\nu t)^{\gamma-1} S \ \text{ in } \ D_\varepsilon^k,$$
$$S = \nu + h(x) - b(C+1)^{\beta-1}\gamma^{-1}(1+|x|^\mu)^{\frac{\alpha(\beta-1)}{\mu}}(1-\nu t)^{\gamma(\beta-1)+1}k^{\frac{\alpha(mp-\beta)-(1+p)}{\alpha}},$$

and therefore

$$S \ge 1 + B, \ \text{ in } \ D_{0\varepsilon}^k = D_\varepsilon^k \cap \{(x,t) : 0 < t \le t_0\}, \tag{2.55}$$

where

$$B = O(k^{mp-1-(1+p)/\alpha}) \ \text{ uniformly for } \ (x,t) \in D_{0\varepsilon}^k \ \text{ as } k \to +\infty.$$

Hence, we have for $0 < \varepsilon << 1$ and $k >> 1$

$$L_k g \geq 0, \text{ in } D^k_{0\varepsilon}, \tag{2.56}$$

$$g(x,0) \geq u_k^{\pm\varepsilon}(x,0), \text{ for } |x| \leq k^{1/\alpha}|x_\varepsilon|, \tag{2.57}$$

$$g(\pm k^{1/\alpha}x_\varepsilon, t) \geq u_k^{\pm\varepsilon}(\pm k^{1/\alpha}x_\varepsilon, t), \text{ for } 0 \leq t \leq t_0. \tag{2.58}$$

From (2.56)-(2.58) and comparison theorem it follows that

$$0 \leq u_k^{\pm\varepsilon}(x,t) \leq g(x,t), \text{ in } \bar{D}^k_{0\varepsilon} \tag{2.59}$$

Let G be an arbitrary fixed compact subset of

$$P = \{(x,t) : x \in \mathbb{R}, 0 < t \leq t_0\}$$

By choosing k to be so large that $G \subset P$, it follows from (2.59) that the sequences $\{u_k^{\pm\varepsilon}\}$ are uniformly bounded in G. From [68, 67], it follows that they are uniformly Hölder continuous in G. From the Arzelá-Ascoli theorem and standard diagonalization argument it follows that there exist functions $v_{\pm\varepsilon}$ such that for some subsequence k'

$$\lim_{k' \to +\infty} u_{k'}^{\pm\varepsilon}(x,t) = v_{\pm\varepsilon}(x,t), \ (x,t) \in P. \tag{2.60}$$

It may be easily checked that $v_{\pm\varepsilon}$ is a solution of the CP (1.7), (1.8) with $u_0 = (C \pm \varepsilon)(-x)_+^\alpha$.

The lemma is proved. □

2.3.1.2 Diffusion and Reaction are in Balance

In the next lemma we analyze a special class of finite travelling wave solutions. By a finite travelling-wave solution with velocity $0 \neq k \in \mathbb{R}$ we mean a solution $u(x,t) = \varphi(kt - x)$, where $\varphi(y) \geq 0$, $\varphi \not\equiv 0$, and $\varphi(y) = 0$ for $y \leq y_0$ for some $y_0 \in \mathbb{R}$.

Lemma 2.3.6. *Let* $0 < \beta < 1, p(m+\beta) > 1+p$. *PDE (1.7) admits a finite travelling-wave solution,* $u(x,t) = \varphi(kt-x)$, *with* $\varphi(y) = 0$ *for* $y \leq 0$; $\varphi(y) > 0$, *for* $y > 0$, *and*

$$\lim_{y \to +\infty} y^{-\frac{1+p}{mp-\beta}} \varphi(y) = C_*. \tag{2.61}$$

The formula (2.61) is formula 2. from Theorem 3.1.1 of Chapter 3. Please refer to the proof of Theorem 3.1.1 for the proof of (2.61).

Lemma 2.3.7. *If* $0 < \beta < 1, \alpha = (1+p)/(mp-\beta)$, *then the minimal solution u to the CP (1.7), (1.10) has a self-similar form (2.15), where the self-similarity function f satisfies* (4.34)*,* (2.18)*. If* $C > C_*$, *then* $f_1(0) = A_1 > 0$, *where* A_1 *depends on* m, p, β, C *and b. If* u_0 *satisfies (1.9) with* $\alpha = (1+p)/(mp-\beta)$, $C > C_*$, *and u be a unique weak solution to CP (1.7), (1.8), then u satisfies*

$$u(0,t) \sim A_1 t^{1/(1-\beta)}, \text{ as } t \to 0^+ \tag{2.62}$$

Proof of Lemma 2.3.7. Let u be a unique minimal solution of the problem (1.7), (1.10) [66, 92]. Rescaled function

$$u_k(x,t) = ku(k^{-1/\alpha}x, k^{\beta-1}t), \ k > 0, \tag{2.63}$$

satisfies (1.7), (1.10), and therefore

$$u(x,t) \le ku(k^{-1/\alpha}x, k^{\beta-1}t), \ k > 0. \tag{2.64}$$

As in the proof of lemma 2.3.4, it follows that (2.64) is true with equality sign. If we choose $k = t^{1/(1-\beta)}$, then (2.64) implies (2.15) with $f_1(\zeta) = u(\zeta, 1)$, where f_1 is a nonnegative and continuous solution of (4.34), (2.18). By [42], PDE (1.7) has a finite speed of propagation property, and minimal solution of (1.7), (1.10) has a finite interface. Therefore, upper bound ξ_* of the support of f is finite; f is positive and smooth for $\xi < \xi_*$ and $f = 0$ for $\xi \ge \xi_*$. Now we prove that if $C > C_*$, then $f_1(0) = A_1 > 0$. We divide the proof into two cases:

Case 1: $p(m+\beta) < 1+p$

It is enough to show that there exists $t_0 > 0$ such that $u(0,t_0) > 0$. Let $g(x,t) = C_1(t-x)_+^{\frac{1+p}{mp-\beta}}$, $C_1 \in (C_*, C)$.

$$Lg = bC_1^{\beta}(t-x)_+^{\frac{\beta(1+p)}{mp-\beta}}\left[1-\left(\frac{C_1}{C_*}\right)^{mp-\beta} + C_1^{1-\beta}\frac{1+p}{b(mp-\beta)}(t-x)_+^{\frac{1+p-p(m+\beta)}{mp-\beta}}\right]$$

Since $C_1 < C$, we can choose $x_1 < 0$ and $\delta > 0$ such that

$$Lg \le 0, \text{ in } Q := \{(x,t) : x_1 \le x < t, 0 < t \le \delta\},$$

$$g(x,0) \le u(x,0), \ x_1 \le x; \ g(x_1,t) \le u(x_1,t), \ 0 \le t \le \delta.$$

Comparison Lemma 2.3.1 implies

$$0 < g(x,t) \le u(x,t), \ x_1 \le x < t, 0 \le t \le \delta.$$

In particular, we have: $u(0,t_0) > 0$ for all $0 < t_0 \leq \delta$, which implies that $f_1(0) = A_1 > 0$.

Case 2: $p(m+\beta) > 1+p$

We apply Lemma 2.3.6 with the forward traveling wave ($k > 0$). By (2.61) for some $M > 0$ we have

$$\varphi(y) < Cy^{\frac{1+p}{mp-\beta}}, \text{ for } y > M. \tag{2.65}$$

We choose

$$K = \max\{\varphi(y) : 0 \leq y \leq M\}, \ \xi = \max\Big\{M; \Big(\frac{K}{C}\Big)^{\frac{mp-\beta}{1+p}}\Big\}, \tag{2.66}$$

and consider a family of traveling-wave solutions to (1.7) of the form: $g(x,t) = \varphi(kt - x - \xi)$. From (2.65), (2.66) it follows that

$$\varphi(-x-\xi) \leq C(-x)_+^{\frac{1+p}{mp-\beta}}, \text{ for any } x \in \mathbb{R}. \tag{2.67}$$

From the comparison theorem it follows that $g \leq u$ for any $x \in \mathbb{R}, t \geq 0$. By choosing $t_0 > 0$ such that $k > \xi t_0^{-1} > 0$, we ensure that

$$0 < g(0,t_0) = \varphi(kt_0 - \xi) \leq u(0,t_0) = t_0^{\frac{1}{1-\beta}} f_1(0), \tag{2.68}$$

which proves that $f_1(0) > 0$. To prove the asymptotic formula (2.62) we proceed as we did in the proof in Lemma 2.3.5. As before, (2.37)-(2.39) follow from (1.9), where $v_{\pm\varepsilon}$ is a solution of the problem

$$v_t - \Big(|(v^m)_x|^{p-1}(v^m)_x\Big)_x + bv^\beta = 0, \ |x| < |x_\varepsilon|, \ 0 < t \leq \delta, \tag{2.69}$$

$$v(x_\varepsilon, t) = (C \pm \varepsilon)(-x_\varepsilon)^\alpha, \ v(-x_\varepsilon, t) = u(-x_\varepsilon, t), 0 \leq t \leq \delta, \tag{2.70}$$

$$v(x,0) = (C \pm \varepsilon)(-x)_+^\alpha, \ |x| \leq |x_\varepsilon|. \tag{2.71}$$

Rescaled function

$$u_k^{\pm\varepsilon}(x,t) = ku_{\pm\varepsilon}\Big(k^{-\frac{1}{\alpha}}x, k^{\beta-1}t\Big),\ k>0,$$

satisfies the Dirichlet problem:

$$v_t = \Big(|(v^m)_x|^{p-1}(v^m)_x\Big)_x - bv^\beta,\ \text{in}\ E_\varepsilon^k = \Big\{|x| < k^{\frac{1}{\alpha}}|x_\varepsilon|,\ 0<t\le k^{1-\beta}\delta\Big\}, \tag{2.72a}$$

$$v(k^{\frac{1}{\alpha}}x_\varepsilon,t) = k(C\pm\varepsilon)(-x_\varepsilon)^\alpha,\ v(-k^{\frac{1}{\alpha}}x_\varepsilon,t) = ku(-x_\varepsilon,k^{\beta-1}t),\ 0\le t\le k^{1-\beta}\delta, \tag{2.72b}$$

$$v(x,0) = (C\pm\varepsilon)(-x)_+^\alpha,\ |x|\le k^{\frac{1}{\alpha}}|x_\varepsilon|. \tag{2.72c}$$

As before in the proof of Lemma 2.3.5 we have:

$$\lim_{k\to+\infty} u_k^{\pm\varepsilon}(x,t) = v_{\pm\varepsilon}(x,t),\ (x,t)\in P := \{(x,t): x\in\mathbb{R},\ 0<t\le t_0\}, \tag{2.73}$$

thus,

$$v_{\pm\varepsilon}(x,t) = t^{\frac{1}{1-\beta}} f_1(\rho; C\pm\varepsilon), \rho<\zeta_*, t\ge 0 \tag{2.74}$$

Taking $x=\eta_\rho(t)=\rho t^{\frac{mp-\beta}{(1+p)(1-\beta)}}$ and $\tau = k^{\beta-1}t$ it follows from (2.73) that

$$u_{\pm\varepsilon}(\eta_\rho(\tau),\tau) \sim \tau^{\frac{1}{1-\beta}} f_1(\rho; C\pm\varepsilon),\ \text{as}\ \tau\to 0^+. \tag{2.75}$$

From (2.39) and (2.75), since $\varepsilon>0$ is arbitrary and $f_1(0)=A_1>0$, the desired asymptotic formula (2.62) follows. The lemma is proved. □

2.3.1.3 Reaction Dominates Diffusion

Lemma 2.3.8. *If* $0<\beta<1$, $\alpha>(1+p)/(mp-\beta)$, *and u be a unique weak solution to CP (1.7)-(1.9), then u satisfies (2.7).*

Proof of Lemma 2.3.8. As before, (2.37)-(2.39) follow from (1.9), where $v_{\pm\varepsilon}$ is a solution of the problem

$$v_t - \Big(|(v^m)_x|^{p-1}(v^m)_x\Big)_x + bv^\beta = 0,\ |x| < |x_\varepsilon|,\ 0 < t \le \delta, \tag{2.76}$$

$$v(x_\varepsilon, t) = (C \pm \varepsilon)(-x_\varepsilon)^\alpha,\ v(-x_\varepsilon, t) = u(-x_\varepsilon, t), 0 \le t \le \delta, \tag{2.77}$$

$$v(x, 0) = (C \pm \varepsilon)(-x)_+^\alpha,\ |x| \le |x_\varepsilon|. \tag{2.78}$$

Rescaled function

$$u_k^{\pm\varepsilon}(x,t) = ku_{\pm\varepsilon}\Big(k^{-\frac{1}{\alpha}}x, k^{\beta-1}t\Big),\ k > 0,$$

satisfies the Dirichlet problem

$$v_t = k^{\frac{1+p-\alpha(mp-\beta)}{\alpha}}\Big(|(v^m)_x|^{p-1}(v^m)_x\Big)_x - bu^\beta,\ \text{in } E_\varepsilon^k, \tag{2.79a}$$

$$v(k^{\frac{1}{\alpha}}x_\varepsilon, t) = k(C \pm \varepsilon)(-x_\varepsilon)^\alpha,\ v(-k^{\frac{1}{\alpha}}x_\varepsilon, t) = ku(-x_\varepsilon, k^{\beta-1}t),\ 0 \le t \le k^{1-\beta}\delta, \tag{2.79b}$$

$$v(x, 0) = (C \pm \varepsilon)(-x)_+^\alpha,\ |x| \le k^{\frac{1}{\alpha}}|x_\varepsilon|, \tag{2.79c}$$

where

$$E_\varepsilon^k := \Big\{|x| < k^{\frac{1}{\alpha}}|x_\varepsilon|,\ 0 < t \le k^{1-\beta}\delta\Big\}.$$

The next step consists in proving the convergence of the sequence $\{u_k^{\pm\varepsilon}\}$ as $k \to +\infty$. This step is identical with the proof given in the similar Lemma 3.4 from [23]. For any fixed $t_0 > 0$, the function $g(x,t) = (C+1)(1+x^2)^{\alpha/2}e^t$ is a uniform upper bound for the sequence $\{u_k^{\pm\varepsilon}\}$ in $E_{0\varepsilon}^k = E_\varepsilon^k \cap P$, where $P = \{(x,t) : 0 < t \le t_0\}$. The sequences $\{u_k^{\pm\varepsilon}\}$ are uniformly Hölder continuous on an arbitrary compact subset of P [57, 68]. As in the proof of the Lemma 3.4 of [23] it is proved that some subsequences $\{u_{k'}^{\pm\varepsilon}\}$ converge to solutions of the reaction equation. This implies

$$u_{\pm\varepsilon}(\eta_\ell(\tau),\tau) \sim \tau^{\frac{1}{1-\beta}}\Big[(C\pm\varepsilon)^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)\Big]^{\frac{1}{1-\beta}}, \text{ as } \tau \to 0^+. \tag{2.80}$$

From (2.39) and (2.80), since $\varepsilon > 0$ is arbitrary, the desired formula (2.7) follows. The lemma is proved. □

2.4 Proofs of the Main Results

2.4.1 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. From Lemma 2.3.5 and (2.3) it follows

$$\liminf_{t\to 0^+} \eta(t) t^{1/(\alpha(mp-1)-(1+p))} \geq \xi_*. \tag{2.81}$$

For any $\varepsilon > 0$, let u_ε be a minimal solution of the CP (1.7), (1.10) with $b = 0$ and with C replaced by $C+\varepsilon$. The second inequality of (2.37) and the first inequality of (2.38) follow from (1.9). Since u_ε is a supersolution of (1.7), from (2.37), (2.38), and a comparison principle, the second inequality of (2.39) follows. By Lemma 2.3.4 we have

$$\eta(t) \leq (C+\varepsilon)^{\frac{mp-1}{1+p-\alpha(mp-1)}} \xi'_* t^{1/(1+p-\alpha(mp-1))}, \; 0 \leq t \leq \delta,$$

and hence

$$\limsup_{t\to 0^+} \eta(t) t^{1/(\alpha(mp-1)-(1+p))} \leq \xi_*. \tag{2.82}$$

From (2.81) and (2.82), (2.1) follows. □

2.4.2 Proof of Theorem 2.1.2

Proof of Theorem 2.1.2. Assume that u_0 is defined by (1.10) and $p(m+\beta) \neq 1+p$. The self-similar form (2.15) follows from Lemma 2.3.7. Let $C > C_*$. For a function

$$g(x,t) = t^{1/(1-\beta)} f_1(\zeta),\ \zeta = xt^{-\frac{mp-\beta}{(1+p)(1-\beta)}}. \tag{2.83}$$

we have

$$Lg = t^{\frac{\beta}{1-\beta}} \mathcal{L}^0 f_1, \tag{2.84}$$

where the operator $\mathcal{L}^0$ is defined by (4.34). By choosing

$$f_1(\zeta) = C_0(\zeta_0 - \zeta)_+^{\gamma_0},\ \ 0 < \zeta < +\infty,$$

with $C_0,\ \zeta_0 > 0$ and $\gamma_0 = (1+p)/(mp-\beta)$ we have

$$\mathcal{L}^0 f_1 = bC_0^{\beta}(\zeta_0 - \zeta)_+^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_0/C_*)^{mp-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)} \zeta_0 (\zeta_0 - \zeta)_+^{\frac{1+p-p(m+\beta)}{mp-\beta}} \right\}. \tag{2.85}$$

For an upper estimation we choose $C_0 = C_2$ and $\zeta_0 = \zeta_2$ (see Appendix 6.1). If $p(m+\beta) > 1+p$, we have

$$\mathcal{L}^0 f_1 \geq bC_2^{\beta}(\zeta_2 - \zeta)_+^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_2/C_*)^{mp-\beta} + \frac{C_2^{1-\beta}}{b(1-\beta)} \zeta_2^{\frac{(1+p)(1-\beta)}{mp-\beta}} \right\} = 0,\ \text{for } 0 \leq \zeta \leq \zeta_2,$$

while if $p(m+\beta) < 1+p$, we have

$$\mathcal{L}^0 f_1 \geq bC_2^{\beta}(\zeta_2 - \zeta)_+^{\frac{\beta(1+p)}{mp-\beta}} \left\{ 1 - (C_2/C_*)^{mp-\beta} \right\} = 0,\ \text{for } 0 \leq \zeta \leq \zeta_2.$$

By (4.44) we have

$$Lg \geq 0, \text{ for } 0 < x < \zeta_2 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \ 0 < t < +\infty, \tag{2.86a}$$

$$Lg = 0, \text{ for } x > \zeta_2 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \ 0 < t < +\infty. \tag{2.86b}$$

Lemma 2.3.1 implies that g is a supersolution of (1.7) in $\{(x,t) : x > 0, \ t > 0\}$. Since

$$g(x,0) = u(x,0) = 0, \text{ for } 0 \leq x < +\infty, \tag{2.87a}$$

$$g(0,t) = u(0,t), \text{ for } 0 \leq x < +\infty, \tag{2.87b}$$

the right-hand side of (2.19) follows. If $p(m+\beta) < 1+p$, to prove the lower estimation we choose $C_0 = C_1, \ \zeta_0 = \zeta_1,$and $\gamma_0 = (1+p)/(mp-\beta)$. From (2.85) and (4.44) we have

$$\mathcal{L}^0 f_1 \leq bC_1^{\beta}(\zeta_1-\zeta)^{\frac{\beta(1+p)}{mp-\beta}}\left\{1-(C_1/C_*)^{mp-\beta}+\frac{C_1^{1-\beta}}{b(1-\beta)}\zeta_1^{\frac{(1+p)(1-\beta)}{mp-\beta}}\right\} = 0, \text{ for } 0 \leq \zeta \leq \zeta_1,$$

$$Lg \leq 0, \text{ for } 0 < x < \zeta_1 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \ 0 < t < +\infty, \tag{2.88a}$$

$$Lg = 0, \text{ for } x > \zeta_1 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \ 0 < t < +\infty. \tag{2.88b}$$

As before from (2.87) and Lemma 2.3.1, the left-hand side of (2.19) follows. If $p(m+\beta) > 1+p$, then to prove the lower estimation we choose $C_0 = C_1, \zeta_0 = \zeta_1$ and $\gamma_0 =$

$p/(mp-1)$. We have

$$\mathcal{L}^0 f_1 \le C_1(1-\beta)^{-1}(\zeta_1-\zeta)^{\frac{1+p-mp}{mp-1}} \times$$
$$\times\left\{\zeta_1 - C_1^{mp-1}\frac{(1-\beta)p(mp)^p}{(mp-1)^{1+p}} + b(1-\beta)C_1^{\beta-1}\zeta_1^{\frac{p(m+\beta)-(1+p)}{mp-1}}\right\} = 0,\ 0<\zeta<\zeta_1,$$

which again implies (2.88). From Lemma 2.3.1, the left-hand side of (2.19) follows.

Let $p(m+\beta) > 1+p$ and $0 < C < C_*$. For $\gamma \in [0,1)$ consider a function

$$g(x,t) = \left[C^{1-\beta}(-x)_+^{\frac{(1+p)(1-\beta)}{mp-\beta}} - b(1-\beta)(1-\gamma)t\right]_+^{\frac{1}{1-\beta}},\ x\in\mathbb{R},\ t>0.$$

We estimate Lg in

$$M := \{(x,t) : -\infty < x < \mu_\gamma(t),\ t>0\},\ \mu_\gamma(t) = -\left[b(1-\beta)(1-\gamma)C^{\beta-1}t\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}}.$$

We have $Lg = bg^\beta S$, where

$$S = \gamma - C^{mp-\beta}\left[1-\left(\frac{-\mu_\gamma(t)}{(-x)_+}\right)^{\frac{(1+p)(1-\beta)}{mp-\beta}}\right]^{\frac{p(m+\beta-1)-\beta}{1-\beta}} \times$$
$$\times\left[R_1 + R_2\left[1-\left(\frac{-\mu_\gamma(t)}{(-x)_+}\right)^{\frac{(1+p)(1-\beta)}{mp-\beta}}\right]^{-1}\right], \qquad (2.89a)$$
$$S\Big|_{t=0} = \gamma - (C/C_*)^{mp-\beta},\ S\Big|_{x=\mu_\gamma(t)} = \gamma, \qquad (2.89b)$$

where $R_1, R_2 > 0$ (see Appendix 6.1). Moreover,

$$S_t \ge 0 \text{ in } M.$$

Thus,

$$\gamma - (C/C_*)^{mp-\beta} \leq S \leq \gamma \text{ in } M.$$

If we take $\gamma = (C/C_*)^{mp-\beta}$ (respectively, $\gamma = 0$), then we have

$$Lg \geq 0 \text{ (respectively,} Lg \leq 0) \text{ in } M, \tag{2.90a}$$

$$Lg = 0, \text{ for } x > \mu_\gamma(t),\ t > 0. \tag{2.90b}$$

From Lemma 2.3.1, the estimation (2.21) follows. Let $p(m+\beta) < 1+p$ and $0 < C < C_*$. First, we establish the following rough estimation:

$$\left[C^{1-\beta}(-x)_+^{\frac{(1+p)(1-\beta)}{mp-\beta}} - b(1-\beta)(1-(C/C_*)^{mp-\beta})t\right]_+^{\frac{1}{1-\beta}} \leq$$
$$\leq u(x,t) \leq C(-x)_+^{\frac{1+p}{mp-\beta}}, \text{ for } x \in \mathbb{R},\ 0 \leq t < +\infty. \tag{2.91}$$

To prove the left-hand side we consider the function, g, as in the case when $p(m+\beta) > 1+p$ with $\gamma = (C/C_*)^{mp-\beta}$. As before, we then derive (2.89a), and since

$$S_t \leq 0, \text{ in} M,$$

we have $S \leq 0$ in M. Hence, (2.90) is valid with $\leq$ in (2.90a). As before, from Lemma 2.3.1, the left-hand side of (2.91) follows. To prove the right-hand side of (2.91) it is enough to observe that

$$Lu_0 = bu_0^\beta(1-(C/C_*)^{mp-\beta}) \geq 0, \text{ for } x \in \mathbb{R},\ t \geq 0.$$

Having (2.91), we can now establish a more accurate estimation, (2.22). Consider a function

$$g(x,t) = C_0\Big(-\zeta_0 t^{\frac{mp-\beta}{(1+p)(1-\beta)}} - x\Big)_+^{\frac{1+p}{mp-\beta}}, \text{ in } G_\ell,$$

$$G_\ell := \Big\{(x,t) : \zeta(t) = -\ell t^{\frac{mp-\beta}{(1+p)(1-\beta)}} < x < +\infty, \ 0 < t < +\infty\Big\},$$

where, $C_0, \ \zeta_0 > 0$ and $\ell > \zeta_0$. Calculating Lg in

$$G_\ell^+ := \Big\{(x,t) : \zeta(t) < x < -\zeta_0 t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \ 0 < t < +\infty\Big\},$$

we have

$$Lg = bg^\beta S, \quad S = 1 - (C_0/C_*)^{mp-\beta} - (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 t^{\frac{p(m+\beta)-(1+p)}{(1+p)(1-\beta)}} \times \\ \times\Big(-\zeta_0 t^{\frac{mp-\beta}{(1+p)(1-\beta)}} - x\Big)^{\frac{1+p-p(m+\beta)}{mp-\beta}}. \tag{2.92}$$

By choosing $C_0 = C_*$, we have

$$Lg \le 0, \text{ in } G_\ell^+; \ Lg = 0, \text{ in } G_\ell \backslash \bar{G}_\ell^+. \tag{2.93}$$

To obtain a lower estimation we choose $\zeta_0 = \zeta_3$ and $\ell = \ell_0$ (see Appendix 6.1). Using (2.91), we have

$$g\Big|_{x=\zeta(t)} = t^{\frac{1}{1-\beta}} C_*(\ell_0 - \zeta_3)^{\frac{1+p}{mp-\beta}} = \Big(b(1-\beta)\theta_* t\Big)^{\frac{1}{1-\beta}} = t^{\frac{1}{1-\beta}} \times \\ \times\Big[C^{1-\beta} \ell_0^{\frac{(1+p)(1-\beta)}{mp-\beta}} - b(1-\beta)\Big(1-(C/C_*)^{mp-\beta}\Big)\Big]^{\frac{1}{1-\beta}} \le u(\zeta(t),t), \ t \ge 0, \tag{2.94a}$$

$$g(x,0) = u(x,0) = 0, \ 0 \le x \le x_0, \tag{2.94b}$$

$$g(x_0,t) = u(x_0,t) = 0, \ t \ge 0, \tag{2.94c}$$

where $x_0 > 0$ is an arbitrary fixed number. By using (2.93) and (2.94), we can apply Lemma 2.3.1 in

$$G'_{\ell_0} := G_{\ell_0} \cap \{x < x_0\}.$$

Since $x_0 > 0$ is arbitrary, the left inequality in (2.22) follows. Since

$$S_x \geq 0, \text{ for } \zeta(t) < x < -\zeta_0 t^{\frac{mp-\beta}{1+p)(1-\beta)}}, \ t > 0,$$

from (2.92) it follows that

$$S \geq S\Big|_{x=\zeta((t)} = 1 - (C_0/C_*)^{mp-\beta} - (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 (\ell - \zeta_0)^{\frac{1+p-p(m+\beta)}{mp-\beta}}.$$

By choosing $C_0 = C_3,\ \zeta_0 = \zeta_4,\ \ell = \ell_1$ (see Appendix 6.1), we have

$$S\Big|_{x=\zeta(t)} = 0,$$

$$Lg \geq 0 \text{ in } G^+_{\ell_1},\ Lg = 0 \text{ in } G_{\ell_1} \backslash \bar{G}^+_{\ell_1},$$

$$u(\zeta(t), t) \leq t^{\frac{1}{1-\beta}} C \ell_1^{\frac{1+p}{mp-\beta}} = t^{\frac{1}{1-\beta}} C_3 (\ell_1 - \zeta_4)^{\frac{1+p}{mp-\beta}} = g(\zeta(t), t),\ t \geq 0,$$

and, for arbitrary $x_0 > 0$, (2.94b) and (2.94c) are valid. By applying Lemma 2.3.1 in G'_{ℓ_1}, due to the arbitrariness of $x_0 > 0$, we derive the right-hand side of (2.22). From (2.19), (2.21), and (2.22) it follows that

$$\zeta_1 t^{\frac{mp-\beta}{(1+p)(1-\beta)}} \leq \eta(t) \leq \zeta_2 t^{\frac{mp-\beta}{(1+p)(1-\beta)}},\ 0 \leq t < +\infty,$$

where the constants ζ_1 and ζ_2 are chosen according to relevant estimations for u. From (2.16) and the respective estimations (2.19), (2.21), and (2.22), the estimation (2.20) follows. If u_0 satisfies (1.9) with $\alpha = (1+p)/(mp-\beta)$ and with $C \neq C_*$, then the asymptotic formulae (2.4) and (2.5) may be proved as the similar estimations (2.1) and (2.3) were in Lemma 2.3.4. □

2.4.3 Proof of Theorem 2.1.3

Proof of Theorem 2.1.3. For for any $\varepsilon > 0$ from (1.9), (2.37) follows. Consider a function

$$g_\varepsilon(x,t) = \Big[(C+\varepsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)(1-\varepsilon)t\Big]_+^{1/(1-\beta)}. \tag{2.95}$$

We estimate Lg in

$$M_1 := \{(x,t) : x_\varepsilon < x < \eta_\ell(t), 0 < t \le \delta_1\},$$

$$\eta_\ell(t) = -\ell t^{1/(\alpha(1-\beta)},\ \ell(\varepsilon) = (C+\varepsilon)^{-1/\alpha}\big[b(1-\beta)(1-\varepsilon)\big]^{1/\alpha(1-\beta)},$$

where $\delta_1 > 0$ is chosen such that $\eta_{\ell(\varepsilon)}(\delta_1) = x_\varepsilon$. We have

$$Lg_\varepsilon = bg_\varepsilon^\beta\{\varepsilon + S\},$$

$$S = -b^{-1}(\alpha m)^p(C+\varepsilon)^{mp-\beta}(-x)_+^{\alpha(mp-\beta)-(1+p)}\Big\{g|x|^{-\alpha}\Big/(C+\varepsilon)\Big\}^{p(m+\beta)-(1+p)}S_1,$$

$$S_1 = \left\{\alpha p(m+\beta-1) + p(\alpha(1-\beta)-1)\Big[g|x|^{-\alpha}\Big/(C+\varepsilon)\Big]^{1-\beta}\right\}.$$

If $p(m+\beta) \ge 1+p$, we can choose $x_\varepsilon < 0$ such that

$$|S| < \frac{\varepsilon}{2},\ \text{in } M_1.$$

Thus we have

$$Lg_\varepsilon > b(\varepsilon/2)g_\varepsilon^\beta \text{ (respectively, } Lg_{-\varepsilon} < -b(\varepsilon/2)g_{-\varepsilon}^\beta), \text{ in } M_1, \tag{2.96}$$

$$Lg_{\pm\varepsilon} = 0, \text{ for } x > \eta_{\ell(\pm\varepsilon)}(t),\ 0 < t \leq \delta_1, \tag{2.97}$$

$$g_\varepsilon(x,0) \ \geq \ u_0(x) \text{ (respectively, } g_{-\varepsilon}(x,0) \leq u_0(x)),\ x \geq x_\varepsilon. \tag{2.98}$$

Since u and g are continuous functions, $\delta = \delta(\varepsilon) \in (0, \delta_1]$, may be chosen such that

$$g_\varepsilon(x_\varepsilon, t) \geq u(x_\varepsilon, t) \text{ (respectively, } g_{-\varepsilon}(x_\varepsilon, t) \leq u(x_\varepsilon, t)),\ 0 \leq t \leq \delta.$$

From Lemma 2.3.1 it follows that

$$g_{-\varepsilon} \leq u \leq g_\varepsilon,\ x \geq x_\varepsilon,\ 0 \leq t \leq \delta, \tag{2.99a}$$

$$\eta_{\ell(-\varepsilon)}(t) \leq \eta(t) \leq \eta_{\ell(\varepsilon)},\ 0 \leq t \leq \delta, \tag{2.99b}$$

which imply (2.6) and (2.7). Let $p(m+\beta) < 1 + p$. In this case the left-hand side of (2.99) may be proved similarly. Moreover, we can replace $1+\varepsilon$ with 1 in $g_{-\varepsilon}$ and $\eta_\ell(-\varepsilon)$. For a sharp upper estimation, consider a function

$$g(x,t) = C_6\Big(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\Big)_+^\alpha, \text{ in } G_{\ell,\delta},$$

$$G_{\ell,\delta} := \{(x,t) : \eta_\ell(t) < x < +\infty,\ 0 < t < \delta\},$$

where $\ell \in (\ell_*, +\infty)$, C_6 and ζ_5 are given in Appendix 6.1. From (2.7) it follows that for all $\ell > \ell_*$ and for all sufficiently small $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, \ell) > 0$ such that

$$u(\eta_\ell(t), t) \leq t^{\frac{1}{1-\beta}}[C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\varepsilon)]^{\frac{1}{1-\beta}},\ 0 \leq t \leq \delta. \tag{2.100}$$

Calculating Lg in

$$G^+_{\ell,\delta} = \left\{(x,t) : \eta_\ell(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}},\ 0 < t < \delta\right\},$$

we derive

$$Lg = bg^\beta S,\ S = 1 - (b(1-\beta))^{-1}\zeta_5 C_6^{1/\alpha}\left\{gt^{1/(\beta-1)}\right\}^{(\alpha(1-\beta)-1)/\alpha}$$

$$-\frac{p}{b}(\alpha m)^p(\alpha m - 1)C_6^{(1+p)/\alpha} g^{mp-\beta-((1+p)/\alpha)}.$$

Since

$$S_x \geq 0,\ \text{in } G^+_{l,\delta},$$

$$S \geq S\big|_{x=\eta_\ell(t)} = 1 - (b(1-\beta))^{-1}\zeta_5 C_6^{1-\beta}(\ell - \zeta_5)^{\alpha(1-\beta)-1} -$$

$$-t^{\frac{\alpha(mp-\beta)-(1+p)}{\alpha(1-\beta)}} b^{-1} p(\alpha m)^p(\alpha m - 1)C_6^{mp-\beta}(\ell - \zeta_5)^{\alpha(mp-\beta)-(1+p)},$$

we have

$$S \geq \varepsilon - t^{\frac{\alpha(mp-\beta)-(1+p)}{\alpha(1-\beta)}} b^{-1} p(\alpha m)^p(\alpha m - 1)C_6^{mp-\beta}(\ell - \zeta_5)^{\alpha(mp-\beta)-(1+p)},\ \text{in } G^+_{\ell,\delta}.$$

By choosing $\delta = \delta(\varepsilon) > 0$ sufficiently small we have

$$Lg \geq b(\varepsilon/2)g^\beta,\ \text{in } G^+_{\ell,\delta}. \tag{2.101}$$

By applying (2.100) and Lemma 2.3.1 in $G'_{\ell,\delta} = G_{\ell,\delta} \cap \{x < x_0\}$ we have

$$Lg = 0, \text{ in } G'_{\ell,\delta} \backslash \bar{G}^{+}_{\ell,\delta}, \tag{2.102a}$$

$$u(\eta_\ell(t),t) \le t^{\frac{1}{1-\beta}} [C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\varepsilon)]^{\frac{1}{1-\beta}} =$$

$$= C_6(\ell - \zeta_5)^{\alpha} t^{\frac{1}{\alpha(1-\beta)}} = g(\eta_\ell(t),t), \ 0 \le t \le \delta, \tag{2.102b}$$

$$u(x_0,t) = g(x_0,t) = 0, \ 0 \le t \le \delta, \tag{2.102c}$$

$$u(x,0) \ = g(x,0) \ = 0, \ 0 \le x \le x_0. \tag{2.102d}$$

Since $x_0 > 0$ is arbitrary, from (2.102) and Lemma 2.3.1, it follows that for all $\ell > \ell_*$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \ell) > 0$ such that

$$u(x,t) \le C_6\Big(- \zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\Big)^{\alpha}_{+}, \text{ in } \bar{G}_{\ell,\delta.} \tag{2.103}$$

In view of (2.7) (which is valid along $x = \eta_\ell(t)$), δ may be chosen so small that

$$-\ell t^{1/\alpha(1-\beta)} \le \eta(t) \le -\zeta_5 t^{1/\alpha(1-\beta)}, \ 0 \le t \le \delta. \tag{2.104}$$

Since $\ell > \ell_*$ and $\varepsilon > 0$ are arbitrary, (2.6) follows from (2.104). □

The proofs of Theorem 2.1.4 and estimations (2.23)-(2.32) are almost identical to the similar proofs given in [23].

Chapter 3

Traveling-Wave Solutions to the Nonlinear Double Degenerate Reaction-Diffusion Equation

3.1 Introduction and the Main Result

In this section we consider equation (1.7)

$$u_t = \left(|(u^m)_x|^{p-1}(u^m)_x\right)_x - bu^\beta,\ x \in \mathbb{R},\ t > 0,$$

with $mp > 1$, $(m, p > 0)$, $0 < \beta < 1$, and $b > 0$. We present asymptotic results for the finite traveling-wave solutions to (1.7): $u(x,t) = \varphi(kt - x)$, where $\varphi(y) \geq 0$, $\varphi \not\equiv 0$, and $\varphi(y) \to 0^+$ as $y \to -\infty$; $\varphi(y) = 0$ for $y \leq y_0$, for some $y_0 \in \mathbb{R}$. By translation we select $y_0 = 0$. Finite traveling-wave solutions are weak solutions to (1.7), (1.8) in sense of Definition 1.2.1.

There exists a finite traveling-wave solution to equation (1.7) if we can find a function $\varphi \in \mathbb{R}^+$ that satisfies the following initial-value problem (IVP)

$$\begin{cases} \left(|(\varphi^m)'|^{p-1}(\varphi^m)'\right)' - k\varphi' - b\varphi^\beta = 0, \\ \varphi(0) = (\varphi^m)'(0) = 0, \end{cases} \tag{3.1}$$

we extend $\varphi(y)$ by 0 on for all $y < 0$. Note that all derivatives are understood in the weak sense.

The following is the main result of this Chapter.

Theorem 3.1.1. *There exists a finite traveling-wave solution, $u(x,t) = \varphi(kt - x)$, to equation* (1.7)*, with $\varphi(0) = 0$ if $k \neq 0$. Further we have,*

1. $\lim_{y\to 0^+} y^{-\frac{1+p}{mp-\beta}}\varphi(y) = C_* = \left[\frac{b(mp-\beta)^{1+p}}{(m(1+p))^p p(m+\beta)}\right]^{\frac{1}{mp-\beta}}$, if $p(m+\beta) < 1+p$;
2. $\lim_{y\to +\infty} y^{-\frac{1+p}{mp-\beta}}\varphi(y) = C_*$, if $p(m+\beta) > 1+p$;
3. $\lim_{y\to +\infty} y^{-\frac{p}{mp-1}}\varphi(y) = \left(\frac{mp-1}{mp}\right)^{\frac{p}{mp-1}} k^{\frac{1}{mp-1}}$, if $k > 0$, $p(m+\beta) < 1+p$;
4. $\lim_{y\to 0^+} y^{-\frac{p}{mp-1}}\varphi(y) = \left(\frac{mp-1}{mp}\right)^{\frac{p}{mp-1}} k^{\frac{1}{mp-1}}$, if $k > 0$, $p(m+\beta) > 1+p$;
5. $\lim_{y\to +\infty} y^{-\frac{1}{1-\beta}}\varphi(y) = \left((1-\beta)\left(-\frac{b}{k}\right)\right)^{\frac{1}{1-\beta}}$, if $k < 0$, $p(m+\beta) < 1+p$;
6. $\lim_{y\to 0^+} y^{-\frac{1}{1-\beta}}\varphi(y) = \left((1-\beta)\left(-\frac{b}{k}\right)\right)^{\frac{1}{1-\beta}}$, if $k < 0$, $p(m+\beta) > 1+p$.

The existence of traveling-wave solutions with interfaces for the nonlinear diffusion equation ((1.7) with $p = 1$) is considered in [65] and for the parabolic p-Laplacian equation ((1.7) with $m = 1$) in [80].

The organization of this section is as follows: In Section 3.1.1 we formulate and prove some preliminary results which are necessary for the proof of main result and in

Section 3.2 we prove the main result, Theorem 3.1.1.

3.1.1 Preliminary Results: Traveling-Wave Solutions and Phase-Space Analysis

In this section we'll apply phase-space analysis to find finite traveling-wave solutions for (1.7). We aim to analyze the phase portrait for problem (3.1). Consider the following lemma.

Lemma 3.1.2. *If the function $\varphi(y)$ is a positive solution to problem* (3.1)*, then $\varphi(y)$ is increasing for all $y > 0$.*

The proof of Lemma 3.1.2 follows as in the proof of the analagous result for the p-Laplacian equation in [80].

There exists a unique $\varphi(y) > 0$ that satisfies the IVP (3.1) for $y > 0$. To prove this, we introduce the following change of variable

$$\Theta = \varphi \text{ and } \Upsilon = ((\varphi^m)')^p,$$

it follows that

$$\Theta' = \frac{1}{m}\Theta^{1-m}\Upsilon^{\frac{1}{p}} \text{ and } \Upsilon' = b\Theta^{\beta} + \frac{k}{m}\Theta^{1-m}\Upsilon^{\frac{1}{p}}.$$

(Θ, Υ) starts from (0,0) at $y = 0$, exists for any $y \in \mathbb{R}^+$, and are contained in the first quadrant: $Q_1 = \{(\Theta, \Upsilon) : \Theta, \Upsilon > 0\}$ for $y > 0$. There exists a unique solution, or trajectory,

$\Upsilon(\Theta)$. To prove this, we divide Υ' by Θ' and derive the following problem

$$\begin{cases} \frac{d\Upsilon}{d\Theta} = f(\Theta, \Upsilon) = k + bm\Theta^{m+\beta-1}\Upsilon^{-\frac{1}{p}}, \\ \Upsilon(0) = 0. \end{cases} \tag{3.2}$$

As done in [80], we find the nonzero solutions or trajectories, $\Upsilon(\Theta)$, to (3.2), in two steps. First, we prove the global existence of the solution of the following perturbed IVP

$$\begin{cases} \frac{d\Upsilon}{d\Theta} = f(\Theta, \Upsilon) = k + bm\Theta^{m+\beta-1}\Upsilon^{-\frac{1}{p}}, \\ \Upsilon(0) = \varepsilon, \varepsilon > 0. \end{cases} \tag{3.3}$$

The fact that the function $f(\Theta, \Upsilon)$ is locally Lipshitz continuous in $\mathbb{R}^+ \times (\varepsilon, +\infty)$ implies that there exists a unique local solution Υ_ε to problem (3.3). For both $k > 0$ and for $k < 0$ with $p(m+\beta) > 1 + p$, the proof of the existence of a global solution to (3.3) follows as in the proof of the existence of a global solution to the analgous IVP for the p-Laplacian equation in [80].

Let $k < 0$ with $p(m+\beta) < 1 + p$. The difference from the previous case is that

$$\widetilde{C} : (0, +\infty) \to (+\infty, 0),$$

see Figure 3.1b. Since

$$\frac{d\Upsilon_\varepsilon}{d\Theta} > 0 \text{ if } \Upsilon_\varepsilon < \left(-\frac{k}{bm}\Theta^{1-m-\beta}\right)^{-p},$$

i.e., Υ_ε is increasing to the left of $\widetilde{C}$. Then Υ_ε must cross $\widetilde{C}$ with horizontal tangent, after that Υ_ε will be strictly decreasing. It follows that Υ_ε is a global solution to (3.3) if $k < 0$.

We now prove the global existence of solution to the perturbed problem

$$\begin{cases} \frac{d\Upsilon}{d\Theta} = f(\Theta,\Upsilon) = k + bm\Theta^{m+\beta-1}\Upsilon^{-\frac{1}{p}}, \\ \Upsilon(\varepsilon) = 0, \varepsilon > 0. \end{cases} \tag{3.4}$$

To do this, we consider the following Cauchy problem for the inverse function of Υ, denoted as v

$$\begin{cases} \frac{dv}{dt} = g(v,t) = \frac{1}{f(v,t)} = \frac{t^{\frac{1}{p}}}{kt^{\frac{1}{p}} + bmv^{m+\beta-1}}, \\ v(0) = \varepsilon, \varepsilon > 0. \end{cases} \tag{3.5}$$

Since the right hand side of (3.5) is Lipshitz continuous, there exists a local solution, v_ε, to the CP (3.5). For $k > 0$ and for $k < 0$ with $p(m+\beta) > 1 + p$, as for (3.3), the proof of the existence of a global solution to (3.4) follows as in the proof of the existence of a global solution to the analgous IVP for the p-Laplacian equation in [80].

Let $k < 0$ with $p(m+\beta) < 1 + p$. As before, we define the curve where $f(v,t) = 0$ by $\widetilde{C}$. We denote the region to the left of $\widetilde{C}$ as $Q_l = \{(v,t) : f(v,t) > 0\}$ and to the region to the right of $\widetilde{C}$ as $Q_r = \{(v,t) : f(v,t) < 0\}$, see Figure 3.1d. Since v is increasing in Q_l it must cross $\widetilde{C}$ with vertical tangent, however, this is impossible. Let t_ε be such that $v(t_\varepsilon) = M_\varepsilon \in \widetilde{C}$. Consider the function w such that

$$w : [\varepsilon, M_\varepsilon] \to [0, t_\varepsilon].$$

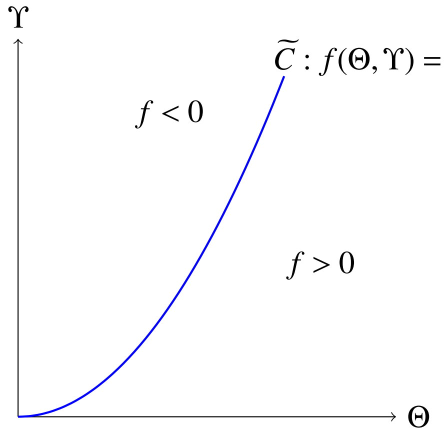
Then w is the inverse function of v in $[0, t_\varepsilon]$ and so solves the following problem

$$\begin{cases} \frac{dw}{dt} = k + bmw^{m+\beta-1}t^{-\frac{1}{p}} = f(w,t), \\ w(\varepsilon) = 0,\ w(M_\varepsilon) = t_\varepsilon,\ \varepsilon > 0. \end{cases} \tag{3.6}$$

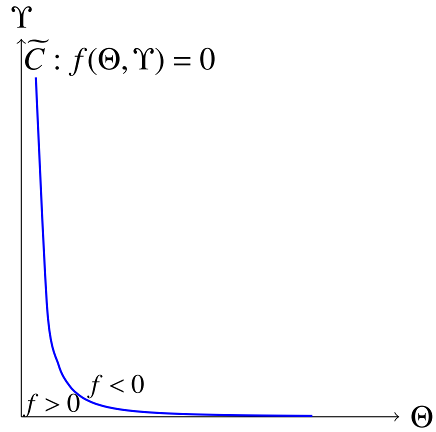
Let $\widehat{C}$ denote the curve where $f(w,t) = 0$. So w enters the region to the right of $\widehat{C}$ with horizontal tangent and since if $t > M_\varepsilon$, then $w(t)$ is decreasing, we have that w cannot cross $\widehat{C}$ again since it must cross with horizontal tangent, which is a contradiction. It follows that the solution, w, to problem (3.6) is global and so there exists a global solution to problem (3.4) if $k < 0$.

Lemma 3.1.3. *The problem* (3.2) *has a unique global solution.*

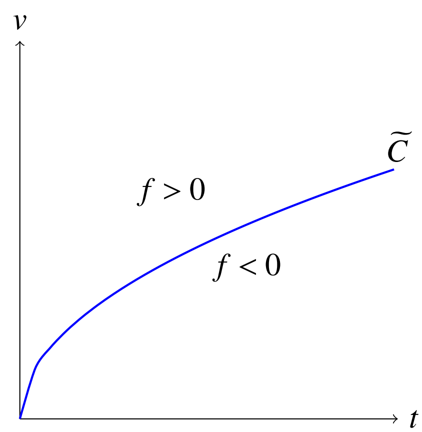
The proof of Lemma 3.1.3 follows as in the proof of existence and uniqueness of solution for the analagous problem for the p-Laplacian equation in [80].



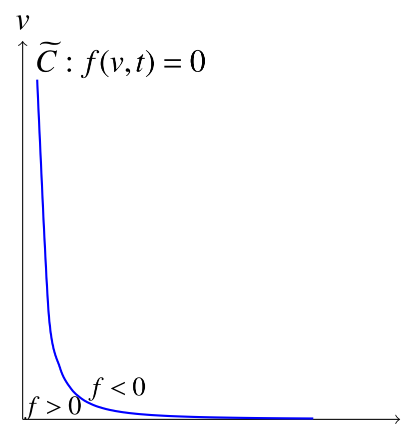
(a) $\Upsilon(\Theta) : p(m+\beta) > 1+p$



(b) $\Upsilon(\Theta) : p(m+\beta) < 1+p$



(c) $v(t) : p(m+\beta) > 1+p$



(d) $v(t) : p(m+\beta) < 1+p$

Figure 3.1: Trajectories $\Upsilon(\Theta)$ and $v(t)$

Let $\Upsilon = ((\varphi^m)')^p$ be a solution of the problem (3.2). For the problem

$$\frac{d\varphi}{dy} = \frac{1}{m}(\varphi(y))^{1-m}\Upsilon^{\frac{1}{p}}(\varphi(y)),\ \varphi(0) = 0, \tag{3.7}$$

there exists a unique maximal solution defined on $(-\infty, \ell)$ such that

$$\lim_{y \to \ell^-} \varphi(y) = +\infty.$$

By (3.7) we have that $(\varphi^m)'(0) = \Upsilon^{\frac{1}{p}}(0) = 0$, so we can continue φ by zero on $(-\infty, 0)$. On the other side, φ is strictly increasing, and

$$\lim_{y \to \ell^-} \varphi(y) = +\infty,$$

if ℓ is finite. By (3.7) and the boundedness of $\Upsilon^{-\frac{1}{p}}$, the above limit also holds if $\ell = +\infty$.

The solution of (3.7) defined on $(-\infty, \ell)$ satisfies the following problem

$$\begin{cases} (|(\varphi^m)'|^{p-1}(\varphi^m)')' - k\varphi' - b\varphi^\beta = 0, \text{ on } (-\infty, \ell), \\ \varphi(0) = (\varphi^m)'(0) = 0. \end{cases} \tag{3.8}$$

We now prove that the solution to problem (3.8) is global. To proceed, we require the following lemma.

Lemma 3.1.4. *Let Υ be a solution of the problem* (3.2)*. Then we have*

1. $\Upsilon(\Theta) \sim \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} \Theta^{\frac{p(m+\beta)}{1+p}}$, *as* $\Theta \to 0^+$ *if* $p(m+\beta) < 1+p$.
2. $\Upsilon(\Theta) \sim \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} \Theta^{\frac{p(m+\beta)}{1+p}}$, *as* $\Theta \to +\infty$ *if* $p(m+\beta) > 1+p$.
3. $\Upsilon(\Theta) \sim k\Theta$, *as* $\Theta \to +\infty$ *if* $k > 0$, $p(m+\beta) < 1+p$.
4. $\Upsilon(\Theta) \sim k\Theta$, *as* $\Theta \to 0^+$ *if* $k > 0$, $p(m+\beta) > 1+p$.
5. $\Upsilon(\Theta) \sim (-\frac{k}{bm})^{-p}\Theta^{p(m+\beta-1)}$, *as* $\Theta \to +\infty$ *if* $k < 0$, $p(m+\beta) < 1+p$.
6. $\Upsilon(\Theta) \sim (-\frac{k}{bm})^{-p}\Theta^{p(m+\beta-1)}$, *as* $\Theta \to 0^+$ *if* $k < 0$, $p(m+\beta) > 1+p$.

Proof of Lemma 3.1.4. We begin by proving formulas 1. and 2. from Lemma 3.1.4. We apply nonlinear scaling as follows: we choose $\Upsilon_l(\Theta) = l\Upsilon(l^\gamma\Theta)$, with $l > 0$ and γ to be determined.

$$\Upsilon_l(\Theta) = l\Upsilon(l^\gamma\Theta) \iff \Upsilon(\Theta) = l^{-1}\Upsilon_l(l^{-\gamma}\Theta).$$

We set $Z = l^\gamma\Theta$. It follows from (3.2) that

$$\begin{aligned}\frac{d\Upsilon_l}{d\Theta} &= l^{1+\gamma}\frac{d\Upsilon}{dZ} = l^{1+\gamma}\Big(k + bmZ^{m+\beta-1}\Upsilon^{-\frac{1}{p}}\Big) \\ &= kl^{1+\gamma} + bml^{1+\gamma}l^{\gamma(m+\beta-1)}l^{\frac{1}{p}}\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}.\end{aligned} \tag{3.9}$$

We choose γ such that

$$1+\gamma+\gamma(m+\beta-1)+\tfrac{1}{p} = 0 \implies \gamma = -\tfrac{1+p}{p(m+\beta)}.$$

So we have that

$$\frac{d\Upsilon_l}{d\Theta} = kl^{\frac{p(m+\beta)-(1+p)}{p(m+\beta)}} + bm\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}. \tag{3.10}$$

From our previous results we that there exists a unique solution to (3.10). To prove formula 1., since $p(m+\beta) < 1+p$, we set

$$\lim_{l\to+\infty}\Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta),$$

where $\widetilde{\Upsilon}(\Theta)$ solves

$$\begin{cases}\frac{d\Upsilon}{d\Theta} = bm\Theta^{m+\beta-1}\Upsilon^{-\frac{1}{p}}, \\ \Upsilon(0) = 0.\end{cases} \tag{3.11}$$

The existence of the above limit follows from a similar argument used to prove an analogous limit in the proof of formula 3. The ODE in (3.11) is separable. Separating variables and integrating we have that

$$\widetilde{\Upsilon}(\Theta) = \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}\Theta^{\frac{p(m+\beta)}{1+p}}. \tag{3.12}$$

Recall that $Z = l^{\gamma}\Theta \implies \Theta = l^{-\gamma}Z$. So we have that

$$\Theta^{\frac{p(m+\beta)}{1+p}} = l^{-\frac{\gamma p(m+\beta)}{1+p}} Z^{\frac{p(m+\beta)}{1+p}} = lZ^{\frac{p(m+\beta)}{1+p}}.$$

It follows that

$$\begin{aligned}\lim_{l\to+\infty} \Upsilon_l(\Theta) = \lim_{l\to+\infty} l\Upsilon(Z) &= \widetilde{\Upsilon}(\Theta)\\ &= \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} lZ^{\frac{p(m+\beta)}{1+p}}\\ \implies \lim_{Z\to 0^+} \frac{\Upsilon(Z)}{Z^{\frac{p(m+\beta)}{1+p}}} &= \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}.\end{aligned}$$

Therefore,

$$\Upsilon(\Theta) \sim \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} \Theta^{\frac{p(m+\beta)}{1+p}}, \text{ as } \Theta \to 0^+.$$

Note that formula 2, where $p(m+\beta) > 1+p$, follows from the same procedure by setting

$$\lim_{l\to 0^+} \Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta).$$

To prove formulas 3. and 4. we let $k > 0$ and proceed as in the proof of formulas 1. and 2. We choose the same scale as follows

$$\Upsilon_l(\Theta) = l\Upsilon(l^{\gamma}\Theta) \iff \Upsilon(\Theta) = l^{-1}\Upsilon_l(l^{-\gamma}\Theta).$$

We set $Z = l^{\gamma}\Theta$. It follows from (3.2) that

$$\begin{aligned}\frac{d\Upsilon_l}{d\Theta} = l^{1+\gamma}\frac{d\Upsilon}{dZ} &= l^{1+\gamma}\Big(k + bmZ^{m+\beta-1}\Upsilon^{-\frac{1}{p}}\Big) \qquad (3.13)\\ &= kl^{1+\gamma} + bml^{1+\gamma} l^{\gamma(m+\beta-1)} l^{\frac{1}{p}} \Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}.\end{aligned}$$

Now, we choose γ such that

$$1+\gamma = 0 \implies \gamma = -1.$$

So we have that

$$\frac{d\Upsilon_l}{d\Theta} = k + bml^{\frac{1+p-p(m+\beta)}{p}}\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}. \tag{3.14}$$

From our previous results we that there exists a unique solution to (3.14). To prove formula 3., since $p(m+\beta) < 1+p$, we set

$$\lim_{l\to 0^+}\Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta).$$

To prove the existence of this limit, let $0 \le \Gamma < \Delta < +\infty$. We show

- $\{\Upsilon_l\}$ is uniformly bounded, i.e., $|\Upsilon_l(\Theta)| \le C$, for all $\Theta \in [\Gamma, \Delta]$ and l, where C is independent of l.
- $\{\Upsilon_l\}$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$ such that for all $\Theta, \Theta_0 \in [\Gamma, \Delta]$ we have

$$|\Theta - \Theta_0| < \delta \implies |\Upsilon_l(\Theta) - \Upsilon_l(\Theta_0)| < \varepsilon, \forall\, l.$$

First we prove that $\{\Upsilon_l\}$ is uniformly bounded. Since we want to pass l to zero, we fix $l \in (0,1]$. So we have that

$$\frac{d\Upsilon_l}{d\Theta} = k + bml^{\frac{1+p-p(m+\beta)}{p}}\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}} \le k + bm\Theta^{m+\beta-1}\Upsilon_1^{-\frac{1}{p}} = \frac{d\Upsilon_1}{d\Theta}.$$

Choosing $\Gamma = 0$ we have that $\Upsilon_l(0) = \Upsilon_1(0) = 0$, so by applying the comparison theorem we have

$$0 \leq \Upsilon_l(\Theta) \leq \Upsilon_1(\Theta), \forall \Theta \in [0, \Delta], \forall l \in (0, 1].$$

It remains to show that $\frac{d\Upsilon_l}{d\Theta}$ is uniformly bounded. Let $\Theta \in [\Gamma, \Delta]$. Since $k > 0$ we have that

$$\frac{d\Upsilon_l}{d\Theta} \geq k \implies \Upsilon_l(\Theta) \geq k\Theta \implies \Upsilon_l(\Gamma) \geq k\Gamma > 0 \implies \Upsilon_l^{-\frac{1}{p}}(\Gamma) \leq (k\Gamma)^{-\frac{1}{p}}.$$

So we have

$$\frac{d\Upsilon_l}{d\Theta} = k + bml^{\frac{1+p-p(m+\beta)}{p}} \Theta^{m+\beta-1} \Upsilon_l^{-\frac{1}{p}}(\Theta) \leq k + bm\Delta^{m+\beta-1}(k\Gamma)^{-\frac{1}{p}} < +\infty.$$

This holds for all $l \in (0, 1]$. Since $\frac{d\Upsilon_l}{d\Theta}$ is uniformly bounded on $[\Gamma, \Delta]$ it follows that $\Upsilon_l(\Theta)$ is uniformly bounded on $[\Gamma, \Delta]$.

Now we need to show that $\{\Upsilon_l\}$ is equicontinuous on $[\Gamma, \Delta]$. Let $\Theta, \Theta_0 \in [\Gamma, \Delta]$. We need to show that for any $\varepsilon > 0$, there exists $\delta = \delta_\varepsilon > 0$ such that

$$|\Theta - \Theta_0| < \delta \implies |\Upsilon_l(\Theta) - \Upsilon_l(\Theta_0)| < \varepsilon, \forall l.$$

By Lagrange's mean value theorem, for all $\theta \in [0, 1]$, we have

$$|\Upsilon_l(\Theta) - \Upsilon_l(\Theta_0)| = \left| \frac{d\Upsilon_l(\Theta_0 + \theta(\Theta - \Theta_0))}{d\Theta}(\Theta - \Theta_0) \right| \leq C|\Theta - \Theta_0| < C\delta.$$

Choosing $\delta = \frac{\varepsilon}{C}$ ensures that $|\Upsilon_l(\Theta) - \Upsilon_l(\Theta_0)| < \varepsilon, \forall l$. So $\{\Upsilon_l\}$ is equicontinuous on $[\Gamma, \Delta]$.

Since $\{\Upsilon_l\}$ is both uniformly bounded and equicontinuous on $[\Gamma, \Delta]$, and since $[\Gamma, \Delta]$ is an arbitrary compact subset of $[0, +\infty)$, there exists $\widetilde{\Upsilon}(\Theta)$ such that for some subsequence l' we have

$$\lim_{l' \to 0^+} \Upsilon_{l'}(\Theta) = \widetilde{\Upsilon}(\Theta), \ \forall \Theta > 0.$$

Where $\widetilde{\Upsilon}(\Theta)$ solves

$$\begin{cases} \frac{d\Upsilon}{d\Theta} = k, \Theta > 0, \\ \\ \Upsilon(0) = 0. \end{cases} \tag{3.15}$$

So $\widetilde{\Upsilon}(\Theta) = k\Theta$, and we have

$$\lim_{l \to 0^+} \Upsilon_l(\Theta) = \lim_{l \to 0^+} l\Upsilon(l^\gamma \Theta) = k\Theta, \Theta > 0.$$

Recall that $Z = l^\gamma \Theta \implies \Theta = l^{-\gamma} Z$. So we have that

$$\lim_{l \to 0^+} \Upsilon_l(\Theta) = \lim_{l \to 0^+} l\Upsilon(Z) = \widetilde{\Upsilon}(\Theta) = klZ$$
$$\implies \lim_{Z \to +\infty} \frac{\Upsilon(Z)}{Z} = k.$$

Therefore,

$$\Upsilon(\Theta) \sim k\Theta, \text{ as } \Theta \to +\infty.$$

Note that formula 4., where $p(m+\beta) > 1 + p$, follows from the same procedure by setting

$$\lim_{l \to +\infty} \Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta).$$

To prove formulas 5. and 6. we let $k < 0$ and proceed as in the proof of the previous formulas. We choose the same scale as follows

$$\Upsilon_l(\Theta) = l\Upsilon(l^\gamma \Theta) \iff \Upsilon(\Theta) = l^{-1}\Upsilon_l(l^{-\gamma}\Theta).$$

We set $Z = l^{\gamma}\Theta$. It follows from (3.2) that

$$\frac{d\Upsilon_l}{d\Theta} = l^{1+\gamma}\frac{d\Upsilon}{dZ} = l^{1+\gamma}\Big(k + bmZ^{m+\beta-1}\Upsilon^{-\frac{1}{p}}\Big) \tag{3.16}$$
$$= kl^{1+\gamma} + bml^{1+\gamma}l^{\gamma(m+\beta-1)}l^{\frac{1}{p}}\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}.$$

Now, we choose γ such that

$$1+\gamma = 1+\gamma+\gamma(m+\beta-1)+\tfrac{1}{p} \implies \gamma = -\tfrac{1}{p(m+\beta-1)}.$$

So we have that

$$l^{\frac{1+p-p(m+\beta)}{p(m+\beta-1)}}\frac{d\Upsilon_l}{d\Theta} = k + bm\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}. \tag{3.17}$$

From our previous results we that there exists a unique solution to (3.17). To prove formula 5., since $p(m+\beta) < 1+p$, we set

$$\lim_{l\to 0^+}\Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta).$$

As before, we have to show that the above limit exists. In this case, it's enough to prove that $\{\Upsilon_l\}$ is uniformly bounded on any compact interval, $[\Gamma,\Delta]$. From the equation we have that

$$k + bm\Theta^{m+\beta}\Upsilon_l^{-\frac{1}{p}} \geq 0 \implies 0 \leq \Upsilon_l(\Theta) \leq \Big(-\frac{k}{bm}\Big)^{-p}\Theta^{p(m+\beta-1)},\ \Theta > 0.$$

It remains to show that $\frac{d\Upsilon_l}{d\Theta}$ is uniformly bounded on $[\Gamma,\Delta]$. Consider

$$l^{\frac{1+p-p(m+\beta)}{p(m+\beta-1)}}\frac{d\Upsilon_l}{d\Theta} = k + bm\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}} \implies \frac{d\Upsilon_l}{d\Theta} = l^{\frac{p(m+\beta)-(1+p)}{p(m+\beta-1)}}\Big(k + bm\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}}\Big),$$

$$(l+1)^{\frac{1+p-p(m+\beta)}{p(m+\beta-1)}}\frac{d\Upsilon_{l+1}}{d\Theta} = k + bm\Theta^{m+\beta-1}\Upsilon_{l+1}^{-\frac{1}{p}}$$

$$\implies \frac{d\Upsilon_{l+1}}{d\Theta} = (l+1)^{\frac{p(m+\beta)-(1+p)}{p(m+\beta-1)}}\left(k + bm\Theta^{m+\beta-1}\Upsilon_{l+1}^{-\frac{1}{p}}\right) \le l^{\frac{p(m+\beta)-(1+p)}{p(m+\beta-1)}}\left(k + bm\Theta^{m+\beta-1}\Upsilon_{l+1}^{-\frac{1}{p}}\right).$$

Define $Z(\Theta) := \Upsilon_{l+1}(\Theta) - \Upsilon_l(\Theta)$. By mean value theorem, for all $\theta \in [0,1]$, we have

$$\frac{dZ}{d\Theta} \le l^{\frac{p(m+\beta)-(1+p)}{p(m+\beta-1)}} bm\Theta^{m+\beta-1}\left(\Upsilon_{l+1}^{-\frac{1}{p}} - \Upsilon_l^{-\frac{1}{p}}\right) =$$

$$= -l^{\frac{p(m+\beta)-(1+p)}{p(m+\beta-1)}}\frac{bm}{p}\Theta^{m+\beta-1}(\Upsilon_l + \theta(\Upsilon_{l+1} - \Upsilon_l))^{-\frac{1+p}{p}} Z$$

$$\implies l^{\frac{1+p-p(m+\beta}{p(m+\beta-1)}}\frac{dZ}{d\Theta} \le -\frac{bm}{p}\Theta^{m+\beta-1}(\Upsilon_l + \theta(\Upsilon_{l+1} - \Upsilon_l))^{-\frac{1+p}{p}} Z.$$

Since $Z(0) = 0$, it follows from the comparison theorem that $\Upsilon_{l+1}(\Theta) \le \Upsilon_l(\Theta)$, $\Theta \in [\Gamma, \Delta]$. Hence $\{\Upsilon_l\}$ is a monotonically decreasing sequence as $l \to 0^+$, and since $\Upsilon_l(\Theta) > 0$, for all $\Theta > 0$, there exists $\widetilde{\Upsilon}(\Theta)$ such that

$$\lim_{l\to 0^+} \Upsilon_l(\Theta) = \widetilde{\Upsilon}(\Theta).$$

Now, for any $\nu \in C_0^\infty(\Gamma, \Delta)$, we appeal to the integral identity

$$\int_\Gamma^\Delta l^{\frac{1+p-p(m+\beta)}{p(m+\beta-1)}}\Upsilon_l \nu' + (k + bm\Theta^{m+\beta-1}\Upsilon_l^{-\frac{1}{p}})\nu d\Theta = 0.$$

Letting $l \to 0^+$ we have

$$\int_\Gamma^\Delta (k + bm\Theta^{m+\beta-1}\widetilde{\Upsilon}^{-\frac{1}{p}})\nu d\Theta = 0.$$

Since ν is arbitrary we necessarily have that

$$k + bm\Theta^{m+\beta-1}\widetilde{\Upsilon}^{-\frac{1}{p}} = 0.$$

Solving for $\widetilde{\Upsilon}$ we have that

$$\widetilde{\Upsilon}(\Theta) = \left(-\frac{k}{bm}\right)^{-p}\Theta^{p(m+\beta-1)}. \tag{3.18}$$

Recall that $Z = l^{\gamma}\Theta \implies \Theta = l^{-\gamma}Z$. So we have that

$$\begin{aligned}\lim_{l\to 0^+}\Upsilon_l(\Theta) = \lim_{l\to 0^+} l\Upsilon(Z) = \widetilde{\Upsilon}(\Theta) = \left(-\frac{k}{bm}\right)^{-p} lZ^{p(m+\beta-1)} \\ \implies \lim_{Z\to+\infty}\frac{\Upsilon(Z)}{Z^{p(m+\beta-1)}} = \left(-\frac{k}{bm}\right)^{-p}.\end{aligned}$$

Therefore,

$$\Upsilon(\Theta) \sim (-\tfrac{k}{bm})^{-p}\Theta^{p(m+\beta-1)}, \text{ as } \Theta \to +\infty.$$

The proof of formula 6. follows from a similar argument. □

3.2 Proof of the Main Result

Using the results above we now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. As long as $\varphi(y) \neq 0$ ($\Upsilon(\varphi(y)) \neq 0$), we write (3.7) as

$$m\varphi^{m-1}\Upsilon^{-\frac{1}{p}}(\varphi(y))d\varphi(y) = dy \tag{3.19}$$

We will prove formula 2. from Theorem 3.1.1, the proof of formula 1. is similar. Since $p(m+\beta) > 1 + p$, from Lemma 3.1.4 we know that

$$\Upsilon(\Theta) \sim \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} \Theta^{\frac{p(m+\beta)}{1+p}}, \text{ as } \Theta \to +\infty.$$

Hence, for any $\varepsilon > 0$, there exists an $\Theta_\varepsilon >> 1$ such that

$$\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} - \varepsilon\right)\Theta^{\frac{p(m+\beta)}{1+p}} \leq \Upsilon(\Theta) \leq \left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} + \varepsilon\right)\Theta^{\frac{p(m+\beta)}{1+p}}, \Theta_\varepsilon >> 1.$$

Raising both sides by the power $-\frac{1}{p}$ then multiplying by $m\Theta^{m-1}$ we have

$$m\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} + \varepsilon\right)^{-\frac{1}{p}} \Theta^{-\frac{m+\beta}{1+p}+m-1} \leq m\Theta^{m-1}\Upsilon^{-\frac{1}{p}} \leq$$
$$m\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} - \varepsilon\right)^{-\frac{1}{p}} \Theta^{-\frac{m+\beta}{1+p}+m-1}, \Theta_\varepsilon >> 1.$$

Integrating over $(0, \varphi(y))$ where $(0, y) \subset (0, \ell)$ we have

$$\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} + \varepsilon\right)^{-\frac{1}{p}} (\varphi(y))^{\frac{mp-\beta}{1+p}} \leq m\int_0^{\varphi(y)} \Theta^{m-1}\Upsilon^{-\frac{1}{p}}(\Theta)d\Theta \leq$$
$$\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} - \varepsilon\right)^{-\frac{1}{p}} (\varphi(y))^{\frac{mp-\beta}{1+p}}.$$

Rewriting

$$\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} + \varepsilon\right)^{-\frac{1}{p}} \leq \frac{m\int_0^{\varphi(y)} \Theta^{m-1}\Upsilon^{-\frac{1}{p}}(\Theta)d\Theta}{(\varphi(y))^{\frac{mp-\beta}{1+p}}} \leq \frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} - \varepsilon\right)^{-\frac{1}{p}}.$$

By (3.19) we know that

$$m\int_0^{\varphi(y)} \Theta^{m-1}\Upsilon^{-\frac{1}{p}}(\Theta)d\Theta = y. \tag{3.20}$$

Using this fact and raising the last inequality to the power $-\frac{1+p}{mp-\beta}$ we have

$$\left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} - \varepsilon\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}} \leq y^{-\frac{1+p}{mp-\beta}}\varphi(y) \leq \left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}} + \varepsilon\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}}.$$

Note that the limit as $y \to +\infty$ may not exist for the inequality above, but $\liminf$ and $\limsup$ do exist as $y \to +\infty$. So we have

$$\left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}+\varepsilon\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}} \leq \liminf_{y\to+\infty} y^{-\frac{1+p}{mp-\beta}}\varphi(y) \leq \limsup_{y\to+\infty} y^{-\frac{1+p}{mp-\beta}}\varphi(y) \leq$$

$$\left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}-\varepsilon\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}}.$$

Now, passing $\varepsilon \to 0^+$, we have

$$\lim_{y\to+\infty} y^{-\frac{1+p}{mp-\beta}}\varphi(y) = \left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}}.$$

Simplifying we have that

$$\left(\frac{m(1+p)}{mp-\beta}\left(\left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{p}{1+p}}\right)^{-\frac{1}{p}}\right)^{-\frac{1+p}{mp-\beta}} = \left[\frac{bm(1+p)}{p(m+\beta)}\right]^{\frac{1}{mp-\beta}}\left[\frac{m(1+p)}{mp-\beta}\right]^{-\frac{1+p}{mp-\beta}} =$$

$$\left[\frac{b(mp-\beta)^{1+p}}{(m(1+p))^p p(m+\beta)}\right]^{\frac{1}{mp-\beta}} = C_*.$$

So, in fact, we have that

$$\lim_{y\to+\infty} y^{-\frac{1+p}{mp-\beta}}\varphi(y) = C_*,$$

which proves formula 2.

Now we prove formula 4., the proof of formula 3. is similar. From Lemma 3.1.4 we know that

$$\Upsilon(\Theta) \sim k\Theta, \text{ as } \Theta \to 0^+ \text{ if } k > 0,\ p(m+\beta) > 1+p.$$

Hence, for any $\varepsilon > 0$, there exists $\Theta_\varepsilon > 0$ such that

$$(k-\varepsilon)\Theta \leq \Upsilon(\Theta) \leq (k+\varepsilon)\Theta, \quad \Theta \leq \Theta_\varepsilon,$$

$$(k-\varepsilon)(\varphi(y)) \leq ((\varphi^m(y))')^p \leq (k+\varepsilon)(\varphi(y))^{\frac{p(m+\beta)}{1+p}}, \quad \varphi(y) \leq \Theta_\varepsilon,$$

$$(k-\varepsilon)^{1/p}(\varphi(y))^{\frac{1}{p}} \leq m\varphi^{m-1}\varphi'(y) \leq (k+\varepsilon)^{1/p}(\varphi(y))^{\frac{1}{p}}, \quad \varphi(y) \leq \Theta_\varepsilon,$$

$$(k-\varepsilon)^{1/p}dy \leq m(\varphi(y))^{m-1-\frac{1}{p}}d\varphi \leq (k+\varepsilon)^{1/p}dy, \quad \varphi(y) \leq \Theta_\varepsilon.$$

Integrating we get

$$(k-\varepsilon)^{1/p}y \le \frac{mp}{mp-1}(\varphi(y))^{\frac{mp-1}{p}} \le (k+\varepsilon)^{1/p}y, \quad \varphi(y) \le \Theta_\varepsilon,$$

$$\left[\frac{mp-1}{mp}(k-\varepsilon)^{1/p}\right]^{\frac{p}{mp-1}} \le y^{-\frac{p}{mp-1}}\varphi(y) \le$$

$$\le \left[\frac{mp-1}{mp}(k+\varepsilon)^{1/p}\right]^{\frac{p}{mp-1}}, \quad \varphi(y) \le \Theta_\varepsilon.$$

So we have

$$\left[\frac{mp-1}{mp}(k-\varepsilon)^{1/p}\right]^{\frac{p}{mp-1}} \le \liminf_{y\to 0^+} y^{-\frac{p}{mp-1}}\varphi(y) \le$$

$$\le \limsup_{y\to 0^+} y^{-\frac{p}{mp-1}}\varphi(y) \le \left[\frac{mp-1}{mp}(k+\varepsilon)^{1/p}\right]^{\frac{p}{mp-1}}.$$

Passing $\varepsilon \to 0^+$ we have

$$\lim_{y\to 0^+} y^{-\frac{p}{mp-1}}\varphi(y) = \left[\frac{mp-1}{mp}\right]^{\frac{p}{mp-1}} k^{\frac{1}{mp-1}},$$

which proves formula 4.

Finally, we prove formula 6. The proof of formula 5. is similar. From Lemma 3.1.4 we know that

$$\Upsilon(\Theta) \sim (-\tfrac{k}{bm})^{-p}\Theta^{p(m+\beta-1)}, \text{ as } \Theta \to 0^+ \text{ if } k<0,\ p(m+\beta) > 1+p.$$

Hence, for any $\varepsilon > 0$, there exists $\Theta_\varepsilon > 0$ such that

$$\left(\left(-\frac{k}{bm}\right)^{-p} - \varepsilon\right)\Theta^{p(m+\beta-1)} \le \Upsilon(\Theta) \le \left(\left(-\frac{k}{bm}\right)^{-p} + \varepsilon\right)\Theta^{p(m+\beta-1)}, \quad \Theta \le \Theta_\varepsilon,$$

$$\left(\left(-\frac{k}{bm}\right)^{-p} - \varepsilon\right)(\varphi(y))^{p(m+\beta-1)} \le ((\varphi^m(y))')^p \le \left(\left(-\frac{k}{bm}\right)^{-p} + \varepsilon\right)(\varphi(y))^{p(m+\beta-1)}, \quad \varphi(y) \le \Theta_\varepsilon,$$

$$\left(\left(-\frac{k}{bm}\right)^{-p}-\varepsilon\right)^{1/p}(\varphi(y))^{(m+\beta-1)} \le m\varphi^{m-1}\varphi'(y) \le \left(\left(-\frac{k}{bm}\right)^{-p}+\varepsilon\right)^{1/p}(\varphi(y))^{(m+\beta-1)},$$

$$\left(\left(-\frac{k}{bm}\right)^{-p}-\varepsilon\right)^{1/p}dy \le m(\varphi(y))^{-\beta}d\varphi \le \left(\left(-\frac{k}{bm}\right)^{-p}+\varepsilon\right)^{1/p}dy, \quad \varphi(y) \le \Theta_\varepsilon.$$

Integrating we get

$$\left(\left(-\frac{k}{bm}\right)^{-p}-\varepsilon\right)^{1/p}y \le \frac{m}{1-\beta}(\varphi(y))^{1-\beta} \le \left(\left(-\frac{k}{bm}\right)^{-p}+\varepsilon\right)^{1/p}y, \quad \varphi(y) \le \Theta_\varepsilon,$$

$$\left[\frac{1-\beta}{m}\left(\left(-\frac{k}{bm}\right)^{-p}-\varepsilon\right)^{1/p}\right]^{\frac{1}{1-\beta}} \le y^{-\frac{1}{1-\beta}}\varphi(y) \le$$

$$\le \left[\frac{1-\beta}{m}\left(\left(-\frac{k}{bm}\right)^{-p}+\varepsilon\right)^{1/p}\right]^{\frac{1}{1-\beta}}, \quad \varphi(y) \le \Theta_\varepsilon.$$

So we have

$$\left[\frac{1-\beta}{m}\left(\left(-\frac{k}{bm}\right)^{-p}-\varepsilon\right)^{1/p}\right]^{\frac{1}{1-\beta}} \le \liminf_{y\to 0^+} y^{-\frac{1}{1-\beta}}\varphi(y) \le$$

$$\le \limsup_{y\to 0^+} y^{-\frac{1}{1-\beta}}\varphi(y) \le \left[\frac{1-\beta}{m}\left(\left(-\frac{k}{bm}\right)^{-p}+\varepsilon\right)^{1/p}\right]^{\frac{1}{1-\beta}}.$$

Passing $\varepsilon \to 0^+$ we have

$$\lim_{y\to 0^+} y^{-\frac{1}{1-\beta}}\varphi(y) = \left((1-\beta)\left(-\frac{b}{k}\right)\right)^{\frac{1}{1-\beta}},$$

which proves formula 6.

Note that letting $y \to \ell^-$ in (3.20), we obtain $\ell = +\infty$, if and only if

$$\int_0^{+\infty} \Upsilon^{-\frac{1}{p}}(z)dz = +\infty. \tag{3.21}$$

By Lemma 3.1.4, we know that the integral above (3.21) is true, then the solution to the problem (3.8) is global. It follows that there exists a finite traveling-wave solution of the form: $u(x,t) = \varphi(kt - x)$, to equation (1.7). □

Chapter 4

Evolution of Interfaces for the Nonlinear Double Degenerate Parabolic Equation with Fast Diffusion

We present a full qualitative analysis of the asymptotic behavior of the interfaces (when they exist) and local asymptotic behavior of the solution near them for the problem (1.7), (1.9) in the fast diffusion case ($0 < mp < 1$) in this chapter. In the cases where there is an infinite speed of propagation ($\eta(t) = +\infty$), we classify the asymptotic behavior of the solution at infinity. The results of Chapter 4 have recently been submitted for publication, [24].

4.1 Main Results

We assume that u is a unique weak solution of the CP (1.7), (1.9), throughout this section. The main results are classified according to regions I-V, respectively, in Figure 4.1, the (α,β) parameter space diagram, below. We refer to Appendix 6.2 for the explicit values of relevant constants that appear throughout this section.

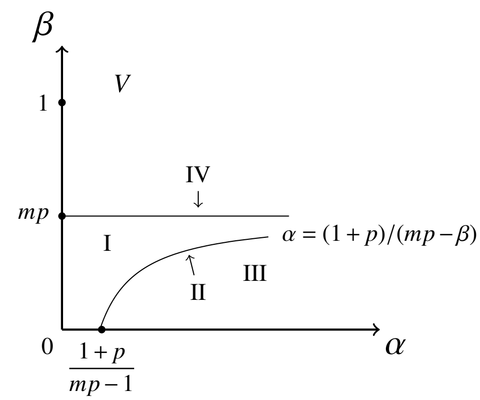


Figure 4.1: (α,β) parameter space diagram for the interface development for the CP (1.7), (1.9) with $0 < mp < 1$, which is presented in [24].

- **Region I**: $b > 0, 0 < \beta < mp$ and $0 < \alpha < (1+p)/(mp-\beta)$.

  The interface initially expands and there exists a number $\delta > 0$ such that

$$z_1 t^{\frac{mp-\beta}{(1-\beta)(1+p)}} \leq \eta(t) \leq z_2 t^{\frac{mp-\beta}{(1-\beta)(1+p)}}, 0 \leq t \leq \delta, \tag{4.1}$$

  Further, for any $\sigma \in \mathbb{R}$, there is a number $f(\sigma) > 0$ (depending on C, m, and p) such that

$$u(\chi_\sigma(t), t) \sim f(\sigma) t^{\frac{\alpha}{1+p-\alpha(mp-1)}}, \text{ as } t \to 0^+, \tag{4.2}$$

  where $\chi_\sigma(t) = \sigma t^{\frac{1}{1+p-\alpha(mp-1)}}$.

- **Region II**: $b > 0, 0 < \beta < mp, \alpha = (1+p)/(mp-\beta)$, and

$$C_* = \left[\frac{b(mp-\beta)^{1+p}}{(m(1+p))^p p(m+\beta)}\right]^{\frac{1}{mp-\beta}}.$$

Then the interface shrinks or expands accordingly as $C < C_*$ or $C > C_*$ and we have that

$$\eta(t) \sim z_* t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, \text{ as } t \to 0^+, \tag{4.3}$$

where $z_* \lesseqgtr 0$ if $C \lesseqgtr C_*$, and for any $\sigma < z_*$, there is a number $f_1(\sigma) > 0$ such that

$$u(z_\sigma(t), t) \sim t^{1/(1-\beta)} f_1(\sigma), \text{ as } t \to 0^+, \tag{4.4}$$

where $z_\sigma(t) = \sigma t^{\frac{mp-\beta}{(1-\beta)(1+p)}}$.

- **Region III**: $b > 0, 0 < \beta < mp$, and $\alpha > (1+p)/(mp-\beta)$.

Then the interface initially shrinks and we have that

$$\eta(t) \sim -\tau_* t^{\frac{1}{\alpha(1-\beta)}}, \text{ as } t \to 0^+, \tag{4.5}$$

where $\tau_* = C^{-1/\alpha}(b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$ and, for any $\tau > \tau_*$, we have

$$u(\eta_\tau(t), t) \sim \left[C^{1-\beta}\tau^{\alpha(1-\beta)} - b(1-\beta)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \text{ as } t \to 0^+, \tag{4.6}$$

where $\eta_\tau(t) = -\tau t^{\frac{1}{\alpha(1-\beta)}}$.

- **Region IV**: $b > 0$, $0 < \beta = mp < 1$, and $\alpha > 0$.

In this case the solution travels with an infinite speed of propagation: there is no interface. For arbitary $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$t^{\frac{1}{1-mp}}\vartheta(x) \leq u(x,t) \leq (t+\varepsilon)^{\frac{1}{1-mp}}\vartheta(x),\ x > 0,\ 0 \leq t \leq \delta, \tag{4.7}$$

where $\vartheta = \vartheta(x) > 0$ is a solution of the stationary problem

$$\begin{cases} (|(\vartheta^m)'|^{p-1}(\vartheta^m)')' - \frac{1}{1-mp}\vartheta - b\vartheta^{mp} = 0,\ x > 0, \\ \vartheta(0) = 1,\ \vartheta(+\infty) = 0. \end{cases} \tag{4.8}$$

Further, we have the estimation

$$\ln u(x,t) \sim -\frac{1}{m}\left(\frac{b}{p}\right)^{1/(1+p)} x,\ \text{as } x \to +\infty,\ 0 \leq t \leq \delta. \tag{4.9}$$

- **Region V**: This region is divided into cases V(a), V(b) and V(b).

  - **V(a)**: Either $b > 0, \beta > mp$ or $b < 0, \beta \geq 1$ or $b = 0$, and

    $$\mathcal{D} = \left[\frac{(m(1+p))^p(m+1)}{(1-mp)^p}\right]^{\frac{1}{1-mp}}.$$

    In this case the solution travels with an infinite speed of propagation: there is no interface; and (4.2) holds. If $b > 0, \beta \geq \frac{p(1-m)+2}{1+p}$ or $b < 0, \beta \geq 1$ or $b = 0$, then there exists a number $\delta > 0$ such that

    $$u(x,t) \sim \mathcal{D}t^{\frac{1}{1-mp}}x^{\frac{1+p}{1-mp}},\ \text{as } x \to +\infty,\ t \in (0,\delta], \tag{4.10}$$

– **V(b)**: $b > 0$ and $1 \leq \beta < \frac{p(1-m)+2}{1+p}$.

Then,

$$\lim_{t\to 0^+} \lim_{x\to+\infty} \frac{u(x,t)}{t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}} = \mathcal{D}. \tag{4.11}$$

– **V(c)**: $b > 0$ and $0 < mp < \beta < 1$.

Then

$$u(x,t) \sim C_* x^{\frac{1+p}{mp-\beta}}, \text{ as } x \to +\infty, \tag{4.12}$$

holds for any $t \in (0,\delta]$ for some δ.

4.2 Additional Details of the Results

In this section we outline some additional, essential details of Results I-V. We refer to Appendix 6.2 for the explicit values of relevant constants that appear throughout this section.

- *Region I*. The solution u satisfies the following estimation

$$C_1 t^{\frac{1}{1-\beta}} (z_1 - z)_+^{\frac{1+p}{mp-\beta}} \leq u(x,t) \leq C_* t^{\frac{1}{1-\beta}} (z_2 - z)_+^{\frac{1+p}{mp-\beta}}, 0 < t \leq \delta, \tag{4.13}$$

where $z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}$. The left-hand side of (4.13) is valid for $0 \leq x < +\infty$ and the right-hand side is valid in the curved region: $x \geq \tau_0 t^{(mp-\beta)/(1-\beta)(1+p)}$. $C_1, z_1, z_2,$

and τ_0, are positive constants depending on m, p, β, and b. Moreover, the function f is such that

$$f(\sigma) = C^{\frac{1+p}{1+p-\alpha(mp-1)}} f_0(C^{\frac{mp-1}{1+p-\alpha(mp-1)}}\sigma),\ f_0(\sigma) = w(\sigma, 1), \sigma \in \mathbb{R}, \tag{4.14}$$

where w is a minimal solution of the CP (1.7), (1.10) with $C = 1$ and $b = 0$. If u_0 is given by (1.10), then the right-hand sides of (4.13) and (4.1) are valid for all $t > 0$.

- *Region II*. Assume u solves the CP (1.7), (1.10). If $C = C_*$ in u_0, then $u = u_0$ is the stationary solution to the CP. If $C \neq C_*$, then the minimal solution of the CP is given by

$$u(x,t) = t^{\frac{1}{1-\beta}} f_1(z),\ z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}, \tag{4.15}$$

and

$$\eta(t) = z_* t^{\frac{mp-\beta}{(1-\beta)(1+p)}},\ t \geq 0, \tag{4.16}$$

If $C > C_*$, the interface initially expands and we have the following estimation

$$C'\Big(z' t^{\frac{mp-\beta}{(1-\beta)(1+p)}} - x\Big)_+^{\frac{1+p}{mp-\beta}} \leq u \leq C''\Big(z'' t^{\frac{mp-\beta}{(1-\beta)(1+p)}} - x\Big)_+^{\frac{1+p}{mp-\beta}}, \tag{4.17}$$

$$z' \leq z_* \leq z'',\ 0 \leq x < +\infty,\ t > 0, \tag{4.18}$$

where $C' = C_2, C'' = C_*, z' = z_3$, and $z'' = z_4$. While, if $0 < C < C_*$, then the interface shrinks and there exists a number $\tau_1 > 0$ such that for all $\tau \leq \tau_1$ there exists a number $\varrho > 0$ such that

$$u\Big(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\Big) = \varrho t^{\frac{1}{1-\beta}},\ t \geq 0, \tag{4.19}$$

and u and z_* satisfy the estimates (4.17) and (4.18) with $C' = C_*, C'' = C_3, z' = -z_5,$ and $z'' = -z_6$.

- *Region IV*. The explicit solution of the problem (4.8) is given by the following formula

$$\vartheta(x) = \mathcal{J}^{-1}(x), 0 \le x < +\infty, \tag{4.20}$$

where $\mathcal{J}^{-1}(\cdot)$ is the inverse function of

$$\mathcal{J}(z) = \int_z^1 ms^{-1}\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)}s^{1-mp}\right]^{-\frac{1}{1+p}} ds,\ 0 < z \le 1. \tag{4.21}$$

The function $\vartheta(x)$ satisfies the asymptotic formula

$$\ln\vartheta(x) \sim -\frac{1}{m}\left(\frac{b}{p}\right)^{1/(1+p)} x,\ \text{as } x \to +\infty, \tag{4.22}$$

and the global estimation

$$0 < \vartheta(x) \le \exp\left(-\frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}} x\right), x > 0. \tag{4.23}$$

From (4.23), it follows that

$$\frac{\vartheta(x)}{e^{-\gamma x}} \to +\infty,\ \text{as } x \to +\infty,\ \text{if } \gamma > \frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}}. \tag{4.24}$$

From (4.7) and (4.24), it follows that

$$\lim_{t\to 0^+}\lim_{x\to+\infty} u(x,t)\exp\left(-\frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}} x\right) = 0, \tag{4.25}$$

and respectively

$$\frac{u(x,t)}{e^{-\gamma x}} \to +\infty, \text{ as } x \to +\infty, 0 \le t \le \delta(\varepsilon), \text{ if } \gamma > \frac{1}{m}\left(\frac{b}{p}\right)^{\frac{1}{1+p}}. \qquad (4.26)$$

- *Region V*. If $\beta \ge 1$, then for arbitrary $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that we have the following estimation

$$C_5 t^{\frac{\alpha}{1+p-\alpha(mp-1)}} (\chi_1 + \chi)^{\frac{1+p}{mp-1}} \le u(x,t) \le C_6 t^{\frac{\alpha}{1+p-\alpha(mp-1)}} (\chi_2 + \chi)^{\frac{1+p}{mp-1}}, \qquad (4.27)$$

where $\chi = x t^{\frac{-1}{1+p-\alpha(mp-1)}}$, for all $x \in [0,\infty)$ and $0 \le t \le \delta(\varepsilon)$. C_5, C_6, χ_1, and χ_2, are positive constants depending on m, p, β, b, and ε.

If $b > 0$ and $\beta \ge 1$, we have the upper estimation

$$u(x,t) \le \mathcal{D} t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}, 0 < x < +\infty, 0 < t < +\infty. \qquad (4.28)$$

While if $b < 0$ and $\beta \ge 1$, then for arbitrary $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that the following estimation is satisfied

$$u(x,t) \le \mathcal{D}(1-\varepsilon)^{\frac{1}{mp-1}} t^{\frac{1}{1-mp}} x^{\frac{1+p}{mp-1}}, \text{ for } \varkappa t^{\frac{1}{1+p+\alpha(1-mp)}} < x < +\infty, 0 < t \le \delta, \qquad (4.29)$$

where

$$\varkappa = \left[\frac{A_0 + \varepsilon}{\mathcal{D}(1-\varepsilon)^{\frac{1}{mp-1}}}\right]^{\frac{mp-1}{1+p}}. \qquad (4.30)$$

If $b > 0$ and $mp < \beta < 1$, then there exists $\delta > 0$ such that

$$t^{\frac{1}{1-\beta}} C_*(1-\varepsilon)(z_8 + z)^{\frac{1+p}{mp-\beta}} \le u(x,t) \le C_* x^{\frac{1+p}{mp-\beta}}, 0 < x < +\infty, 0 < t \le \delta, \qquad (4.31)$$

where $z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}$, $\varepsilon > 0$ is an arbitrary sufficiently small number, and z_8 is a positive constant depending on m, p, β, b, and ε.

If $b = 0$ and $\alpha > 0$, then the minimal solution to the CP (1.7), (1.10) has the self-similar form

$$u(x,t) = t^{\frac{\alpha}{1+p+\alpha(1-mp)}} f(\chi), \chi = xt^{\frac{-1}{1+p+\alpha(1-mp)}}. \tag{4.32}$$

where f satisfies (4.14). Further, following global estimation is valid

$$\mathcal{D}t^{\frac{\alpha}{1+p+\alpha(1-mp)}}(\chi_3+\chi)^{\frac{1+p}{mp-1}} \leq u(x,t) \leq C_7 t^{\frac{\alpha}{1+p+\alpha(1-mp)}}(\chi_4+\chi)^{\frac{1+p}{mp-1}}, \tag{4.33}$$

$$0 \leq x < +\infty, 0 < t < +\infty,$$

where C_7, χ_3, and χ_4, are positive constants depending on m and p.

Explicit solution (4.29) provides sharper upper bound than (4.33) as $x \to +\infty$. From (4.33) and (4.29), asymptotic result (4.10) easily follows. In a similar way asymptotic result (4.10) follows in the local case, (1.9).

4.3 Asymptotic Properties of Solutions With $0 < mp < 1$

In this section we establish a series of lemmas that describe preliminary estimations for the CP. The proof of these results is based on nonlinear scaling.

4.3.0.1 Diffusion Weakly Dominates Reaction

Lemma 4.3.1. *If $b = 0$ and $\alpha > 0$, then the global formula* (4.32) *holds,*

$$u(x,t) = t^{\alpha/(1+p-\alpha(mp-1))} f(\chi), \chi = xt^{-\frac{1}{1+p-\alpha(mp-1)}},$$

where u is a minimal solution of the CP (1.7)*,* (1.10) *and the self-similarity function f satisfies* (4.14)*,*

$$\begin{cases} (|(f^m(\chi))'|^{p-1}(f^m(\chi))')' + (1+p-\alpha(mp-1))^{-1}(\chi f'(\chi) - \alpha f(\chi)) = 0, \\ f(\chi) \sim C(-\chi)^{\alpha}, \text{ as } \chi \downarrow -\infty, \text{ and } f(\chi) \to 0, \text{ as } \chi \uparrow +\infty. \end{cases}$$

If u_0 satisfies (1.9)*, and u is the unique weak solution to CP* (1.7)*,* (1.8)*, then u satisfies* (4.2)*.*

The proof coincides with that given for Lemma 2.3.4.

Lemma 4.3.2. *Let u be a weak solution to the CP* (1.7)*,* (1.8)*, with u_0 satisfying the condition* (1.9)*. Let one of the following cases be valid*

$$\begin{cases} b > 0, 0 < \beta < mp, 0 < \alpha < (1+p)/(mp-\beta) & \textit{Case 1}, \\ b > 0, \beta \geq mp, \alpha > 0 & \textit{Case 2}, \\ b < 0, \beta \geq 1, \alpha > 0 & \textit{Case 3}. \end{cases}$$

Then, for any $\sigma \in \mathbb{R}$, u satisfies (4.2) *with the same function f as in Lemma 4.3.1.*

The proof coincides with the proof of Lemma 2.3.5. A minor change is required in order to prove Case 3. On the lines $x = -x_\varepsilon$ and $x = -k^{1/\alpha}x_\varepsilon$, due to the infinite speed of propagation property, we choose

$$u_{\pm\varepsilon}(-x_\varepsilon, t) = u(-x_\varepsilon, t), 0 \leq t \leq \delta,$$

and

$$u_k^{\pm\varepsilon}(-k^{1/\alpha}x_\varepsilon, t) = u(-x_\varepsilon, k^{(\alpha(mp-1)-(1+p))/\alpha}, t), 0 \leq t \leq k^{(1+p-\alpha(mp-1))/\alpha}\delta,$$

rather than a zero boundary condition. The rest of the proof of Case 3 coincides with the proof of Lemma 2.3.5 when $b < 0$.

4.3.0.2 Diffusion and Reaction are in Balance

Lemma 4.3.3. *If $b > 0$, $0 < \beta < mp < 1$, and $\alpha = (1+p)/(mp-\beta)$, then the minimal solution u to the CP* (1.7), (1.9) *has the self-similar form* (4.15)*, where the self-similarity function f_1 satisfies*

$$\begin{cases} \mathcal{L}^0 f_1 \equiv (|(f_1^m)'|^{p-1}(f_1^m)')' + \frac{mp-\beta}{(1+p)(1-\beta)} z f_1' - \frac{1}{1-\beta} f_1 - b f_1^{\beta} = 0, \ z \in \mathbb{R}, \\ f_1(z) \sim C(-z)^{(1+p)/(mp-\beta)}, \ \text{as } z \downarrow -\infty, \ \text{and } f_1(z) \to 0, \ \text{as } z \uparrow +\infty. \end{cases} \tag{4.34}$$

There exists $\tau_1, \varrho > 0$ such that for any $\tau \in (-\infty, -\tau_1)$ we have

$$u\Big(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\Big) = \varrho t^{\frac{1}{1-\beta}}, \ t \geq 0. \tag{4.35}$$

If $0 < C < C_*$, then we have

$$0 < \varrho < C_*(-\tau)^{\frac{1+p}{mp-\beta}}, \tag{4.36}$$

while if $C > C_*$, then $f_1(0) = A_1(m,p,\beta,C,b) = A_1 > 0$.

Proof of Lemma 4.3.3. We define

$$u_k(x,t) = ku(k^{\frac{\beta-mp}{1+p}} x, k^{\beta-1} t), \ k > 0, \tag{4.37}$$

(see Lemma 6 of [20]). The function (4.37) satisfies the CP (1.7), (1.10). We consider u to be a unique minimal solution of CP (1.7), (1.10) such that

$$u(x,t) \leq ku(k^{\frac{\beta-mp}{1+p}} x, k^{\beta-1} t), \ k > 0. \tag{4.38}$$

By changing the variables in (4.38) as

$$y = k^{\frac{\beta - mp}{1+p}} x, \, \ell = k^{\beta - 1} t, \tag{4.39}$$

we derive (4.38) with opposite inequality and with k replaced with k^{-1}. Since $k > 0$ is arbitrary, (4.38) follows with "=". Taking $k = t^{1/(1-\beta)}$, (4.37) implies (4.15) with $f_1(z) = u(z,1)$, where f_1 is a solution to problem (4.34). This proves the first part of the lemma. The second part of the lemma is proved in the same way as the second part of Lemma 3.3 of [6]. □

Lemma 4.3.4. *Let $b > 0, 0 < \beta < mp < 1$, and $\alpha = (1+p)/(mp - \beta)$, and let u be the minimal solution to the CP* (1.7)*,* (1.9)*. Then u satisfies*

$$u\Big(\tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}}, t\Big) \sim \varrho t^{\frac{1}{1-\beta}}, \text{ as } t \to 0^+, \tag{4.40}$$

where $\tau_1, \varrho > 0$ are the same as in Lemma 4.3.3. Furthermore, if $0 < C < C_*$, then

$$0 < \varrho < C_*(-\tau)^{\frac{1+p}{mp-\beta}}. \tag{4.41}$$

If $C > C_*$, then

$$u(0,t) \sim A_1 t^{\frac{1}{1-\beta}}, \text{ as } t \to 0^+; \, f_1(0) = A_1 > 0. \tag{4.42}$$

The proof of Lemma 4.3.4 follows as a localization of the proof of Lemma 4.3.3, exactly as local results were proven for Lemma 2.3.5.

4.3.0.3 Reaction Dominates Diffusion

Lemma 4.3.5. *If* $b > 0, 0 < \beta < mp < 1,$ *and* $\alpha > (1+p)/(mp-\beta)$*, then the unique weak solution* u *to the CP* (1.7)*,* (1.9) *satisfies* (4.6)*.*

The proof of Lemma 4.3.5 coincides with the proof of Lemma 2.3.8.

4.4 Proofs of the Main Results

From Lemma 4.3.2, the asymptotic formulas (4.2) and (4.14) follow. For any $\varepsilon > 0$, from (4.2), we have

$$(A_0-\varepsilon)t^{\alpha/(1+p-\alpha(mp-1))} \leq u(0,t) \leq (A_0+\varepsilon)t^{\alpha/(1+p-\alpha(mp-1))}, 0 \leq t \leq \delta_1(\varepsilon), \tag{4.43}$$

for some $\delta_1 = \delta_1(\varepsilon) > 0$ and where $A_0 = f(0) > 0$.

The proof of results for Region I follows exactly as the proof of result (1) from [6] for $b \neq 0$, by choosing

$$r(x,t) = t^{\frac{1}{1-\beta}} f_1(z), z = xt^{\frac{\beta-mp}{(1-\beta)(1+p)}}, \tag{4.44}$$

$$f_1 = C_0(z_0-z)_+^{\frac{1+p}{mp-\beta}}, 0 < z < +\infty \tag{4.45}$$

with $C_0, z_0 > 0$ to be determined. To prove the left-hand sides of estimates (4.13) and (4.1), we choose $C_0 = C_1$ and $z_0 = z_1$ (see the Appendix 6.2) and apply comparison theorem. We prove the right-hand sides of the estimations (4.13) and (4.1) by choosing $C_0 = C_*$, $z_0 = z_2$ and τ_0 and applying comparison theorem in the curved region $G_{\tau_0,\delta}$, where

$$G_{\tau,\delta} = \{(x,t) : z_\tau(t) = \tau t^{\frac{mp-\beta}{(1+p)(1-\beta)}} < x < +\infty, 0 < t \leq \delta\}.$$

Region II. Assume that u_0 is defined as (1.10). The self-similar solution (4.15) follows from Lemma 4.3.3. The proof of estimation (4.17) when $C > C_*$ (also when u_0 is given through (1.9)) coincides with the proof given in [20]. Let $0 < C < C_*$. The formula (4.19) follows from Lemma 4.3.3. The proof of the right-hand side of (4.17) (also when u_0 is given through (1.9)) coincides with the proof given in [20] and the proof of the left-hand side of (4.17) follows in the same way as the proof of the analogous estimate from result (3) of [6]. □

Region III. The asymptotic estimate (4.6) follows from Lemma 4.3.4. The proof of the asymptotic estimate (4.5) coincides with the proof of Theorem 2.1.3. □

Region IV. The asymptotic estimation (4.2) is proved in Lemma 4.3.2. From (4.2), (4.43) follows. The proof of estimate (4.7) follows in the same way as the analogous estimate in result (4) from [6].

Intergration of (4.8) implies (4.20). By rescaling x with $\varepsilon^{-1}x$, $\varepsilon > 0$ from (4.20) we have

$$\frac{x}{\varepsilon} = \int_{\vartheta(\frac{x}{\varepsilon})}^{1} \frac{m}{s}\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)} s^{1-mp}\right]^{-\frac{1}{1+p}} ds,\ s > 0.$$

Letting $r = -\varepsilon \ln s$ implies

$$x = \mathcal{J}[\Lambda_\varepsilon(x)], \tag{4.46}$$

where

$$\mathcal{J}(s) = \int_0^s m\left[\frac{b}{p} + \frac{m(1+p)}{p(1-mp)(1+m)} e^{r(mp-1)/\varepsilon}\right]^{-\frac{1}{1+p}} dr,$$

and

$$\Lambda_\varepsilon(x) = -\varepsilon \ln \vartheta\Big(\frac{x}{\varepsilon}\Big).$$

From (4.46) it follows that

$$\Lambda_\varepsilon(x) = \mathcal{J}^{-1}(x), \tag{4.47}$$

where $\mathcal{J}^{-1}$ denotes the inverse function of $\mathcal{J}$.

Since $0 < mp < 1$ it follows that

$$\lim_{\varepsilon\to 0^+}\mathcal{J}(y) = m(b/p)^{-\frac{1}{1+p}}y,\ \lim_{\varepsilon\to 0^+}\mathcal{J}^{-1}(y) = m^{-1}(b/p)^{\frac{1}{1+p}}y, \tag{4.48}$$

for $y \geq 0$, uniformly, on bounded subsets. From (4.47) and (4.48) it follows that

$$-\lim_{\varepsilon\to 0^+}\varepsilon\ln\vartheta\left(\frac{x}{\varepsilon}\right) = m^{-1}(b/p)^{\frac{1}{1+p}}x. \tag{4.49}$$

The asymptotic formula (4.22) follows by choosing $y = x/\varepsilon$. Inequality (4.23), as well as estimations (4.24),(4.25),(4.26) follow from (4.20) and (4.21). □

Region V. Let either $b > 0, \beta > mp$ or $b < 0, \beta \geq 1$. The proof of this case follows in the same way as the analogous case of result (5) of [6] Formula (4.2) is a consequence of Lemma 4.3.2. Choose $\varepsilon > 0$, sufficiently small. From (4.2), (4.43) holds for some $\delta_1 = \delta_1(\varepsilon) > 0$. Let $\beta \geq 1$. Consider

$$r(x,t) = t^{\frac{\alpha}{1+p+\alpha(1-mp)}}f(\chi), \chi = xt^{\frac{-1}{1+p+\alpha(1-mp)}}. \tag{4.50}$$

We have

$$Lr = t^{\frac{\alpha mp-1-p}{1+p+\alpha(1-mp)}}L_1 f, \tag{4.51}$$

where

$$\begin{aligned} L_1 f = \frac{\alpha}{1+p+\alpha(1-mp)}f - \frac{1}{1+p+\alpha(1-mp)}\chi f' - \\ -(|(f^m)'|^{p-1}(f^m)')' + bt^{\frac{1+p-\alpha(mp-\beta)}{1+p+\alpha(1-mp)}}f^\beta. \end{aligned} \tag{4.52}$$

As a function f we select

$$f(\chi) = C_0(\chi_0 + \chi)^{\frac{1+p}{mp-1}}, \chi \geq 0, \tag{4.53}$$

where C_0 and χ_0 are positive constants.

From (4.43) and Lemma 1 of [20], the right-hand side of (4.27) follows with $\delta = \delta_2$, such that

$$\delta_2 = \delta_1, \text{ if } b > 0$$

and

$$\delta_2 = \min(\delta_1, \delta_3), \text{ if } b < 0,$$

with

$$\delta_3 = \left[\frac{\alpha\varepsilon(A_0+\varepsilon)^{1-\beta}}{(1+\varepsilon)(-b(1+p+\alpha(1-mp)))}\right]^{\frac{1+p+\alpha(1-mp)}{1+p+\alpha(\beta-mp)}}.$$

To prove a lower bound in this case we take $C_0 = C_5$ and $\chi_0 = \chi_1$.

The proof of the left-hand side of (4.27) if either $b > 0, \beta < \frac{p(1-m)+2}{1+p}$ or $b < 0, \beta \geq 1$ or $b > 0, \beta \geq \frac{p(1-m)+2}{1+p}$, follows in the same way as the analogous estimate in result (5) of [6].

If $b > 0$ and $\beta \geq 1$, then the proof of estimates (4.10), (4.11), and (4.29) follow as the analogous proof from result (5) of [6]. While if $b > 0$ and $0 < mp < \beta < 1$ the left-hand side of (4.31) follows as left-hand side of (4.13) was previously proved by choosing $f_1(z) = C_*(1-\varepsilon)(z_8+z)_+^{\frac{1+p}{mp-\beta}}$ in (4.44). The proof of (4.12) follows in the same way as the proof of the analogous result from result (5) of [6].

Now, let $b = 0$. Let u_0 be given by (1.10). Formula (4.32) is a consequence of Lemma 4.3.1. In order to prove inequality (4.33), choose the function g as in (4.50); satisfying (4.51) with $b = 0$. Choose the function f as (4.53). The proof of estimate (4.33) follows in the same way as the proof of the analogous result from result (5) of [6]. To prove an upper estimate we choose $C_0 = C_7$ and $\chi_0 = \chi_4$ and to prove a lower estimation we choose $C_0 = \mathcal{D}$ and $\chi_0 = \chi_3$. $\square$

Chapter 5

Conclusions

A full qualitative analysis of the short-time behavior of the interface function and local solution near the interface function or at infinity (when the interface does not exist) for the Cauchy problem for the nonlinear double degenerate parabolic equation of turbulent filtration with absorption

$$u_t = (|(u^m)_x|^{p-1}(u^m)_x)_x - bu^\beta,\ x \in \mathbb{R},\ 0 < t < T, \beta > 0,$$

with

$$u(x,0) = u_0(x) \sim C(-x)_+^\alpha, \text{ as } x \to 0^-, \text{ for some } C > 0,\ \alpha > 0,$$

is successfully pursued in this dissertation in the case of both slow ($mp > 1$) and fast ($0 < mp < 1$) diffusion with either $b \geq 0$ with $0 < \beta < 1$ or $b < 0$ with $\beta \geq 1$.

The following is a summary of the main results.

Slow Diffusion Case: $mp > 1$.

- If $b > 0$, $0 < \beta < 1$, and $0 < \alpha < (1+p)/(mp-\beta)$, then the diffusion term dominates over the reaction term and the interface expands forward, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim \psi t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$

where $\psi = \psi(C, m, p, \alpha) > 0$.

- If $b > 0$, $0 < \beta < 1$, and $\alpha = (1+p)/(mp-\beta)$, then the diffusion and reaction terms are in balance, borderline case, and there is a critical constant C_*, such that the interface expands forward if $C > C_*$ or shrinks backward if $C < C_*$, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim \zeta_* t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$

where $\zeta_* = \zeta_*(C, m, p) \lesseqgtr 0$ if $C \lesseqgtr C_*$.

- If $b > 0$, $0 < \beta < 1$, and $\alpha > (1+p)/(mp-\beta)$, then the reaction term *strongly* dominates over the diffusion term and the interface shrinks backwards, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim -\tau_* t^{1/\alpha(1-\beta)}, \text{ as } t \to 0^+,$$

where $\tau_* = \tau_*(C, m, p, \alpha, \beta) > 0$.

- If $b \in \mathbb{R}$, $\beta \geq 1$, and $\alpha \geq (1+p)/(mp-1)$, then interface initially remains stationary.

Fast Diffusion Case: $0 < mp < 1$.

- If $b > 0$, $0 < \beta < mp$, and $0 < \alpha < (1+p)/(mp-\beta)$, then the diffusion term dominates over the reaction term and the interface expands forward, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim \psi t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$

where $\psi = \psi(C, m, p, \alpha) > 0$.

- If $b > 0$, $0 < \beta < mp$, and $\alpha = (1+p)/(mp-\beta)$, then the diffusion and reaction terms are in balance, borderline case, and there is a critical constant C_*, such that the interface expands forward if $C > C_*$ or shrinks backward if $C < C_*$, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim z_* t^{(mp-\beta)/(1-\beta)(1+p)}, \text{ as } t \to 0^+,$$

where $z_* = z_*(C, m, p) \lesseqgtr 0$ if $C \lesseqgtr C_*$.

- If $b > 0$, $0 < \beta < mp$, with $\alpha > (1+p)/(mp-\beta)$, then the reaction term *strongly* dominates over the diffusion term and the interface shrinks backwards, with the asymptotic formula for the interface function given by the following expression

$$\eta(t) \sim -\tau_* t^{1/\alpha(1-\beta)}, \text{ as } t \to 0^+,$$

where $\tau_* = \tau_*(C, m, p, \alpha, \beta) > 0$.

- If either $b > 0$, $0 < \beta = mp < 1$, and $\alpha > 0$, or $b > 0, \beta > mp > 0$ or $b < 0, \beta \geq 1$, then the diffusion term *strongly* dominates over the reaction term and there is no interface ($\eta(t) = +\infty$): the solution travels with an infinite speed of propagation.

If $b > 0$, $0 < \beta = mp < 1$ and $\alpha > 0$ the solution decays as an exponential function at infinity; while if either $b > 0, \beta > mp > 0$ or $b < 0, \beta \geq 1$, then the solution decays as a power-type function at infinity.

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Chapter 6

Appendix

6.1

Here we give explicit values of the constants used in Chapter 2 ($mp > 1$).

I. $0 < \beta < 1$ and $\alpha < (1+p)/(mp - \min\{1,\beta\})$.

$$\xi_1 = p^{\frac{1}{1+p}}\Big(\alpha(mp-1)\Big)^{-\frac{1}{1+p}}, \xi_2 = 1, \; p(mp-1)^{-1} \le \alpha < (1+p)(mp-1)^{-1},$$

$$\xi_1 = 1, \xi_2 = p^{\frac{1}{1+p}}\Big(\alpha(mp-1)\Big)^{-\frac{1}{1+p}}, \; 0 < \alpha \le p(mp-1)^{-1}.$$

II. $0 < \beta < 1$ and $\alpha = (1+p)/(mp-\beta)$.

$$f_1(0) = A_1 > 0,$$

$$\zeta_1 = A_1^{\frac{mp-1}{1+p}}(1 + b(1-\beta)A_1^{\beta-1})^{-\frac{1}{1+p}}(p(mp)^p(1-\beta))^{\frac{1}{1+p}}(mp-1)^{-1},$$

$$C_1 = A_1\zeta_1^{-\frac{p}{mp-1}}, \; p(m+\beta) > 1+p, \; C > C_*,$$

$$\zeta_1 = A_1^{\frac{mp-1}{1+p}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{1+p}}((m(1+p))^p p(m+\beta)(1-\beta))^{\frac{1}{1+p}}(mp-\beta)^{-1},$$

$$C_1 = A_1\zeta_1^{-\frac{1+p}{mp-\beta}},\ p(m+\beta) < 1+p,\ C > C_*,$$

$$\zeta_2 = A_1^{\frac{mp-1}{1+p}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{1+p}}((m(1+p))^p p(m+\beta)(1-\beta))^{\frac{1}{1+p}}(mp-\beta)^{-1},$$

$$C_2 = A_1\zeta_2^{-\frac{1+p}{mp-\beta}},\ p(m+\beta) > 1+p,\ C > C_*,$$

$$\zeta_2 = (A_1/C_*)^{\frac{mp-\beta}{1+p}},\ C_2 = C_*,\ p(m+\beta) < 1+p, C > C_*,$$

$$\zeta_1 = -C^{-\frac{mp-\beta}{1+p}}(b(1-\beta))^{\frac{mp-\beta}{(1+p)(1-\beta)}},\ p(m+\beta) > 1+p,\ 0 < C < C_*,$$

$$\zeta_2 = -C^{-\frac{mp-\beta}{1+p}}\left(b(1-\beta)(1-(C/C_*)^{mp-\beta})^{\frac{mp-\beta}{(1+p)(1-\beta)}}\right),\ p(m+\beta) < 1+p,\ 0 < C < C_*,$$

$$R_1 = (m(1+p))^p p(1+p-p(m+\beta))(b(mp-\beta)^{1+p})^{-1},$$

$$R_2 = (m(1+p))^p(1+p)p(m+\beta-1)(b(mp-\beta)^{1+p})^{-1},$$

$$\theta_* = \left[1-\left(C/C_*\right)^{mp-\beta}\right]\left[\left(C_*/C\right)^{\frac{(mp-\beta)(1-\beta)}{1+p-p(m+\beta)}}-1\right]^{-1},$$

$$\ell_0 = C_*^{-\frac{mp-\beta}{1+p}}(C_*/C)^{\frac{(mp-\beta)(1-\beta)}{1+p-p(m+\beta)}}(b(1-\beta)\theta_*)^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\zeta_3 = C_*^{-\frac{mp-\beta}{1+p}}\left[(C_*/C)^{\frac{(mp-\beta)(1-\beta)}{1+p-p(m+\beta)}}-1\right](b(1-\beta)\theta_*)^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\ell_1 = C^{-\frac{mp-\beta}{1+p}}\left[b(1-\beta)(\delta_*\Gamma)^{-1}\left(1-\delta_*\Gamma-(1-\delta_*\Gamma)^{-p}(C/C_*)^{mp-\beta}\right)\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\zeta_4 = \delta_*\Gamma\ell_1,\ \Gamma = 1-(C/C_*)^{\frac{mp-\beta}{1+p}},\ C_3 = C(1-\delta_*\Gamma)^{-\frac{1+p}{mp-\beta}},$$

where $\delta_* \in (0,1)$ satisfies

$$g(\delta_*) = \max_{[0,1]} g(\delta),\ g(\delta) = \delta^{\frac{1+p-p(m+\beta)}{mp-\beta}}\left(1-\delta\Gamma-(1-\delta\Gamma)^{-p}(C/C_*)^{mp-\beta}\right).$$

III. $0 < \beta < 1$ and $\alpha > (1+p)/(mp-\beta)$.

$$\zeta_5 = (\ell_*/\ell)^{\alpha(1-\beta)}(1-\varepsilon)\ell,$$

$$C_6 = [1-(\ell_*/\ell)^{\alpha(1-\beta)}(1-\varepsilon)]^{-\alpha}[C^{1-\beta} - \ell^{-\alpha(1-\beta)}b(1-\beta)(1-\varepsilon))]^{1/(1-\beta)}.$$

V. $b = 0$ and $\alpha > 0$.

$$w(0,1) = A_0 > 0,$$

$$\bar{C} = \left[\frac{(mp-1)^{1+p}}{p(m+1)(m(1+p))^p}\right]^{\frac{1}{mp-1}},$$

$$\gamma_\varepsilon = \frac{p(m+1)(m(1+p))^p(C+\varepsilon)^{mp-1}}{(mp-1)^p} + \varepsilon,$$

$$\xi_3 = A_0^{\frac{mp-1}{1+p}}\left[\frac{(mp)^p(1+p-\alpha(mp-1))}{(mp-1)^p}\right]^{\frac{1}{1+p}} C^{\frac{mp-1}{1+p-\alpha(mp-1)}}\xi_1,$$

$$\xi_4 = A_0^{\frac{mp-1}{1+p}}\left[\frac{(mp)^p(1+p-\alpha(mp-1))}{(mp-1)^p}\right]^{\frac{1}{1+p}} C^{\frac{mp-1}{1+p-\alpha(mp-1)}}\xi_2,$$

$$C_4 = C^{(1+p)/(1+p-\alpha(mp-1))}A_0\xi_3^{p/(1-mp)},\ C_5 = C^{(1+p)/(1+p-\alpha(mp-1))}A_0\xi_4^{p/(1-mp)}.$$

6.2

Here we give explicit values of the constants used in Chapter 4 ($0 < mp < 1$).

I. $0 < \beta < mp$ and $0 < \alpha < (1+p)/(mp-\beta)$.

$$z_1 = (b(1-\beta))^{\frac{mp-1}{(1+p)(1-\beta)}}(m(1+p))^{\frac{p}{1+p}}(p(m+\beta))^{\frac{1}{1+p}}(mp-\beta)^{\frac{p(m+\beta-1)-1}{(1+p)(1-\beta)}}(1-mp)^{\frac{1-mp}{(1+p)(1-\beta)}},$$

$$C_1 = \left(\frac{1-\beta}{1-mp}\right)^{\frac{1}{mp-\beta}} C_*,$$

$$\tau_0 = \left(\frac{1-mp}{mp-\beta}\right)^{\frac{mp-1}{1-\beta}} \left(\frac{\mathcal{D}}{C_*}\right)^{\frac{(mp-1)(\beta-mp)}{(1+p)(1-\beta)}}, z_2 = \tau_0 \frac{1-\beta}{mp-\beta}.$$

II. $0 < \beta < mp$ and $\alpha = (1+p)/(mp-\beta)$.

$$C_2 = A_1 z_3^{\frac{1+p}{\beta-mp}}, A_1 = f_1(0) > 0,$$

$$z_3 = (m(1+p))^{\frac{p}{p+1}} (m+\beta)^{\frac{1}{1+p}} p^{\frac{1}{1+p}} (mp-\beta)^{-1} (1-\beta)^{\frac{1}{1+p}} A_1^{\frac{m-1}{1+p}} \left[b(1-\beta)A_1^{\beta-1}+1\right]^{\frac{-1}{1+p}},$$

$$z_4 = \left(\frac{A_1}{C_*}\right)^{\frac{mp-\beta}{1+p}}, z_5 = \tau_1 - \left(\frac{\varrho}{C_*}\right)^{\frac{1+p}{mp-\beta}}, C_3 = C\left(\frac{1}{1-\delta_*\Gamma}\right)^{\frac{1+p}{mp-\beta}},$$

$$\Gamma = 1 - \left(\frac{C}{C_*}\right)^{\frac{mp-\beta}{1+p}}, z_6 = \delta_*\Gamma\tau_2, \text{ with } \delta_* \text{ such that } g(\delta_*) = \max_{\delta\in(0,1)} g(\delta),$$

$$g(\delta) = (\delta\Gamma)^{\frac{1+p-p(m+\beta)}{(1+p)(1-\beta)}} \left[1-\delta\Gamma - \left(\frac{C}{C_*}\right)^{mp-\beta}\left(\frac{1}{1-\delta\Gamma}\right)^p\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\tau_2 = C^{\frac{\beta-mp}{1+p}} \left[\frac{b(1-\beta)}{\delta_*\Gamma}\left(1-\delta_*\Gamma - \left(\frac{C}{C_*}\right)^{mp-\beta}\left(\frac{1}{1-\delta_*\Gamma}\right)^p\right)\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}}.$$

V. $\beta > mp$.

$$C_5 = (1-\varepsilon)^{\frac{1}{1-mp}}\mathcal{D},$$

$$C_6 = \left(\frac{\alpha(1-mp)^{p+1}}{\varkappa_b(1+p+\alpha(1-mp))(m(1+p))^p p(m+1)}\right)^{\frac{1}{mp-1}},$$

$$\chi_1 = (A_0-\varepsilon)^{(mp-1)/(1+p)}(1-\varepsilon)^{1/(1+p)}\mathcal{D}^{(1-mp)/(1+p)}, \text{ if } b>0,\ 1\le\beta<(p(1-m)+2)/(1+p),$$

$$\chi_1 = (A_0-\varepsilon)^{(mp-1)/(1+p)}\mathcal{D}^{(1-mp)/(1+p)}, \text{ if } b>0, \beta\ge(p(1-m)+2)/(1+p) \text{ or } b<0, \beta\ge 1,$$

$$\chi_2 = \left(\frac{A_0+\varepsilon}{C_6}\right)^{\frac{mp-1}{1+p}}, A_0 = f(0) > 0,\ \varkappa_b = \begin{cases} 1, \text{ if } b>0, \\ 1+\varepsilon, \text{ if } b<0, \end{cases}$$

$$z_8 = \left[b(1-\beta)C_*^{\beta-1}(1-\varepsilon)^{mp-1}(1-(1-\varepsilon)^{\beta-mp})\right]^{\frac{mp-\beta}{(1+p)(1-\beta)}},$$

$$\chi_3 = (A_0/\mathcal{D})^{(mp-1)/(1+p)},$$

$$\chi_4 = \chi_3(1+(1+p)/\alpha(1-mp))^{1/(1+p)},$$

$$C_7 = \mathcal{D}(1+(1+p)/\alpha(1-mp))^{1/(1+mp)}.$$