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Two-Point Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations

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By

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"Two-Point Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations"
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ABSTRACT

Title: *Two-Point Boundary Value Problems for Higher Order Nonlinear Hyperbolic Equations*
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Two–point boundary value problems in a multidimensional box for higher order nonlinear hyperbolic equations are considered. The concepts of a strongly isolated solution, and locally and globally strong well–posedness of a nonlinear boundary value problem are introduced.

For general two–point boundary value problems and periodic problems there are established:
(i) Necessary and sufficient conditions of locally and globally strong well–posedness;
(ii) Unimprovable Sufficient conditions of solvability.

For the Dirichlet and Periodic type problems for equations of even order there are established:
(i) Effective sufficient conditions of solvability and locally strong well–posedness;
(ii) Unimprovable sufficient conditions of solvability for the case, where the right-hand side of equation has arbitrary growth order with respect to certain phase variables;
(iii) sufficient conditions of solvability and locally strong well–posedness for the case, where the righthand side of equation is Hölder continuous with respect to certain principal phase variables.

For initial–boundary value problems there are established:
(i) Necessary and sufficient conditions of locally and globally strong well–posedness;
(ii) Unimprovable sufficient conditions of local and global solvability.

CONTENTS

List of Notations

- $\mathbf{m} = (m_1, \ldots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.
- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) < \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \iff \alpha_i \le \beta_i \ (i = 1, \ldots, n)$ and $\boldsymbol{\alpha} \ne \boldsymbol{\beta}$.
- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \le \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \iff \boldsymbol{\alpha} < \boldsymbol{\beta}$, or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.
- $\mathbf{0} = (0, \ldots, 0), \quad \mathbf{1} = (1, \ldots, 1), \quad \mathbf{1_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$.
- $\operatorname{supp} \boldsymbol{\alpha} = \{i \mid \ \alpha_i > 0\}, \quad \|\boldsymbol{\alpha}\| = |\alpha_1| + \cdots + |\alpha_n|$.
- $\boldsymbol{\Xi} = \big\{\boldsymbol{\sigma} \big| \ \mathbf{0} < \boldsymbol{\sigma} < \mathbf{1}\big\}$.
- $\widehat{\boldsymbol{\alpha}} = \mathbf{m} - \boldsymbol{\alpha}$. If $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$, then $\widehat{\boldsymbol{\sigma}} = \mathbf{1} - \boldsymbol{\sigma}$.
- $\mathbf{m}_{\boldsymbol{\sigma}} = (\sigma_1 m_1, \ldots, \sigma_n m_n)$. It is clear that $\widehat{\mathbf{m}}_{\boldsymbol{\sigma}} = \mathbf{m} - \mathbf{m}_{\boldsymbol{\sigma}} = \mathbf{m}_{\widehat{\boldsymbol{\sigma}}}$.
- $\mathbf{m}_i = (0, \ldots, m_i, \ldots, 0)$, $\mathbf{1}_i = (0, \ldots, 1, \ldots, 0)$.
- $\boldsymbol{\Upsilon}_{\mathbf{m}} = \big\{\boldsymbol{\alpha} \le \mathbf{m} : \alpha_i = m_i \ \text{ for some } \ i \in \{1, \ldots, n\}\big\}$.
- If $\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}$, then $\mathbf{1}_{\boldsymbol{\alpha}} = (\chi_{m_1}(\alpha_1), \ldots, \chi_{m_n}(\alpha_n))$, where

$$\chi_m(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha = m \\ 0 & \text{if} \quad \alpha < m \end{cases} .$$

- $\mathcal{D}^{\mathbf{m}}[u] = \Big(u^{(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\alpha} \le \mathbf{m}}, \quad \widetilde{\mathcal{D}}^{\mathbf{m}}[u] = \Big(u^{(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\alpha} < \mathbf{m}},$

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} .$$

- $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n), \quad \boldsymbol{\omega}_{\mathbf{i}} = (0, \ldots, \omega_i, \ldots, 0)$.
- $\Omega = [0, \omega_1] \times \cdots \times [0, \omega_n], \quad \Omega_{\mathbf{i}} = [0, \omega_i]$.
- $\widehat{\Omega}_{\mathbf{i}} = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$.
- $\mathbf{x}_{\boldsymbol{\sigma}} = (\sigma_1 x_1, \ldots, \sigma_n x_n)$. $\mathbf{x}_{\boldsymbol{\sigma}}$ will be identified with $(x_{i_1}, \ldots, x_{i_l})$, as well as the set $\Omega_{\boldsymbol{\sigma}} = [0, \sigma_1 \omega_1] \times \cdots \times [0, \sigma_n \omega_n]$ will be identified with the set $[0, \omega_{i_1}] \times \cdots \times [0, \omega_{i_l}]$, where $\{i_1, \cdots i_l\} = \operatorname{supp} \boldsymbol{\sigma}$. Furthermore, $\mathbf{x}_{\boldsymbol{\sigma}}$ will be identified with $(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{0}}_{\boldsymbol{\sigma}})$, and $\mathbf{x}$ will be identified with $(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}})$, or with $(\mathbf{x}_{\boldsymbol{\sigma}}, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}})$.

- $C(\Omega)$ is the Banach space of continuous functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{C(\Omega)} = \max\{|u(\mathbf{x})| : \mathbf{x} \in \Omega\}.$$

- $C_{\boldsymbol{\omega}}(\Omega)$ is the subspace of functions $u \in C(\Omega)$ such that

$$u(\mathbf{0}_i, \widehat{\mathbf{x}}_i) = u(\boldsymbol{\omega}_i, \widehat{\mathbf{x}}_i) \quad (i = 1, \ldots, n).$$

- $C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

- $C_0^{\mathbf{m}}(\Omega)$ is subspace $C^{\mathbf{m}}(\Omega)$ consisting of functions having a compact support, i.e.

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})\Big|_{\mathbf{x} \in \partial\Omega} = 0 \qquad (\boldsymbol{\alpha} \le \mathbf{m}).$$

- $\widetilde{C}^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} < \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

- If $u_0 \in C^{\mathbf{m}}(\Omega)$ and $r > 0$, then

$$\mathbf{B}^{\mathbf{m}}(u_0; r) = \left\{u \in C^{\mathbf{m}}(\Omega) : \ \|u - u_0\|_{C^{\mathbf{m}}} \le r\right\}.$$

- If $u_0 \in \widetilde{C}^{\mathbf{m}}(\Omega)$ and $r > 0$, then

$$\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0; r) = \left\{u \in \widetilde{C}^{\mathbf{m}}(\Omega) : \ \|u - u_0\|_{\widetilde{C}^{\mathbf{m}}} \le r\right\}.$$

- $C^{\mathbf{m},k}(\Omega \times \mathbb{R}^{\boldsymbol{\beta}})$ the space of continuous functions $u(\mathbf{x}, \mathbf{Z})$ such that $u(\cdot, \mathbf{Z}) \in C^{\mathbf{m}}(\Omega)$ for every $\mathbf{Z} \in \mathbb{R}^{\boldsymbol{\beta}}$ and $u(\mathbf{x}, \cdot) \in C^k(\mathbb{R}^{\boldsymbol{\beta}})$ for every $\mathbf{x} \in \mathbb{R}^n$.
- $C_{\boldsymbol{\omega}}^{\mathbf{m}}(\Omega)$ is the Banach space of continuous functions $u \in C_{\boldsymbol{\omega}}(\Omega)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})} \in C_{\boldsymbol{\omega}}(\Omega)$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C_{\boldsymbol{\omega}}^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

- $C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e., functions that are ω_i-periodic with respect to the variable x_i $(i = 1, \dots, n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \leq \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

- $\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} < \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

In certain cases it is more convenient to operate with spaces $C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\Omega)$ rather than spaces $C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \geq \mathbf{0})$. It is clear that the space $C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\Omega)$ is a "restriction" of the space $C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ on the domain Ω, and vice versa, the latter space is a "$\boldsymbol{\omega}$–periodic continuation" of the former one. More precisely, $u \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\Omega)$ if and only if $\boldsymbol{\omega}$–periodic continuation u belongs to $C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$.

- If $u_0 \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $r > 0$, then

$$\mathbf{B}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r) = \big\{u \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n) : \ \|u - u_0\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} \leq r\big\}.$$

- If $u_0 \in \widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ and $r > 0$, then

$$\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r) = \big\{u \in \widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n) : \ \|u - u_0\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}} \leq r\big\}.$$

- $C^{\mathbf{m},k}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\beta}})$ the space of continuous functions $u(\mathbf{x}, \mathbf{Z})$ such that $u(\cdot, \mathbf{Z}) \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ for every $\mathbf{Z} \in \mathbb{R}^{\boldsymbol{\beta}}$ and $u(\mathbf{x}, \cdot) \in C^k(\mathbb{R}^{\boldsymbol{\beta}})$ for every $\mathbf{x} \in \mathbb{R}^n$.
- $L(\Omega)$ is the Banach space of Lebesgue integrable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \iint\limits_{\Omega} |u(\mathbf{x})| \, d\mathbf{x}.$$

- $L_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of locally Lebesgue integrable $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L_{\boldsymbol{\omega}}} = \int\limits_{\Omega} |u(\mathbf{x})|\, d\mathbf{x}.$$

- $L^{\infty}(\Omega)$ is the spaces of essentially bounded measurable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L^{\infty}} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})|.$$

- $L^{\infty}_{\boldsymbol{\omega}}(\Omega)$ is the spaces of essentially bounded $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L^{\infty}_{\boldsymbol{\omega}}} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})|.$$

- $AC([0,\omega])$ is the Banach space of absolutely continuous functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC} = |u(0)| + \int\limits_{0}^{\omega} \left|u'(x)\right| dx.$$

- $AC(\Omega)$ is the Banach space of absolutely continuous functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC(\Omega)} = |u(\mathbf{0})| + \sum_{\boldsymbol{\sigma}\leq\mathbf{1}} \iint\limits_{\Omega_{\boldsymbol{\sigma}}} |u^{(\boldsymbol{\sigma})}(\mathbf{x}_{\boldsymbol{\sigma}})|\, d\mathbf{x}_{\boldsymbol{\sigma}}$$

- $AC^{\mathbf{m}-\mathbf{1}}(\Omega)$ is the Banach space of functions $u \in C^{(\mathbf{m}-\mathbf{1})}(\Omega)$, having absolutely continuous derivative $u^{(\mathbf{m}-\mathbf{1})}$, endowed with the norm

$$\|u\|_{AC^{\mathbf{m}-\mathbf{1}}(\Omega)} = \|u\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} + \|u^{(\mathbf{m}-\mathbf{1})}\|_{AC(\Omega)}.$$

- $AC_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the Banach space of locally absolutely continuous $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC_{\boldsymbol{\omega}}} = |u(\mathbf{0})| + \sum_{\boldsymbol{\sigma}\leq\mathbf{1}} \int\limits_{\Omega_{\boldsymbol{\sigma}}} |u^{(\boldsymbol{\sigma})}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{0}}_{\boldsymbol{\sigma}})|\, d\mathbf{x}_{\boldsymbol{\sigma}}$$

- $AC_{\boldsymbol{\omega}}^{\mathbf{m-1}}(\mathbb{R}^{\boldsymbol{n}})$ is the Banach space of functions $u \in AC_{\boldsymbol{\omega}}(\mathbb{R}^n)$, having locally absolutely continuous derivative $u^{(\mathbf{m-1})}$, endowed with the norm

$$\|u\|_{AC_{\boldsymbol{\omega}}^{\mathbf{m-1}}} = \|u\|_{C_{\boldsymbol{\omega}}^{\mathbf{m-1}}} + \|u^{(\mathbf{m-1})}\|_{AC_{\boldsymbol{\omega}}}.$$

Introduction

In the present dissertation for the higher order nonlinear hyperbolic equation

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{0.1}$$

we investigate boundary conditions

$$a_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i})$$
$$\text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \;\; (k = 1, \dots, m_i; \;\; i = 1, \dots, n). \tag{0.2}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_\mathbf{i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\widehat{\Omega}_\mathbf{i} = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $\mathbf{m} = (m_1, \dots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\mathbf{m}^i = (m_1, \dots, m_i, 0, \dots, 0)$ ($\mathbf{m}^i = (0, \dots, 0)$ if $i = 0$), and $\mathbf{1}_i = (0, \dots, 1, \dots, 0)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,,$$

$\mathcal{D}^{\mathbf{m}}[u] = \left(u^{(\boldsymbol{\alpha})}\right)_{\boldsymbol{\alpha} \le \mathbf{m}}$, $\widetilde{\mathcal{D}}^{\mathbf{m}}[u] = \left(u^{(\boldsymbol{\alpha})}\right)_{\boldsymbol{\alpha} < \mathbf{m}}$ and $f \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$.

By a solution of problem (0.1),(0.2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (0.1) and boundary conditions (0.2) everywhere in Ω.

Problem (0.1),(0.2) does not belong to the classical boundary value problems of mathematical physics, with the exception of Darboux and Goursat initial value problems for second order hyperbolic equations (the case $n = 2$, $\mathbf{m} = (1, 1)$).

Beginning from the 1960ies, two–dimensional problems on periodic solutions, as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see [4-17, 41–55]). These problems naturally led to the initial–boundary value problems in a rectangle with general boundary conditions:

$$w^{(1,1)} = P_0(x,y)w + P_1(x,y)w^{(1,0)} + P_2(x,y)w^{(0,1)} + q(x,y), \tag{0.3}$$

$$w(0,y) = \varphi(y), \qquad h\big(w^{(1,0)}(x,\cdot)\big)(x) = \psi(x), \tag{0.4}$$

where $P_i \in C([0,a]\times[0,b];\mathbb{R}^{n\times n})$ $(i = 0,1,2)$, $q \in C([0,a]\times[0,b];\mathbb{R}^n)$, $\varphi \in C^1([0,b];\mathbb{R}^n)$, $\psi \in C([0,a];\mathbb{R}^n)$ and $h : C([0,b]) \to C([0,a])$ is a bounded linear operator. A complete theory of problem (0.3), (0.4) was constructed in [24].

Initial–boundary value problems with integral boundary conditions for quasi–linear systems were studied in [1–3].

The initial–periodic boundary value problems for quasi-linear and nonlinear systems

$$w^{(1,1)} = F\big(x,y,w^{(1,0)},w^{(0,1)},w\big), \tag{0.5}$$

$$w(0,y) = \varphi(y), \quad w^{(1,0)}(x,0) = w^{(1,0)}(x,b), \tag{0.6}$$

were studied in [22, 23, 30, 32].

Nonlocal boundary value problems, in particular the Dirichlet problem and problems on doubly periodic solutions, were studied in [20, 25, 26, 27, 31].

Two–dimensional initial–boundary value problems for linear equations

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} + q(x,y), \tag{0.7}$$

$$\begin{gathered} u^{(j,0)}(0,y) = \varphi_j(y) \quad (j = 0,\dots,m-1), \\ h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k(x) \quad (k = 1,\dots,n). \end{gathered} \tag{0.8}$$

was studied in [28] and [29].

Same problems for the nonlinear hyperbolic equation

$$u^{(m,n)} = f\big(x,y,u^{(m,0)},\dots,u^{(m,n-1)},u^{(0,n)},\dots,u^{(m-1,n)},u^{(m-1,n-1)},\dots,u\big), \tag{0.9}$$

was studied in [33].

The linear case of problem (0.1), (0.2), i.e. the case of the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{0.10}$$

was studied in [36, 37], where necessary and sufficient conditions of well–posedness and $\boldsymbol{\alpha}$-well–posedness of problem (0.10),(0.2) we established.

Investigation of problem (0.1),(0.2) required a fundamental modification of methods developed in [36] and [37]. In particular, there were introduced concepts of *strong well—posedness* and *strong* $(u_0; r)$*-well—posedness*, and established new a priori estimates for solutions of linear problem (0.10),(0.2) with measurable and essentially bounded coefficients.

The work is organized as follows: general two–point boundary value problems are studied in Chapter I; problems on periodic solutions are studied in Chapter II; initial–boundary value problems are studied in Chapter III.

CHAPTER I
Two–Point Boundary Value Problems

1. Formulation of the Main Results

Let $m_1, \dots, m_n$ be positive integers. In the n-dimensional box $\Omega = [0, \omega_1] \times \cdots \times [0, \omega_n]$ for the nonlinear hyperbolic equation

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{1.1}$$

consider the boundary conditions

$$a_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i})$$
$$\text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (k = 1, \dots, m_i;\ \ i = 1, \dots, n). \tag{1.2}$$

Here $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_\mathbf{i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\widehat{\Omega}_\mathbf{i} = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $\mathbf{m} = (m_1, \dots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\mathbf{m}^i = (m_1, \dots, m_i, 0, \dots, 0)$ ($\mathbf{m}^i = (0, \dots, 0)$ if $i = 0$), and $\mathbf{1}_i = (0, \dots, 1, \dots, 0)$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,,$$

$\mathcal{D}^{\mathbf{m}}[u] = \Big(u^{(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\alpha} \le \mathbf{m}}$, $\widetilde{\mathcal{D}}^{\mathbf{m}}[u] = \Big(u^{(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\alpha} < \mathbf{m}}$ and $f \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$.

By a solution of problem (1.1),(1.2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (1.1) and boundary conditions (1.2) everywhere in Ω.

Remark 1.1. Conditions (1.2) are not equivalent to the conditions

$$a_{ik}\, u^{((k-1)\mathbf{1}_i)}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{((k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ik}(\widehat{\mathbf{x}}_\mathbf{i})$$
$$\text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (k = 1, \dots, m_i;\ \ i = 1, \dots, n). \tag{$\widetilde{1.2}$}$$

since the latter require the additional consistency conditions

$$a_{ik}\,\varphi_{jl}^{((k-1)\mathbf{1}_i)}(\mathbf{0}_i,\,\widehat{\mathbf{x}}_{\mathbf{ij}})+b_{ik}\,\varphi_{jl}^{((k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\,\widehat{\mathbf{x}}_{\mathbf{ij}})$$
$$=a_{jl}\,\varphi_{ik}^{((l-1)\mathbf{1}_j)}(\mathbf{0}_j,\,\widehat{\mathbf{x}}_{\mathbf{ij}})+b_{jl}\,\varphi_{ik}^{((l-1)\mathbf{1}_j)}(\boldsymbol{\omega}_j,\,\widehat{\mathbf{x}}_{\mathbf{ij}})$$
$$(k=1,\dots,m_i;\quad l=1,\dots,m_j;\quad i,j=1,\dots,n).$$

However, in the well–posed case the homogeneous conditions

$$a_{ik}\,u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\,\widehat{\mathbf{x}}_{\mathbf{i}})+b_{ik}\,u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\,\widehat{\mathbf{x}}_{\mathbf{i}})=0$$
$$\text{for }\ \widehat{\mathbf{x}}_{\mathbf{i}}\in\widehat{\Omega}_{\mathbf{i}}\ \ (k=1,\dots,m_i;\quad i=1,\dots,n)\qquad(1.2_0)$$

are equivalent to the homogeneous conditions

$$a_{ik}\,u^{((k-1)\mathbf{1}_\mathbf{i})}(\mathbf{0}_\mathbf{i},\,\widehat{\mathbf{x}}_{\mathbf{i}})+b_{ik}\,u^{((k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\,\widehat{\mathbf{x}}_i)=0$$
$$\text{for }\ \widehat{\mathbf{x}}_i\in\widehat{\Omega}_{\mathbf{i}}\ \ (k=1,\dots,m_i;\quad i=1,\dots,n).\qquad(\widetilde{1.2_0})$$

1.1. **Strong Well–Posedness.** Along with problem (1.1),(1.2) consider the perturbed problem

$$u^{(\mathbf{m})}=f(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])+\widetilde{f}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]),\tag{1.3}$$

$$a_{ik}\,u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\,\widehat{\mathbf{x}}_{\mathbf{i}})+b_{ik}\,u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\,\widehat{\mathbf{x}}_{\mathbf{i}})=\varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}})+\widetilde{\varphi}_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}})$$
$$\text{for }\ \widehat{\mathbf{x}}_{\mathbf{i}}\in\widehat{\Omega}_{\mathbf{i}}\ \ (k=1,\dots,m_i;\quad i=1,\dots,n).\qquad(1.4)$$

A vector function $\big(\widetilde{f};\widetilde{\varphi}_{1\,1},\dots,\widetilde{\varphi}_{1\,m_1},\dots,\widetilde{\varphi}_{n\,1},\dots,\widetilde{\varphi}_{n\,m_n}\big)$ is said to be an *admissible perturbation*, if $\widetilde{f}\in C(\Omega\times\mathbb{R}^{\mathbf{m}})$ is locally Lipschitz continuous with respect to the *principal* phase variables, and $\widetilde{\varphi}_{ik}\in C^{\widehat{\mathbf{m}}_i}(\Omega_{\mathbf{i}})$ $(k=1,\dots,m_i;\ i=1,\dots,n)$.

Definition 1.1. Let u_0 be a solution of problem $(1.1),(1.2)$, and $r>0$. We say that problem $(1.1),(1.2)$ is (u_0,r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem $(1.1),(1.2)$ in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;r)$;

(II) there exist a positive constant δ_0 and an increasing continuous $\varepsilon:[0,\delta_0]\to[0,+\infty)$ such that $\varepsilon(0)=0$ and for any $\delta\in(0,\delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f};\widetilde{\varphi}_{1\,1},\dots,\widetilde{\varphi}_{1\,m_1},\dots,\widetilde{\varphi}_{n\,1},\dots,\widetilde{\varphi}_{n\,m_n}\big)$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})| \le \delta_0 \quad \text{for} \quad (\mathbf{x},\mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{1.5}$$

$$|\widetilde{f}(\mathbf{x},\mathbf{Z})| \le \delta \quad \text{for} \quad (\mathbf{x},\mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}}, \tag{1.6}$$

$$\sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\widetilde{\varphi}_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} \le \delta, \tag{1.7}$$

problem (1.3), (1.4) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;\varepsilon(\delta))$.

Definition 1.2. Let u_0 be a solution of problem (1.1), (1.2), and $r > 0$. We say that problem (1.1), (1.2) is *strongly* (u_0,r)*-well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem (1.1), (1.2) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0,r)$;

(II) there exist a positive constants δ_0 and M such that for any $\delta \in (0,\delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f};\widetilde{\varphi}_{1\,1},\dots,\widetilde{\varphi}_{1\,m_1},\dots,\widetilde{\varphi}_{n\,1},\dots,\widetilde{\varphi}_{n\,m_n}\big)$ satisfying conditions (1.5)–(1.7), problem (1.3), (1.4) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;M\delta)$.

Definition 1.3. Problem (1.1), (1.2) is called well–posed (strongly well–posed), if it is (u_0,r)-well–posed (strongly (u_0,r)-well–posed) for every $r > 0$.

Definition 1.4. A solution u_0 of problem (1.1), (1.2) is called *strongly isolated*, if problem (1.1), (1.2) is strongly (u_0,r)-well–posed for some $r > 0$.

The linear case of equation (1.1), i.e. the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{1.8}$$

was studied in [21, 25–29, 36, 37].

Definition 1.5. Problem (1.8), (1.2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_\mathbf{i}}(\widehat{\Omega}_i)} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of q and φ_{ik} $(k = 1, \dots, m_i;\ i = 1, \dots, n)$.

Remark 1.2. Notice that for the linear problem (1.8), (1.2) (u_0, r)-well–posedness is equivalent to the strong well–posedness. Furthermore, for problem (1.8), (1.2) Definitions 1.1 and 1.2 are equivalent to Definition 1.5.

1.2. **Necessary and Sufficient Conditions of Strong Well–Posedness.**

Theorem 1.1. *Let problem* (1.1), (1.2) *be solvable for arbitrary* $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_\mathbf{i}}(\Omega_\mathbf{i})$ $(k = 1, \dots, m_i;\ i = 1, \dots n)$. *Then for each* $i \in \{1, \dots, n-1\}$ *the problem*

$$z^{(m_i)} = 0, \qquad a_{ik}\, z^{(k-1)}(0) + b_{ik}\, z^{(k-1)}(\omega_i) = 0 \ \ (k = 1, \dots, m_i) \tag{1.9}$$

has only the trivial solution.

Theorem 1.2. *Let problem* (1.9) *have only the trivial solution for every* $i \in \{1, 2, \dots, n-1\}$. *Furthermore, let the function* $f(\mathbf{x}, Z)$ *be continuously differentiable with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ *such that:*

(A_1) $$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le f_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} < \mathbf{m}); \tag{1.10}$$

(A_2) *For every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi} \cup \{\mathbf{1}\}$, $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ *and arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}} \in L^\infty(\Omega_{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$ *satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \quad \text{for} \quad \mathbf{y} \in \Omega_{\boldsymbol{\sigma}} \quad (\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}), \tag{1.11}$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol\sigma})} = \sum_{\boldsymbol\alpha<\mathbf{m}_{\boldsymbol\sigma}} \rho_{\boldsymbol\alpha}(\mathbf{y})v^{(\boldsymbol\alpha)}, \tag{1.12}$$

$$a_{ik}\, v^{((k-1)\mathbf{1}_i)}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{y}}_\mathbf{i}) + b_{ik}\, v^{((k-1)\mathbf{1}_i)}(\boldsymbol\omega_\mathbf{i},\, \widehat{\mathbf{y}}_\mathbf{i}) = 0 \;\; (k = 1,\dots,m_i; i \in \operatorname{supp}\boldsymbol\sigma). \tag{1.13}$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol\sigma}-\mathbf{1}_{\boldsymbol\sigma}}(\Omega_{\boldsymbol\sigma})$. *Then problem* (1.1), (1.2) *is strongly well–posed.*

Consider the quasi–linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol\alpha<\mathbf{m}} p_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]) \tag{1.14}$$

Corollary 1.1. *Let the homogeneous problem*

$$u^{(\boldsymbol m)} = \sum_{\boldsymbol\alpha<\boldsymbol m} p_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)}, \tag{1.15}$$

$$a_{ik}\, u^{((k-1)\mathbf{1}_\mathbf{i})}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{((k-1)\mathbf{1}_i)}(\boldsymbol\omega_i,\, \widehat{\mathbf{x}}_i) = 0$$

$$\text{for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \;\; (k = 1,\dots,m_i;\;\; i = 1,\dots,n) \tag{1.16}$$

be well–posed, let q *be a continuously differentiable function with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol\alpha} \in C(\Omega)$ ($\boldsymbol\alpha \le \mathbf{m} - \mathbf{1}$; $i = 1, 2$) *such that:*

$$(A_1) \quad P_{1\boldsymbol\alpha}(\mathbf{x}) \le \frac{\partial q(\mathbf{x}, \mathbf{Z})}{\partial z_{\boldsymbol\alpha}} \le P_{2\boldsymbol\alpha}(\mathbf{x}) \;\; \text{for} \;\; (\mathbf{x}, \mathbf{Z}) \in \Omega\times\mathbb{R}^{\mathbf{m}} \;\; (\boldsymbol\alpha \le \mathbf{m}-\mathbf{1}); \tag{1.17}$$

(A_2) *for arbitrary measurable functions* $\rho_{\boldsymbol\alpha} \in L^\infty(\Omega)$ ($\boldsymbol\alpha \le \mathbf{m} - \mathbf{1}$) *satisfying the inequalities*

$$P_{1\boldsymbol\alpha}(\mathbf{x}) \le \rho_{\boldsymbol\alpha}(\mathbf{x}) \le P_{2\boldsymbol\alpha}(\mathbf{x}) \;\; \text{for} \;\; \mathbf{x} \in \Omega \;\; (\boldsymbol\alpha \le \mathbf{m} - \mathbf{1}), \tag{1.18}$$

the problem

$$u^{(\boldsymbol m)} = \sum_{\boldsymbol\alpha<\boldsymbol m} p_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)} + \sum_{\boldsymbol\alpha\le\mathbf{m-1}} \rho_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)}, \tag{1.19}$$

$$a_{ik}\, u^{((k-1)\mathbf{1}_\mathbf{i})}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{((k-1)\mathbf{1}_i)}(\boldsymbol\omega_i,\, \widehat{\mathbf{x}}_i) = 0$$

$$\text{for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \;\; (k = 1,\dots,m_i;\;\; i = 1,\dots,n) \tag{1.20}$$

has only the trivial solution.

Then problem (1.14), (1.2) *is strongly well–posed.*

Theorem 1.3. *Let the function* $f(\mathbf{x}, \mathbf{Z})$ *be continuously differentiable with respect to the phase variables, and let* u_0 *be a solution of problem* (1.1), (1.2). *Then problem* (1.1), (1.2) *is strongly* (u_0, r)*-well–posed for some* $r > 0$ *if and only if the linear homogeneous problem* (1.15), (1.16) *is well–posed, where*

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) = f_{\boldsymbol{\alpha}}(\mathbf{x}, \widetilde{\mathcal{D}}^{\boldsymbol{m}}[u_0(\mathbf{x})]) \quad (\boldsymbol{\alpha} < \boldsymbol{m}).$$

Remark 1.3. In Theorem 1.2 continuous differentiability of the function $f(\mathbf{x}, \mathbf{Z})$ with respect to the phase variables $\mathbf{Z}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorem more cumbersome. However, Lipschitz continuity of the function $f(\mathbf{x}, \mathbf{Z})$ with respect to the *principal* phase variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}})$ is essential and cannot be replaced by Hölder continuity with the exponent $\gamma \in (0, 1)$.

As an example, in the rectangle $\Omega = [0, \pi] \times [0, \pi]$ consider the two–dimensional problem

$$u^{(2,2)} = -\delta^{1-\gamma} \sin^{1-\gamma}(x_1) |u^{(0,2)}|^{\gamma} \operatorname{sgn}(u^{(0,2)}) - q(x_2) \sin(x_1), \tag{1.21}$$

$$u(0, x_2) = 0, \quad u(\pi, x_2) = 0, \qquad u(x_1, 0) = 0, \quad u(x_2, \pi) = 0. \tag{1.22}$$

where $\delta \geq 0$, $\gamma \in (0, 1)$.

If $\delta = 0$, the linear problem

$$u^{(2,2)} = -q(x_2) \sin(x_1),$$

$$u(0, x_2) = 0, \quad u(\pi, x_2) = 0, \quad u(x_1, 0) = 0, \quad u(x_1, \pi) = 0$$

is well posed, and its solution u is given by the formula

$$u(x_1,x_2) = -\int_0^\pi \int_0^\pi g(x_1,s_1)g(x_2,s_2)q(s_2)\sin(s_1)\,ds_1\,ds_2$$
$$= \sin x_1 \int_0^\pi g(x_2,s_2)q(s_2)\,ds_2,$$

where g is the Green's function of the problem

$$z'' = 0,$$
$$z(0) = 0, \quad z(\pi) = 0,$$

notice that

$$g(x,s) \le 0 \quad \text{for} \quad (x,s) \in \Omega,$$

and

$$\int_0^\pi g(x,s)\sin(s)\,ds = -\sin(x).$$

Let $\delta > 0$, and $q \in C([0,\pi])$ be such that

$$q(\frac{\pi}{4}) = -1,\ q(\frac{3\pi}{4}) = 1,\ q(\frac{\pi}{2}) = 0,\ \text{and}\ \begin{cases} q(x) < 0 & \text{for}\ \ x \in (0,\frac{\pi}{2}) \\ q(x) > 0 & \text{for}\ \ x \in (\frac{\pi}{2},\pi) \end{cases}. \tag{1.23}$$

Our goal is to show that problem (1.21), (1.22) has no classical solution. Indeed, let u be a solution of problem (1.21), (1.22). Set $w = u^{(0,2)}$. Then w is a solution of the problem

$$w^{(2,0)} = -\delta^{1-\gamma}\sin^{1-\gamma}(x_1)|w|^\gamma \mathrm{sgn}(w) - q(x_2)\sin(x_1), \tag{1.24}$$
$$w(0) = 0, \qquad\qquad w(\pi) = 0. \tag{1.25}$$

Consequently, w admits the representation

$$\begin{aligned} w(s_1, x_2) = &- \int_0^\pi g(x_1, s_1)\sin(s_1)\, ds_1\, q(x_2) \\ &- \delta^{1-\gamma} \int_0^\pi g(x_1, s_1) \sin^{1-\gamma}(s_1)\, |w|^\gamma \operatorname{sgn} w(s_1, x_2)\, ds_1 \\ &= \sin(x_1) q(x_2) - \delta^{1-\gamma} \int_0^\pi g(x_1, s_1) \sin^{1-\gamma}(s_1)\, |w|^\gamma \operatorname{sgn} w(s_1, x_2)\, ds_1. \end{aligned}$$

Set

$$v(x_1, x_2) = \frac{w(x_1, x_2)}{\sin(x_1)}.$$

Then we have

$$v(x_1, x_2) = q(x_2) - \frac{\delta^{1-\gamma}}{\sin(x_1)} \int_0^\pi g(x_1, s_1)\sin(s_1) |v(s_1, x_2)|^\gamma \operatorname{sgn} v(s_1, x_2)\, ds_1, \quad (1.26)$$

and

$$\begin{aligned} |v(x_1, x_2)| &\le q(x_2) + \frac{\delta^{1-\gamma}}{\sin(x_1)} \int_0^\pi |g(x_1, s_1)| \sin(s_1) |v(s_1, x_2)|^\gamma\, ds_1 \\ &= q(x_2) + \frac{\delta^{1-\gamma}}{\sin(x_1)} \int_0^\pi |g(x_1, s_1)| \sin(s_1)\, ds_1 \sup\{|v^*(s_1, x_2)| : s_1 \in (0, \pi)\}^\gamma \\ &= q(x_2) + \delta^{1-\gamma} \sup\{|v^*(s_1, x_2)| : s_1 \in (0, \pi)\}^\gamma, \end{aligned}$$

and, consequently,

$$\sup\{|v(s_1, x_2)| : s_1 \in (0, \pi)\} \le \frac{q(x_2) + \delta^{1-\gamma}}{1 - \delta^{1-\gamma}} \le 2(|q(x_2)| + 1)$$

for $\delta \in [0, 4^{\frac{1}{\gamma-1}}]$. Hence, in view of (1.23) and (1.26), we have

$$v\Big(x_1, \frac{3\pi}{4}\Big) = 1 - \frac{\delta^{1-\gamma}}{\sin(x_1)} \int\limits_0^\pi g(x_1, s_1)\sin(s_1)\Big|v\Big(s_1, \frac{3\pi}{4}\Big)\Big|^\gamma \operatorname{sgn} v\Big(s_1, \frac{3\pi}{4}\Big)\, ds_1$$

$$\geq 1 - \frac{\delta^{1-\gamma}}{\sin(x_1)} \int\limits_0^\pi |g(x_1, s_1)|\sin(s_1)\, ds_1\, 2(1+1)$$

$$= 1 - 4\delta^{1-\gamma} > 0 \quad \text{for} \quad x_1 \in (0, \pi),$$

and

$$v\Big(x_1, \frac{3\pi}{4}\Big) = 1 - \frac{\delta^{1-\gamma}}{\sin(x_1)} \int\limits_0^\pi g(x_1, s_1)\sin(s_1) v^\gamma\Big(s_1, \frac{3\pi}{4}\Big)\, ds_1$$

$$= 1 + \frac{\delta^{1-\gamma}}{\sin(x_1)} \int\limits_0^\pi |g(x_1, s_1)|\sin(s_1) v^\gamma\Big(s_1, \frac{3\pi}{4}\Big)\, ds_1 \geq 1 > \delta \;\text{ for }\; x_1 \in (0, \pi). \quad (1.27)$$

Let us prove that if $\delta \in (0, 4^{\frac{1}{\gamma-1}}]$, then

$$v(x_1, x_2) > \delta \quad \text{for} \quad x_1 \in (0, \pi), \quad x_2 \in \Big(\frac{\pi}{2}, \pi\Big). \quad (1.28)$$

Assume the contrary. Then, in view of (1.27), there exists $x_2^* \in (\frac{\pi}{2}, \pi)$ and $x_1^* \in (0, \pi)$ such that

$$v(x_1, x_2^*) \geq \delta \quad \text{for} \quad x_1 \in (0, \pi), \qquad \text{and} \qquad v(x_1^*, x_2^*) = \delta. \quad (1.29)$$

(1.26) and (1.29) imply

$$v(x_1, x_2^*) = q(x_2^*) - \frac{\delta^{1-\gamma}}{\sin(x_1)} \int\limits_0^\pi g(x_1, s_1)\sin(s_1) v^\gamma(s_1, x_2)\, ds_1$$

$$\geq q(x_2^*) - \frac{\delta^{1-\gamma}}{\sin(x_1)}\, \delta^\gamma \int\limits_0^\pi g(x_1, s_1)\sin(s_1)\, ds_1 = q(x_2^*) + \delta > \delta \quad \text{for} \quad x_1 \in (0, \pi).$$

The obtained contradiction prove the validity of (1.28).

Similarly one can show that

$$v(x_1, x_2) < -\delta \quad \text{for} \quad x_1 \in (0, \pi), \quad x_2 \in \Big(0, \frac{\pi}{2}\Big). \quad (1.30)$$

(1.28) and (1.30) immediately imply the solution w of problem $(1.24), (1.25)$ is discontinuous along the line $x_2 = \frac{\pi}{2}$. Consequently, problem $(1.21), (1.22)$ has no classical solution for $\delta \in \left(0, 4^{\frac{1}{\gamma-1}}\right]$, since $u^{(0,2)}(x_1, x_2)$ is discontinuous along the line $x_2 = \frac{\pi}{2}$.

This is the result of the fact that the righthand side of equation (1.21) is not Lipschitz continuous with respect to the principal phase variables, but instead is a Hölder continuous function with the exponent $\gamma \in (0, 1)$.

Remark 1.4. The aforementioned example explains why in an *admissible perturbation* $\left(\widetilde{f}; \widetilde{\varphi}_{1\,1}, \ldots, \widetilde{\varphi}_{1\,m_1}, \ldots, \widetilde{\varphi}_{n\,1}, \ldots, \widetilde{\varphi}_{n\,m_n}\right)$ the function $\widetilde{f} \in C(\Omega \times \mathbb{R}^{\mathbf{m+1}})$ is required to be locally Lipschitz continuous with respect to the *principal* phase variables .

Also, the example demonstrates that just the inequality (1.6), without inequality (1.5), does not guarantee even solvability of a perturbed problem.

1.3. **Solvability and Locally Strong Well–Posedness.** Consider the equations

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]) \tag{1.31}$$

and

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u])u^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]). \tag{1.32}$$

Theorem 1.4. *Let the function f satisfy all of the conditions of Theorem 1.2, and let $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ be such that*

$$\lim_{\|\mathbf{Z}\|\to+\infty} \frac{|q(\mathbf{x}, \mathbf{Z})|}{\|\mathbf{Z}\|} = 0 \quad \textit{uniformly on} \quad \Omega. \tag{1.33}$$

Then problem $(1.31), (1.2)$ is solvable.

Theorem 1.5. *Let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ satisfying condition (A_2) of Theorem 1.2 such that*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m-1}} \quad (\boldsymbol{\alpha} < \mathbf{m}). \tag{1.34}$$

Furthermore, let the function $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy* (1.33). *Then problem* (1.32), (1.2) *is solvable.*

Theorem 1.6. *Let* $p_{\boldsymbol{\alpha}} \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ $(\boldsymbol{\alpha} < \mathbf{m})$, *the function* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *be Lipschitz continuous with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ *satisfying condition* (A_2) *of Theorem 1.2 such that*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{1.35}$$

and

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) + \frac{\partial q(\mathbf{x}, \mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}} \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} \leq \mathbf{m} - \mathbf{1}). \tag{1.36}$$

Then problem (1.32), (1.2) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ $(\boldsymbol{\alpha} < \mathbf{m})$ *are locally Lipschitz continuous with respect to the phase variables, then there exists* $\varepsilon > 0$ *such that if*

$$\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} \leq \varepsilon, \qquad |q(\mathbf{x}, \mathbf{Z})| \leq \varepsilon \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}},$$

then every solution of problem (1.32), (1.2) *is strongly isolated.*

1.4. **Dirichlet Type Boundary Value Problems for Equations of Even Order.** For the equations

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\, u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]), \tag{1.37}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\, u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q(\mathbf{x}, u), \tag{1.38}$$

and

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} \Big(p_{\boldsymbol{\alpha}}\big(\mathbf{x}, \mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\, u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]) \tag{1.39}$$

consider the boundary conditions

$$\begin{aligned} & u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i},\widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k=1,\dots,m_i;\ i=1,\dots,n), \\ & u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k=1,\dots,m_i;\ i=1,\dots,n), \end{aligned} \tag{1.40}$$

and

$$\begin{aligned} & u^{(2\mathbf{m}^{i-1}+2(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k=1,\dots,m_i;\ i=1,\dots,n), \\ & u^{(2\mathbf{m}^{i-1}+2(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k=1,\dots,m_i;\ i=1,\dots,n). \end{aligned} \tag{1.41}$$

Theorem 1.7. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\Omega\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\,\boldsymbol{\beta}\le\boldsymbol{m})$, *and let the quadratic form be nonnegative defined:*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},\mathbf{Z}) z_{\boldsymbol{\alpha}} z_{\boldsymbol{\beta}} \ge 0 \quad \textit{for} \quad \mathbf{x}\in\Omega. \tag{1.42}$$

Furthermore, let the function $q \in C(\Omega\times\mathbb{R}^{\mathbf{m}})$ *satisfy the equality*

$$\limsup_{\|\mathbf{Z}\|\to+\infty} \frac{(-1)^{\|\boldsymbol{m}\|} q(\mathbf{x},\mathbf{Z})\ \operatorname{sgn} z_{\mathbf{0}}}{\|\mathbf{Z}\|} = 0 \quad \text{uniformly on} \quad \Omega. \tag{1.43}$$

Then problem (1.37), (1.40) *is solvable.*

Corollary 1.2. *Let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}(\Omega\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha}\le\boldsymbol{m})$ *be such that the inequalities*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z}) \ge 0 \quad (\boldsymbol{\alpha}\le\mathbf{m}) \tag{1.44}$$

hold. Furthermore, let the function $q \in C(\Omega\times\mathbb{R}^{\mathbf{m}})$ *satisfy equality* (1.43)*. Then problem* (1.39), (1.40) *is solvable.*

Theorem 1.8. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\Omega\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\,\boldsymbol{\beta}\le\boldsymbol{m})$, *and let the quadratic form* (1.42) *be nonnegative defined. Furthermore, let the function* $q\in C(\Omega\times\mathbb{R})$ *satisfy the inequality*

$$(-1)^{\|\boldsymbol{m}\|-1}\big(q(\mathbf{x},z_1)-q(\mathbf{x},z_2)\big)\ \operatorname{sgn}(z_1-z_2)\ge 0. \tag{1.45}$$

Then problem (1.38), (1.40) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|+1}(\Omega\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\,\boldsymbol{\beta}\le\boldsymbol{m})$, *then there exists* $\varepsilon>0$ *such that if*

$$|q(\mathbf{x},z)|\le\varepsilon \quad \textit{for} \quad (\mathbf{x},z)\in\Omega\times\mathbb{R},$$

then every solution of problem (1.38), (1.40) *is strongly isolated.*

Theorem 1.9. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *and let the quadratic form* (1.42) *be nonnegative defined. Furthermore, let the function* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy inequality* (1.43). *Then problem* (1.37), (1.41) *is solvable.*

Corollary 1.3. *Let* $p_{\boldsymbol{\alpha}} \in C_0^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$, *and let the inequalities* (1.44) *hold. Furthermore, let the function* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy inequality* (1.43). *Then problem* (1.39), (1.41) *is solvable.*

Theorem 1.10. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *and let the quadratic form* (1.42) *be nonnegative defined. Furthermore, let the function* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy inequality* (1.45). *Then problem* (1.38), (1.41) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|+1}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *then there exists* $\varepsilon > 0$ *such that if*

$$|q(\mathbf{x}, z)| \leq \varepsilon \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R},$$

then every solution of problem (1.38), (1.41) *is strongly isolated.*

Remark 1.5. In Theorems 1.7–1.10 and Corollaries 1.2 and 1.3 the functions $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ and $q(\mathbf{x}, \mathbf{Z})$ satisfy one sided inequalities (1.42), (1.43) and (1.44). Consequently, Theorems 1.7–1.10 and Corollaries 1.2 and 1.3 cover the case, where the functions $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ and $q(\mathbf{x}, \mathbf{Z})$ have arbitrary growth order with respect to certain phase variables.

Consider the particular cases of equation (1.39)

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, u^{(\mathbf{2}\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]) \tag{1.46}$$

and

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, u^{(\mathbf{2}\boldsymbol{\alpha})} + q(\mathbf{x}, u). \tag{1.47}$$

Theorem 1.11. *Let* $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$, *and let the inequalities hold:*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \geq 0 \quad \textit{for} \quad \widehat{\mathbf{x}}_{\boldsymbol{\alpha}} \in \widehat{\Omega}_{\boldsymbol{\alpha}} \quad (\boldsymbol{\alpha} \leq \mathbf{m}). \tag{1.48}$$

Furthermore, let the function $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy equality* (1.43). *Then problem* (1.46), (1.41) *is solvable.*

Theorem 1.12. *Let* $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$, *and let inequalities* (1.48) *hold. Furthermore, let the function* $q \in C(\Omega \times \mathbb{R})$ *satisfy inequality* (1.45). *Then problem* (1.47), (1.41) *is strongly well–posed.*

1.5. **Periodic Type Boundary Value Problems.** For the equations (1.37), (1.38) and (1.39) consider the following boundary conditions of periodic type

$$u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) - c_{ik} u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, 2m_i;\ i = 1, \dots, n), \tag{1.49}$$

where $c_{ik} \neq 0$ $(k = 1, \dots, 2m_i;\ i = 1, \dots, n)$, and $\varphi_{ik} \in C^{\mathbf{2\widehat{m}}_i}(\Omega_i)$ $(k = 1, \dots, 2m_i;$ $i = 1, \dots, n)$.

If $c_{ik} = 1$, then (1.49) are nonhomogeneous periodic conditions. On the other hand, problem (1.8) with periodic boundary conditions has nontrivial solutions for every $i \in \{1, \dots, n-1\}$. Consequently, by Theorem 1.1, problem (1.1), (1.49) with $c_{ik} = 1$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ is ***not*** well–posed in the sense of Definition 1.1. This is the reason why the periodic problems are studied separately in Chapter II.

Theorem 1.13. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{\boldsymbol{\beta}, \|\boldsymbol{\beta}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha}, \boldsymbol{\beta} \leq \boldsymbol{m})$,

$$c_{ik} \neq 1 \quad (k = 1, \dots, 2m_i;\ i = 1, \dots, n), \tag{1.50}$$

$$c_{ik}\, c_{i\, 2m_i+1-k} = 1 \quad (k = 1, \dots, 2m_i;\ i = 1, \dots, n), \tag{1.51}$$

and let the quadratic form (1.42) *be nonnegative defined. Then problem* (1.37), (1.49) *is solvable.*

Corollary 1.4. *Let* (1.50), (1.51) *hold, and let the functions* $p_{\boldsymbol{\alpha}} \in C_0^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ *and* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy inequalities* (1.44) *and equality* (1.43). *Then problem* (1.39), (1.49) *is solvable.*

Theorem 1.14. *Let* (1.50), (1.51) *hold, let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *and let the quadratic form* (1.42) *be nonnegative defined. Furthermore, let the function* $q \in C(\Omega \times \mathbb{R})$ *satisfy inequality* (1.45). *Then problem* (1.38), (1.49) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|+1}(\Omega \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *then there exists* $\varepsilon > 0$ *such that if*

$$|q(\mathbf{x}, z)| \leq \varepsilon \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R},$$

then every solution of problem (1.38), (1.49) *is strongly isolated.*

Theorem 1.15. *Let* (1.50), (1.51) *hold, and let the functions* $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ *and* $q \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ *satisfy inequalities* (1.48) *and equality* (1.43). *Then problem* (1.46), (1.49) *is solvable.*

Theorem 1.16. *Let* (1.50), (1.51) *hold, and let the functions* $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ *and* $q \in C(\Omega \times \mathbb{R})$ *satisfy inequalities* (1.48) *and* (1.45). *Then problem* (1.47), (1.49) *is strongly well–posed.*

1.6. **Equations with Hölder Continuous Righthand Side.** In Remarks 1.3 and 1.4 it was shown that, generally speaking, problem (1.1), (1.2) may not have a classical solution, if the function $f(\mathbf{x}, \mathbf{Z})$ is not Lipschitz continuous with respect to the principal phase variables, but instead is Hölder continuous with the exponent $\gamma \in (0, 1)$.

In this subsection we show that Lipschitz continuity is not the necessary condition for the existence of a classical solution. More precisely, we study the case where the righthand side of the equation is Hölder continuous with respect to certain principal variable.

For the equations

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})} + p_k\big(\mathbf{x}, u^{(2\mathbf{m_k})}\big) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]) \tag{1.52}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})} + p_k\big(\mathbf{x}, u^{(2\mathbf{m_k})}\big) + q(\mathbf{x}), \tag{1.53}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\ u^{(2\boldsymbol{\alpha})} + p_{\boldsymbol{\sigma}}\big(\mathbf{x}, u^{(2\mathbf{m}_{\boldsymbol{\sigma}})}\big) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]) \tag{1.54}$$

and

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\ u^{(2\boldsymbol{\alpha})} + p_{\boldsymbol{\sigma}}\big(\mathbf{x}, u^{(2\mathbf{m}_{\boldsymbol{\sigma}})}\big) + q(\mathbf{x}) \tag{1.55}$$

consider the boundary conditions (1.40).

Theorem 1.17. *Let $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$ and $q \in C(\widehat{\Omega}_{\boldsymbol{\alpha}} \times \mathbb{R})$ satisfy conditions* (1.33) *and* (1.48)*. Furthermore, let there exist $M > 0$ and $\gamma \in (0,1)$ such that the function $p_i \in C(\Omega \times \mathbb{R})$ satisfies the inequalities*

$$|p_k(\mathbf{x}, z)| \le M(1 + |z|^{\gamma}) \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R}, \tag{1.56}$$

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{m}_{\mathbf{k}}\|-1}\big(p_k(\mathbf{x}, z_1) - p_k(\mathbf{x}, z_2)\big)\ \mathrm{sgn}(z_1 - z_2) \ge 0$$
$$\textit{for} \quad (\mathbf{x}, z_i) \in \Omega \times \mathbb{R} \quad (i = 1, 2). \tag{1.57}$$

Then problem (1.52), (1.40) *is solvable.*

Theorem 1.18. *Let the functions $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$ and $p_k \in C(\Omega \times \mathbb{R})$ satisfy all of the conditions of Theorem 1.17 for some positive constants M and $\gamma \in (0,1)$. Then for an arbitrary $q \in C(\Omega)$ problem* (1.53), (1.40) *is uniquely solvable and strongly well–posed.*

Theorem 1.19. *Let there exist $M > 0$ and $\gamma \in (0,1)$ such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}} \geq 0 \quad (\boldsymbol{\alpha} \leq \mathbf{m}), \tag{1.58}$$

$$|p_{\boldsymbol{\sigma}}(\mathbf{x}, z)| \leq M(1 + |z|\gamma) \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R}, \tag{1.59}$$

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{m}_{\boldsymbol{\sigma}}\|-1}\big(p_{\boldsymbol{\sigma}}(\mathbf{x}, z_1) - p_{\boldsymbol{\sigma}}(\mathbf{x}, z_2)\big)\ \mathrm{sgn}(z_1 - z_2) \geq 0$$
$$\textit{for} \quad (\mathbf{x}, z_i) \in \Omega \times \mathbb{R} \quad (i = 1, 2). \tag{1.60}$$

Then problem $(1.54), (1.40)$ *is solvable.*

Theorem 1.20. *Let the functions $p_{\boldsymbol{\alpha}} \in \mathbb{R}$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ and $p_{\boldsymbol{\sigma}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R})$ satisfy all of the conditions of Theorem 1.19 for some positive constants M and $\gamma \in (0,1)$. Then for an arbitrary $q \in C(\Omega)$ problem* $(1.55), (1.40)$ *is uniquely solvable and strongly well–posed.*

Remark 1.6. Consider the following equations with the Holder continuous righthand sides:

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\leq\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1}\, |p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})\ u^{(2\boldsymbol{\alpha})}$$
$$+ (-1)^{\|\widehat{\mathbf{m}}_{\mathbf{1}}\|-1}\big|u^{(2\mathbf{m}_{\mathbf{1}})}\big|^{\gamma}\, \mathrm{sgn}\, u^{(2\mathbf{m}_{\mathrm{i}})} + \ln\big(1 + u^2 + |u^{(2\mathbf{m}-\mathbf{1})}|^4\big), \tag{1.61}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\leq\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1}\, |p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})\ u^{(2\boldsymbol{\alpha})}$$
$$+ (-1)^{\|\widehat{\mathbf{m}}_{\mathbf{1}}\|-1}\big|u^{(2\mathbf{m}_{\mathbf{1}})}\big|^{\gamma}\, \mathrm{sgn}\, u^{(2\mathbf{m}_{\mathbf{1}})} + q(\mathbf{x}), \tag{1.62}$$

and

$$u^{(2\mathbf{m})} = (-1)^{\|\widehat{\mathbf{m}}_{\sigma}\|-1}\big|u^{(2\mathbf{m}_{\sigma})}\big|^{\gamma}\, \mathrm{sgn}\, u^{(2\mathbf{m}_{\mathbf{1}})} + q(\mathbf{x}). \tag{1.63}$$

Here $p_{\boldsymbol{\alpha}} \in C(\widehat{\Omega}_{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ and $q \in C(\Omega)$ are arbitrary functions.

By Theorem 1.17, problem $(1.61), (1.40)$ is solvable.

By Theorem 1.18, problem $(1.62), (1.40)$ is uniquely solvable and well–posed.

By Theorem 1.20, problem $(1.63), (1.40)$ is uniquely solvable and well–posed.

2. Auxiliary Statements

2.1. **Some facts from the theory of ODE.** Consider the boundary value problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)} + q(t), \tag{2.1}$$

$$h_k(z) = c_k \quad (k = 1, \dots, m), \tag{2.2}$$

and its corresponding homogeneous problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)}, \tag{2.1$_0$}$$

$$h_k(z) = 0 \quad (k = 1, \dots, m), \tag{2.2$_0$}$$

where $p_k \in C([0,\omega])$ $(k = 0, \dots, m-1)$, $q \in C([0,\omega])$, $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$, and $h_k : C^{m-1}([0,\omega]) \to \mathbb{R}$ $(k = 1, \dots, m)$ are bounded linear functionals.

Lemma 2.1. *The following facts are equivalent:*
(A_1) *problem* (2.1), (2.2) *is solvable for arbitrary* $q \in C(\Omega)$ *and* $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$*;*
(A_2) *problem* (2.1), (2.2_0) *is solvable for arbitrary* $q \in C([0,\omega])$*;*
(A_3) *problem* (2.1_0), (2.2) *is solvable for arbitrary* $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$*;*
(A_4) *problem* (2.1_0), (2.2_0) *has only the trivial solution.*

Lemma 2.1 is a well–known fact from the theory of ordinary differential equation (e.g. see Theorem 1.1 from [19]). If problem (2.1_0), (2.2_0) has only the trivial solution then a solution of problem (2.1),(2.2) admits the representation

$$z(t) = \Gamma(c_1, \dots, c_m)(t) + \mathcal{G}(q)(t),$$

where $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ and $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ are bounded linear operators. Moreover, the operator $\mathcal{G}$ admits the representation

$$\mathcal{G}(q)(t) = \int_0^\omega g(t,\tau) q(\tau)\, d\tau,$$

where $g : [0,\omega] \times [0,\omega] \to \mathbb{R}$ is called the **Green's function of problem** $(2.1_0), (2.2_0)$ (for more about Green's functions see [19]).

Definition 2.1. $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ is called the **Green's operator** of problem $(2.1_0), (2.2_0)$.

Definition 2.2. $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ is called the **Green's boundary operator** of problem $(2.1_0), (2.2_0)$.

2.2. **A priori estimates for solutions of linear problems.** Consider the linear problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{2.3}$$

$$a_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i}) \quad \text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (k = 1,\dots,m_i;\ i = 1,\dots,n). \tag{2.4}$$

Along with the problem (2.3), (2.4), for each $\boldsymbol{\sigma} \in \Xi$, in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x})v^{(\boldsymbol{\alpha})}, \tag{$2.3_{\boldsymbol{\sigma}}$}$$

$$a_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad \text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (k = 1,\dots,m_i;\ i \in \operatorname{supp}\boldsymbol{\sigma}). \tag{$2.4_{\boldsymbol{\sigma}}$}$$

Problem $(2.3_{\boldsymbol{\sigma}})$,$(2.4_{\boldsymbol{\sigma}})$ is called a $\boldsymbol{\sigma}$*–associate problem*, or an associate problem of level $\|\boldsymbol{\sigma}\|$.

The following result was proved in [37].

Lemma 2.2. *Let problem* (1.9) *have only the trivial solution for every* $i \in \{1, 2, \dots, n-1\}$*, let* $p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} \le \mathbf{m})$*, and let for each* $\boldsymbol{\sigma} \in \Xi$ $\boldsymbol{\sigma}$*–associated problem* $(2.3_{\boldsymbol{\sigma}})$,$(2.4_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$*. Then* *there exists a positive constant M such that an arbitrary solution u of problem* (2.3), (2.4) *admits the estimate*

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \leq M\, \|u\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)}. \tag{2.5}$$

Moreover, problem (2.3),(2.4) *is well–posed if and only if the homogeneous problem*

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{2.3$_0$}$$

$$a_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i,\, \widehat{\mathbf{x}}_\mathbf{i}) = 0$$
$$\textit{for } \ \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (k = 1, \dots, m_i;\ i = 1, \dots, n) \tag{2.4$_0$}$$

has only the trivial solution.

Lemma 2.3. *Let $p_l \in L^\infty(\Omega)$ $(l = 1, 2, \dots)$, $p \in L^\infty(\Omega)$, and let there exist $M > 0$ such that*

$$\|p_l\|_{L^\infty(\Omega)} \leq M \qquad (l = 1, 2, \dots),$$

$$\lim_{l\to\infty} \int_0^{x_1} \cdots \int_0^{x_n} \big(p_l(s_1, \dots, s_n) - p(s_1, \dots, s_n)\big)\, ds_n \dots ds_1 = 0 \quad \textit{uniformly on} \quad \Omega.$$

Then

$$\lim_{l\to\infty} \int_0^{x_1} \cdots \int_0^{x_n} \big(p_l(s_1, \dots, s_n) - p(s_1, \dots, s_n)\big) z(s_1, \dots, s_n)\, ds_n \dots ds_1 = 0 \quad \textit{uniformly} \quad \textit{on} \quad \Omega \tag{2.6}$$

for arbitrary $z \in C(\Omega)$.

Proof. For $z \in C^{\mathbf{1}}(\Omega)$, (2.6) can be easily proved by utilizing integration by parts multiple times.

Now let $z \in C(\Omega)$, and let $z_k \in C^{\mathbf{1}}(\Omega)$ $(k = 1, 2, \dots)$ be such that

$$\lim_{k\to\infty} \|z - z_k\|_{C(\Omega)} = 0 \tag{2.7}$$

Then

$$\limsup_{l\to\infty}\left|\int_0^{x_1}\cdots\int_0^{x_n}\big(p_l(s_1,\dots,s_n)-p(s_1,\dots,s_n)\big)z(s_1,\dots,s_n)\,ds_n\dots ds_1\right|$$
$$=\limsup_{l\to\infty}\left|\int_0^{x_1}\cdots\int_0^{x_n}\big(p_l(s_1,\dots,s_n)-p(s_1,\dots,s_n)\big)z_k(s_1,\dots,s_n)\,ds_n\dots ds_1\right|$$
$$+\limsup_{l\to\infty}\left|\int_0^{x_1}\cdots\int_0^{x_n}\big(p_l(s_1,\dots,s_n)-p(s_1,\dots,s_n)\big)\right.$$
$$\left.\times\big(z(s_1,\dots,s_n)-\widetilde{z}(s_1,\dots,s_n)\big)\,ds_n\dots ds_1\right|$$
$$\le 2M|\Omega|\|z-z_k\|_{C(\Omega)}. \quad (2.8)$$

(2.7) and (2.8) immediately imply (2.6). □

Lemma 2.4. *let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x})\in C(\Omega)$ $(\boldsymbol{\alpha}<\mathbf{m};\ i=1,2)$ *such that:*

(A_1) $$P_{1\boldsymbol{\alpha}}(\mathbf{x})\le p_{\boldsymbol{\alpha}}(\mathbf{x})\le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad \mathbf{x}\in\Omega \quad (\boldsymbol{\alpha}<\mathbf{m}); \quad (2.9)$$

(A_2) *For every* $\boldsymbol{\sigma}\in\boldsymbol{\Xi}\cup\{\mathbf{1}\}$, [1] $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}\in\Omega_{\widehat{\boldsymbol{\sigma}}}$ *and arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}}\in L^\infty(\Omega_{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}})$ *satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}})\le\rho_{\boldsymbol{\alpha}}(\mathbf{y})\le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \quad \textit{for} \quad \mathbf{y}\in\Omega_{\boldsymbol{\sigma}} \quad (\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}), \quad (2.10)$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}}\rho_{\boldsymbol{\alpha}}(\mathbf{y})v^{(\boldsymbol{\alpha})}, \quad (2.11)$$

$$a_{ik}\,v^{((k-1)\mathbf{1}_i)}(\mathbf{0}_\mathbf{i},\,\widehat{\mathbf{y}}_\mathbf{i})+b_{ik}\,v^{((k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\,\widehat{\mathbf{y}}_\mathbf{i})=0 \ \ (k=1,\dots,m_i;\ i\in\operatorname{supp}\boldsymbol{\sigma}). \quad (2.12)$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}}-\mathbf{1}_{\boldsymbol{\sigma}}}(\Omega_{\boldsymbol{\sigma}})$*. Then problem* (2.1), (2.2) *is well–posed. Moreover, there exists a positive constant* M_0 *depending only on* $P_{i\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha}<$

[1]Notice that if $\boldsymbol{\sigma}=\mathbf{1}$, then problem $(2.3_{\boldsymbol{\sigma}}), (2.4_{\boldsymbol{\sigma}})$ is problem $(2.3_0), (2.4_0)$.

$\mathbf{m}$; $i = 1, 2)$, a_{ik}, b_{ik} $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ *and* ω_i $(i = 1, \dots, n)$ *such that a solution* u *of problem* (2.3), (2.4) *admits the estimate*

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M_0\Big(\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} + \|q\|_{C(\Omega)}\Big). \tag{2.13}$$

Proof. The lemma is true for the case $n = 1$, i.e. for ordinary differential equations (see Theorem 1.2 and Corollary 3.6 from [19]).

Let $n \ge 2$, and let us assume that Lemma 2.4 is true for $n-1$ dimensional problems, but is false for n–dimensional problems.

Then, by Banach–Steinhaus theorem, there exist $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})$ $(k = 1, \dots, m_i;$ $i = 1, \dots, n)$, $q \in C(\Omega)$, and $\widetilde{p}_{l\,\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m})$ $(l = 1, 2, \dots)$ satisfying the inequalities

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le \widetilde{p}_{l\,\boldsymbol{\alpha}}(\mathbf{x}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \ \ (\boldsymbol{\alpha} < \mathbf{m};\ \ l = 1, 2, \dots), \tag{2.14}$$

such that

$$\|u_l\|_{C^{\mathbf{m}}(\Omega)} = \eta_l, \qquad \lim_{l \to \infty} \eta_l = +\infty, \tag{2.15}$$

where u_l is a solution of the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{l\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{2.3$_l$}$$

satisfying conditions (2.4).

Due to inequalities (2.14), without loss of generality, one may assume that there exist measurable functions $\widetilde{p}_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} < \mathbf{m})$ satisfying inequalities (2.9) such that

$$\lim_{l \to \infty} \int_0^{x_1} \cdots \int_0^{x_n} \big(p_{l\boldsymbol{\alpha}}(s_1, \dots, s_n) - p_{\boldsymbol{\alpha}}(s_1, \dots, s_n)\big)\, ds_n \dots ds_1$$
$$= 0 \quad \text{uniformly on} \quad \Omega. \tag{2.16}$$

Set

$$\widetilde{u}_l(\mathbf{x}) = \frac{u_l(\mathbf{x})}{\eta_l}. \tag{2.17}$$

Then, by our assumption about $n-1$–dimensional problems and the estimate (2.5) from Lemma 2.2, we have

$$\|\widetilde{u}_l\|_{C^{\mathbf{m}}(\Omega)} = 1, \tag{2.18}$$

$$\|\widetilde{u}_l\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\|\widetilde{u}_l\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} + \frac{1}{\eta_l}\sum_{i=1}^{n}\sum_{k=1}^{m_i}\|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} + \frac{1}{\eta_l}\|q\|_{C(\Omega)}\Big), \tag{2.19}$$

where M is a positive constant independent of φ_{ik}, q and l.

In view of (2.18) and (2.19), by Arzela–Ascoli lemma, without loss of generality, one may assume that there exists $u_0 \in C^{\mathbf{m}}(\Omega)$ such that

$$\lim_{l\to+\infty}\|\widetilde{u}_l - u_0\|_{C^{\mathbf{m}}(\Omega)} = 0, \quad \text{and} \quad \|u_0\|_{C^{\mathbf{m}}(\Omega)} = 1. \tag{2.20}$$

Taking into account (2.16), (2.20) and Lemma 2.3, we conclude that u_0 is a nontrivial solution of problem (2.3_0),(2.4_0). The obtained contradiction completes the proof of the lemma. □

3. Proofs of the Main Results

Proof of Theorem 1.1. Let $\varphi_{ik}(\widehat{\mathbf{x}}_i) \equiv 0$ $(k = 1, \ldots, m_i;\ i = 2, \ldots, n)$, and let

$$\varphi_{1k}(\widehat{\mathbf{x}}_1) = c_k \varphi(x_2) \quad (k = 1, \ldots, m_1), \tag{3.1}$$

where $c_1, \ldots, c_{m_1}$ are arbitrary real numbers and

$$a_{21}\, \varphi(0) + b_{21}\, \varphi(\omega_2) = 1. \tag{3.2}$$

Let u be an arbitrary solution of problem (1.1),(1.2). Set

$$z(\widehat{\mathbf{x}}_2) = a_{21}\, u(\mathbf{0_2}, \widehat{\mathbf{x}}_2) + b_{21}\, u(\boldsymbol{\omega_2}, \widehat{\mathbf{x}}_2).$$

Then z is a solution of the problem

$$z^{(m_1)} = 0, \tag{3.3}$$

$$a_{1k}\, z^{(k-1)}(0) + b_{1k}\, z^{(k-1)}(\omega_1) = c_k \quad (k = 1, \ldots, m_1). \tag{3.4}$$

Consequently, problem (3.3), (3.4) is solvable for arbitrary boundary values c_1, $\ldots$, c_{m_1}. By Lemma 2.1, this is equaivalent to the fact that the homogeneous problem

$$z^{(m_1)} = 0, \tag{3.3_0}$$

$$a_{1k}\, z^{(k-1)}(0) + b_{1k}\, z^{(k-1)}(\omega_1) = 0 \quad (k = 1, \ldots, m_1). \tag{3.4_0}$$

has only the trivial solution.

In order to complete the proof of the theorem one needs to consider

$$z(\widehat{\mathbf{x}}_{j+1}) = a_{j+1\,1}\, u^{(\mathbf{m}^j)}(\mathbf{0_{j+1}}, \widehat{\mathbf{x}}_{j+1}) + b_{j+1\,1}\, u^{(\mathbf{m}^j)}(\boldsymbol{\omega}_{\mathbf{j+1}}, \widehat{\mathbf{x}}_{j+1}),$$

and choose $\varphi_{ik}(\widehat{\mathbf{x}}_i) \equiv 0$ $(k = 1, \ldots, m_i;\ i \neq j)$,

$$\varphi_{jk}(\widehat{\mathbf{x}}_j) = c_k \varphi(x_{j+1}) \quad (k = 1, \ldots, m_j),$$

where $c_1, \ldots, c_{m_j}$ are arbitrary real numbers and

$$a_{j+1\,1}\, \varphi(0) + b_{j+1\,1}\, \varphi(\omega_{j+1}) = 1. \quad \square$$

Proof of Theorem 1.2. Consider the equation

$$u^{(\mathbf{m})} = (1-\lambda) \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{1,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \lambda f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) \tag{3.5}$$

where $\lambda \in [0,1]$. Let u be a solution of problem $(3.5), (1.2)$ for some $\lambda_0 \in [\lambda_0, 1)$. Then, due to the continuous differentiability of the function f with respect to the phase variables, u is a solution of the linearized equation

$$u^{(\mathbf{m})} = (1-\lambda) \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{1,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \lambda \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}[u](\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{3.6}$$

where

$$p_{\boldsymbol{\alpha}}[u](\mathbf{x}) = \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}, t\, \widetilde{\mathcal{D}}^{\mathbf{m}}[u(\mathbf{x})])\, dt \quad (\boldsymbol{\alpha} < \mathbf{m}), \quad q(\mathbf{x}) = f(\mathbf{x}, \mathbf{0}).$$

In view of conditions (A_1) and (A_2), by Lemma 2.4, u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M_0 \Big(\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)} + \|q\|_{C(\Omega)} \Big), \tag{3.7}$$

where the constant $M_0 > 0$ depends on the functions $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\Omega)$ ($\boldsymbol{\alpha} < \mathbf{m}$; $i = 1, 2$) only, and is independent of $\lambda \in [0,1]$ and q. Hence, in view of condition (A_1) and estimate (3.7), it follows that solvability of problem $(3.5), (1.2)$ for some $\lambda_0 \in [0,1)$ implies strong well–posedness of problem $(3.5), (1.2)$ for $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0) \cap [0,1]$, where

$$\varepsilon = \frac{1}{2M_0} \Big(1 + \sum_{\boldsymbol{\alpha}<\mathbf{m}} \big(\|p_{1,\boldsymbol{\alpha}}\|_{C(\Omega)} + \|p_{2,\boldsymbol{\alpha}}\|_{C(\Omega)} \big) \Big)^{-1}.$$

But this implies solvability and strong well posedness of problem $(3.5), (1.2)$ for every $\lambda \in [0,1]$, in particular $\lambda = 1$, since problem $(3.5), (1.2)$ is well–posed for $\lambda = 0$. $\square$

Corollary 1.1 immediately follows from Theorem 1.2.

Proof of Theorem 1.3. Let u_0 be a solution of problem $(1.1), (1.2)$. Assume first that problem $(1.1), (1.2)$ is (u_0, r)–well-posed for some $r > 0$. Consider the equation

$$u^{(\mathbf{m})} = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) + q(\mathbf{x}) + \Bigg(\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)$$
$$+ f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)\Bigg)\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big), \quad (3.8)$$

where $\Xi_\delta : [0, +\infty) \to [0, +\infty)$ is a non–increasing smooth function such that

$$\Xi_r(t) = \begin{cases} 1 & \text{for} \quad t \in [0, r] \\ \frac{r+1}{t} & \text{for} \quad t \in (r+1, +\infty) \end{cases} \quad (3.9)$$

and $\|q\|_{C(\Omega)} < \delta$.

In view of continuous differentiability of f, if $u \in \widetilde{\mathbf{B}}^{\mathbf{m}}(u_0; \delta)$, then

$$\Big|\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)\Big| = o(\delta) \quad (3.10)$$

as $\delta \to 0$.

Therefore, in view of strong (u_0, r)–well-posedness of problem (1.1), (1.2), every solution of problem (3.8), (1.4) admits the estimate

$$\|u - u_0\|_{\widetilde{C}^{m},n(\Omega)} \le M(o(\delta) + \|q\|_{C(\Omega)}).$$

Choosing $\delta > 0$ sufficiently small, we achieve that $\|u - u_0\|_{\widetilde{C}^{\mathbf{m}}(\Omega)} < \delta$, but then

$$\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big) = 1,$$

and thus equation (3.8) receives the form

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q(\mathbf{x}) + f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big)$$

Denote $u - u_0$ by v. Then $\|v\|_{\widetilde{C}^{\mathbf{m}}(\Omega)} < \delta$, and v is a solution of the linear problem

$$v^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \quad (3.11)$$

$$a_{ik}\, v^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0}_i, \,\widehat{\mathbf{x}}_{\mathbf{i}}) + b_{ik}\, v^{(\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_i, \,\widehat{\mathbf{x}}_{\mathbf{i}}) = \widehat{\varphi}_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}})$$
$$\text{for } \ \widehat{\mathbf{x}}_{\mathbf{i}} \in \widehat{\Omega}_{\mathbf{i}} \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n), \quad (3.12)$$

where $\widetilde{\varphi}_{ik}$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ and q are arbitrary functions satisfying the inequalities

$$\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\widetilde{\varphi}_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} \leq \delta, \qquad \|q\|_{C(\Omega} < \delta.$$

Consequently, problem (1.15), (1.16) is well–posed.

Now assume that problem (1.15), (1.16) is well–posed, and let us prove that problem (1.2), (1.2) is strongly (u_0, r)-well–posed for some $r > 0$.

Let $\big(\widetilde{f}; \widetilde{\varphi}_{1\,1}, \dots, \widetilde{\varphi}_{1\,m_1}, \dots, \widetilde{\varphi}_{n\,1}, \dots, \widetilde{\varphi}_{n\,m_n}\big)$ be an arbitrary admissible perturbation. Consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big), \tag{3.13}$$

where

$$q\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big) + \Bigg(f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big) + \widetilde{f}\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) - \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)\Bigg)\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big).$$

Due to the continuous differentiability of $f(\mathbf{x}, \mathbf{Z})$ with respect to the phase variables, for $u \in \widetilde{\mathbf{B}}^{\mathbf{m}}(u_0; \delta)$ we have

$$\Big|f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big) - \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)\Big| = o(\delta) \quad \text{as} \quad \delta \to 0$$

and

$$\big|q_{\boldsymbol{\alpha}}(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u])\big| \leq \delta_0 + o(\delta) \quad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}),$$

where

$$q_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) = \frac{\partial q(\mathbf{x}, \mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}.$$

By Lemma 2.4 and Theorem 1.4 (see the proof of Theorem 1.4 below), for sufficiently small $\delta_0 > 0$ problem (3.13), (1.4) is solvable and its every solution admits the estimate

$$\|u - u_0\|_{C^{\mathbf{m}}(\Omega)} \leq M_0(\delta_0 + o(\delta)).$$

Choosing δ_0 and δ sufficiently small, one can achieve the estimate

$$\|u - u_0\|_{C^{\mathbf{m}}(\Omega)} < \delta.$$

But then

$$\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big) = 1,$$

and every solution of equation (3.13) is a solution of equation (1.3) too. $\square$

Proof of Theorem 1.4. Let u be a solution of problem (1.31), (1.2). Then, in view of continuous differentiability of f with respect to the phase variables, u is also a solution of the linearized equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}[u](\mathbf{x}) u^{(\boldsymbol{\alpha})} + Q[u](\mathbf{x}), \tag{3.14}$$

where

$$p_{\boldsymbol{\alpha}}[u](\mathbf{x}) = \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}, t\,\widetilde{\mathcal{D}}^{\mathbf{m}}[u(\mathbf{x})])\,dt \quad (\boldsymbol{\alpha} < \mathbf{m}),$$

$$Q[u](\mathbf{x}) = q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u(\mathbf{x})]) + f(\mathbf{x}, \mathbf{0}).$$

In view of conditions (A_1), (A_2) and (1.33), by Lemma 2.4, u admits the estimate

$$\begin{aligned}\|u\|_{C^{\mathbf{m}}(\Omega)} &\le M_0\Big(\sum_{i=1}^n \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)} + \|Q\|_{C(\Omega)}\Big) \\ &\le M_0\Big(\sum_{i=1}^n \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)} + M_1 + \frac{1}{2M_0}\|u\|_{C^{\mathbf{m-1}}(\Omega)}\Big), \end{aligned} \tag{3.15}$$

where $M_0 > 0$ is the constant from Lemma 2.4.

Consequently, every solution of problem (1.31), (1.2) admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le 2M_0\Big(M_1 + \Big(\sum_{i=1}^n \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)}\Big). \tag{3.16}$$

Consider the "truncated" equation

$$u^{(\mathbf{m})} = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) + q\Big(\mathbf{x}, \Xi_r\big(\|\mathcal{D}^{\mathbf{m-1}}[u]\|\big)\mathcal{D}^{\mathbf{m-1}}[u]\Big)), \tag{3.17}$$

where $\Xi_r : [0, +\infty) \to [0, +\infty)$ is a non–increasing smooth function given by (3.9).

It is clear that every solution of problem (3.17), (1.2) also admits estimate (3.16).

In order to prove solvability of problem (3.17), (1.2), consider the equation

$$u^{(\mathbf{m})} = f\Big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\Big) + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\Big), \tag{3.18}$$

where $z \in C^{\mathbf{m-1}}(\Omega)$ is an arbitrary function.

By Theorem 1.2 problem (3.18),(1.2) has a unique solution u. This defines the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$. By Lemma 2.4, $\mathcal{A}_0$ is a nonlinear continuous operator mapping $C^{\mathbf{m-1}}(\Omega)$ into the ball $\mathbf{B}^{\mathbf{m}}(0; R)$ of radius

$$R = 2M_0\Big(M_1 + \sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_\mathbf{i}}(\widehat{\Omega}_i)}\Big).$$

Hence, $\mathcal{A}_0$ is a nonlinear compact operator mapping $C^{\mathbf{m-1}}(\Omega)$ into the ball $\mathbf{B}^{\mathbf{m-1}}(0; R)$. By Schauder's fixed point theorem, $\mathcal{A}_0$ has a fixed point $u \in \mathbf{B}^{\mathbf{m}}(0; R)$. Consequently, problem (3.17),(1.2) has a solution u admitting estimate (3.16).

Choosing r large enough we ensure that u is a solution of problem (1.31), (1.2) too. $\square$

Proof of Theorem 1.5. The proof is similar to the proof of Theorem 1.4. Therefore, instead of a detailed proof, we present just the main steps of the proof.

Step 1: Consider the "truncated" equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u])u^{(\boldsymbol{\alpha})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[u]\Big). \tag{3.19}$$

In view (1.33) and (1.34), by Lemma 2.4, every solution of problem (3.19), (1.2) admit the estimate (3.16), where the constants M_0 and M_1 are same as in the proof of Theorem 1.4.

Step 2: For and arbitrary $z \in C^{\mathbf{m-1}}(\Omega)$ consider the linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})])\, u^{(\boldsymbol{\alpha})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\Big). \quad (3.20)$$

By Lemma 2.4, problem $(3.20), (1.2)$ has a unique solution u. Thus the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$ is defined.

Step 3: The operator A_0 is a nonlinear continuous operator mapping $C^{\mathbf{m-1}}(\Omega)$ into $\mathbf{B}^{\mathbf{m}}(0; R)$, where

$$R = M_0 \max\Big\{\big|q\big(\mathbf{x}, \mathbf{Z}\big)\big| : x \in \Omega,\ \|Z\| \le r+1\Big\}.$$

Consequently, $A_0 : C^{\mathbf{m-1}}(\Omega) \to \mathbf{B}^{\mathbf{m-1}}(0; R)$ is a compact operator.

Step 4: By Schauder's fixed point theorem, A_0 has a fixed point $u \in \mathbf{B}^{\mathbf{m}}(0; R)$, which is a solution of problem $(3.19), (1.2)$.

Step 4: Since u is a solution of problem $(3.19), (1.2)$, it admits the estimate (3.16). Choosing

$$r > 2M_0\Big(M_1 + \Big(\sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)}\Big),$$

we get that u is a solution of problem $(1.32), (1.2)$ too. □

Proof of Theorem 1.6. Solvability of problem $(1.32), (1.2)$ easily follows from Theorem 1.2. Let $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ $(\boldsymbol{\alpha} < \mathbf{m})$ be locally Lipschitz continuous with respect to the phase variables. In order to prove the theorem, it remains to show that $(1.32), (1.2)$ is strongly well–posed if

$$\sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} \le \varepsilon, \qquad |q(\mathbf{x}, \mathbf{Z})| \le \varepsilon \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \qquad (3.21)$$

for sufficiently small $\varepsilon > 0$.

Let (3.21) hold, and let u_0 and u_1 be solutions of problem $(1.32), (1.2)$. Then, by lemma 2.4,

$$\|u_i\|_{C^{\mathbf{m}}(\Omega)} \le M_0\, \varepsilon \quad (i = 0, 1), \qquad (3.22)$$

and

$$(u_1 - u_0)^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}[u_1](\mathbf{x})(u_1 - u_0)^{(\boldsymbol{\alpha})}$$
$$+ \sum_{\boldsymbol{\alpha}\le\mathbf{m\text{-}1}} q_{\boldsymbol{\alpha}}[u_1, u_0](\mathbf{x})(u_1 - u_0)^{(\boldsymbol{\alpha})}$$
$$+ \sum_{\boldsymbol{\alpha}<\mathbf{m}} \big(p_{\boldsymbol{\alpha}}[u_1(\mathbf{x})](\mathbf{x}) - p_{\boldsymbol{\alpha}}[u_0(\mathbf{x})](\mathbf{x})\big)\, u_0^{(\boldsymbol{\alpha})}(\mathbf{x}), \quad (3.23)$$

where

$$p_{\boldsymbol{\alpha}}[u_i](\mathbf{x}) = p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u_i(\mathbf{x})]) \quad (\boldsymbol{\alpha} < \mathbf{m};\ i = 0, 1),$$
$$q_{\boldsymbol{\alpha}}[u_1, u_0](\mathbf{x}) = \int_0^1 q_{\boldsymbol{\alpha}}\big(\mathbf{x}, (1-t)\, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})] + t\, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_1(\mathbf{x})]\big)\, dt \quad (\boldsymbol{\alpha} \le \mathbf{m} - 1).$$

In view of (3.22) and local Lipschitz continuity of $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ ($\boldsymbol{\alpha} < \mathbf{m}$) with respect to the phase variables, we there exists $K > 0$ such that

$$\left|p_{\boldsymbol{\alpha}}[u_1(\mathbf{x})](\mathbf{x}) - p_{\boldsymbol{\alpha}}[u_0(\mathbf{x})](\mathbf{x})\right| \left|u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\right|$$
$$\le K M_0\, \varepsilon \left|u_1^{(\boldsymbol{\alpha})}(\mathbf{x}) - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\right| \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1}). \quad (3.24)$$

(1.35), (1.36) and (3.24) and Lemma 2.4 imply that for sufficiently small $\varepsilon > 0$ $u_0(\mathbf{x}) \equiv u_1(\mathbf{x})$, and u_0 is a strongly isolated solution. $\square$

Proof of Theorem 1.7. The proof of Theorem 1.7 is similar to the proofs of Theorems 1.4 and 1.5. Therefore, instead of a detailed proof, we present just the main steps of the proof.

Step 1: Consider the "truncated" equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\ u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})}$$
$$+ q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[u]\Big). \quad (3.25)$$

Let u be an arbitrary solution of problem (3.25), (1.40). Multiply equation (3.25) by u and integrate over Ω. After integrating by parts multiple times and taking into account (1.40), we arrive at

$$(-1)^{\|\mathbf{m}\|}\int\limits_{\Omega}\left|u^{(\mathbf{m})}(\mathbf{x})\right|^2 d\mathbf{x} =$$

$$\int\limits_{\Omega}\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}}(-1)^{\|\boldsymbol{\beta}\|}p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x},\Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)u^{(\boldsymbol{\alpha})}(\mathbf{x})u^{(\boldsymbol{\beta})}(\mathbf{x})\,d\mathbf{x}$$

$$+\int\limits_{\Omega}q\Big(\mathbf{x},\Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[u]\Big)u(\mathbf{x})\,d\mathbf{x} \quad (3.26)$$

In view of (1.42) and (1.43), for arbitrary $\varepsilon > 0$ there exists $M_1 > 0$ such that

$$\int\limits_{\Omega}\left|u^{(\mathbf{m})}(\mathbf{x})\right|^2 d\mathbf{x} \le \int\limits_{\Omega}\big(M_1 + \varepsilon u^2(\mathbf{x})\big)\,d\mathbf{x}. \quad (3.27)$$

(3.27) and the boundary conditions (1.40) imply the estimate

$$\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\int\limits_{\Omega}\left|u^{(\boldsymbol{\alpha})}(\mathbf{x})\right|^2 d\mathbf{x} \le M_2, \quad (3.28)$$

and

$$\|u\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} \le M_3, \quad (3.29)$$

where the positive constants M_2 and M_3 are independent of $r > 0$ and depend on M_1 and ε only.

Step 2: For and arbitrary $z \in C^{\mathbf{m}}(\Omega)$ consider the linear equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}}\Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x},\Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[z(\mathbf{x})]\big)\;u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})}$$

$$+\,q\Big(\mathbf{x},\Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\Big). \quad (3.30)$$

By Lemma 2.4, problem (3.30), (1.40) has a unique solution u. Thus the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$ is defined.

Step 3: The operator $\mathcal{A}_0$ is a nonlinear continuous operator mapping $C^{\mathbf{m}}(\Omega)$ into $\mathbf{B}^{2\mathbf{m}}(0; R)$, where

$$R = M_0 \max\Big\{\big|q\big(\mathbf{x},\mathbf{Z}\big)\big| : x \in \Omega,\ \|\mathbf{Z}\| \le r+1\Big\}.$$

Consequently, $\mathcal{A}_0 : C^{\mathbf{m}}(\Omega) \to \mathbf{B}^{\mathbf{m}}(0; R)$ is a compact operator.

Step 4: By Schauder's fixed point theorem, $\mathcal{A}_0$ has a fixed point $u \in \mathbf{B}^{\mathbf{2m}}(0; R)$, which is a solution of problem (3.30), (1.40).

Step 4: Since u is a solution of problem (3.30), (1.40), it admits the estimate (3.29). Choosing $r > M_3$ we get that u is a solution of problem (1.37), (1.40) too. □

Corollary 1.2 immediately follows from Theorem 1.7

The proofs of Theorems 1.8, 1.10, 1.12 and 1.14 are similar to the proofs of Theorems 1.6 and 1.7.

Corollary 1.3 immediately follows from Theorem 1.9.

Corollary 1.4 immediately follows from Theorem 1.13.

The proofs of Theorems 1.9, 1.11, 1.13 and 1.15 are similar to the proof of Theorem 1.7.

Proof of Theorem 1.18. Let u be a solution of problem (1.53),(1.40). Then u admits the representation

$$u(\mathbf{x}) = \mathcal{G}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}), \tag{3.31}$$

where $\mathcal{G}$ is the Green's operator of the linear problem

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})}, \tag{3.32}$$

$$\begin{aligned} &u^{(\mathbf{2m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, m_i;\ i = 1, \dots, n), \\ &u^{(\mathbf{2m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, m_i;\ i = 1, \dots, n). \end{aligned} \tag{3.33}$$

Furthermore, $u^{(\mathbf{m_k})}$ is a solution of the $n-1$–dimensional problem

$$\begin{gathered} v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, v^{(2\boldsymbol{\alpha}-\mathbf{m_k})} + p_k\big(\mathbf{x}, v\big) + Q[u](\mathbf{x}), \\ v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \ldots, m_i;\ i \neq k), \\ v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \ldots, m_i;\ i \neq k), \end{gathered} \tag{3.34}$$

where

$$Q[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha}<\mathbf{m_k}} \sum_{\boldsymbol{\beta}\le\widehat{\mathbf{m}}_\mathbf{k}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big)\, u^{(2\boldsymbol{\alpha}+2\boldsymbol{\beta})} + q(\mathbf{x}). \tag{3.35}$$

Consider the nonlinear equation

$$v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, v^{(2\boldsymbol{\alpha}-\mathbf{m_k})} + p_k\big(\mathbf{x}, v\big) + Q(\mathbf{x}). \tag{3.36}$$

If p_k is a smooth function, then unique solvability of problem (3.36), (1.40) follows from Corollary 1.1 and inequality (1.57). This defines the operator $\mathcal{A}_0 : Q \to v$. Utilising (1.56) and (1.57), it is not difficult to show that $\mathcal{A}_0 : C(\omega) \to C^{\widehat{\mathbf{m}}_\mathbf{k}}(\Omega)$ is a continuous operator satisfying the inequalities

$$\|\mathcal{A}_0(Q)\|_{C^{\widehat{\mathbf{m}}_\mathbf{k}}} \le K\|Q\|_{C(\Omega)}, \tag{3.37}$$

$$\|\mathcal{A}_0(Q_1) - \mathcal{A}_0(Q_2)\|_{C^{\widehat{\mathbf{m}}_\mathbf{k}}} \le K\|Q_1 - Q_2\|_{C(\Omega)}, \tag{3.38}$$

where the constant K depends on the norm of Green's operator of the $n-1$–dimensional problem

$$\begin{gathered} v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, v^{(2\boldsymbol{\alpha}-\mathbf{m_k})}, \\ v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \ldots, m_i;\ i \neq k), \\ v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \ldots, m_i;\ i \neq k), \end{gathered}$$

and the constants M and γ from (1.56) only, and does not depend the differential properties of p_k. Because of this, one can easily show that for nonsmooth p_k problem (3.36), (1.40) is still uniquely solvable and the operator $\mathcal{A}_0$ satisfies (3.37) and (3.38).

Now Let u be a solution of problem $(1.53),(1.40)$. The u admits the representation

$$u(\mathbf{x}) = \mathcal{G}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}),$$
$$u^{(\mathbf{2m_k})}(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}),$$

where

$$\mathcal{A}(u)(\mathbf{x}) = \mathcal{A}_0(Q[u])(\mathbf{x}).$$

In view of (3.37) and (3.38), it is clear that

$$\mathcal{A} : C^{\widehat{\mathbf{m}}_\mathbf{k}-\mathbf{1_k}}(\Omega) \to C^{\widehat{\mathbf{m}}_\mathbf{k}-\mathbf{1_k}}(\Omega)$$

is a compact operator

It is clear that problem $(1.53),(1.40)$ is equivalent to the systems of the operator equations

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}) \quad (\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}),$$
$$u^{(\mathbf{2m_k})}(\mathbf{x}) = \mathcal{A}\big(u\big)(\mathbf{x}).$$

and

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k\big(\mathcal{A}(u)\big) + q\big)(\mathbf{x}) \quad (\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}). \tag{3.39}$$

The operator

$$\mathcal{B}[u] = \Big(\mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k\big(\mathcal{A}(u)\big) + q\big)(\mathbf{x})\Big)_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}}$$

is a *bounded* nonlinear compact operator mapping a closed ball of radius R of the space

$$\prod_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}} C^{\widehat{\mathbf{m}}_\mathbf{k}}(\Omega)$$

into itself, where R is a sufficiently large number depending on constants K, M and γ.

By Schauder's fixed point theorem $\mathcal{B}$ has a fixed point. Therefore equation (3.39), and consequently, problem $(1.53),(1.40)$ are solvable. But solvability of problem $(1.53), (1.40)$ along with (3.38) implies its unique solvability and strong well–posedness. $\square$

Theorem 1.17 follows from Theorems 1.2 and 1.18.

Theorems 1.19 and 1.20 can be proved similarly.

CHAPTER II
The Problem on Periodic Solutions

4. Formulation of the Main Results

As we have already noted, the periodic boundary conditions

$$u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_\mathbf{i}) - u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_\mathbf{i})$$
$$= \varphi_{ik}^{(2\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i}) \quad (k=1,\dots,2m_i;\ i=1,\dots,n)$$

do not satisfy the conditions of Theorem 1.1, since problem (1.8) with periodic boundary conditions has nontrivial solutions for every $i \in \{1,\dots,n-1\}$. Consequently, problem (1.1),(1.2) nonhomogeneous periodic conditions is *not* well–posed in the sense of Definition 1.2.

Therefore it makes sense to study periodic problem with the homogeneous boundary conditions only. Furthermore, instead of studying periodic problems in the domain Ω, it is more convenient to study problems on periodic solutions in $\mathbb{R}^n$:

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{4.1}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_\mathbf{i}) = u(\mathbf{x}) \quad (i=1,\dots,n), \tag{4.2}$$

where $\boldsymbol{\omega}_\mathbf{i} = (0,\dots,\omega_i,\dots,0)$, $\boldsymbol{\omega} = (\omega_1,\dots,\omega_n)$ and $f \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$.

4.1. **Strong Well–Posedness.** Along with problem (4.1), (4.1) consider the perturbed problem

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]) + \widetilde{f}(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{4.3}$$

A function $\widetilde{f} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$ is said to be an *admissible perturbation*, if it is locally Lipschitz continuous with respect to the *principal* phase variables.

Definition 4.1. Let the function $f(\mathbf{x},\mathbf{Z})$ be continuously differentiable with respect to the *principal* phase variables, let u_0 be a solution of problem (4.1), (4.2), and $r > 0$. We say that problem (4.1), (4.2) to is (u_0, r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem (4.1), (4.2) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r)$;

(II) there exist a positive constant δ_0 and an increasing continuous $\varepsilon : [0, \delta_0] \to [0, +\infty)$ such that $\varepsilon(0) = 0$ and for any $\delta \in (0, \delta_0]$ and an arbitrary admissible perturbation $\widetilde{f}$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})| \le \delta_0 \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{4.4}$$

$$|\widetilde{f}(\mathbf{x}, \mathbf{Z})| \le \delta \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}}, \tag{4.5}$$

problem (4.3), (4.2) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; \varepsilon(\delta))$.

Definition 4.2. Let the function $f(\mathbf{x}, \mathbf{Z})$ be continuously differentiable with respect to the *principal* phase variables, let u_0 be a solution of problem (4.1), (4.2), and $r > 0$. We say that problem (4.1), (4.2) to is *strongly* (u_0, r)*-well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem (4.1), (4.2) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0, r)$;

(II) there exist positive constants δ_0 and M such that for any $\delta \in (0, \delta_0]$ and an arbitrary admissible perturbation $\widetilde{f}$ satisfying conditions (4.4), (4.5), problem (4.3), (4.2) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; M\delta_0)$.

Definition 4.3. Problem (4.1), (4.2) is called well–posed (strongly well–posed), if it is (u_0, r)-well–posed (strongly (u_0, r)-well–posed) for every $r > 0$.

Definition 4.4. A solution u_0 of problem (4.1), (4.2) is called *strongly isolated*, if problem (4.1), (4.2) is strongly (u_0, r)-well–posed for some $r > 0$.

The problem on periodic solutions for the linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{4.6}$$

was studied in [25, 36].

Definition 4.5. Problem (4.6), (4.2) is called *well-posed*, if it is uniquely solvable for $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\sum_{i=1}^{n}\sum_{k=1}^{m_i}\|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\widehat{\Omega}_i)} + \|q\|_{C(\Omega)}\Big),$$

where M is a positive constant independent of q and φ_{ik} ($k = 1, \dots, m_i$; $i = 1, \dots, n$).

Remark 4.1. Notice that for the linear problem (4.6), (4.2) (u_0, r)-well–posedness is equivalent to the strong well–posedness. Furthermore, for problem (4.6), (4.2) Definitions 4.1 and 4.2 are equivalent to Definition 4.5.

4.2. **Necessary and Sufficient Conditions of Strong Well–Posedness.**

Theorem 4.1. *Let the function $f(\mathbf{x}, Z)$ be continuously differentiable with respect to the phase variables, and let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ ($\boldsymbol{\alpha} < \mathbf{m}$; $i = 1, 2$) such that:*

(A_1) $\quad P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le f_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x})$ *for* $(\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}}$ $(\boldsymbol{\alpha} < \mathbf{m})$; (4.7)

(A_2) *For every $\boldsymbol{\sigma} \in \Xi \cup \{\mathbf{1}\}$, $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ and arbitrary measurable functions $\rho_{\boldsymbol{\alpha}} \in L^{\infty}_{\boldsymbol{\omega}}(\mathbb{R}^{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$ satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \quad \text{for} \quad \mathbf{y} \in \mathbb{R}^{\boldsymbol{\sigma}} \quad (\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}), \tag{4.8}$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{y}) v^{(\boldsymbol{\alpha})}, \tag{4.9}$$

$$v(\mathbf{y} + \boldsymbol{\omega}_{\mathbf{i}}) = v(\mathbf{y}) \ \ (i \in \operatorname{supp} \boldsymbol{\sigma}). \tag{4.10}$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}}-\mathbf{1}_{\boldsymbol{\sigma}}}_{\boldsymbol{\omega}}(\mathbb{R}^{\boldsymbol{\sigma}})$.

Then problem (4.1), (4.2) is strongly well–posed.

Along with the quasi–linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m}-\mathbf{1}}[u]) \tag{4.11}$$

consider the linear homogeneous equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}. \tag{4.12}$$

Corollary 4.1. *Let the homogeneous problem* (4.12), (4.2) *be well–posed, let* $q(\mathbf{x}, Z)$ *be a continuously differentiable function with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \leq \mathbf{m}-\mathbf{1};\ i = 1,2)$ *such that:*

$$(A_1) P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq \frac{\partial q(\mathbf{x}, \mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}} \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \mathbb{R}^n \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \leq \mathbf{m}-\mathbf{1}); \tag{4.13}$$

(A_2) *for arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}} \in L^{\infty}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \leq \mathbf{m}-\mathbf{1})$ *satisfying the inequalities*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq \rho_{\boldsymbol{\alpha}}(\mathbf{x}) \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \mathbb{R}^n \ \ (\boldsymbol{\alpha} \leq \mathbf{m}-\mathbf{1}), \tag{4.14}$$

the problem

$$v^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m}-\mathbf{1}} \rho_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{4.15}$$

$$v(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}) = v(\mathbf{x}) \quad (i = 1, \dots, n). \tag{4.16}$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}}-\mathbf{1}_{\boldsymbol{\sigma}}}_{\boldsymbol{\omega}}(\mathbb{R}^{\boldsymbol{\sigma}})$. *Then problem* (4.11), (4.2) *is strongly well–posed.*

Theorem 4.2. *Let the function* $f(\mathbf{x}, Z)$ *be continuously differentiable with respect to the phase variables, and let* u_0 *be a solution of problem* (4.1), (4.2). *Then problem* (4.1), (4.2) *is strongly* (u_0, r)*-well–posed for some* $r > 0$, *if and only if the linear homogeneous problem* (4.12), (4.2) *is well–posed, where*

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) = f_{\boldsymbol{\alpha}}(\mathbf{x}, \widetilde{\mathcal{D}}^{\boldsymbol{m}}[u_0(\mathbf{x})]) \quad (\boldsymbol{\alpha} < \boldsymbol{m}).$$

Remark 4.2. In Theorem 4.1 continuous differentiability of the function $f(\mathbf{x}, \mathbf{Z})$ with respect to the phase variables $\mathbf{Z}$ can be replaced by Lipschitz continuity, although that will make the formulation of the theorem more cumbersome. However, Lipschitz continuity of the function $f(\mathbf{x},\mathbf{Z})$ with respect to the *principal* phase variables $z_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}$) is essential and cannot be replaced by Hölder continuity with the exponent $\gamma \in (0,1)$.

As an example, consider the two–dimensional periodic problem

$$u^{(2,2)} = u^{(2,0)} + u^{(0,2)} - \delta^{1-\gamma}\big|u^{(0,2)} - u\big|^{\gamma}\operatorname{sgn}\big(u^{(0,2)} - u\big) - u - \sin x_2, \qquad (4.17)$$

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2), \qquad u(x_1, x_2 + 2\pi) = u(x_1, x_2) \qquad (4.18)$$

where $\delta \geq 0$ and $\gamma \in (0,1)$.

Let u be a solution of problem (4.17), (4.18). Set:

$$v(x_1, x_2) = u^{(0,2)}(x_1, x_2) - u(x_1, x_2). \qquad (4.19)$$

Then v is a solution of the problem

$$v^{(2,0)} = v - \delta^{1-\gamma}\big|v\big|^{\gamma}\operatorname{sgn}\big(v\big) - \sin x_2, \qquad (4.20)$$

$$v(x_1 + 2\pi, x_2) = v(x_1, x_2). \qquad (4.21)$$

If $\delta = 0$, then it is clear that problem (4.20), (4.21) is a uniquely solvable linear periodic problem with the solution

$$v(x_1, x_2) \equiv \sin x_2,$$

and problem (4.17), (4.18) is a well–posed linear problem with the solution

$$u(x_1, x_2) \equiv u(x_2) = \int\limits_{x_2 - 2\pi}^{x_2} \frac{\cosh(x_2 - t - \pi)}{2\sinh(\pi)} \sin t\, dt.$$

Let us show that problem (4.17), (4.18) has no classical solutions for sufficiently small $\delta > 0$. For that it is sufficient to show that for sufficiently small $\delta > 0$ problem (4.20), (4.21) has no solution that is continuous with respect to x_2.

Problem (4.20), (4.21) is a periodic problem for an ordinary differential equation depending on the parameter x_2. It has a solution $v(x_1, x_2) \equiv v^*(x_2)$, where, $v^*(x_2)$ is the root of the equation

$$v - \delta^{1-\gamma}\big|v\big|^{\gamma}\operatorname{sgn}\big(v\big) - \sin x_2 = 0. \qquad (4.22)$$

One can easily show that problem (4.20), (4.21) is solvable for every x_2 if $\delta \in (0,1)$. Moreover, if $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$, then problem (4.20), (4.21) is uniquely solvable for $x_2 = \frac{\pi}{2}$, and its solution is positive. The latter fact implies that $v^*(\frac{\pi}{2}) > \delta$.

Let $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$, and let $v(x_1, x_2)$ be a solution of problem (4.20), (4.21) that is a continuous function of x_2. Then $v(x_1, \frac{\pi}{2}) = v^*(\frac{\pi}{2}) > \delta$. Due to continuity there exists $\varepsilon > 0$ such that

$$v(x_1, x_2) \geq \delta \quad \text{for} \quad x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{4.23}$$

But then problem (4.20), (4.21) is uniquely solvable for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Indeed, let $v_1(x_1) \geq \delta$ and $v_2(x_1) \geq \delta$ be arbitrary solutions of problem (4.20), (4.21) for some $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. Then $v(x_1) = v_2(x_1) - v_1(x_1)$ is a solution of the problem

$$v'' = (1 - \theta(x_1))v, \qquad v(x_1 + 2\pi) = v(x_1), \tag{4.24}$$

where

$$\theta(x_1) = \gamma \int_0^1 \frac{\delta^{1-\gamma}}{\Big(t\, v_1(x_1) + (1-t)\, v_2(x_1)\Big)^{1-\gamma}}\, dt \leq \gamma < 1. \tag{4.25}$$

The latter inequality implies that problem (4.24) has only the trivial solution, i.e. problem (4.20), (4.21) is uniquely solvable. Consequently, $v(x_1, x_2) = v^*(x_2)$ for $x_2 \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon]$. However, it is easy to see that a positive root of equation (18) is strictly bigger than δ for $x_2 \in (0, \pi)$. Hence

$$v(x_1, x_2) = v^*(x_2) > \delta \quad \text{for} \quad x_2 \in \Big[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\Big] \subset (0, \pi). \tag{4.26}$$

From (4.23)–(4.26) one can easily deduce that

$$v(x_1, x_2) = v^*(x_2) > \delta \quad \text{for} \quad x_2 \in (0, \pi). \tag{4.27}$$

Similarly one can show that

$$v(x_1, x_2) = v^*(x_2) < -\delta \quad \text{for} \quad x_2 \in (-\pi, 0). \tag{4.28}$$

(4.27) and (4.28) imply that $v^*(0+) = \delta$ and $v^*(0-) = -\delta$. Thus $v(x_1, x_2) \equiv v^*(x_2)$ is discontinuous at 0. Consequently, problem (4.17), (4.18) has no classical solutions for sufficiently small $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$.

This is the result of the fact that the righthand side of equation (4.17) is not Lipschitz continuous with respect to the principal phase variables, but instead is a Hölder continuous function with the exponent $\gamma \in (0,1)$.

Remark 4.3. In the rectangle $\Omega = [-\pi,\pi]\times[-\pi,\pi]$ consider the two–dimensional periodic problem

$$u^{(2,2)} = u^{(2,0)} + u^{(0,2)} - \delta^{1-\gamma}\big|u^{(0,2)} - u\big|^{\gamma}\,\mathrm{sgn}\,\big(u^{(0,2)} - u\big) - u - q(x_2), \qquad (4.29)$$

$$u^{(j,0)}(-\pi, x_2) = u^{(j,0)}(\pi, x_2), \quad u^{(0,k)}(x_1, -\pi) = u^{(0,k)}(x_1, \pi) \quad (j,k = 0,1), \quad (4.30)$$

where $\delta \geq 0$ and $\gamma \in (0,1)$.

In Remark 4.2 it was shown that if $\delta \in \big(0, 2^{\frac{1}{\gamma-1}}\big)$ and $q(x_2) = \sin x_2$, then problem (4.29), (4.30) has a unique weak solution $u \in AC^{\mathbf{1}}(\Omega)$ such that $u^{(0,2)}(x_1,x_2)$ is discontinuous along the lines $x_2 = k\pi$ $(k = 0, \pm 1)$.

In fact, for an arbitrary $\varepsilon \in (0,1/2)$ there exists $q \in C^{\infty}([-\pi,\pi])$ such that problem (4.29), (4.30) has a weak solution $u \in AC^{\mathbf{1}}(\Omega)$ and $u^{(0,2)}(x_1,x_2)$ is discontinuous on the set of the measure greater than $|\Omega| - \varepsilon$, where $|\Omega| = 4\pi^2$ is the measure of the rectangle Ω.

Indeed, let $\varepsilon \in (0,1/2)$ be an arbitrary positive number. Let us build a Cantor set of positive measure (also called a fat Cantor set) by removing certain open subintervals from the interval $[-\pi,\pi]$.

Step one: remove the middle open interval I_{11} of width $\varepsilon/2$ from the interval $[-\pi,\pi]$.

Step two: remove open subintervals I_{21} and I_{22} of the width $\varepsilon^2/2^2$ from the middle of each of two remaining intervals.

Step n: remove open subintervals I_{nk} $(k = 1,\dots,2^{n-1})$ of the width $\varepsilon^n/2^n$ from the middle of each of the 2^{n-1} remaining intervals.

Continuing indefinitely with this removal, we arrive at the Cantor set $\mathbf{K}$ of the positive measure

$$|\mathbf{K}| = 2\pi - \sum_{+\infty} 2^n \frac{\varepsilon^{n+1}}{2^{n+1}} = 2\pi - \frac{1}{2}\frac{\varepsilon}{1-\varepsilon} > 2\pi - \varepsilon.$$

Let $q \in C^\infty([-\pi,\pi])$ be such that

$$q(t) = 0 \quad \text{for} \quad t \in \mathbf{K};$$

$$(-1)^{n-1} q(t) > 0 \quad \text{for} \quad t \in I_{nk} \quad (k = 1, \dots, 2^{n-1};\ n = 1, 2, \dots).$$

Let for each $x_2 \in [-\pi,\pi] \backslash \mathbf{K}$, $v^*(x_2)$ be the root of the equation

$$v - \delta^{1-\gamma} \big|v\big|^\gamma \operatorname{sgn}\big(v\big) - q(x_2) = 0,$$

and let $v^*(x_2) = 0 \quad \text{for} \quad x_2 \in \mathbf{K}$. Then, it is easy to see that $v^* : [-\pi,\pi] \to \mathbb{R}$ is a measurable function, it is continuous on each interval $I_{nk} \quad (k = 1, \dots, 2^{n-1};\ n = 1, 2, \dots)$, and

$$(-1)^{n-1} v^*(x_2) > \delta \quad \text{for} \quad x_2 \in I_{nk} \quad (k = 1, \dots, 2^{n-1};\ n = 1, 2, \dots).$$

From the latter inequalities it immediately follows that v^* is discontinuous on the set $\mathbf{K}$.

Let $g(t,\tau)$ be the Green's function of the periodic problem

$$z'' = z, \qquad z^{(k-1)}(-\pi) = z^{(k-1)}(\pi) \quad (k = 1, 2).$$

Then it is not difficult to verify that:

(I) the function

$$u(x_1, x_2) = -\int_{-\pi}^{\pi} g(x_2, s_2) v^*(s_2)$$

is a weak solution of problem (4.29), (4.30);

(II) $u^{(0,2)}(x_1, x_2) = u(x_1, x_2) - v^*(x_2)$ is discontinuous on the set $J = [-\pi,\pi] \times \mathbf{K}$ and

$$|J| = 2\pi\,|\mathbf{K}| > 2\pi(2\pi - \varepsilon) > 4\pi^2 - \varepsilon.$$

Remark 4.4. The aforementioned examples explain why in an *admissible perturbation* the function $\widetilde{f} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m+1}})$ is required to be locally Lipschitz continuous with respect to the *principal* phase variables .

Also, the examples demonstrate that just the inequality (4.5), without inequality (4.4), does not guarantee even solvability of a perturbed problem.

4.3. **Solvability and Locally Strong Well–Posedness.** Consider the equations

$$u^{(\mathbf{m})} = f(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]) \tag{4.31}$$

and

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u])u^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u]). \tag{4.32}$$

Theorem 4.3. *Let the function f satisfy all of the conditions of Theorem 4.1 and let $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ be such that*

$$\lim_{\|\mathbf{Z}\|\to+\infty} \frac{|q(\mathbf{x}, \mathbf{Z})|}{\|\mathbf{Z}\|} = 0 \tag{4.33}$$

uniformly on Ω. Then problem (4.31), (4.2) *is solvable.*

Theorem 4.4. *Let $p_{\boldsymbol{\alpha}} \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ $(\boldsymbol{\alpha} < \mathbf{m})$, and let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ satisfying condition (A_2) of Theorem 4.1 such that*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}-\mathbf{1}} \quad (\boldsymbol{\alpha} < \mathbf{m}). \tag{4.34}$$

Furthermore, let the function $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ satisfy (4.33) *uniformly on Ω. Then problem* (4.32), (4.2) *is solvable.*

Theorem 4.5. *Let the function $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ be Lipschitz continuous with respect to the phase variables, and let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ satisfying condition (A_2) of Theorem 4.1 such that*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{4.35}$$

and

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) + \frac{\partial q(\mathbf{x}, \mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}} \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1}). \tag{4.36}$$

Then problem (4.32), (4.2) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})$ $(\boldsymbol{\alpha}<\mathbf{m})$ *are locally Lipschitz continuous with respect to the phase variables, then there exists* $\varepsilon>0$ *such that if*

$$|q(\mathbf{x},\mathbf{Z})|\le\varepsilon \quad \textit{for} \quad (\mathbf{x},\mathbf{Z})\in\Omega\times\mathbb{R}^{\mathbf{m}},$$

then every solution of problem (4.32), (4.2) *is strongly isolated.*

4.4. **Equations of Even Order.** For the equations

$$u^{(2\mathbf{m})}=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}}\Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\,u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})}+q(\mathbf{x},\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]), \tag{4.37}$$

$$u^{(2\mathbf{m})}=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}}\Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\,u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})}+q(\mathbf{x},u), \tag{4.38}$$

and

$$u^{(2\mathbf{m})}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\Big(p_{\boldsymbol{\alpha}}\big(\mathbf{x},\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\,u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\alpha})}+q(\mathbf{x},\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]) \tag{4.39}$$

Theorem 4.6. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\in C_{\boldsymbol{\omega}}^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\,\boldsymbol{\beta}\le\boldsymbol{m})$*, and let there exist* $\delta>0$ *such that*

$$\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}}(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1}p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},\mathbf{Z})v_{\boldsymbol{\alpha}}v_{\boldsymbol{\beta}}\ge\delta\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}}v_{\boldsymbol{\alpha}}^2 \quad \textit{for} \quad (\mathbf{x},\mathbf{Z})\in\Omega\times\mathbb{R}^{\mathbf{m}}. \tag{4.40}$$

Furthermore, let the function $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ *satisfy the equality*

$$\limsup_{\|\mathbf{Z}\|\to+\infty}\frac{(-1)^{\|\boldsymbol{m}\|}q(\mathbf{x},\mathbf{Z})\,\operatorname{sgn}z_{\mathbf{0}}}{\|\mathbf{Z}\|}=0 \quad \textit{uniformly on} \quad \Omega. \tag{4.41}$$

Then problem (4.37), (4.2) *is solvable.*

Corollary 4.2. *Let* $p_{\boldsymbol{\alpha}}\in C_{\boldsymbol{\omega}}^{\boldsymbol{\alpha},\|\boldsymbol{\alpha}\|}(\mathbb{R}^n\times\mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha}\le\boldsymbol{m})$*, and let there exist* $\delta>0$ *such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1}p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})\ge\delta \quad \textit{for} \quad (\mathbf{x},\mathbf{Z})\in\Omega\times\mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha}\le\mathbf{m}). \tag{4.42}$$

Furthermore, let the function $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R}^{\mathbf{m}})$ *satisfy equality* (4.41). *Then problem* (4.39), (4.2) *is solvable.*

Theorem 4.7. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\alpha}})$ $(\boldsymbol{\alpha},\, \boldsymbol{\beta} \leq \boldsymbol{m})$, *and let there exist* $\delta > 0$ *such that* (4.40) *holds. Furthermore, let the function* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R})$ *satisfy the inequality*

$$(-1)^{\|\boldsymbol{m}\|-1}\big(q(\mathbf{x}, z_1) - q(\mathbf{x}, z_2)\big)\ \mathrm{sgn}(z_1 - z_2) \geq 0. \tag{4.43}$$

Then problem (4.38), (4.2) *is solvable. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta},\|\boldsymbol{\beta}\|+1}_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\boldsymbol{\alpha}})$, *then there exists* $\varepsilon > 0$ *such that if*

$$|q(\mathbf{x}, z)| \leq \varepsilon \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R},$$

then every solution of problem (4.38), (4.2) *is strongly isolated.*

Remark 4.5. In Theorems 4.6 and 4.7 and Corollary 4.2 the functions $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ and $q(\mathbf{x}, \mathbf{Z})$ satisfy one sided inequalities (4.40), (4.41) and (4.42). Consequently, Theorems 4.6 and 4.7 and Corollary 4.2 cover the case, where the functions $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ and $q(\mathbf{x}, \mathbf{Z})$ have arbitrary growth order with respect to certain phase variables.

Consider the particular cases of equation (4.38)

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]) \tag{4.44}$$

and

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}, u). \tag{4.45}$$

Corollary 4.3. *Let* $p_{\boldsymbol{\alpha}} \in C_{\widehat{\boldsymbol{\omega}}_{\boldsymbol{\alpha}}}(\mathbb{R}^{\widehat{\mathbf{1}}_{\boldsymbol{\alpha}}})(\boldsymbol{\alpha} \leq \boldsymbol{m})$, *and let there exist* $\delta > 0$ *such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \geq \delta \quad \textit{for} \quad \widehat{\mathbf{x}}_{\boldsymbol{\alpha}} \in \widehat{\Omega}_{\boldsymbol{\alpha}} \quad (\boldsymbol{\alpha} \leq \mathbf{m}). \tag{4.46}$$

Furthermore, let the function $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R}^{\mathbf{m}})$ *satisfy equality* (4.41). *Then problem* (4.44), (4.2) *is solvable.*

Corollary 4.4. *Let* $p_{\boldsymbol{\alpha}} \in C_{\widehat{\boldsymbol{\omega}}_{\boldsymbol{\alpha}}}(\mathbb{R}^{\widehat{\mathbf{1}}_{\boldsymbol{\alpha}}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$*, and let there exist* $\delta > 0$ *such that* (4.46) *hold. Furthermore, let the function* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R})$ *satisfy the inequality* (4.43)*. Then problem* (4.45), (4.2) *is strongly well–posed.*

4.5. **Equations with Hölder Continuous Righthand Side.** In Remarks 4.2, 4.3 and 4.4 it was shown that, generally speaking, problem (4.1), (4.2) may not have a classical solution, if the function $f(\mathbf{x}, \mathbf{Z})$ is not Lipschitz continuous with respect to the principal phase variables, but instead is Hölder continuous with the exponent $\gamma \in (0, 1)$.

In this subsection we show that Lipschitz continuity is not the necessary condition for the existence of a classical solution. More precisely, we study the case where the righthand side of the equation is Hölder continuous with respect to certain principal variable.

Consider the equations:

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, u^{(2\boldsymbol{\alpha})} + p_k\big(\mathbf{x}, u^{(\mathbf{2m_k})}\big) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]) \tag{4.47}$$

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\, u^{(2\boldsymbol{\alpha})} + p_k\big(\mathbf{x}, u^{(\mathbf{2m_k})}\big) + q(\mathbf{x}), \tag{4.48}$$

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\, u^{(2\boldsymbol{\alpha})} + p_{\boldsymbol{\sigma}}\big(\mathbf{x}, u^{(\mathbf{2m}_{\boldsymbol{\sigma}})}\big) + q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u]) \tag{4.49}$$

and

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}\, u^{(2\boldsymbol{\alpha})} + p_{\boldsymbol{\sigma}}\big(\mathbf{x}, u^{(\mathbf{2m}_{\boldsymbol{\sigma}})}\big) + q(\mathbf{x}). \tag{4.50}$$

Theorem 4.8. *Let* $p_{\boldsymbol{\alpha}} \in C_{\widehat{\boldsymbol{\omega}}_{\boldsymbol{\alpha}}}(\mathbb{R}^{\widehat{\mathbf{1}}_{\boldsymbol{\alpha}}})$ $(\boldsymbol{\alpha} \leq \boldsymbol{m})$ *and* $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R})$ *satisfy conditions* (4.33) *and* (4.46) *for some* $\delta > 0$*. Furthermore, let there exist* $M > 0$ *and* $\gamma \in (0, 1)$ *such that the function* $p_i \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times \mathbb{R})$ *satisfies the inequalities*

$$|p_k(\mathbf{x}, z)| \leq M(1 + |z|^{\gamma}) \quad \textit{for} \quad (\mathbf{x}, z) \in \Omega \times \mathbb{R}, \tag{4.51}$$

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{m}_{\mathbf{k}}\|-1}\big(p_k(\mathbf{x},z_1)-p_k(\mathbf{x},z_2)\big)\ \operatorname{sgn}(z_1-z_2)\geq 0$$
$$\textit{for}\quad (\mathbf{x},z_i)\in\Omega\times\mathbb{R}\quad (i=1,2). \quad (4.52)$$

Then problem (4.47), (4.2) *is solvable.*

Theorem 4.9. *Let the functions* $p_{\boldsymbol{\alpha}}\in C_{\widehat{\boldsymbol{\omega}}_{\boldsymbol{\alpha}}}(\mathbb{R}^{\widehat{\mathbf{1}}_{\boldsymbol{\alpha}}})$ $(\boldsymbol{\alpha}\leq\boldsymbol{m})$ *and* $p_k\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R})$ *satisfy all of the conditions of Theorem 4.8 for some positive constants* δ, M *and* $\gamma\in(0,1)$. *Then for an arbitrary* $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ *problem* (4.48), (4.2) *is uniquely solvable and strongly well–posed.*

Theorem 4.10. *Let there exist* $\delta>0$, $M>0$ *and* $\gamma\in(0,1)$ *such that*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1}p_{\boldsymbol{\alpha}}\geq\delta\quad(\boldsymbol{\alpha}\leq\mathbf{m}), \quad (4.53)$$

$$|p_{\boldsymbol{\sigma}}(\mathbf{x},z)|\leq M(1+|z|^{\gamma})\quad\textit{for}\quad(\mathbf{x},z)\in\Omega\times\mathbb{R}, \quad (4.54)$$

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{m}_{\boldsymbol{\sigma}}\|-1}\big(p_{\boldsymbol{\sigma}}(\mathbf{x},z_1)-p_{\boldsymbol{\sigma}}(\mathbf{x},z_2)\big)\ \operatorname{sgn}(z_1-z_2)\geq 0$$
$$\textit{for}\quad (\mathbf{x},z_i)\in\Omega\times\mathbb{R}\quad (i=1,2). \quad (4.55)$$

Then problem (4.49), (4.2) *is solvable.*

Theorem 4.11. *Let the functions* $p_{\boldsymbol{\alpha}}\in\mathbb{R}$ $(\boldsymbol{\alpha}\leq\boldsymbol{m})$ *and* $p_{\boldsymbol{\sigma}}\in C_{\boldsymbol{\omega}}(\mathbb{R}^n\times\mathbb{R})$ *satisfy all of the conditions of Theorem 4.10 for some positive constants* δ, M *and* $\gamma\in(0,1)$. *Then for an arbitrary* $q\in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ *problem* (4.50), (4.2) *is uniquely solvable and strongly well–posed.*

Remark 4.6. Consider the following equations with the Holder continuous righthand sides:

$$u^{(2\mathbf{m})}=\sum_{\boldsymbol{\alpha}\leq\mathbf{m}}(-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1}(1+|p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}))\ u^{(2\boldsymbol{\alpha})}$$
$$+(-1)^{\|\widehat{\mathbf{m}}_{\mathbf{1}}\|-1}\big|u^{(2\mathbf{m}_{\mathbf{1}})}\big|^{\gamma}\operatorname{sgn}u^{(2\mathbf{m}_{\mathbf{i}})}+\ln\big(1+u^2+|u^{(2\mathbf{m}-\mathbf{1})}|^4\big), \quad (4.56)$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1}(1+|p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})) \, u^{(2\boldsymbol{\alpha})} + (-1)^{\|\widehat{\mathbf{m}}_{\mathbf{1}}\|-1}\big|u^{(2\mathbf{m}_{\mathbf{1}})}\big|^{\gamma} \operatorname{sgn} u^{(2\mathbf{m}_{\mathbf{1}})} + q(\mathbf{x}), \quad (4.57)$$

and

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} (-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1} \, u^{(2\boldsymbol{\alpha})} + (-1)^{\|\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}\|-1}\big|u^{(2\mathbf{m}_{\boldsymbol{\sigma}})}\big|^{\gamma} \operatorname{sgn} u^{(2\mathbf{m}_{\mathbf{1}})} + q(\mathbf{x}). \quad (4.58)$$

Here $p_{\boldsymbol{\alpha}} \in C_{\widehat{\boldsymbol{\omega}}_{\boldsymbol{\alpha}}}(\mathbb{R}^{\widehat{\mathbf{1}}_{\boldsymbol{\alpha}}})$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$ and $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ are arbitrary functions.

By Theorem 4.8, problem (4.56), (4.2) is solvable.

By Theorem 4.9, problem (4.57), (4.2) is uniquely solvable and well–posed.

By Theorem 4.11, problem (4.58), (4.2) is uniquely solvable and well–posed.

5. Auxiliary Statements

Consider the linear problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{5.1}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \quad (i = 1, \dots, n). \tag{5.2}$$

Along with the problem (5.1), (5.2), for each $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$, in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{$5.1_{\boldsymbol{\sigma}}$}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \quad (i \in \operatorname{supp} \boldsymbol{\sigma}). \tag{$5.2_{\boldsymbol{\sigma}}$}$$

Problem $(5.1_{\boldsymbol{\sigma}})$,$(5.2_{\boldsymbol{\sigma}})$ is called a $\boldsymbol{\sigma}$–*associate problem*, or an associate problem of level $\|\boldsymbol{\sigma}\|$.

The following result was proved in [38].

Lemma 5.1. *Let* $p_{\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \le \mathbf{m})$, *and let for each* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$–*associated problem* $(5.1_{\boldsymbol{\sigma}})$,$(5.2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$. *Then there exists a positive constant* M *such that an arbitrary solution* u *of the homogeneous equation*

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} \tag{5.1_0}$$

satisfying conditions (5.2) *admits the estimate*

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M \, \|u\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)}. \tag{5.3}$$

Moreover, problem (5.1),(5.2) *is well–posed if and only if problem* (5.1_0), (5.2) *has only the trivial solution.*

Lemma 5.2. *let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ *such that:*

(A_1) $$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le p_{\boldsymbol{\alpha}}(\mathbf{x}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega \quad (\boldsymbol{\alpha} < \mathbf{m}); \tag{5.4}$$

(A_2) *For every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi} \cup \mathbf{1}$,[2] $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ *and arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}} \in L^{\infty}_{\boldsymbol{\omega_\sigma}}(\mathbb{R}^{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$ *satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) \quad \textit{for} \quad \mathbf{y} \in \Omega_{\boldsymbol{\sigma}} \quad (\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}), \tag{5.5}$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{y}) v^{(\boldsymbol{\alpha})}, \tag{5.6}$$

$$v(\mathbf{y} + \boldsymbol{\omega}_{\mathbf{i}}) = v(\mathbf{y}) \quad (i \in \operatorname{supp} \boldsymbol{\sigma}) \tag{5.7}$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}} - \mathbf{1}_{\boldsymbol{\sigma}}}_{\boldsymbol{\omega_\sigma}}(\mathbb{R}^{\boldsymbol{\sigma}})$. *Then problem* (5.1), (5.2) *is well–posed. Moreover, there exists a positive constant* M_0 *depending only on* $P_{i\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ *and* ω_i $(i = 1, \dots, n)$, *such that a solution* u *of problem* (5.1), (5.2) *admits the estimate*

$$\|u\|_{C^m_{\boldsymbol{\omega}}(\mathbb{R}^n)} \le M_0 \, \|q\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)}. \tag{5.8}$$

Proof. The lemma is true for the case $n = 1$, i.e. for ordinary differential equations (see Theorem 1.2 and Corollary 3.6 from [19]).

Let $n \ge 2$, and let us assume that Lemma 5.2 is true for $n - 1$ dimensional problems, but is false for n–dimensional problems.

Then, by Banach–Steinhaus theorem, there exist $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$, and $\widetilde{p}_{l\,\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m})$ $(l = 1, 2, \dots)$ satisfying the inequalities

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le \widetilde{p}_{l\,\boldsymbol{\alpha}}(\mathbf{x}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \ \ (\boldsymbol{\alpha} < \mathbf{m};\ \ l = 1, 2, \dots), \tag{5.9}$$

such that

$$\|u_l\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} = \eta_l, \qquad \lim_{l \to \infty} \eta_l = +\infty, \tag{5.10}$$

where u_l is a solution of the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{l\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{5.1$_l$}$$

satisfying conditions (5.2).

[2]Notice that if $\boldsymbol{\sigma} = \mathbf{1}$, then problem $(5.1_{\boldsymbol{\sigma}}), (5.2_{\boldsymbol{\sigma}})$ is problem $(5.1_0), (5.2)$.

Due to inequalities (5.9), by Arzela–Ascoli lemma, without loss of generality, one may assume that there exist measurable functions $\widetilde{p}_{\boldsymbol{\alpha}} \in L^{\infty}_{\boldsymbol{\omega}_{\boldsymbol{\sigma}}}(\mathbb{R}^{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m})$ satisfying inequalities (5.4), such that

$$\lim_{l\to\infty} \int\limits_0^{x_1} \cdots \int\limits_0^{x_n} \big(p_{l\boldsymbol{\alpha}}(s_1,\ldots,s_n) - p_{\boldsymbol{\alpha}}(s_1,\ldots,s_n)\big)\, ds_n \ldots ds_1$$
$$= 0 \quad \text{uniformly on} \quad \Omega. \tag{5.11}$$

Set

$$\widetilde{u}_l(\mathbf{x}) = \frac{u_l(\mathbf{x})}{\eta_l}. \tag{5.12}$$

Then, by our assumption about $n-1$–dimensional problems and the estimate (5.3) from Lemma 5.1, we have

$$\|\widetilde{u}_l\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} = 1, \tag{5.13}$$

$$\|\widetilde{u}_l\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} \le M\Big(\|\widetilde{u}_l\|_{C^{\mathbf{m}-\mathbf{1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} + \frac{1}{\eta_l}\|q\|_{C(\Omega)}\Big), \tag{5.14}$$

where M is a positive constant independent of q and l.

In view of (5.13) and (5.14), by Arzela–Ascoli lemma, without loss of generality, one may assume that there exists $u_0 \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ such that

$$\lim_{l\to+\infty} \|\widetilde{u}_l - u_0\|_{C^{\mathbf{m}}(\Omega)} = 0, \quad \text{and} \quad \|u_0\|_{C^{\mathbf{m}}(\Omega)} = 1. \tag{5.15}$$

Taking into account (5.11), (5.15) and Lemma 2.3, we conclude that u_0 is a nontrivial solution of problem (5.1_0),(5.2). The obtained contradiction completes the proof of the lemma. ☐

6. Proof of the Main Results

Proof of Theorem 4.1. Consider the equation

$$u^{(\mathbf{m})} = (1-\lambda)\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{1,\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \lambda f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) \tag{6.1}$$

where $\lambda \in [0,1]$. Let u be a solution of problem (6.1), (4.2) for some $\lambda_0 \in [\lambda_0, 1)$. Then, due to the continuous differentiability of the function f with respect to the phase variables, u is a solution of the linearized equation

$$u^{(\mathbf{m})} = (1-\lambda)\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{1,\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \lambda\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}[u](\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{6.2}$$

where

$$p_{\boldsymbol{\alpha}}[u](\mathbf{x}) = \int_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}, t\,\widetilde{\mathcal{D}}^{\mathbf{m}}[u(\mathbf{x})])\,dt \quad (\boldsymbol{\alpha}<\mathbf{m}), \quad q(\mathbf{x}) = f(\mathbf{x}, \mathbf{0}).$$

In view of conditions (A_1) and (A_2), by Lemma 5.2, u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M_0\,\|q\|_{C(\Omega)}, \tag{6.3}$$

where the constant $M_0 > 0$ depends on the functions $P_{i\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ ($\boldsymbol{\alpha} < \mathbf{m}$; $i = 1, 2$) only, and is independent of $\lambda \in [0,1]$ and q. Hence, in view of condition (A_1) and estimate (6.3), it follows that solvability of problem (6.1), (4.2) for some $\lambda_0 \in [0,1)$ implies strong well–posedness of problem (6.1), (4.2) for $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0) \cap [0,1]$, where

$$\varepsilon = \frac{1}{2M_0}\Big(1 + \sum_{\boldsymbol{\alpha}<\mathbf{m}}\big(\|p_{1,\boldsymbol{\alpha}}\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)} + \|p_{2,\boldsymbol{\alpha}}\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)}\big)\Big)^{-1}.$$

But this implies solvability and strong well posedness of problem (6.1), (4.2) for every $\lambda \in [0,1]$, in particular $\lambda = 1$, since problem (6.1), (4.2) is well–posed for $\lambda = 0$. □

Corollary 4.1 is the direct consequence of Theorem 4.1.

Proof of Theorem 4.2. Let u_0 be a solution of problem (4.1), (4.2). Assume first that problem (4.1), (4.2) is (u_0, r)–well-posed for some $r > 0$. Consider the equation

$$u^{(\mathbf{m})} = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) + q(\mathbf{x}) + \Bigg(\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)$$
$$+ f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)\Bigg)\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big), \quad (6.4)$$

where $\Xi_\delta : [0, +\infty) \to [0, +\infty)$ is a non–increasing smooth function given by (3.9) and $\|q\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)} < \delta$.

In view of continuous differentiability of f, if $u \in \widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0; \delta)$, then

$$\Big| \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big) - f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)\Big| = o(\delta) \quad (6.5)$$

as $\delta \to 0$.

Therefore, in view of strong (u_0, r)–well-posedness of problem (4.1), (4.2), every solution of problem (6.4), (1.2) admits the estimate

$$\|u - u_0\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} \leq M(o(\delta) + \|q\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)}).$$

Choosing $\delta > 0$ sufficiently small, we achieve that $\|u - u_0\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} < \delta$, but then

$$\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}} \big|u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big) = 1,$$

and thus equation (6.4) receives the form

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q(\mathbf{x}) + f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})]\big)$$

Denote $u - u_0$ by v. Then $\|v\|_{\widetilde{C}^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} < \delta$, and v is a solution of the linear equation

$$v^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})} + q(\mathbf{x})$$

satisfying conditions (4.2). Consequently, problem (4.12), (4.2) is well–posed.

Now assume that problem (4.12), (4.2) is well–posed, and let us prove that problem (4.1), (4.2) is strongly (u_0, r)-well–posed for some $r > 0$.

Let $\widetilde{f}$ be an arbitrary admissible perturbation. Consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})} - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big), \quad (6.6)$$

where

$$q\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big)$$
$$+\bigg(f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)-f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big)+\widetilde{f}\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)$$
$$-\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})}-u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)\bigg)\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}}\big|u^{(\boldsymbol{\alpha})}-u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big).$$

Due to the continuous differentiability of $f(\mathbf{x},\mathbf{Z})$ with respect to the phase variables, for $u\in\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_0;\delta)$ we have

$$\Big|f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big)-f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(x)]\big)-\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\boldsymbol{\alpha}}(\mathbf{x})\big(u^{(\boldsymbol{\alpha})}-u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big)\Big|=o(\delta)\quad\text{as}\quad\delta\to 0$$

and

$$\big|q_{\boldsymbol{\alpha}}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])\big|\le\delta_0+o(\delta)\quad(\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{m}}),$$

where

$$q_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{Z})=\frac{\partial q(\mathbf{x},\mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}.$$

By Lemma 5.2 and Theorem 4.3 (see the proof of Theorem 4.3 below), for sufficiently small $\delta_0>0$ problem (6.6), (4.2) is solvable and its every solution admits the estimate

$$\|u-u_0\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)}\le M_0(\delta_0+o(\delta)).$$

Choosing δ_0 and δ sufficiently small, one can achieve the estimate

$$\|u-u_0\|_{C^{\mathbf{m}}(\Omega)}<\delta.$$

But then

$$\Xi_\delta\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}}\big|u^{(\boldsymbol{\alpha})}-u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big|\Big)=1,$$

and every solution of equation (6.6) is a solution of equation (4.12) too. □

Proof of Theorem 4.3. Let u be a solution of problem (4.31), (4.2). Then, in view of continuous differentiability of f with respect to the phase variables, u is also a solution of the linearized equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}[u](\mathbf{x})u^{(\boldsymbol{\alpha})} + Q[u](\mathbf{x}), \tag{6.7}$$

where

$$p_{\boldsymbol{\alpha}}[u](\mathbf{x}) = \int\limits_0^1 f_{\boldsymbol{\alpha}}(\mathbf{x}, t\,\widetilde{\mathcal{D}}^{\mathbf{m}}[u(\mathbf{x})])\,dt \quad (\boldsymbol{\alpha} < \mathbf{m}),$$

$$Q[u](\mathbf{x}) = q(\mathbf{x}, \mathcal{D}^{\mathbf{m-1}}[u(\mathbf{x})]) + f(\mathbf{x}, \mathbf{0}).$$

In view of conditions (A_1), (A_2) and (4.33), by Lemma 5.2, u admits the estimate

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} \le M_0\|Q\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n} \le M_0\Big(M_1 + \frac{1}{2M_0}\|u\|_{C^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)}\Big), \tag{6.8}$$

where $M_0 > 0$ is the constant from Lemma 5.2.

Consequently, every solution of problem (4.31), (4.2) admits the estimate

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} \le 2M_0M_1. \tag{6.9}$$

Consider the "truncated" equation

$$u^{(\mathbf{m})} = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m-1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m-1}}[u]\Big)), \tag{6.10}$$

where $\Xi_r : [0, +\infty) \to [0, +\infty)$ is a non–increasing smooth function given by (3.9). It is clear that every solution of problem (6.10), (4.2) also admits estimate (6.9).

In order to prove solvability of problem (3.17), (1.2), consider the equation

$$u^{(\mathbf{m})} = f\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]\big) + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m-1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m-1}}[z(\mathbf{x})]\Big), \tag{6.11}$$

where $z \in C^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is an arbitrary function.

By Theorem 4.1 problem (6.11),(4.2) has a unique solution u. This defines the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$. By Lemma 5.2, $\mathcal{A}_0$ is a nonlinear continuous operator mapping $C^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ into the ball $\mathbf{B}^{\mathbf{m}}_{\boldsymbol{\omega}}(0; 2M_0M_1)$. Hence, $\mathcal{A}_0$ ia a nonlinear compact operator mapping $C^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ into the ball $\mathbf{B}^{\mathbf{m-1}}_{\boldsymbol{\omega}}(0; 2M_0M_1)$. By Schauder's fixed point theorem, $\mathcal{A}_0$ has a fixed point $u \in \mathbf{B}^{\mathbf{m}}(0; 2M_0M_1)$. Consequently, problem (6.10),(4.2) has a solution u admitting estimate (6.9).

Choosing $r > 2M_0M_1$ we ensure that u is a solution of problem (4.31), (4.2) too. $\square$

Proof of Theorem 4.4. The proof is similar to the proof of Theorem 4.3. Therefore, instead of a detailed proof, we present just the main steps of the proof.

Step 1: Consider the "truncated" equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u])u^{(\boldsymbol{\alpha})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[u]\Big). \tag{6.12}$$

In view (4.33) and (4.34), by Lemma 5.2, every solution of problem (6.12), (4.2) admit the estimate (6.9), where the constants M_0 and M_1 are same as in the proof of Theorem 4.3.

Step 2: For and arbitrary $z \in C^{\mathbf{m}-\mathbf{1}}(\Omega)$ consider the linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})])\, u^{(\boldsymbol{\alpha})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m\text{-}1}}[z(\mathbf{x})]\Big). \tag{6.13}$$

By Lemma 5.2, problem (6.13), (4.2) has a unique solution u. Thus the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$ is defined.

Step 3: The operator A_0 is a nonlinear continuous operator mapping $C^{\mathbf{m}-\mathbf{1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ into $\mathbf{B}^{\mathbf{m}}_{\boldsymbol{\omega}}(0; R)$, where

$$R = M_0 \max\Big\{\big|q\big(\mathbf{x}, \mathbf{Z}\big)\big| : x \in \Omega,\ \|Z\| \le r+1\Big\}.$$

Consequently, $A_0 : C^{\mathbf{m}-\mathbf{1}}_{\boldsymbol{\omega}}(\mathbb{R}^n) \to \mathbf{B}^{\mathbf{m}-\mathbf{1}}_{\boldsymbol{\omega}}(0; R)$ is a compact operator.

Step 4: By Schauder's fixed point theorem, A_0 has a fixed point $u \in \mathbf{B}^{\mathbf{m}}_{\boldsymbol{\omega}}(0; R)$, which is a solution of problem (6.12), (4.2).

Step 4: Since u is a solution of problem (6.12), (4.2), it admits the estimate (6.9). Choosing

$$r > 2M_0\Big(M_1 + \sum_{i=1}^{n}\sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_\mathbf{i}}(\widehat{\Omega}_i)}\Big),$$

we get that u is a solution of problem (4.32), (4.2) too. $\square$

Proof of Theorem 4.5. Solvability of problem (4.32), (4.2) easily follows from Theorem 4.1. Let $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ $(\boldsymbol{\alpha} < \mathbf{m})$ be locally Lipschitz continuous with respect to the phase variables. In order to prove the theorem, it remains to show that (4.32), (4.2) is strongly well–posed if

$$|q(\mathbf{x}, \mathbf{Z})| \le \varepsilon \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \tag{6.14}$$

for sufficiently small $\varepsilon > 0$.

Let (6.14) hold, and let u_0 and u_1 be solutions of problem (4.32), (4.2). Then, by lemma 5.2,

$$\|u_i\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)} \le M_0\,\varepsilon \quad (i = 0, 1), \tag{6.15}$$

and

$$\begin{aligned}(u_1 - u_0)^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} & p_{\boldsymbol{\alpha}}[u_1](\mathbf{x})(u_1 - u_0)^{(\boldsymbol{\alpha})} \\ &+ \sum_{\boldsymbol{\alpha}\le\mathbf{m\text{-}1}} q_{\boldsymbol{\alpha}}[u_1, u_0](\mathbf{x})(u_1 - u_0)^{(\boldsymbol{\alpha})} \\ &+ \sum_{\boldsymbol{\alpha}<\mathbf{m}} \big(p_{\boldsymbol{\alpha}}[u_1(\mathbf{x})](\mathbf{x}) - p_{\boldsymbol{\alpha}}[u_0(\mathbf{x})](\mathbf{x})\big)\, u_0^{(\boldsymbol{\alpha})}(\mathbf{x}), \end{aligned}\tag{6.16}$$

where

$$p_{\boldsymbol{\alpha}}[u_i](\mathbf{x}) = p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}^{\mathbf{m\text{-}1}}[u_i(\mathbf{x})]) \quad (\boldsymbol{\alpha} < \mathbf{m};\ i = 0, 1),$$

$$q_{\boldsymbol{\alpha}}[u_1, u_0](\mathbf{x}) = \int_0^1 q_{\boldsymbol{\alpha}}\big(\mathbf{x}, (1-t)\,\widetilde{\mathcal{D}}^{\mathbf{m}}[u_0(\mathbf{x})] + t\,\widetilde{\mathcal{D}}^{\mathbf{m}}[u_1(\mathbf{x})]\big)\,dt \quad (\boldsymbol{\alpha} \le \mathbf{m} - 1).$$

In view of (6.15) and local Lipschitz continuity of $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})$ $(\boldsymbol{\alpha} < \mathbf{m})$ with respect to the phase variables, we there exists $K > 0$ such that

$$\begin{aligned}\big|p_{\boldsymbol{\alpha}}[u_1(\mathbf{x})](\mathbf{x}) - p_{\boldsymbol{\alpha}}[u_0(\mathbf{x})](\mathbf{x})\big|\ \Big|u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\Big| \\ \le K M_0\,\varepsilon \left|u_1^{(\boldsymbol{\alpha})}(\mathbf{x}) - u_0^{(\boldsymbol{\alpha})}(\mathbf{x})\right| \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1}).\end{aligned} \tag{6.17}$$

(4.35), (4.36) and (6.17) and Lemma 5.2 imply that for sufficiently small $\varepsilon > 0$ $u_0(\mathbf{x}) \equiv u_1(\mathbf{x})$, and u_0 is a strongly isolated solution. □

Proof of Theorem 4.6. The proof of Theorem 4.6 is similar to the proofs of Theorems 4.3 and 4.4. Therefore, instead of a detailed proof, we present just the main steps of the proof.

Step 1: Consider the "truncated" equation

$$u^{(\mathbf{2m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\le\mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)\ u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]\Big). \quad (6.18)$$

Let u be an arbitrary solution of problem (6.18), (4.2). Multiply equation (6.18) by u and integrate over Ω. After integrating by parts multiple times and taking into account (4.2), we arrive at

$$(-1)^{\|\mathbf{m}\|}\int\limits_{\Omega}\big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\,d\mathbf{x} = \int\limits_{\Omega}\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<\mathbf{2m}}(-1)^{\|\boldsymbol{\beta}\|}p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[u]\big)u^{(\boldsymbol{\alpha})}(\mathbf{x})u^{(\boldsymbol{\beta})}(\mathbf{x})\,d\mathbf{x} + \int\limits_{\Omega} q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]\big\|\big)\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[u]\Big)u(\mathbf{x})\,d\mathbf{x} \quad (6.19)$$

In view of (4.40) and (4.41), for arbitrary $\varepsilon > 0$ there exists $M_1 > 0$ such that

$$\int\limits_{\Omega}\big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\,d\mathbf{x} + \delta\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\int\limits_{\Omega}\big|u^{(\boldsymbol{\alpha})}(\mathbf{x})\big|^2 \le \int\limits_{\Omega}\big(M_1 + \varepsilon u^2(\mathbf{x})\big)\,d\mathbf{x}. \quad (6.20)$$

(6.20) and (4.2) imply the estimate

$$\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\int\limits_{\Omega}\big|u^{(\boldsymbol{\alpha})}(\mathbf{x})\big|^2\,d\mathbf{x} \le M_2, \quad (6.21)$$

and

$$\|u\|_{C^{\mathbf{m}-\mathbf{1}}_{\boldsymbol{\omega}}(\Omega)} \le M_3, \quad (6.22)$$

where the positive constants M_2 and M_3 are independent of $r > 0$ and depend on M_1 and ε only.

Step 2: For and arbitrary $z \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ consider the linear equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\leq\mathbf{m}} \Big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\boldsymbol{\alpha}-\mathbf{1}_{\boldsymbol{\alpha}}}[z(\mathbf{x})]\big)\ u^{(\boldsymbol{\alpha})}\Big)^{(\boldsymbol{\beta})} + q\Big(\mathbf{x}, \Xi_r\big(\big\|\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[z(\mathbf{x})]\big\|\big)\mathcal{D}^{\mathbf{m}\text{-}\mathbf{1}}[z(\mathbf{x})]\Big). \quad (6.23)$$

By Lemma 5.2, problem (6.23), (4.2) has a unique solution u. Thus the operator $\mathcal{A}_0 : z(\mathbf{x}) \to u(\mathbf{x})$ is defined.

Step 3: The operator $\mathcal{A}_0$ is a nonlinear continuous operator mapping $C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ into $\mathbf{B}^{2\mathbf{m}}_{\boldsymbol{\omega}}(0; R)$, where

$$R = M_0 \max\Big\{\big|q\big(\mathbf{x}, \mathbf{Z}\big)\big| : x \in \Omega,\ \|\mathbf{Z}\| \leq r + 1\Big\}.$$

Consequently, $\mathcal{A}_0 : C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n) \to \mathbf{B}^{\mathbf{m}}(0; R)$ is a compact operator.

Step 4: By Schauder's fixed point theorem, $\mathcal{A}_0$ has a fixed point $u \in \mathbf{B}^{2\mathbf{m}}_{\boldsymbol{\omega}}(0; R)$, which is a solution of problem (6.23), (4.2).

Step 4: Since u is a solution of problem (6.23), (4.2), it admits the estimate (6.22). Choosing $r > M_3$ we get that u is a solution of problem (4.37), (4.2) too. □

Corollary 4.2 immediately follows from Theorem 4.6

The proof of Theorems 4.7 is similar to the proofs of Theorems 1.6 and 1.7.

Corollaries 4.3 and 4.4 are direct consequences of Theorems 4.6 and 4.7.

Proof of Theorem 4.9. Let u be a solution of problem (4.48),(4.2). Then u admits the representation

$$u(\mathbf{x}) = \mathcal{G}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}), \quad (6.24)$$

where $\mathcal{G}$ is the Green's operator of the linear problem

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\leq\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ u^{(2\boldsymbol{\alpha})}, \quad (6.25)$$

$$u(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \quad (i = 1, \dots, n). \quad (6.26)$$

Furthermore, $u^{(\mathbf{m_k})}$ is a solution of the $n-1$–dimensional problem

$$v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ v^{(2\boldsymbol{\alpha}-\mathbf{m_k})} + p_k\big(\mathbf{x}, v\big) + Q[u](\mathbf{x}), \tag{6.27}$$
$$u(\mathbf{x}+\boldsymbol{\omega}_\mathbf{i}) = u(\mathbf{x}) \quad (i \ne k),$$

where

$$Q[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha}<\mathbf{m_k}} \sum_{\boldsymbol{\beta}\le\widehat{\mathbf{m}}_\mathbf{k}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\big)\ u^{(2\boldsymbol{\alpha}+2\boldsymbol{\beta})} + q(\mathbf{x}). \tag{6.28}$$

Consider the nonlinear equation

$$v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ v^{(2\boldsymbol{\alpha}-\mathbf{m_k})} + p_k\big(\mathbf{x}, v\big) + Q(\mathbf{x}). \tag{6.29}$$

If p_k is a smooth function, then unique solvability of problem (6.29), (4.2) follows from Corollary 4.1 and inequality (4.52). This defines the operator $\mathcal{A}_0$: $Q \to v$. Utilising (4.51) and (4.52), it is not difficult to show that $\mathcal{A}_0 : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}}(\mathbb{R}^n)$ is a continuous operator satisfying the inequalities

$$\|\mathcal{A}_0(Q)\|_{C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}}(\mathbb{R}^n)} \le K\|Q\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)}, \tag{6.30}$$
$$\|\mathcal{A}_0(Q_1) - \mathcal{A}_0(Q_2)\|_{C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}}(\mathbb{R}^n)} \le K\|Q_1 - Q_2\|_{C_{\boldsymbol{\omega}}(\mathbb{R}^n)}, \tag{6.31}$$

where the constant K depends on the norm of Green's operator of the $n-1$–dimensional problem

$$v^{(2\widehat{\mathbf{m}}_\mathbf{k})} = \sum_{\mathbf{m_k}\le\boldsymbol{\alpha}\le\mathbf{m}} p_{\boldsymbol{\alpha}}\big(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}\big)\ v^{(2\boldsymbol{\alpha}-\mathbf{m_k})},$$
$$v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, m_i;\ i \ne k),$$
$$v^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, m_i;\ i \ne k),$$

and the constants M and γ from (4.51) only, and does not depend on the differential properties of p_k. Because of this, one can easily show that for nonsmooth p_k problem (6.29), (4.2) is still uniquely solvable and the operator $\mathcal{A}_0$ satisfies (6.30) and (6.31).

Now Let u be a solution of problem (4.48), (4.2). The u admits the representation

$$u(\mathbf{x}) = \mathcal{G}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}),$$
$$u^{(2\mathbf{m_k})}(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}),$$

where

$$\mathcal{A}(u)(\mathbf{x}) = \mathcal{A}_0(Q[u])(\mathbf{x}).$$

In view of (6.30) and (6.31), it is clear that

$$\mathcal{A} : C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}-\mathbf{1_k}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}-\mathbf{1_k}}(\mathbb{R}^n)$$

is a compact operator

It is easy to see that problem (4.48), (4.2) is equivalent to the systems of the operator equations

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k(u^{2(\mathbf{m_k})}) + q\big)(\mathbf{x}) \quad (\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}),$$
$$u^{(2\mathbf{m_k})}(\mathbf{x}) = \mathcal{A}\big(u\big)(\mathbf{x}).$$

and

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k\big(\mathcal{A}\big(u\big)\big) + q\big)(\mathbf{x}) \quad (\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}). \tag{6.32}$$

The operator

$$\mathcal{B}[u] = \Big(\mathcal{G}^{(\boldsymbol{\alpha})}\big(p_k\big(\mathcal{A}\big(u\big)\big) + q\big)(\mathbf{x})\Big)_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}}$$

is a *bounded* nonlinear compact operator mapping a closed ball of radius R of the space

$$\prod_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_\mathbf{k}} C_{\boldsymbol{\omega}}^{\widehat{\mathbf{m}}_\mathbf{k}}(\mathbb{R}^n)$$

into itself, where R is a sufficiently large number depending on constants K, M and γ.

By Schauder's fixed point theorem, the operator $\mathcal{B}$ has a fixed point. Therefore equation (6.32), and consequently, problem (4.48), (4.2) are solvable. But solvability of problem (4.48), (4.2) along with (6.31) implies its unique solvability and strong well–posedness. □

Theorem 4.8 follows from Theorems 4.1 and 4.7.

Theorems 4.10 and 4.11 can be proved similarly.

CHAPTER III
Initial–Boundary Value Problems

7. Formulation of the Main Results

Let $\Omega = [0,\omega_1] \times \cdots \times [0,\omega_n]$, $T = [0,\tau_1] \times \cdots \times [0,\tau_k]$, $T_{\mathbf{i}} = [0,\tau_i]$,

$$\widehat{T}_{\mathbf{i}} = [0,\tau_1] \times \cdots \times [0,\tau_{i-1}] \times [0,\tau_{i+1}] \times \cdots \times [0,\tau_k],$$

$$T_{\mathbf{t}} = [0,t_1] \times \cdots \times [0,t_l], \qquad \widehat{T}_{\mathbf{i},\,\mathbf{t}} = \widehat{T}_{\mathbf{i}} \cap T_{\mathbf{t}}.$$

In the $n+k$-dimensional box $\Omega \times T$ for the nonlinear hyperbolic equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = f(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]), \tag{7.1}$$

consider the initial–boundary conditions

$$a_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) + b_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) = \varphi_{ij}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t})$$
$$\text{for } (\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) \in \widehat{\Omega}_{\mathbf{i}} \times T \ \ (j = 1, \dots, m_i; \ \ i = 1, \dots, n), \tag{7.2}$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = \psi_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}})$$
$$\text{for } (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i; \ i = 1, \dots, k). \tag{7.3}$$

If $n = 0$, then initial–boundary conditions (7.2), (7.3) are transformed into the initial conditions

$$u^{(\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_i, \widehat{\mathbf{t}}_i\big) = \psi_{ik}^{(\mathbf{m}^{i-1})}\big(\widehat{\mathbf{t}}_i\big)$$
$$\text{for } \widehat{\mathbf{t}}_{\mathbf{i}} \in \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i; \ i = 1, \dots, k). \tag{7.4}$$

Two-dimensional initial-boundary value problems were studied in [21–23,28–30,32,33].

7.1. **Strong Well–Posedness.** Along with problem (7.1)–(7.3) consider the perturbed problem

$$u^{(\mathbf{m}+\boldsymbol{l})} = f(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]) + \widetilde{f}(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]), \tag{7.5}$$

$$a_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) + b_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t})$$
$$= \varphi_{ij}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) + \widetilde{\varphi}_{ij}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t})$$
$$\text{for } (\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) \in \widehat{\Omega}_{\mathbf{i}} \times T \ \ (j = 1, \dots, m_i;\ i = 1, \dots, n), \quad (7.6)$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = \psi_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) + \widetilde{\psi}_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}})$$
$$\text{for } (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i;\ i = 1, \dots, k). \quad (7.7)$$

A vector function $\big(\widetilde{f}; \widetilde{\varphi}_{1\,1}, \dots, \widetilde{\varphi}_{1\,m_1}, \dots, \widetilde{\psi}_{k\,1}, \dots, \widetilde{\psi}_{k\,l_k}\big)$ is said to be an *admissible perturbation*, if $\widetilde{f} \in C(\Omega \times \mathbb{R}^{\mathbf{m}})$ is locally Lipschitz continuous with respect to the *principal* phase variables, $\widetilde{\varphi}_{ij} \in C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\Omega_{\mathbf{i}} \times T)$ $(j = 1, \dots, m_i;\ i = 1, \dots, n)$, and $\widetilde{\psi}_{ij} \in C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega \times \widehat{T}_{\mathbf{i}})$ $(j = 1, \dots, m_i;\ i = 1, \dots, k)$.

Due to the exceptional nature of the initial conditions, namely its property of locality, it makes sense to adapt the concepts of well–posedness to initial–boundary value problems.

Definition 7.1. Let u_0 be a solution of problem (7.1)–(7.3), and $r > 0$. We say that problem (7.1)–(7.3) to is (u_0, r)-*well-posed*, if for arbitrary $t_i \in (0, \tau_i]$ $(i = 1, \dots, k)$:

(I) $u_0(\mathbf{x}, \mathbf{t})$ is the unique solution of problem (7.1)–(7.3) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega \times T_{\mathbf{t}}](u_0; r)$;

(II) there exist a positive constant δ_0 and an increasing continuous $\varepsilon : [0, \delta_0] \to [0, +\infty)$ such that $\varepsilon(0) = 0$ and for any $\delta(\mathbf{t}) \in (0, \delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f}; \widetilde{\varphi}_{1\,1}, \dots, \widetilde{\varphi}_{1\,m_1}, \dots, \widetilde{\psi}_{k\,1}, \dots, \widetilde{\psi}_{k\,l_k}\big)$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\tau},\mathbf{Z})| \leq \delta_0 \quad \text{for} \quad (\mathbf{x},\boldsymbol{\tau},\mathbf{Z}) \in \Omega \times T_{\mathbf{t}} \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}+\boldsymbol{l}}), \tag{7.8}$$

$$|\widetilde{f}(\mathbf{x},\boldsymbol{\tau},\mathbf{Z})| \leq \delta(\mathbf{t}) \quad \text{for} \quad (\mathbf{x},\boldsymbol{\tau},\mathbf{Z}) \in \Omega \times T_{\mathbf{t}} \times \mathbb{R}^{\mathbf{m}}, \tag{7.9}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} \|\widetilde{\varphi}_{ij}\|_{C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\widehat{\Omega}_{\mathbf{i}}\times T_{\mathbf{t}})} + \sum_{i=1}^{k}\sum_{j=1}^{l_i} \|\widetilde{\psi}_{ij}\|_{C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega\times \widehat{T}_{\mathbf{i},\mathbf{t}})} \leq \delta(\mathbf{t}), \tag{7.10}$$

problem (7.5)–(7.7) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega\times T_{\mathbf{t}}](u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega \times T_{\mathbf{t}}]\big(u_0;\varepsilon(\delta(\mathbf{t}))\big)$.

Definition 7.2. Let u_0 be a solution of problem (7.1)–(7.3), and $r > 0$. We say that problem (7.1)–(7.3) to is *strongly* (u_0,r)*-well-posed*, if for arbitrary $x_i \in (0,\omega_i]$ $(i = 1,\dots,k)$:

(I) $u_0(\mathbf{x})$ is the unique solution of problem (7.1)–(7.3) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega \times T_{\mathbf{t}}](u_0;r)$;

(II) there exist a positive constants δ_0 and M such that for any $\delta(\mathbf{t}) \in (0,\delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f};\widetilde{\varphi}_{1\,1},\dots,\widetilde{\varphi}_{1\,m_1},\dots,\widetilde{\psi}_{k\,1},\dots,\widetilde{\psi}_{k\,l_k}\big)$ satisfying conditions (7.8)–(7.10), problem (7.1)–(7.3) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega\times T_{\mathbf{t}}](u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega \times T_{\mathbf{t}}](u_0;M\delta(\mathbf{t}))$.

Definition 7.3. Problem (7.1)–(7.3) is called *well–posed* (*strongly well–posed*) if it has a unique solution u_0 and it is (u_0,r)–well–posed (strongly (u_0,r)–well-posed) for every $r > 0$.

Definition 7.4. A solution u_0 of problem (7.1)–(7.3) is called *strongly isolated*, if problem (7.1)–(7.3) is strongly (u_0,r)-well–posed for some $r > 0$.

The linear case of equation (1.1), i.e. the equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}+\boldsymbol{l}} p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t})u^{(\boldsymbol{\alpha})} + q(\mathbf{x},\mathbf{t}) \tag{7.11}$$

was studied in [21, 28, 29, 36].

Definition 7.5. Problem (7.11), (7.2), (7.3) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_{ij} \in C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\widehat{\Omega}_{\mathbf{i}} \times T_{\mathbf{t}})$ $(j = 1, \dots, m_i;\ i = 1, \dots, n)$, $\widetilde{\psi}_{ij} \in C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega \times \widehat{T}_{\mathbf{i},\mathbf{t}})$ $(j = 1, \dots, l_i;\ i = 1, \dots, k)$ and $q \in C(\Omega \times T)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega_{\mathbf{x}\boldsymbol{l}})} \le M\Big(\sum_{i=1}^{n}\sum_{j=1}^{m_i} \|\widetilde{\varphi}_{ik}\|_{C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\widehat{\Omega}_{\mathbf{i}}\times T_{\mathbf{t}})} + \sum_{i=1}^{k}\sum_{j=1}^{l_i} \|\widetilde{\psi}_{ij}\|_{C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega\times\widehat{T}_{\mathbf{i},\mathbf{t}})} + \|q\|_{C(\Omega\times T_{\mathbf{t}})}\Big),$$

where M is a positive constant independent of φ_{ij} $(j = 1, \dots, m_i;\ i = 1, \dots, n)$, ψ_{ij} $(j = 1, \dots, l_i;\ i = 1, \dots, k)$, and q.

Remark 7.1. Notice that for the linear problem (7.11), (7.2), (7.3) (u_0, r)-well–posedness is equivalent to the strong well–posedness. Furthermore, for problem (7.11), (7.2), (7.3) Definitions 7.10 and 7.2 are equivalent to Definition 7.5.

7.2. Necessary and Sufficient Conditions of Strong Well–Posedness.

Theorem 7.1. *Let the function $f(\mathbf{x}, \mathbf{t}, Z)$ be continuously differentiable with respect to the phase variables, and let problem* (7.1)–(7.3) *be strongly (u_0, r)-well–posed for some $r > 0$. Then the linear homogeneous problem*

$$u^{(\boldsymbol{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}+\boldsymbol{l}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) u^{(\boldsymbol{\alpha})}. \qquad (7.11_0)$$

$$a_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\mathbf{0}_i,\ \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) + b_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\ \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) = 0$$
$$\textit{for}\ \ (\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) \in \widehat{\Omega}_{\mathbf{i}} \times T\ \ (j = 1, \dots, m_i;\ \ i = 1, \dots, n), \quad (7.2_0)$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = 0,$$
$$\textit{for}\ \ (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega_{\mathbf{i}} \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i;\ i = 1, \dots, k) \quad (7.3_0)$$

is well–posed, where $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) = f_{\boldsymbol{\alpha}}\big(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\boldsymbol{m}+\boldsymbol{l}}[u_0(\mathbf{x}, \mathbf{t})]\big)$ $(\boldsymbol{\alpha} < \boldsymbol{m} + \boldsymbol{l})$.

Theorem 7.2. *Let the function $f(\mathbf{t}, Z)$ be locally Lipschitz continuous with respect to the principal phase variables. Then there exists $\delta > 0$ such that problem* (7.1),(7.4) *has a solution in $T_{\boldsymbol{\delta}} = [0,\delta] \times \cdots \times [0,\delta]$. Furthermore, if f is locally Lipschitz continuous with respect to all phase variables, then such solution is unique.*

Corollary 7.1. *Let the function $f(\mathbf{t}, Z)$ be locally Lipschitz continuous with respect to the principal phase variables, and let there exist $M > 0$ such that*

$$|f(\mathbf{t}, \mathbf{Z})| \leq M(1 + \|\mathbf{Z})\|) \quad \text{for} \quad (\mathbf{t}, \mathbf{Z}) \in T \times \mathbb{R}^{\mathbf{m}}. \tag{7.12}$$

Then problem (7.1),(7.4) *is solvable in T. Furthermore, if f is locally Lipschitz continuous with respect to all phase variables, then problem* (7.1),(7.4) *is strongly well–posed.*

Remark 7.2. In Theorem 7.2 Lipschitz continuity of $f(\mathbf{t}, Z)$ with respect to the principal phase variables is essential and, generally speaking, it cannot be replaced by Hölder continuity with exponent $\gamma \in (0,1)$. Indeed, in the rectangle $[0,1] \times [0,2]$ consider the initial value problem

$$u^{(1,1)} = \frac{1}{1-\gamma} |u^{(0,1)}|^{\gamma} \operatorname{sgn}(u^{(0,1)}), \tag{7.13}$$

$$u(0, t_2) = \frac{1}{2}(t_2 - 1)^2 \text{ for } t_2 \in [0,2], \quad u^{(1,0)}(t_1, 0) = 0 \text{ for } t_1 \in [0,1], \tag{7.14}$$

where $\gamma \in (0,1)$ is an arbitrary number. Problem $(7.13), (7.14)$ has the unique *absolutely continuous* solution

$$u(t_1, t_2) = \frac{1}{2} + \int\limits_0^{t_2} (t_1 + |s-1|^{1-\gamma})^{\frac{1}{1-\gamma}} \operatorname{sgn}(s-1)\, ds,$$

which is not a classical solution because

$$u^{(0,1)}(t_1, t_2) = (t_1 + |t_2 - 1|^{1-\gamma})^{\frac{1}{1-\gamma}} \operatorname{sgn}(t_2 - 1)$$

is discontinuous along the line $t_2 = 1$.

Set: $\mathbf{0}_n = (\underbrace{0,\ldots,0}_{n}),\quad \mathbf{1}_n = (\underbrace{1,\ldots,1}_{n}),\quad \Xi_n = \{\boldsymbol{\sigma} \mid \mathbf{0}_n < \boldsymbol{\sigma} < \mathbf{1}_n\}.$

Theorem 7.3. *Let the function $f(\mathbf{x},\mathbf{t},\mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let there exist functions $P_{i\boldsymbol{\alpha}} \in C(\Omega \times T)$ ($\boldsymbol{\alpha} < \mathbf{m}+\boldsymbol{l}$; $i = 1,2$) such that:*

$$(A_1)\qquad P_{1\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t}) \le f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t},\mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t})$$
$$\textit{for}\quad (\mathbf{x},\mathbf{t},\mathbf{Z}) \in \Omega \times T \times \mathbb{R}^{\mathbf{m}} \quad (\boldsymbol{\alpha} < \mathbf{m}+\boldsymbol{l}); \tag{7.15}$$

(A_2) *For every $\boldsymbol{\sigma} \in \Xi_n \cup \{\mathbf{1}_n\}$, $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ and arbitrary measurable functions $\rho_{\boldsymbol{\alpha}} \in L^\infty(\Omega_{\boldsymbol{\sigma}})$ ($\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}$) satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}+\boldsymbol{l}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}},\mathbf{t}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}+\boldsymbol{l}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}},\mathbf{t}) \;\; \textit{for}\; \mathbf{y} \in \Omega_{\boldsymbol{\sigma}}\; (\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}), \tag{7.16}$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{y}) v^{(\boldsymbol{\alpha})}, \tag{7.17}$$

$$a_{ij} v^{((j-1)\mathbf{1}_i)}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{y}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{y}}_\mathbf{i}) = 0 \;\; (j = 1,\ldots,m_i;\; i \in \operatorname{supp} \boldsymbol{\sigma}) \tag{7.18}$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}}-\mathbf{1}_{\boldsymbol{\sigma}}}(\Omega_{\boldsymbol{\sigma}})$. Then problem (7.1)–(7.3) *is strongly well–posed.*

Consider the equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = f(\mathbf{x},\mathbf{t},\widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]) + q(\mathbf{x},\mathbf{t},\mathcal{D}^{\mathbf{m}}[u],\mathcal{D}^{\mathbf{m}+\boldsymbol{l}-\mathbf{1}}[u]). \tag{7.19}$$

Theorem 7.4. *Let the function $f(\mathbf{x},\mathbf{t},\mathbf{Z})$ satisfy all of the conditions of Theorem 7.3, and let the function $q \in C(\Omega \times T \times \mathbb{R}^{\mathbf{m}})$ be locally Lipschitz continuous with respect to the principal phase variables and*

$$\lim_{\|\mathbf{Z}\| \to +\infty} \frac{|q(\mathbf{x},\mathbf{Z})|}{\|\mathbf{Z}\|} = 0 \tag{7.20}$$

uniformly on Ω. Then problem (7.19), (7.2), (7.3) *is solvable. Moreover, if $q(\mathbf{x},\mathbf{t},\mathbf{Z})$ is locally Lipschitz continuous with respect to all phase variables, then problem* (7.19), (7.2), (7.3) *is strongly well–posed.*

7.3. **Local Solvability and Continuation of a Solution.** Set:

$$\Psi(\mathbf{x}) = \left(\left(\psi_{ij}^{(\mathbf{j^{i-1}} + \boldsymbol{\alpha} + \boldsymbol{\beta})}(\mathbf{x}, \mathbf{0}_k) \right)_{1 \le j \le l_i}^{\boldsymbol{\alpha} \le \mathbf{m};\, \boldsymbol{\beta} \le \widehat{\boldsymbol{l}}^{\mathbf{i}}} \right)_{i=1}^{k},$$

$$p\Big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big) = f\Big(\mathbf{x}, \mathbf{0}, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big).$$

Theorem 7.5. *Let the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(\mathbf{m})} = p\Big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big), \tag{7.21}$$

$$a_{ij}\, v^{(\mathbf{m^{i-1}} + (j-1)\mathbf{1_i})}(\mathbf{0_i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, u^{(\mathbf{m^{i-1}} + (j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ij}^{(\mathbf{m^{i-1}} + \boldsymbol{l})}(\widehat{\mathbf{x}}_\mathbf{i}, \mathbf{0}_k)$$

$$\textit{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (j = 1, \ldots, m_i; \ \ i = 1, \ldots, n). \tag{7.22}$$

Furthermore, let for every $\boldsymbol{\sigma} \in \Xi_n$ and $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \widehat{\Omega}_{\boldsymbol{\sigma}}$, the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) v^{(\boldsymbol{\alpha})}, \tag{7.23}$$

$$a_{ij} v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \ \ (j = 1, \ldots, m_i;\ i \in \operatorname{supp} \boldsymbol{\sigma}), \tag{7.24}$$

where

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x}) = f_{\boldsymbol{\alpha} + \boldsymbol{l}}\Big(\mathbf{x}, \mathbf{0}_k, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v_0(\mathbf{x})]\Big) \qquad (\boldsymbol{\alpha} < \boldsymbol{m}),$$

has only the trivial solution.

Then there exist $\delta_i \in (0, T_i]$ $(i = 1, \ldots, k)$ such that in the domain $\Omega \times T_{\boldsymbol{\delta}}$ problem (7.1)–(7.3) *has a unique solution u satisfying the condition*

$$u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{0}_k) = v_0(\mathbf{x}). \tag{7.25}$$

Corollary 7.2. *Let $n = 1$, the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(m_1)} = p\big(x_1, v, \ldots, v^{(m_1 - 1)}\big), \tag{7.26}$$

$$a_{1j}\, v^{(j-1)}(0) + b_{1j}\, v^{(j-1)}(\omega_1) = \varphi_{1j}^{(\boldsymbol{l})}(\mathbf{0}_k) \quad (j = 1, \ldots, m_1). \tag{7.27}$$

Then there exist $\delta_i \in (0, T_i]$ $(i = 1, \ldots, k)$ *such that in the domain* $\Omega \times T_{\boldsymbol{\delta}}$ *problem* (7.1)–(7.3) *has a unique solution* u *satisfying the condition*

$$u^{(\boldsymbol{l})}(x_1, \mathbf{0}_k) = v_0(x_1). \tag{7.28}$$

Remark 7.3. In Theorem 7.5 and Corollary 7.2 the requirement of strong isolation of the solution v_0 is essential, and it cannot be replaced by well-posedness of problem (7.21), (7.22). In order to illustrate this, consider the problem

$$u^{(1,1)} = \big(u^{(0,1)}\big)^3 - t^2\, u^{(0,1)}, \tag{7.29}$$

$$u(\omega, t) - u(0, t) = \int\limits_0^t s\,\sin\frac{1}{s}\,ds, \qquad u(x, 0) = 0. \tag{7.30}$$

For this case problem (7.21), (7.22) is the following one:

$$v' = v^3, \quad v(\omega) - v(0) = 0. \tag{7.31}$$

By Corollary 4.2 and Theorem 4.4 from [19], problem (7.31) has a unique solution $v_0(y) \equiv 0$ and is well–posed. On the other hand, it is clear, that $v_0(y) \equiv 0$ is not strongly isolated.

Our goal is to show that problem (7.29), (7.30) has no solution in the rectangle $[0, \omega] \times [0, \delta]$ no matter how small $\delta > 0$ is.

Assume the contrary that problem (7.29), (7.30) has a solution u in $[0, \omega] \times [0, \delta]$ for some $\delta > 0$. Then for an arbitrarily fixed $t \in (0, \delta]$, the function $v(\cdot) = u^{(0,1)}(\cdot, t)$ is a solution of the problem

$$v' = v^3 - t^2\, v, \tag{7.32}$$

$$v(\omega) - v(0) = t \sin\frac{1}{t} \tag{7.33}$$

containing the parameter $t \in (0, \delta]$. Moreover, if problem (7.29), (7.30) has a solution, then v is a solution (7.32), (7.33) depending continuously on the parameter $t \in (0, \delta]$.

For every fixed $t \in (0, \delta]$ equation (7.32) has three constant solutions: $v_0(x) = 0$, $v_1(x) = t$ and $v_2(x) = -t$. Due to the existence and uniqueness theorem, no nonconstant solution v of equation (7.32) intersects v_0, v_1 or v_2, and thus $v'(x) \neq 0$ for $x \in [0, \omega]$. Let

$$k > \frac{1}{2\pi\delta} \quad \text{and} \quad t \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Then $v(\omega) > v(0)$ and $v'(x) > 0$ for $x \in [0, \omega]$. Therefore, either

$$v(x) > t \quad \text{for} \quad x \in [0, \omega],$$

or

$$v(x) \in (-t, 0) \quad \text{for} \quad x \in [0, \omega].$$

If $t = \frac{1}{\frac{\pi}{2} + 2\pi k}$, then $v(\omega) - v(0) = t$, and consequently

$$v(x) \notin (-t, 0) \quad \text{for} \quad x \in [0, \omega].$$

From the aforesaid, in view of continuity of $u^{(0,1)}$ in $[0, \omega] \times [0, \delta]$, it follows that

$$u^{(0,1)}(x, t) > t \quad \text{for} \quad t \in \Big(\frac{1}{\pi + 2\pi k}, \frac{1}{2\pi k}\Big).$$

Similarly, one can show that

$$u^{(0,1)}(x, t) < -t \quad \text{for} \quad t \in \Big(\frac{1}{2\pi(k+1)}, \frac{1}{\pi + 2\pi k}\Big).$$

However, the latter two inequalities imply that $u^{(0,1)}(x, t)$ is discontinuous along the lines $t = \frac{1}{\pi k}$ $(k = 1, 2, \dots)$. Thus we have proved that problem (7.29), (7.30) has no solution in $[0, \omega] \times [0, \delta]$ for any $\delta > 0$.

Definition 7.6. We say that the family of problems

$$u^{(\mathbf{m})} = f_\lambda(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{7.34}$$

$$a_{ik}\, u^{(\mathbf{m}^{i-1} + (k-1)\mathbf{1}_i)}(\mathbf{0}_i, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ik}\, u^{(\mathbf{m}^{i-1} + (k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{\lambda\, ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i})$$
$$\text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \;\; (k = 1, \dots, m_i; \;\; i = 1, \dots, n), \tag{7.35}$$

$(\lambda \in \Lambda)$ is *uniformly strongly* (u_λ, r)*–well-posed* for some $r > 0$, if:

(I) u_λ is the unique solution of problem (7.34), (7.35) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_\lambda; r)$;

(II) There exist positive numbers M and δ_0 *independent of* λ such that for arbitrary $\delta \in (0, \delta_0)$, and an arbitrary admissible perturbation $\big(\widetilde{f}_\lambda, \widetilde{\varphi}_{\lambda\,1\,1}, \dots, \widetilde{\varphi}_{\lambda\,1\,m_1}, \dots, \widetilde{\varphi}_{\lambda\,n\,1}, \dots, \widetilde{\varphi}_{\lambda\,n\,m_n}\big)$ satisfying the conditions

$$|\widetilde{f}_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})| \le \delta_0 \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}),$$

$$|\widetilde{f}_\lambda(\mathbf{x}, \mathbf{Z})| \le \delta \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}},$$

$$\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\widetilde{\varphi}_{\lambda\,ik}\|_{C^{\widehat{\mathbf{m}}_i}(\widehat{\Omega}_{\mathbf{i}})} \le \delta,$$

the problem

$$u^{(\mathbf{m})} = \widetilde{f}_\lambda(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]),$$

$$a_{ik}\, u^{(\mathbf{m}^{i-1} + (k-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_{\mathbf{i}}) + b_{ik}\, u^{(\mathbf{m}^{i-1} + (k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\, \widehat{\mathbf{x}}_{\mathbf{i}})$$
$$= \widetilde{\varphi}_{\lambda\,ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}) \ \text{ for } \ \widehat{\mathbf{x}}_{\mathbf{i}} \in \widehat{\Omega}_{\mathbf{i}} \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n),$$

has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_\lambda; r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_\lambda; M\,\delta)$.

Definition 7.7. A family of solutions $\{u_\lambda\}_{\lambda \in \Lambda}$ is said to be *uniformly strongly isolated* if the family of problems (7.34), (7.35) ($\lambda \in \Lambda$) is *uniformly strongly* (u_λ, r)*–well-posed* for some $r > 0$.

Definition 7.8. Let $\mathbf{I} = [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times I_s \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$, where $\eta_j \in (0, \tau_j]$ $(j = 1, \dots, s-1, s+1, k)$ and

$$\text{either } \ I_s = [0, \eta_s), \ \ \eta_s \in (0, \tau_s], \qquad \text{or } \ I_s = [0, \eta_s], \ \ \eta_s \in (0, \tau_s),$$

and let u be a solution of problem (7.1)–(7.3) in the domain $\Omega \times \mathbf{I}$. We say that u is *continuable* with respect to the variable t_s, if there exist $\zeta_s \in [\eta_s, \tau_s]$ for $I_s = [0, \eta_s)$, or $\zeta_s \in (\eta_s, \tau_s]$ for $I_s = [0, \eta_s]$, and a solution u^* of problem (7.1)–(7.3) in the domain $\Omega \times [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times [0, \zeta_s] \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$ such that

$$u^*(\mathbf{x}, \mathbf{t}) = u(\mathbf{x}, \mathbf{t}) \quad \text{for} \quad (\mathbf{x}, \mathbf{t}) \in \Omega \times \mathbf{I}.$$

Otherwise u is called *non–continuable.*

Theorem 7.6. *Let the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be locally Lipschitz continuous with respect to the principal phase variables, let* $\mathbf{I} = [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times I_s \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$, *where*

$$\text{either } I_s = [0, \eta_s), \ \ \eta_s \in (0, \tau_s], \qquad \text{or } I_s = [0, \eta_s], \ \ \eta_s \in (0, \tau_s),$$

let u be a non-continuable with respect to the t_i variable solution of problem (7.1)–(7.3) *defined on the set* $\Omega\times\mathbf{I}$, *and let for each $\boldsymbol{t} \in I$ $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{t_0})$ be a solution of the problem*

$$v^{(\mathbf{m})} = P[u(\mathbf{x}, \boldsymbol{t}); \boldsymbol{t}]\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big), \tag{7.36}$$

$$a_{ij}\, v^{(\mathbf{m^{i-1}}+(j-1)\mathbf{1_i})}(\mathbf{0_i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, u^{(\mathbf{m^{i-1}}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ij}^{(\mathbf{m^{i-1}}+\boldsymbol{l})}(\widehat{\mathbf{x}}_\mathbf{i}, \boldsymbol{t})$$

$$\text{for } \widehat{\mathbf{x}}_\mathbf{i} \in \widehat{\Omega}_\mathbf{i} \ \ (j = 1, \dots, m_i; \ \ i = 1, \dots, n) \tag{7.37}$$

where

$$P[u(\mathbf{x}, \boldsymbol{t}); \boldsymbol{t}]\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big) = f\Big(\mathbf{x}, \boldsymbol{t}, \mathcal{D}^{\mathbf{m}} \circ \widetilde{\mathcal{D}}^{\boldsymbol{l}}[u(\mathbf{x}, \boldsymbol{t})], \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big).$$

If the family of solutions $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{t_0})$ ($\boldsymbol{t} \in I$) is uniformly strongly isolated, then either $I_s = [0, \tau_s]$, or $I_s = [0, \eta_s)$ and

$$\lim_{t_s \to \eta_s} \|u(t_s, \cdot)\|_{\widetilde{C}^{\mathbf{m}+\boldsymbol{l}}(\Omega\times\widehat{\mathbf{I}}_\mathbf{s})} = +\infty, \tag{7.38}$$

where $\widehat{\mathbf{I}}_\mathbf{s} = [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$.

Corollary 7.3. *Let $\mathbf{I}$ be the set from Theorem 7.6, and let u be a non-continuable with respect to the t_i variable solution of problem* (7.1)–(7.3) *defined on the set* $\Omega\times\mathbf{I}$. *Furthermore, the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let there exist functions $P_{s\boldsymbol{\alpha}} \in C(\Omega \times T)$ ($\boldsymbol{\alpha} < \mathbf{m} + \boldsymbol{l}$; $i = 1, 2$) such that conditions (A_1) and (A_2) of Theorem 7.3 hold. Then either $I_s = [0, \tau_s]$, or $I_s = [0, \eta_s)$ and* (7.38) *hods.*

7.4. **Initial–Periodic Problems.** In this subsection consider the initial–periodic problem

$$u^{(\mathbf{m})} = f(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{7.39}$$

$$u(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}, \mathbf{t}) = u(\mathbf{x}, \mathbf{t}) \quad \text{for} \quad (\mathbf{x}, \mathbf{t}) \in \mathbb{R}^n \times T \ \ (i = 1, \dots, n), \tag{7.40}$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = 0$$
$$\text{for} \ \ (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i; \ i = 1, \dots, k), \tag{7.41}$$

where $\boldsymbol{\omega}_{\mathbf{i}} = (0, \dots, \omega_i, \dots, 0)$, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ and $f \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times T \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$, and $\psi_{ij} \in C_{\boldsymbol{\omega}}^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\mathbb{R}^n \times \widehat{T}_{\mathbf{i}})$ $(j = 1, \dots, l_i; \ i = 1, \dots, k)$.

Along with equation m (7.39) and the initial conditions (7.41) consider the equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = f(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]) + \widetilde{f}(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]), \tag{7.42}$$

and the conditions

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = \psi_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) + \widetilde{\psi}_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}})$$
$$\text{for} \ \ (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega_{\mathbf{i}} \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i; \ i = 1, \dots, k). \tag{7.43}$$

A vector function $\big(\widetilde{f}; \widetilde{\psi}_{1\,1}, \dots, \widetilde{\psi}_{1\,l_1}, \dots, \widetilde{\psi}_{k\,1}, \dots, \widetilde{\psi}_{k\,l_k}\big)$ is said to be an *admissible perturbation*, if $\widetilde{f} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times T \times \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$ is locally Lipschitz continuous with respect to the *principal* phase variables, and $\widetilde{\psi}_{ij} \in C_{\boldsymbol{\omega}}^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\mathbb{R}^n \times \widehat{T}_{\mathbf{i}})$ $(j = 1, \dots, l_i;$ $i = 1, \dots, k)$.

Definition 7.9. Let u_0 be a solution of problem (7.39)–(7.41), and $r > 0$. We say that problem (7.39)–(7.41) is (u_0, r)-*well-posed*, if for arbitrary $t_i \in (0, \tau_i]$ $(i = 1, \dots, k)$:

(I) $u_0(\mathbf{x}, \mathbf{t})$ is the unique solution of problem (7.39)–(7.41) in the ball $\widetilde{\mathbf{B}}_{\boldsymbol{\omega}}^{\mathbf{m}+\boldsymbol{l}}[\mathbb{R}^n \times T_{\mathbf{t}}](u_0; r)$;

(II) there exist a positive constant δ_0 and an increasing continuous $\varepsilon : [0, \delta_0] \to [0, +\infty)$ such that $\varepsilon(0) = 0$ and for any $\delta(\mathbf{t}) \in (0, \delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f};\widetilde{\psi}_{1\,1},\ldots,\widetilde{\psi}_{1\,l_1},\ldots,\widetilde{\psi}_{k\,1},\ldots,\widetilde{\psi}_{k\,l_k}\big)$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\tau},\mathbf{Z})| \le \delta_0 \quad \text{for} \quad (\mathbf{x},\boldsymbol{\tau},\mathbf{Z}) \in \Omega \times T_{\mathbf{t}} \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}+\boldsymbol{l}}), \tag{7.44}$$

$$|\widetilde{f}(\mathbf{x},\boldsymbol{\tau},\mathbf{Z})| \le \delta(\mathbf{t}) \quad \text{for} \quad (\mathbf{x},\boldsymbol{\tau},\mathbf{Z}) \in \Omega \times T_{\mathbf{t}} \times \mathbb{R}^{\mathbf{m}}, \tag{7.45}$$

$$\sum_{i=1}^{k}\sum_{j=1}^{l_i} \|\widetilde{\psi}_{ij}\|_{C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega\times\widehat{T}_{\mathbf{i},\,\mathbf{t}})} \le \delta(\mathbf{t}), \tag{7.46}$$

problem (7.42), (7.40),(7.43) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}_{\boldsymbol{\omega}}[\mathbb{R}^n \times T_{\mathbf{t}}](u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}_{\boldsymbol{\omega}}[\mathbb{R}^n \times T_{\mathbf{t}}]\big(u_0;\varepsilon(\delta(\mathbf{t}))\big)$.

Definition 7.10. Let u_0 be a solution of problem (7.39)–(7.41), and $r > 0$. We say that problem (7.39)–(7.41) is *strongly* (u_0,r)*-well-posed*, if for arbitrary $t_i \in (0,\eta_i]$ $(i = 1,\ldots,k)$:

(I) $u_0(\mathbf{x})$ is the unique solution of problem (7.39)–(7.41) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}[\Omega \times T_{\mathbf{t}}](u_0;r)$;

(II) there exist a positive constants δ_0 and M such that for any $\delta(\mathbf{t}) \in (0,\delta_0]$ and an arbitrary admissible perturbation $\big(\widetilde{f};\widetilde{\psi}_{1\,1},\ldots,\widetilde{\psi}_{1\,l_1},\ldots,\widetilde{\psi}_{k\,1},\ldots,\widetilde{\psi}_{k\,l_k}\big)$ satisfying conditions (7.44)–(7.46), problem (7.42), (7.40),(7.43) has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}_{\boldsymbol{\omega}}[\mathbb{R}^n \times T_{\mathbf{t}}](u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}+\boldsymbol{l}}_{\boldsymbol{\omega}}[\mathbb{R}^n \times T_{\mathbf{t}}](u_0;M\delta(\mathbf{t}))$.

Definition 7.11. Problem (7.39)–(7.41) is called *well–posed* (*strongly well–posed*) if it has a unique solution u_0 and it is (u_0,r)–well–posed (strongly (u_0,r)–well-posed) for every $r > 0$.

Definition 7.12. A solution u_0 of problem (7.39)–(7.41) is called *strongly isolated*, if problem (7.39)–(7.41) is strongly (u_0,r)-well–posed for some $r > 0$.

Theorem 7.7. *Let the function* $f \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times T \times \mathbb{R}^{\mathbf{m}+\mathbf{1}})$ *be continuously differentiable with respect to the phase variables, and let problem* (7.39)–(7.41) *be strongly* (u_0, r)*-well–posed for some* $r > 0$*. Then the linear homogeneous problem*

$$u^{(\boldsymbol{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}+\boldsymbol{l}} p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) u^{(\boldsymbol{\alpha})},$$
$$u(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}, \mathbf{t}) = u(\mathbf{x}, \mathbf{t}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{t}) \in \mathbb{R}^n \times T \quad (i = 1, \dots, n),$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = 0,$$
$$\textit{for} \ \ (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i; \ i = 1, \dots, k)$$

is well–posed, where $p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) = f_{\boldsymbol{\alpha}}\big(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\boldsymbol{m}+\boldsymbol{l}}[u_0(\mathbf{x}, \mathbf{t})]\big)$ $(\boldsymbol{\alpha} < \boldsymbol{m} \ + \boldsymbol{l})$.

Theorem 7.8. *Let the function* $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ *be continuously differentiable function with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n \times T)$ $(\boldsymbol{\alpha} < \mathbf{m} + \boldsymbol{l}; \ i = 1, 2)$ *such that:*

(A_1) $\quad P_{1\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) \le f_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t})$

$$\textit{for} \quad (\mathbf{x}, \mathbf{t}, \mathbf{Z}) \in \Omega \times T \times \mathbb{R}^{\mathbf{m}+\boldsymbol{l}+\mathbf{1}} \quad (\boldsymbol{\alpha} < \mathbf{m} + \boldsymbol{l});$$

(A_2) *For every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}_n$*,* $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ *and arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}} \in L^{\infty}_{\boldsymbol{\omega}}(\mathbb{R}^{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$ *satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}+\boldsymbol{l}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}+\boldsymbol{l}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t}) \ \ \textit{for}\, \mathbf{y} \in \Omega_{\boldsymbol{\sigma}} \ (\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}),$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{y}) v^{(\boldsymbol{\alpha})},$$
$$v(\mathbf{y} + \boldsymbol{\omega}_{\mathbf{i}}) = v(\mathbf{y}) \ \ (i \in \operatorname{supp} \boldsymbol{\sigma}).$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}}-\mathbf{1}_{\boldsymbol{\sigma}}}_{\boldsymbol{\omega}}(\mathbb{R}^{\boldsymbol{\sigma}})$*. Then problem* (7.39)–(7.41) *is strongly well–posed.*

Set:

$$\Psi(\mathbf{x}) = \left(\left(\psi_{ij}^{(\mathbf{j^{i-1}}+\boldsymbol{\alpha}+\boldsymbol{\beta})}(\mathbf{x}, \mathbf{0}_k) \right)_{1\le j\le l_i}^{\boldsymbol{\alpha}\le\widehat{\mathbf{m}}^{\mathbf{i}};\, \boldsymbol{\beta}\le\widehat{\boldsymbol{l}}^{\mathbf{i}}} \right)_{i=1}^{k},$$

$$p\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big) = f\Big(\mathbf{x}, \mathbf{0}, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big).$$

Theorem 7.9. *Let the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(\mathbf{m})} = p\big(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big), \tag{7.47}$$

$$v(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}, \mathbf{t}) = v(\mathbf{x}, \mathbf{t}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{t}) \in \mathbb{R}^n \times T \ \ (i = 1, \ldots, n). \tag{7.48}$$

Furthermore, let for every $\boldsymbol{\sigma} \in \Xi_n$ and $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \widehat{\Omega}_{\boldsymbol{\sigma}}$, the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}) v^{(\boldsymbol{\alpha})}, \tag{7.49}$$

$$v(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}, \mathbf{t}) = v(\mathbf{x}, \mathbf{t}) \quad (i \in \operatorname{supp} \boldsymbol{\sigma}), \tag{7.50}$$

where $\rho_{\boldsymbol{\alpha}}(\mathbf{x}) = f_{\boldsymbol{\alpha}}\Big(\mathbf{x}, \mathbf{0}_k, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v_0(\mathbf{x})]\Big)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$, have only the trivial solution. Then there exist $\delta_i \in (0, T_i]$ $(i = 1, \ldots, k)$ such that in the domain $\mathbb{R}^n \times T_{\boldsymbol{\delta}}$ problem (7.39)–(7.41) *has a unique solution u satisfying the condition*

$$u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{0}_k) = v_0(\mathbf{x}). \tag{7.51}$$

Corollary 7.4. *Let $n = 1$, the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be continuously differentiable function with respect to the phase variables, and let v_0 be a strongly isolated solution of the problem*

$$v^{(m_1)} = p\big(x_1, v, \ldots, v^{(m_1-1)}\big), \qquad v(x_1 + \omega_1) = v(x_1).$$

Then there exist $\delta_i \in (0, T_i]$ $(i = 1, \ldots, k)$ such that in the domain $\mathbb{R}^n \times T_{\boldsymbol{\delta}}$ problem (7.39)–(7.41) *has a unique solution u satisfying the condition*

$$u^{(\boldsymbol{l})}(x_1, \mathbf{0}_k) = v_0(x_1).$$

Remark 7.4. Conditions of Theorem 7.9 do not guarantee unique solvability of problem (7.39)–(7.41). Indeed, consider the problem

$$u^{(1,1)} = \sin(u^{(0,1)}) + t\, f_0(x, t, u^{(1,0)}, u^{(0,1)}, u), \tag{7.52}$$

$$u(x + \omega, t) = u(x, t), \quad u^{(1,0)}(x, 0) = 0, \tag{7.53}$$

where $f_0 \in C_\omega(\mathbb{R} \times [0, \tau] \times \mathbb{R}^3)$ is a continuously differentiable function with respect to the phase variables. For this case problem (7.47), (7.48) has the form

$$v' = \sin v, \qquad v(x + \omega) = v(x).$$

The latter problem has a countable set of strongly isolated solutions $z_k = k\pi$ $(k = 0, \pm 1, \dots)$. By Theorem 7.9, for every integer k there exists $\delta_k > 0$ such that in the domain $\mathbb{R}^n \times [0, \delta_k]$, problem (7.52), (7.53) has a unique solution u_k satisfying the condition

$$u_k^{(1,0)}(x, 0) = k\pi.$$

Definition 7.13. We say that the family of problems

$$u^{(\mathbf{m})} = f_\lambda(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]), \tag{7.54}$$

$$u(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n \quad (i = 1, \dots, n), \tag{7.55}$$

$(\lambda \in \Lambda)$ is *uniformly strongly* (u_λ, r)*–well-posed* for some $r > 0$, if:

(I) u_λ is the unique solution of problem (7.54), (7.55) in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_\lambda; r)$;

(II) There exist positive numbers M and δ_0 *independent of* λ such that for arbitrary $\delta \in (0, \delta_0)$, and an arbitrary admissible perturbation $\big(\widetilde{f}_\lambda; \widetilde{\varphi}_{\lambda\,1\,1}, \dots, \widetilde{\varphi}_{\lambda\,1\,m_1}, \dots, \widetilde{\varphi}_{\lambda\,n\,1}, \dots, \widetilde{\varphi}_{\lambda\,n\,m_n}\big)$ satisfying the conditions

$$|\widetilde{f}_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z})| \le \delta_0 \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}} \ \ (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}),$$
$$|\widetilde{f}_\lambda(\mathbf{x}, \mathbf{Z})| \le \delta \quad \text{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{\mathbf{m}},$$

the problem

$$u^{(\mathbf{m})} = \widetilde{f}_\lambda(\mathbf{x}, \widetilde{\mathcal{D}}^{\mathbf{m}}[u]),$$
$$u(\mathbf{x} + \boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^n \quad (i = 1, \dots, n)$$

has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_\lambda; r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{m}}_{\boldsymbol{\omega}}(u_\lambda; M\,\delta)$.

Definition 7.14. A family of solutions $\{u_\lambda\}_{\lambda\in\Lambda}$ is said to be *uniformly strongly isolated* if the family of problems (7.54), (7.55) ($\lambda \in \Lambda$) is *uniformly strongly* (u_λ, r)*–well-posed* for some $r > 0$.

Definition 7.15. Let $\mathbf{I} = [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times I_s \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$, where $\eta_j \in (0, \tau_j]$ $(j = 1, \dots, s-1, s+1, k)$ and

$$\text{either} \;\; I_s = [0, \eta_s), \;\; \eta_s \in (0, \tau_s], \qquad \text{or} \;\; I_s = [0, \eta_s], \;\; \eta_s \in (0, \tau_s),$$

and let u be a solution of problem (7.39)–(7.41) in the domain $\mathbb{R}^n \times \mathbf{I}$. We say that u is *continuable* with respect to the variable t_i , if there exist $\zeta_s \in [\eta_s, \tau_s]$ for $I_s = [0, \eta_i)$, or $\zeta_s \in (\eta_s, \tau_s]$ for $I_s = [0, \eta_s]$, and a solution u^* of problem (7.39)–(7.41) in the domain $\mathbb{R}^n \times [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times [0, \zeta_s] \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$ such that

$$u^*(\mathbf{x}, \mathbf{t}) = u(\mathbf{x}, \mathbf{t}) \quad \text{for} \quad (\mathbf{x}, \mathbf{t}) \in \Omega \times \mathbf{I}.$$

Otherwise u is called *non–continuable.*

Theorem 7.10. *Let the function $f(\mathbf{x}, \mathbf{t}, \mathbf{Z})$ be locally Lipschitz continuous with respect to the principal phase variables, let $\mathbf{I} = [0, \eta_1] \times \cdots \times [0, \eta_{s-1}] \times I_s \times [0, \eta_{s+1}] \times \cdots \times [0, \eta_k]$, where*

$$\textit{either} \;\; I_s = [0, \eta_s), \;\; \eta_s \in (0, \tau_s], \qquad \textit{or} \;\; I_s = [0, \eta_s], \;\; \eta_s \in (0, \tau_s),$$

let u be a non-continuable with respect to the t_s variable solution of problem (7.39)–(7.41) *defined on the set $\mathbb{R}^n \times \mathbf{I}$, and let $\boldsymbol{t} \in I$ $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{t_0})$ be a* *solution of the problem*

$$v^{(\mathbf{m})} = P[u(\mathbf{x},\boldsymbol{t});\boldsymbol{t}]\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big),$$

$$u(\mathbf{x}+\boldsymbol{\omega}_{\mathbf{i}}) = u(\mathbf{x}) \qquad (i=1,\dots,n),$$

where

$$P[u(\mathbf{x},\boldsymbol{t});\boldsymbol{t}]\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{m}}[v]\big) = f\Big(\mathbf{x},\boldsymbol{t},\mathcal{D}^{\mathbf{m}}\circ\widetilde{\mathcal{D}}^{\boldsymbol{l}}[u(\mathbf{x},\boldsymbol{t})],\widetilde{\mathcal{D}}^{\mathbf{m}}[v]\Big).$$

If the family of solutions $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x},\mathbf{t_0})$ $(\boldsymbol{t}\in I)$ *is uniformly strongly isolated, then either* $I_s=[0,\tau_s]$, *or* $I_s=[0,\eta_s)$ *and*

$$\lim_{t_s\to\eta_s}\|u(t_s,\cdot)\|_{C_{\boldsymbol{\omega}}^{\mathbf{m}+\boldsymbol{l}}(\mathbb{R}^n\times\widehat{\mathbf{I}}_{\mathbf{s}})} = +\infty, \tag{7.56}$$

where $\widehat{\mathbf{I}}_{\mathbf{s}} = [0,\eta_1]\times\cdots\times[0,\eta_{s-1}]\times[0,\eta_{s+1}]\times\cdots\times[0,\eta_k]$.

Remark 7.5. In Theorem 7.10 the requirement of uniform strong isolation of solutions $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x},\mathbf{t_0})$ $(\boldsymbol{t}\in I)$ cannot be replaced by strong isolation. As an example, in the strip $\mathbb{R}\times[0,2]$ consider the problem

$$u^{(1,1)} = (1-t)^4\,u^4\,u^{(0,1)} + \Big(-(1-t)^4+(1-t)^2\pi\sin\frac{2\pi}{1-t}\Big)u^6, \tag{7.57}$$

$$u(x+\omega,t)=u(x,t), \qquad u^{(1,0)}(x,0)=0. \tag{7.58}$$

The problem has a solution

$$u(t) = \frac{1}{1-t+\sin^2\frac{\pi}{1-t}}.$$

It is clear that $u(t)$ is a non-continuable solution of the problem (7.57), (7.58) in the rectangle $[0,\omega)\times[0,1)$. On the other hand, the limit (7.56) does not exist since

$$\lim_{k\to+\infty}\Big|u\Big(1-\frac{1}{\frac12+k\pi}\Big)\Big| + \Big|u'\Big(1-\frac{1}{\frac12+k\pi}\Big)\Big| = 2,$$

$$\lim_{k\to+\infty}\Big|u\Big(1-\frac{1}{k\pi}\Big)\Big| + \Big|u'\Big(1-\frac{1}{k\pi}\Big)\Big| = +\infty.$$

The reason for this is that the family of solutions

$$v(t) = u^{(0,1)}(x,t) = \frac{1-\frac{\pi}{(1-t)^2}\sin\frac{2\pi}{1-t}}{\Big(1-t+\sin^2\frac{\pi}{1-t}\Big)^2}$$

of the problem

$$v' = (1-t)^4 u^4(x,t)\, v + \Big(-(1-t)^4 + (1-t)^2 \pi \sin \frac{2\pi}{1-t} \Big) u^6(x,t),$$
$$v(0) = v(\omega),$$

($t \in [0,1)$) is not uniformly strongly isolated.

Remark 7.6. If the family of solutions $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{t_0})$ ($\boldsymbol{t} \in I$) is not uniformly strongly isolated, then problem (7.39)–(7.41) may have a non–continuable solution in the closed domain $\mathbb{R}^n \times \mathbf{I}$. To see this, in the domain $\mathbb{R} \times [0,2]$ consider the problem

$$u^{(1,1)} = \arctan(|u+1|) u^{(0,1)} + \arctan(u+1), \tag{7.59}$$

$$u(x+\omega, t) = u(x,t), \qquad u^{(1,0)}(x,0) = 0. \tag{7.60}$$

In the domain $\mathbb{R} \times [0,1]$ this problem has a solution $u(x,t) = 1-t$. Let us show that this solution is non–continuable. Assume the contrary, that it can be continued on the domain $\mathbb{R} \times [0,a]$ for some $a > 1$. Without loss of generality one can assume that

$$u^{(0,1)}(x,t) < 0, \;\; u(x,t) < 0 \quad \text{ for } \;\; (x,t) \in \mathbb{R}^n(1,a].$$

Consequently,

$$u^{(1,1)}(x,t) = \arctan(|u(x,t)+1|) u^{(0,1)}(x,t) + \arctan(u(x,t)) < 0$$
$$\text{for } \;\; (x,t) \in \mathbb{R} \times (1,a].$$

However, the latter inequality contradicts to the periodicity of $u^{(0,1)}$ with respect to the first variable. The obtained contradiction proves that u is a solution of problem (7.59), (7.60) that is a non–continuable on the closed set $\mathbb{R} \times [0,1]$.

8. Auxiliary Statements

In the $n+k$-dimensional box $\Omega \times T$ for the linear equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}+\boldsymbol{l}} p_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t})u^{(\boldsymbol{\alpha})} + q(\mathbf{x},\mathbf{t}) \tag{8.1}$$

consider the initial–boundary conditions

$$a_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_\mathbf{i}, \mathbf{t}) + b_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}, \mathbf{t}) = \varphi_{ij}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i}, \mathbf{t})$$
$$\text{for } (\widehat{\mathbf{x}}_\mathbf{i}, \mathbf{t}) \in \widehat{\Omega}_\mathbf{i} \times T \quad (j=1,\dots,m_i;\ \ i=1,\dots,n), \tag{8.2}$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_\mathbf{i}, \mathbf{x}, \widehat{\mathbf{t}}_\mathbf{i}\big) = \psi_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_\mathbf{i})$$
$$\text{for } (\mathbf{x}, \widehat{\mathbf{t}}_\mathbf{i}) \in \Omega \times \widehat{T}_\mathbf{i} \quad (j=1,\dots,l_i;\ i=1,\dots,k). \tag{8.3}$$

Lemma 8.1. *Let problem* (1.9) *have only the trivial solution for every* $i \in \{1,2,\dots,n-1\}$, *let* $p_{\boldsymbol{\alpha}} \in C(\Omega \times T)$ $(\boldsymbol{\alpha} < \mathbf{m}+\boldsymbol{l})$, *and let for every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}_n \cup \{\mathbf{1}_n\}$, $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \widehat{\Omega}_{\boldsymbol{\sigma}}$ *and* $\mathbf{t} \in T$ *the* $\boldsymbol{\sigma}$*–associated problem*

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t})v^{(\boldsymbol{\alpha})}, \tag{8.4}$$

$$a_{ij}v^{((j-1)\mathbf{1}_i)}(\mathbf{0}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (j=1,\dots,m_i;\ i \in \operatorname{supp}\boldsymbol{\sigma}), \tag{8.5}$$

have only the trivial solution. Then problem (8.1), (8.2), (8.3) *is well-posed.*

Proof. It is quite clear that it is enough to prove the lemma in the particular case, where $k=1$, $\varphi_{ij}(\widehat{\mathbf{x}}_\mathbf{i}, \mathbf{t}) \equiv 0$ $(j=1,\dots,m_i; i=1,\dots,n)$ and $\psi_{1j}(\mathbf{x}) \equiv 0$ $(j=1,\dots,l)$.

We prove the lemma by induction. For $n=2$, the validity of Lemma 8.1 was already proved (see [28, Theorem 1.1]). Let $n \geq 3$, and let us assume that the theorem is true for $(n-1)$-dimensional problem.

Let u be a solution of problem (8.1)–(8.3). Then u admits the following representations:

$$u^{(\boldsymbol{l})}(\mathbf{x},t) = \mathcal{G}(Q_0[u])(\mathbf{x},t),$$
$$u^{(\mathbf{m_k})}(\mathbf{x},t) = \mathcal{G}_j(Q_k[u])(\mathbf{x},t) \qquad (i = 1,\dots,n),$$

where

$$Q_0[u](\mathbf{x},t) = \sum_{\boldsymbol{\alpha}\le\mathbf{m}}\sum_{\boldsymbol{\beta}<\boldsymbol{l}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},t)u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} + q(\mathbf{x},t),$$
$$Q_k[u](\mathbf{x},t) = \sum_{\boldsymbol{\alpha}<\mathbf{m_k}}\sum_{\boldsymbol{\beta}\le\widehat{\mathbf{m}}_{\mathbf{k}}+\boldsymbol{l}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},t)u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} + q(\mathbf{x},t) \quad (k = 1,\dots,n),$$

$\mathcal{G}_0$ is the Green's operator of the problem

$$v^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x},\mathbf{t})v^{(\boldsymbol{\alpha})}, \tag{8.6}$$

$$a_{ij}v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i},\,\widehat{\mathbf{x}}_{\mathbf{i}}) + b_{ij}\,v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\,\widehat{\mathbf{x}}_{\mathbf{i}}) = 0 \ \ (j = 1,\dots,m_i;\ i = 1,\dots,n), \tag{8.7}$$

and $\mathcal{G}_k$ $(k = 1,\dots,n-1)$ is Green's operator of the associated problem

$$v^{(\widehat{\mathbf{m}}_{\mathbf{k}}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_{\mathbf{k}}+\boldsymbol{l}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\sigma}},\mathbf{x}_{\boldsymbol{\sigma}},\mathbf{t})v^{(\boldsymbol{\alpha})}, \tag{8.8}$$

$$a_{ij}v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i},\,\widehat{\mathbf{x}}_{\mathbf{i}}) + b_{ij}\,v^{((j-1)\mathbf{1}_k)}(\boldsymbol{\omega}_{\mathbf{i}},\,\widehat{\mathbf{x}}_{\mathbf{i}}) = 0 \ \ (j = 1,\dots,m_i;\ i \ne k), \tag{8.9}$$

$$v^{(\widehat{\mathbf{m}}_{\mathbf{k}}+(j-1)\mathbf{1}_i)}\big(\mathbf{x},0\big) = 0 \ \text{ for } \quad (j = 1,\dots,l). \tag{8.10}$$

Notice that Green's operator $\mathcal{G}_0$ exists because problem (8.6), (8.7) is well–posed according to the conditions of Lemma 8.1. As for Green's operator $\mathcal{G}_k$ $(k \in \{1,\dots,n\})$ exist, i.e. problem (8.6)–(8.7) is well–posed because of our assumption that Lemma 8.1 holds true for $n-1$–dimensional boundary conditions.

Furthermore, by the Dunford–Pettis theorem (see [18, Chapter XI, § 1, Theorem 6]), it is not difficult to see that Green's operators $\mathcal{G}_k$ $(k = 1,\dots,n)$ are integral operators of Volterra type with respect to variable t, i.e.,

$$\mathcal{G}_k(z)(\mathbf{x},t) = \int_0^{\omega_1}\cdots\int_0^{\omega_{k-1}}\int_0^{\omega_{k+1}}\cdots\int_0^{t} G(\widehat{\mathbf{x}}_{\mathbf{k}},t;\widehat{\mathbf{s}}_k;\tau\, z(\mathbf{x}_k,\widehat{\mathbf{s}}_k,\tau)\,d\widehat{\mathbf{s}}_k\,d\tau.$$

Also, notice that $Q_0[u](\mathbf{x},t)$ contains terms $u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})}$ with $(\alpha \le \mathbf{m})$ and $(\boldsymbol{\beta} < \boldsymbol{l})$.

Continuing this process step-by-step, one can reduce problem (8.1)–(8.3) to the equivalent system of integral equations that are of Volterra type with respect to variable t.

Unique solvability of such systems can be proved by means of Picard's method of successive approximations. $\square$

Similar result for linear initial–periodic problems was proved in [36].

Lemmas 2.4 and 8.1 imply

Lemma 8.2. *let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\Omega \times T)$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$ *such that:*

(A_1) $\qquad P_{1\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) \le p_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{t})$ *for* $(\mathbf{x}, \mathbf{t}) \in \Omega \times T \quad (\boldsymbol{\alpha} < \mathbf{m})$;

(A_2) *For every* $\boldsymbol{\sigma} \in \Xi \cup \{\mathbf{1}\}$, $(\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t}) \in \Omega_{\widehat{\boldsymbol{\sigma}}} \times T$ *and arbitrary measurable functions* $\rho_{\boldsymbol{\alpha}} \in L^\infty(\Omega_{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$ *satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t}) \le \rho_{\boldsymbol{\alpha}}(\mathbf{y}) \le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{y}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, \mathbf{t}) \quad \text{for} \quad \mathbf{y} \in \Omega_{\boldsymbol{\sigma}} \ \ (\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}), \tag{8.11}$$

the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} \rho_{\boldsymbol{\alpha}}(\mathbf{y}) v^{(\boldsymbol{\alpha})},$$

$$a_{ik}\, v^{((k-1)\mathbf{1}_i)}(\mathbf{0}_{\mathbf{i}},\, \widehat{\mathbf{y}}_{\mathbf{i}}) + b_{ik}\, v^{((k-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\, \widehat{\mathbf{y}}_{\mathbf{i}}) = 0 \ \ (k = 1, \ldots, m_i;\ i \in \operatorname{supp} \boldsymbol{\sigma}).$$

has only the trivial solution in $AC^{\mathbf{m}_{\boldsymbol{\sigma}} - \mathbf{1}_{\boldsymbol{\sigma}}}(\Omega_{\boldsymbol{\sigma}})$. *Then problem* (8.1)–(8.3) *is well–posed, and there exists a positive constant* M_0 *depending only on* $P_{i\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} < \mathbf{m};\ i = 1, 2)$, a_{ik}, b_{ik} $(k = 1, \ldots, m_i;\ i = 1, \ldots, n)$, ω_i $(i = 1, \ldots, n)$ *and* τ_i $(i = 1, \ldots, k)$ *such that a solution* u *of problem* (8.1)–(8.3) *admits the estimate*

$$\|u\|_{C^{\mathbf{m}}(\Omega \times \widehat{T}_{\mathbf{t}})} \le M_0 \Big(\sum_{i=1}^{n} \sum_{j=1}^{m_i} \|\varphi_{ij}\|_{C^{\widehat{\mathbf{m}}_i + \mathbf{l}}(\widehat{\Omega}_{\mathbf{i}} \times T_{\mathbf{t}})} + \sum_{i=1}^{k} \sum_{j=1}^{l_i} \|\psi_{ij}\|_{C^{\mathbf{m} + \widehat{\mathbf{l}}_{\mathbf{i}}}(\Omega \times \widehat{T}_{\mathbf{i},\,\mathbf{t}})} + \|q\|_{C(\Omega \times \widehat{T}_{\mathbf{t}})} \Big).$$

9. Proof of the Main Results

The proof of Theorem 7.1 is similar to the proof of Theorem 1.2.

Theorem 7.2 is a particular case of Theorem 7.5. See the proof of Theorem 7.5 below.

Corollary 7.1 is the particular case of Theorem 1.3.

The proof of Theorem 7.3 is similar to the proof of Theorem 1.3. The only difference is that Lemma 8.2 should be used instead of Lemma 2.4.

Theorem 7.4 follows from Theorems 1.2 and 1.4.

Proof of Theorem 7.5. It is quite clear that it is enough to prove the theorem in the particular case, where $k = 1$. Set:

$$\Psi(\mathbf{x}) = \Big(\psi_j^{(\boldsymbol{\alpha})}(\mathbf{x})\Big)_{1\le j\le l}^{\boldsymbol{\alpha}\le\mathbf{m}},$$

$$p_{\boldsymbol{\alpha}}(\mathbf{x},t) = f_{\boldsymbol{\alpha}}\Big(\mathbf{x}, t, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v_0(\mathbf{x})]\Big) \qquad (\boldsymbol{\alpha} < \boldsymbol{m}),$$

$$q_0(\mathbf{x},t,) = f\Big(\mathbf{x}, t, \Psi(\mathbf{x}), \widetilde{\mathcal{D}}^{\mathbf{m}}[v_0(\mathbf{x})]\Big)$$

According to the conditions of Theorem (7.5), the linear problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x},0)v^{(\boldsymbol{\alpha})},$$

$$a_{ij}v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \;\; (j = 1,\dots,m_i;\; i = 1,\dots,n),$$

is well–posed, and for every $\boldsymbol{\sigma} \in \boldsymbol{\Xi}_n$ and $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \widehat{\Omega}_{\boldsymbol{\sigma}}$, the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, 0)v^{(\boldsymbol{\alpha})},$$

$$a_{ij}v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i},\, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i},\, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \;\; (j = 1,\dots,m_i;\; i \in \operatorname{supp}\boldsymbol{\sigma}),$$

has only the trivial solution. Therefore, there exists $\delta_0 > 0$ such that for every $t \in [0, \delta_0]$ the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}, t) v^{(\boldsymbol{\alpha})},$$

$$a_{ij} v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \ \ (j = 1, \dots, m_i;\ i = 1, \dots, n),$$

is well–posed, and for every $\boldsymbol{\sigma} \in \Xi_n$ and $\widehat{\mathbf{x}}_{\boldsymbol{\sigma}} \in \widehat{\Omega}_{\boldsymbol{\sigma}}$, the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widehat{\mathbf{x}}_{\boldsymbol{\sigma}}, t) v^{(\boldsymbol{\alpha})},$$

$$a_{ij} v^{((j-1)\mathbf{1}_i)}(\mathbf{0_i}, \widehat{\mathbf{x}}_\mathbf{i}) + b_{ij}\, v^{((j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_\mathbf{i}, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \ \ (j = 1, \dots, m_i;\ i \in \operatorname{supp} \boldsymbol{\sigma})$$

has only the trivial solution.

In the domain $\Omega \times [0, \delta]$ consider the linear equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\mathbf{l})} - v_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \sum_{\boldsymbol{\beta}<\boldsymbol{l}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} - \psi_\beta^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q_0(\mathbf{x}, t). \quad (9.1)$$

By Lemma 8.1 problem (9.1), (7.2), (7.3) is well–posed and has a solution $u_0(\mathbf{x}, t)$.

Now consider the equation

$$u^{(\mathbf{m}+\boldsymbol{l})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\mathbf{l})} - v_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \sum_{\boldsymbol{\beta}<\boldsymbol{l}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} - \psi_\beta^{(\boldsymbol{\alpha})}(\mathbf{x})\big) + q(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]), \quad (9.2)$$

where

$$q(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]) = q_0(\mathbf{x}, t) + \Bigg(f(\mathbf{x}, \mathbf{t}, \widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u]) - \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\mathbf{l})} - v_0^{(\boldsymbol{\alpha})}(\mathbf{x})\big) - \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \sum_{\boldsymbol{\beta}<\boldsymbol{l}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}, t)\big(u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} - \psi_\beta^{(\boldsymbol{\alpha})}(\mathbf{x})\big) - q_0(\mathbf{x}, t)\Bigg) \Xi_\delta\Big(\big\|\widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u - u_0(\mathbf{x}, t)]\big\|\Big)$$

where $\Xi_\delta : [0,+\infty) \to [0,+\infty)$ is the non–increasing smooth function defined by (3.9).

Notice that

$$\sum_{\boldsymbol{\alpha}<\mathbf{m}} \left|u_0^{(\boldsymbol{\alpha}+\mathbf{l})}(\mathbf{x},t) - v_0^{(\boldsymbol{\alpha})}(\mathbf{x})\right| + \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \sum_{\boldsymbol{\beta}<\boldsymbol{l}} \left|u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})}(\mathbf{x},t) - \psi_\beta^{(\boldsymbol{\alpha})}(\mathbf{x})\right|$$
$$\le \mu(\delta_0) \quad \text{for} \quad (\mathbf{x},t) \in \Omega \times [0,\delta_0],$$

where $\mu(\delta) \downarrow 0$ as $\delta \downarrow 0$.

Due to the continuous differentiability of the function $f(\mathbf{x},t,\mathbf{Z})$ with respect to the phase variables, in the ball $\mathbf{B}^{\mathbf{m}+\boldsymbol{l}}\big[\Omega \times [0,\delta_0]\big](u_0;\delta)$ the function q admits the estimate

$$\left|q(\mathbf{x},\mathbf{t},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])\right| \le \mu(\delta_0) + \gamma(\delta), \qquad \left|q_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t},\widetilde{\mathcal{D}}^{\mathbf{m}}[u])\right| \le \mu(\delta_0) + \gamma(\delta),$$

where $\gamma(\delta) = o(\delta)$ as $\delta \to 0$ and

$$q_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{t},\mathbf{Z}) = \frac{\partial q(\mathbf{x},\mathbf{t},\mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}.$$

Notice that the functions μ and β depend only on moduli of continuity of functions $f_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x},t,\mathbf{Z})$ on the set $\Omega\times[0,\delta_0]\times\mathbf{B}(\rho)$, where $\mathbf{B}(\rho) = \{Z \in \mathbb{R}^{\mathbf{m}+\mathbf{l}+\mathbf{1}} : \|Z\| \le \rho\}$, and

$$\rho = 1 + \|v_0\|_{C^{\mathbf{m}}(\Omega)} + \sum_{i=1}^{n}\sum_{j=1}^{m_i} \|\widetilde{\varphi}_{ij}\|_{C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\widehat{\Omega}_{\mathbf{i}}\times[0,\delta_0])} + \sum_{j=1}^{l} \|\widetilde{\psi}_j\|_{C^{\mathbf{m}}(\Omega\times[0,\delta_0])}$$

By Theorem 7.3, in the domain $\Omega\times[0,\delta_0]$ for sufficiently small $\delta_0 > 0$, problem (9.2), (7.2), (7.3) is solvable and its every solution admits the estimate

$$\|u - u_0\|_{C^{\mathbf{m}}(\Omega\times[0,\delta_0])} \le M_0(\mu(\delta_0) + \gamma(\delta)).$$

Choosing δ_0 and δ sufficiently small, one can achieve the estimate

$$\|u - u_0\|_{C^{\mathbf{m}+\boldsymbol{l}}(\Omega\times[0,\delta_0])} < \delta.$$

But then

$$\Xi_\delta\Big(\big\|\widetilde{\mathcal{D}}^{\mathbf{m}+\boldsymbol{l}}[u - u_0(\mathbf{x},t)]\big\|\Big) = 1,$$

and every solution of equation (9.2) is a solution of equation (7.1) too. $\square$

Corollary 7.2 follows from Theorem 7.5 and Lemma 2.4.

Proof of Theorem 7.6. Let u be a non–continuable solution of problem (7.1)–(7.3) defined on the set $\Omega \times \mathbf{I}$. The fact that I_s is open in $[0, \tau_s]$ immediately follows from Theorem 1.5.

Let us assume the contrary: $I_s = [0, \eta_s)$, and there exist $R > 0$ and a sequence $t_s^m \uparrow \eta_s$ as $m \to +\infty$ such that

$$\|u(t_s^m, \cdot)\|_{\widetilde{C}^{\mathbf{m}+\boldsymbol{l}}(\Omega \times \widehat{\mathbf{I}}_{\mathbf{s}})} \le R \quad (m = 1, 2, \dots). \tag{9.3}$$

Set

$$u^{((j-1)\mathbf{1}_s)}\big(\mathbf{t_s}^m, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{s}}\big) = \psi_{m,\,sj}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{s}}) \quad (j = 1, \dots, l_s),$$

and for equation (7.1) consider the initial–boundary conditions

$$a_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\mathbf{0}_i,\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) + b_{ij}\, u^{(\mathbf{m}^{i-1}+(j-1)\mathbf{1}_i)}(\boldsymbol{\omega}_{\mathbf{i}},\, \widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) = \varphi_{ij}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t})$$
$$\text{for } (\widehat{\mathbf{x}}_{\mathbf{i}}, \mathbf{t}) \in \widehat{\Omega}_{\mathbf{i}} \times T \ \ (j = 1, \dots, m_i;\ \ i = 1, \dots, n), \tag{9.4}$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{i-1}+(j-1)\mathbf{1}_i)}\big(\mathbf{0}_{\mathbf{i}}, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}\big) = \psi_{ij}^{(\mathbf{m}+\boldsymbol{l}^{i-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}})$$
$$\text{for } (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{i}}) \in \Omega \times \widehat{T}_{\mathbf{i}} \quad (j = 1, \dots, l_i;\ i = 1, \dots, s-1, s+1, \dots, k). \tag{9.5}$$

$$u^{(\mathbf{m}+\boldsymbol{l}^{s-1}+(j-1)\mathbf{1}_s)}\big(\mathbf{t}_{\mathbf{s}}^m, \mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{s}}\big) = \psi_{m,\,sj}^{(\mathbf{m}+\boldsymbol{l}^{s-1})}(\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{s}})$$
$$\text{for } (\mathbf{x}, \widehat{\mathbf{t}}_{\mathbf{s}}) \in \Omega \times \widehat{T}_{\mathbf{s}} \quad (j = 1, \dots, l_s). \tag{9.6}$$

By Theorem 7.5, problem (7.1), (9.4)–(9.6) has a solution in the domain $\Omega \times [t_s^m, t_s^m + \delta] \times \widehat{\mathbf{I}}_{\mathbf{s}}$, where the number $\delta > 0$ depends on the constant M_0 from Lemma 8.2 and

$$\sup\Big\{\big|f(\mathbf{x}, t, \mathbf{Z})\big| + \sum_{\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}} \big|f_{\boldsymbol{\alpha}+\boldsymbol{l}}(\mathbf{x}, t, \mathbf{Z})\big| : (\mathbf{x}, t) \in \Omega \times T,\ \ \|\mathbf{Z}\| \le 2\rho\Big\},$$

where

$$\rho = 1 + R + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \|\varphi_{ij}\|_{C^{\widehat{\mathbf{m}}_i+\boldsymbol{l}}(\widehat{\Omega}_{\mathbf{i}} \times T)} + \sum_{i \ne s}^{k} \sum_{j=1}^{l_i} \|\psi_{ij}\|_{C^{\mathbf{m}+\widehat{\boldsymbol{l}}_{\mathbf{i}}}(\Omega \times \widehat{T}_{\mathbf{i}})}.$$

Due to uniformly strong isolation of the family of solutions $v_{\boldsymbol{t}}(\mathbf{x}) = u^{(\boldsymbol{l})}(\mathbf{x}, \mathbf{t_0})$ $(\boldsymbol{t} \in I)$, the constant M_0 is independent of t_s^m, that makes $\delta > 0$ independent of t_s^m too. But this leads to the contradiction

$$t_s^m + \delta < \eta_s \quad (m = 1, 2, \dots).$$

The obtained contradiction proves the theorem. □

Corollary 7.3 follows from Theorem 7.6 and Lemma 2.4.

The proof of Theorem 7.7 is similar to the proof of Theorem 7.1.

The proof of Theorem 7.8 is similar to the proof of Theorem 7.3.

The proof of Theorem 7.9 is similar to the proof of Theorem 7.5.

Corollary 7.4 is a particular case Theorem 7.9.

The proof of Theorem 7.10 is similar to the proof of Theorem 7.6.

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