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Optimal Control of Multiphase Free Boundary Problems for Nonlinear Parabolic Equations

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by

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ABSTRACT

Title:

Optimal Control of Multiphase Free Boundary Problems for Nonlinear Parabolic Equations

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Dissertation research is on the optimal control of systems with distributed parameters described by singular nonlinear partial differential equations (PDE) modeling multiphase Stefan type second order parabolic free boundary problems. This type of free boundary problems arise in various applications, such as biomedical engineering problem on the laser ablation of biological tissues, aerospace engineering problem on the ice accretion in aircrafts mid-flight, biomedical problem on the growth of cancerous tumor, and many other phase transition processes in thermophysics and fluid mechanics. The aim of the optimal control of distributed free boundary systems is twofold: identification of functional parameters of the model via solving inverse free boundary problems, or optimizing the performance of such systems via optimal choice of control parameters. Ill-posed nature of inverse free boundary problems, formation of singularities by free boundaries, and irregularity of solutions are major difficulties in modeling and controlling distributed free boundary systems. Dissertation exploits a new approach introduced in *U.G. Abdulla & B. Poggi, Calculus of Variations & PDEs, 59:61, 2020*, which is based on the transformation of the multiphase multidimensional Stefan problem to singular PDE problem with discontinuous coefficient in a fixed domain. Optimal control of second order singular parabolic PDE with principal part in divergence form with bounded measurable coefficients is analyzed. Control parameters are boundary heat flux or density of heat sources, and cost functional is a norm difference of the trace of the solution from the available temperature measurement at the final moment. Existence of the optimal control is proved. Discretization of the optimal control problems via finite differences is pursued. Convergence of the sequence of discrete optimal control problems to continuous optimal control problem both with respect to functional and control is proved. Precisely, it is proved that the sequence of multilinear interpolations of the discrete minimizers converge to the optimal solution of the singular PDE problem in a weak topology of the Hilbert space of weakly differentiable functions. In particular, convergence of the method of finite differences, and existence, uniqueness and stability estimations are established for the singular PDE problem under minimal regularity assumptions on the coefficients expressed in terms of anisotropic Sobolev spaces setting.

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List of Notations

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, $n \geq 1$, $T > 0$. Define $D := \Omega \times [0,T]$. For any $t \in (0,T)$ define $D_t = \Omega \times [0,t]$. The boundary of Ω is notated $\partial\Omega$, and we define the lateral boundary as $S := \partial\Omega \times (0,T]$. For a function f, we use the notation Df to represent the spatial gradient of f.

$L_2(0,T)$ - Space of Lebesgue square-integrable functions. It is a Hilbert space with inner product

$$(u,v) = \int_0^T uv\,dt.$$

$L_2(D)$ - Hilbert space with inner product

$$(u,v) = \int_D uv\,dx\,dt.$$

$L_\infty(0,T)$ - Banach space of essentially bounded functions with the norm

$$\|u\|_{L_\infty[0,T]} = \operatorname*{esssup}_{0 \leq t \leq T} |u(t)|.$$

$L_\infty(D)$ - Banach space of essentially bounded functions with the norm

$$\|u\|_{L_\infty(D)} = \operatorname*{esssup}_{(x,t) \in D} |u(x,t)|.$$

$L_{\infty,1}(D)$ - Banach Space with norm

$$\|u\|_{L_{\infty,1}(D)} = \int_0^T \operatorname*{ess\,sup}_{x\in\Omega} |u(x,t)|\, dt$$

$L_{2,\infty}(D)$ - Banach Space with norm

$$\|u\|_{L_{2,\infty}(D)} = \left(\operatorname*{ess\,sup}_{0\le t\le T} \int_\Omega |u(x,t)|^2\, dx \right)^{1/2}$$

$W_2^k(0,T), k = 1,2,...$ - Hilbert space of all elements of $L_2(0,T)$ whose weak derivatives up to order k exist and belong to $L_2(0,T)$. The inner product is defined as

$$(u,v) = \int_0^T \sum_{s=0}^k \frac{d^s u}{dt^s} \frac{d^s v}{dt^s}\, dt.$$

$W_2^1(\Omega)$ - Hilbert space of all elements of $L_2(\Omega)$ whose weak derivatives up to order k exist and belong to $L_2(\Omega)$. The inner product is defined as

$$(u,v) = \int_\Omega \Big(uv + Du \cdot Dv \Big)\, dx.$$

$W_2^{1,0}(D)$ - Hilbert space of all elements of $L_2(D)$ that have a weak derivative in each x_i direction, $\frac{\partial u}{\partial x_i}$, and such that it belongs to $L_2(D)$. The inner product is defined as

$$(u,v) = \int_D \Big(uv + Du \cdot Dv \Big)\, dx\, dt.$$

$W_2^{1,1}(D)$ - Hilbert space of all elements of $L_2(D)$ with weak derivatives of first order. Also its weak derivatives must belong to $L_2(D)$. The inner product is defined as

$$(u,v) = \int_D \left(uv + Du \cdot Dv + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} \right) dx\,dt.$$

$W_2^{2,1}(D)$ - Hilbert space of all elements of $L_2(D)$ with weak derivatives of the second order in space and weak derivative of the first order in time, all belonging to $L_2(D)$. The inner product is defined as

$$(u,v) = \int_D \left(uv + Du \cdot Dv + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \sum_{i,j=1}^{d} u_{x_i x_j} v_{x_i x_j} \right) dx\,dt.$$

$\overset{\circ}{W}{}_2^{1,1}(D)$ - Linear subspace of elements of $W_2^{1,1}(D)$ with

$$u|_S = 0$$

$W_\infty^{1,0}(D)$ - Space of elements of $L_\infty(D)$ with weak derivative in the x direction existing and belonging to $L_\infty(D)$. It is a Banach space with norm

$$\|u\|_{W_\infty^{1,0}(D)} = \|u\|_{L_\infty(D)} + \|Du\|_{L_\infty(D)}$$

$C^1(\bar{D})$ - Banach space of continuously differentiable functions on $\bar{D}$ with norm

$$\|u\|_{C^1(\bar{D})} = \sup_D |u| + \sup_D |Du| + \sup_D \left| \frac{\partial u}{\partial t} \right|$$

$\overset{\bullet}{C}{}^1(\bar{D})$ - Linear subspace of $C^1(\bar{D})$ which is the closure of the subset of elements of $C^1(\bar{D})$ which has compact suport on D.

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Dedication

I dedicate this work to my four loving parents, my sister, and my future wife. Without you're love and support, this would not have been possible.

Chapter 1

Introduction

1.1 Stefan problem

A free boundary problem is a boundary value problem in which some partial differential equation (PDE) must be solved in a domain which is apriori unknown, and must be determined alongside with the solution of the PDE. Free boundary problems arise in many applications, such as various phase transition processes in fluid mechanics and thermophysics; growth of cancerous tumor or laser ablation of tissues in medicine, etc. The classical example of a free boundary problem is the Stefan Problem. The Stefan problem is a boundary value problem for the heat equation in which the boundary in between two or several phases is changing as a function of time. Several examples of Stefan problem in applications are the melting of ice, or freezing of water, the formation of crystals from liquid, or the laser ablation of skin tissue. The main mathematical feature of the Stefan problem is additional dynamical condition imposed on the unknown free boundary which expresses time evolution of the free boundary in terms of the conservation of energy during the phase transition process. As an example, consider the classical one-phase Stefan problem about the melting of the ice [35]: let a semi-infinite block of ice, initially at melting temperature of 0 degrees, starts melting under the time dependent heat flux $f(t)$ applied on the left end. Then the unknown temperature profile $u(x,t)$ and unknown boundary curve $s(t)$ between water and ice satisfy the following system of equations:

$$u_t = u_{xx}, \qquad 0 < x < s(t), t > 0 \tag{1.1}$$

$$u_x(0,t) = f(t), \qquad t > 0 \tag{1.2}$$

$$u(s(t),t) = 0, \qquad t > 0 \tag{1.3}$$

$$\frac{ds}{dt} = -u_x(s(t),t), \qquad t > 0 \tag{1.4}$$

$$u(x,0) = 0, \qquad x \geq 0 \tag{1.5}$$

$$s(0) = 0 \tag{1.6}$$

Stefan condition (1.4) expresses the fact that the free boundary is pushed forward due to jump of the flux on the free boundary during the phase transition. The 1D Stefan problem has a well-established mathematical theory, and an extensive list of works on it can be found in [79]. The existence and uniqueness of the classical solution of the one dimensional Stefan problem is a well known result [35, 45]. For example, it can be established by reduction to Volterra type integral equations by using Green's functions [45]. Now consider the following formulation of the multiphase multidimensional Stefan problem [63]:

Let $d \in \mathbb{N}, \Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, $T > 0$, and $D := \Omega \times (0,T]$. Given phase transition temperatures $u^1 < u^2 < \cdots < u^J$, find a temperature function $u : D \to \mathbb{R}$ and the phase transition boundaries

$$S^j = \{(x,t) \in D \mid u(x,t) = u^j\}, \quad j = 1,2,\dots,J$$

which satisfy

$$\alpha(u)u_t - \operatorname{div}(k(u)\nabla u) = f(x,t), \qquad (x,t) \in D, \quad u(x,t) \neq u^j, j = \overline{1,J} \tag{1.7}$$

where f characterizes the heat sources, α, k are known positive functions which are smooth on each of the intervals $[u^j, u^{j+1}]$ and have discontinuities of the first kind at the points $u = u^j, j = 1,\dots,J$;

$$[u]\big|_{S^j} = 0, \qquad j = \overline{1,J}, \tag{1.8}$$

$$b_j \cos(\mathbf{n},t) + \sum_{i=1}^{d} [k(u)u_{x_i}] \cos(\mathbf{n},x_i)\big|_{S^j} = 0, \qquad j = \overline{1,J}, \tag{1.9}$$

$$u(x,0) = \phi(x), \qquad x \in \Omega, \tag{1.10}$$

$$u|_S = 0, \tag{1.11}$$

where ϕ is a known function representing the initial temperature, each b_j is a positive number, $\mathbf{n}$ is the normal to the free boundary S^j in the direction of increasing u (that is, along the gradient of u), and the saltus $[u]\big|_{S^j}$ is the difference between the limiting value of u on S^j when approached from the domains $\{(x,t) \mid u < u^j\}$ and $\{(x,t) \mid u > u^j\}$ respectively; $S = \partial\Omega \times (0,T]$ is a lateral boundary of the cylinder D. The Stefan condition (1.9) derives from the conservation law which states that the phase transition boundaries are pushed by the jump in heat flux across the different phases.

The multidimensional case is significantly different from the 1D Stefan problem. In general classical, i.e. smooth solution of the multidimensional Stefan problem exists for short time interval only [66]. Local existence and uniqueness of a classical solution to the multidimensional Stefan problem is established in [65]. In general, the solution may develop singularities and thus a global solution will only exist in the weak sense. In the one phase case, in [46, 61], the Stefan problem is transformed to an obstacle problem and existence and uniqueness of a global weak solution is proved through the method of variational inequalities. Significant progress in regularity of free boundaries of the weak solution in the one-phase case are proved in celebrated papers [31, 32]. A very powerful method of solving the multidimensional multiphase Stefan problem is based on the transformation introduced in [70, 59]. The method transforms the free boundary problem into a nonlinear PDE in a fixed domain with discontinuous coefficients. The following is the description of this transformation.

In order to reformulate the multidimensional multiphase Stefan problem, define

$$F(t) = \int_0^t k(y)\,dy, \tag{1.12}$$

and consider the transformation

$$v(x,t) := F(u(x,t)). \tag{1.13}$$

Then $v^j = F(u^j),\ v^1 < \cdots < v^J$, and our system becomes:

$$\beta(v)v_t - \Delta v = f(x,t), \qquad (x,t) \in D, v(x,t) \neq v^j, \tag{1.14}$$

$$[v]|_{S^j} = 0, \qquad j = \overline{1,J}, \tag{1.15}$$

$$b_j \cos(\mathbf{n},t) + \sum_{i=1}^{d} [v_{x_i}] \cos(\mathbf{n},x_i)\big|_{S^j} = 0, \qquad j = \overline{1,J}, \tag{1.16}$$

$$v\big|_{\Omega\times\{t=0\}} = \Phi := F(\phi), \tag{1.17}$$

$$v|_S = 0, \tag{1.18}$$

with

$$\beta(v) = \frac{\alpha(F^{-1}(v))}{k(F^{-1}(v))} \tag{1.19}$$

where F^{-1} is an inverse function of F. Due to assumptions on both α and k, $\beta(v)$ will also be a piecewise continuously differentiable function with discontinuities of the first kind at each phase transition temperature, $v = v_j$. Now, we can invoke a monotone increasing piecewise smooth function $b(v)$ such that $b'(v) = \beta(v)$ on each of the intervals (v^j, v^{j+1}). The partial differential equation becomes

$$\frac{\partial b(v)}{\partial t} - \Delta v = f(x,t), \qquad (x,t) \in D,\ v(x,t) \neq v^j. \tag{1.20}$$

Note that in each interval (v^j, v^{j+1}), $b(v)$ is defined with accuracy up to a constant summand. Thus we're free to choose the jump of b at the values $v = v^j$. We choose them in such a way that $[b(v)]|_{S^j} = -b_j$ so that upon integration by parts of (1.20) over D, the integrals over the phase transition boundaries cancel out. A function $b(v)$ satisfying these requirements is defined with accuracy up to its values at $v = v_j$, $j = 1,\ldots J$ and one additional constant summand which we fix by assigning its value at some point $v^0 \neq v^j$, $j = 1, ..., J$.

Now consider the following definitions:

Definition 1.1.1. We say that a measurable function $B(x,t,v)$ is *of type* $\mathscr{B}$ if

1. $B(x,t,v) = b(v), \qquad v \neq v^j, \quad \forall j = \overline{1,J}$

2. $B(x,t,v) \in [b(v^j)^-, b(v^j)^+], \qquad v = v^j$ for some j.

From [63], the weak solution to the Stefan problem (1.14)-(1.18) is understood in the following sense:

Definition 1.1.2. $v \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$ is called a *weak solution of the Stefan problem* (1.14)-(1.18) if for some two functions B, B_0 of type $\mathscr{B}$, the integral identity

$$\int_D \Big[-B(x,t,v(x,t))\psi_t + Dv \cdot D\psi - f\psi \Big]\, dxdt - \int_\Omega B_0(x,0,\Phi(x))\psi(x,0)\, dx = 0 \quad (1.21)$$

is satisfied for arbitrary $\psi \in \overset{\circ}{W}{}_2^{1,1}(D)$ with $\psi|_{\Omega\times\{t=T\}} = 0$.

The existence and uniqueness of the weak solution was proved in [58, 70, 63]. Regularity of the weak solution was advanced in important papers [38, 39, 30].

1.2 Identification of Parameters and Optimal Control of Free Boundary Problems

Suppose now that some of the data is unknown, or it involves error due to experimental measurements. It is required to find some missing data, as well as the solution of the

PDE and phase transition boundaries. This is what is known as an inverse free boundary problem. There are two different motivations for inverse problems: creating accurate mathematical models for various systems involving free boundaries, and optimal control problems involving free boundaries. The first motivation arises in the identification of various unknown parameters of mathematical models. In practice, experiments are ran and available data is recorded. The experimental data is then used as the measurement in which we find parameters that minimize the difference between the solution and these measurements. The second motivation arises in optimal control problems. An optimal control is obtained over a set of possible controls that brings the system to the desired position. Inverse free boundary problems have two different categories: inverse free boundary problems with known free boundaries, and inverse free boundary problems where the free boundaries are unknown.

An example of an inverse Stefan problem (ISP) with a known free boundary is found through analyzing and modeling ice accretion on aircrafts mid-flight [48]. Numerous atmospheric hazards of flight make it of utmost importance to understand how to protect the aircraft. Supercooled water droplets may form an ice layer on the surface of the aircraft, which can drastically reduce the aerodynamic flight performance. This can be formulated as a Stefan problem, in considering the boundary between the ice and fluid as the free-boundary, and the surface of the airplane being the fixed boundary. The control set can be considered as the possible heat flux provided from the interior surface of the aircraft, and an optimal control is found in order to produce a safe level of ice accretion on the wing.

The one-dimensional **ISP with known free boundary** can be formulated as follows: given the free boundary, $s(t)$, solve for temperature distribution, $u(x,t)$, and heat flux, $f(t)$, that satisfy (1.1)-(1.6).

The following is the typical formulation of the **multidimensional ISP with known free boundary**: given the free boundaries, S_j, solve for the temperature distribution, $u(x,t)$, and density of the heat sources, $f(x,t)$, that satisfy (1.7)-(1.11).

Inverse free boundary problems are ill-posed in the sense of Hadamard, i.e. there is no guarantee a solution exists, if it does exist it may not be unique, and the solution does not depend continuously on the initial-boundary data. The one-phase inverse Stefan problem in a one dimensional case was first mentioned in [34], where phase transition boundary is known and heat flux on left boundary is to be found. Historically, the variational approach was a dominating method for solving the ISP. First papers in that direction were [28, 29].

A typical variational formulation for the one-dimensional ISP (1.1)-(1.6) is as follows: consider the minimization of the cost functional

$$\mathscr{J}(f) = \int_0^T |u(s(t),t)|^2 \, dt \tag{1.22}$$

over a control set $f \in F$, where u is the solution to (1.1),(1.2), (1.4)-(1.6). The control represents the heat flux on the fixed boundary. Given a control, the state vector is found by solving the Neumann boundary value problem, since the Stefan condition (1.4) on the free boundary becomes a Neumann condition on the fixed boundary. Optimal control will be found to minimize the cost functional, representing L_2-norm difference of the trace of the state vector on the given free boundary from the phase transition temperature.

A typical variational method for solving multidimensional ISP (1.7),(1.11) with given free boundaries consists of the the minimization of the functional

$$\mathscr{J}(f) = \sum_{j=1}^{J} \int_{S_j} |[u]|^2 dS \tag{1.23}$$

over a control set $f \in F$, in which u is the solution to (1.7), (1.9)-(1.11). The control represents the density of the heat sources, and we search for a control that will minimize the saltus of the temperature on the phase transition boundaries calculated from different phases.

In [80], one-dimensional ISP was formulated as an optimal control problem and the existence of the optimal control is proved. In [82], Frechet derivative was derived, convergence of finite difference schemes was proved, and Tikhonov regularization was implemented to address instability. Similar methods were developed in various works [22, 24, 27, 33, 36, 41, 42, 53, 75, 49].

Inverse free boundary problems with unknown free boundaries arise in most of the applications. An important example involves the ablation of cancerous tissues. The application involves solving for the solution of the PDE, the phase transition boundary between the solid and ablated skins, and possibly other unknown parameters. In order to solve this inverse problem, experimental data can be obtained. For example, temperature measurements at the final moment in time, or along the fixed boundary, can be measured throughout an experiment.

The one-dimensional **ISP with unknown free boundary** can be formulated as follows: solve for temperature distribution, $u(x,t)$, free boundary, $s(t)$, and heat flux, $f(t)$, that satisfy (1.1)-(1.6), with the given information on the temperature distribution and position of the free boundary at the final moment $t = T$:

$$u(x,T) = \omega(x), \qquad 0 \le x \le s(T) =: \bar{s}. \tag{1.24}$$

The following is the typical formulation of the **multidimensional and multiphase ISP with unknown free boundary**: solve for temperature distribution, $u(x,t)$, free boundaries, S_j, and density of the heat sources, $f(x,t)$, that satisfy (1.7)-(1.11), with given temperature distribution at the final moment $t = T$:

$$u(x,T) = \nu(x), \qquad x \in \Omega \tag{1.25}$$

ISP with unknown free boundaries, or equivalently, optimal control for Stefan type free boundary problems was analyzed in [23, 43, 54, 55, 56, 57, 62, 64, 69, 67, 73, 74, 78, 49]. Summarizing research development up to 2012, the main methods were based on variational formulation, method of quasisolutions, Tikhonov regularization, Frechet differentiability and iterative gradient type numerical methods, and convergence of difference schemes. For example, a variational formulation of the one-dimensional ISP would be a minimization of the cost functional

$$\mathcal{J}(f) = \beta_1 \int_0^{s(T)} |u(x,T) - \omega(x)|^2\, dx + \beta_2 |s(T) - \bar{s}|^2 \tag{1.26}$$

over a control set $f \in F$, in which u is the solution to (1.1)-(1.6). Typical variational formulation in a multidimensional setting consists on the minimization of the functional

$$\mathcal{J}(f) = \int_\Omega |u(x,T;f) - \nu(x)|^2 dx \tag{1.27}$$

over a control set $f \in F$, in which u is a solution to (1.7)-(1.11). As it is outlined in [1, 2] this approach has two major downsides:

- Commonly used numerical solution via iterative gradient type methods for such models requires identification of the free boundaries at each iteration step. This will incur large computational cost due to the complexity of numerical solution of the full Stefan problem.

- The solution of the inverse Stefan problem does not depend continuously on the phase transition temperature. Slight error or perturbation in the phase transition temperature can cause large changes in the solution. This can be problematic as some media may not have consistent phase transition temperatures. For example in the flight ice accretion problem, ice on the surface of the airplane is created by superdroplets, which can stay in liquid form in the atmosphere in temperatures of down to -40 degrees centigrade.

In [1, 2] a new variational formulation of the one-phase ISP was developed, in which optimal control framework was implemented, where the phase transition boundary is included in the control set along with the boundary heat flux. The sum of the L_2-norm declinations are minimized against the available measurements of temperature on the fixed boundary, available measurements of the free boundary location, and temperature at final moment. Important advantage of the new control theoretic approach is that it can handle situations where the phase transition temperature is not known explicitly, and is available only through measurement with possible error. Another major advantage of the new variational method suggested in [1, 2] is based on the fact that for a given control vector corresponding state vector solves PDE problem in a fixed region instead of full free boundary problem. This allows to reduce significantly computational cost of iterative numerical methods based on gradient type methods in Sobolev spaces. In [3, 4], Frechet differentiability in Sobolev-Besov spaces was proved and the formula for the Frechet gradient and optimality condition are derived. In [5, 8] gradient method was implemented in Hilbert-Besov spaces framework for the numerical solution of the ISP.

The new method developed in [1, 2] is not applicable to inverse multiphase free boundary problems. The reason is that the Stefan condition on the free boundary includes the saltus of the boundary flux from neighbouring phases, and by fixing free boundary as a control parameter the Stefan condition does not become a Neumann or Robin type boundary condition for the PDE. In a recent paper [6], a new method was introduced for the solution of the optimal control of the multiphase Stefan problem. The idea of this method is based on the transformation of the multiphase Stefan problem to a nonlinear PDE problem with discontinous coefficient, but in a fixed domain. The idea turned to be very powerful and adjusts for the deficiencies in the above discussion. State vector satisfies multiphase Stefan problem in a weak formulation in a fixed domain, but with the time derivative term coefficient having jump discontinuities along the phase transition boundaries. In [6] existence of the optimal control and convergence of the sequence of discretized optimal control problems via method of finite differences is proved.

In a recent paper [7] a new method is developed for the solution of the optimal control of multidimensional multiphase Stefan problem (1.20), (1.15)-(1.18). It is required to minimize the cost functional (1.27) over the control set $f \in F$, where the state vector u is a solution of the multidimensional and multiphase Stefan problem (1.14)-(1.18) in the sense of the Definition 1.1.2. The idea of employing weak formulation of the multidimensional multiphase Stefan problem turned to be very effective, for it addresses deficiencies associated with existing methods outlined above. In [7], existence of the optimal control and convergence of the sequence of discretized optimal control problems with respect to both functional and control was proved. The latter means that the sequence of multi-linear interpolations of the solution to the discrete multiphase Stefan problems corresponding to discrete optimal control sequence, converge to the optimal multiphase Stefan problem weakly in $W_2^{1,1}(D)$, strongly in $L_2(D)$, and almost everywhere on D. Moreover, convergence of the method of finite differences for the multidimensional and multiphase Stefan problem is established. The convergence is based on establishing new discrete L_∞ and $W_2^{1,1}$ energy estimates and to employ weak compactness argument in Hilbert space setting.

The goal of this dissertation is to extend the methods of [6, 7] to a broad class of optimal control problems for general second order parabolic operators, in both a single and multidimensional in space setting, representing optimal control problems for singular nonlinear PDEs modeling Stefan-type second-order parabolic free boundary problems in anisotropic media.

1.3 Statement of the Open Problem and Outline of the Main Results

Let $d \in \mathbb{N}, \Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $T > 0, D := \Omega \times (0,T]$ and $v^1 < v^2 < \cdots < v^m$ are given real numbers.

Definition 1.3.1. A map $\beta : \mathbb{R} \to \mathbb{R}$ is called monotone if its graph is monotone, namely

$$(u-v)(x-y) \geq 0, \ \forall (x,u),(y,v) \in graph\ \beta := \{(x,\beta(x)) \in \mathbb{R}^2 : \ x \in \mathbb{R}\},$$

and strictly monotone if this inequality is strict when $x \neq y$.

Definition 1.3.2. A monotone map $\beta : \mathbb{R} \to \mathbb{R}$ is maximal if its graph is not properly contained in the graph of any other monotone map, namely

$$(u-v)(x-y) \geq 0,\ \forall v \in \beta(y), y \in \mathbb{R} \Longrightarrow\ u \in \beta(x).$$

Consider the singular PDE problem:

$$\frac{\partial \beta(v)}{\partial t} - \mathscr{L}v - f(x,t) \ni 0, \qquad (x,t) \in D, \tag{1.28}$$

$$v(x,0) = \Phi(x), \qquad x \in \Omega, \tag{1.29}$$

$$v|_S = 0 \tag{1.30}$$

where $\beta(\cdot)$ is a maximal monotone graph of the form

$$\beta(r) = \begin{cases} \beta_j(r) + \sum\limits_{i=0}^{j-1} \nu_i, & \text{for } v^{j-1} < r < v^j, \\ \left[\beta_j(v^j) + \sum\limits_{i=0}^{j-1} \nu_i, \beta_j(v^j) + \sum\limits_{i=0}^{j} \nu_i\right], & \text{for } r = v^j, \\ \beta_{j+1}(r) + \sum\limits_{i=0}^{j} \nu_i, & \text{for } v^j < r < v^{j+1};\ j = 1,2,...,m \end{cases} \tag{1.31}$$

with a given positive constants $\nu_j, j = 1,...,m$; $\nu_0 = 0$, $v^0 = -\infty$, $v^{m+1} = +\infty$; $\beta_i(\cdot), i = 1,...,m+1$ are monotone increasing Lipschitzian functions in their respective domain of definition, $\beta_j(v^j) = \beta_{j+1}(v^j), j = 1,...,m,$

$$0 < \bar{b} \leq \beta_j'(r),\ j = 1,...,m+1; \tag{1.32}$$

and $\mathscr{L}$ is an elliptic operator

$$\mathscr{L}v = \sum_{i=1}^{d} (a_i(x,t)v_{x_i} + b_i(x,t)v)_{x_i} - \sum_{i=1}^{d} c_i(x,t)v_{x_i} - r(x,t)v \tag{1.33}$$

with bounded and measurable coefficients a_i, b_i, c_i, r and

$$a_i(x,t) \geq a_0 > 0, \quad i = \overline{1,d}, \quad \text{a.e. } (x,t) \in D. \tag{1.34}$$

We are going also consider the following general Neumann type boundary condition instead of the Dirichlet condition (1.30):

$$\frac{\partial v}{\partial \mathscr{N}} + k(x,t)v(x,t) = \Lambda(x,t), \quad x \in S \tag{1.35}$$

where

$$\frac{\partial v}{\partial \mathscr{N}} := \sum_{i=1}^{d} (a_i(x,t)v_{x_i} + b_i(x,t)v) cos(\nu, x_i) \tag{1.36}$$

Λ is the boundary heat flux, and ν is the exterior unit normal of Ω.

Singular PDE problem (1.28)-(1.30) (or (1.28), (1.29),(1.35)) is a distributional formulation of the multiphase Stefan problem if elliptic operator $\mathscr{L}$ coincides with Δ [70, 59, 63]. In the physical context, $v(x,t)$ is a temperature distribution, $f(x,t)$ is a density of heat sources, $\Phi(x)$ is an initial temperature, v^j's are phase transition temperatures; $\beta_j'(v), v^j < v < v^{j+1}, j = 0, ..., m$ express heat conductivities in each phase, positive constants $v^j, j = 1, ..., m$ characterize latent heat of fusion during phase transition, and the coefficients a_i, b_i, c_i, r characterize anisotropic properties of the media. The classical case $m = 1, v^1 = 0$, $\mathscr{L} = \Delta$ is a two-phase Stefan problem describing melting of the ice or freezing of the water [46, 66], whereas more sophisticated applications include biomedical problem about the laser ablation of biomedical tissues, which motivates general elliptic operator $\mathscr{L}$.

Consider optimal control problem on the minimization of the functional

$$\mathscr{J}(f) = \|v|_{\Omega\times\{t=T\}} - \Gamma\|^2_{L_2(\Omega)} \tag{1.37}$$

on a control set

$$\mathscr{F}^R = \left\{f \in L_\infty(D):\ \|f\|_{L_\infty(D)} \le R\right\}. \tag{1.38}$$

where $R > 0$ and $\Gamma \in L_2(\Omega)$ are given, and $v = v(x,t;f)$ is a solution of the singular PDE problem (1.28)-(1.30). Furthermore, this optimal control problem will be referred to as *Problem $\mathscr{I}$*.

We also consider the optimal control problem on the minimization of the functional (1.37) on a control set

$$\mathscr{G}^R = \left\{\Lambda \in W_2^1(S):\ \|\Lambda\|_{W_2^1(S)} \le R\right\}.$$

where $R > 0$ and $\Gamma \in L_2(\Omega)$ are given, and $v = v(x,t;\Lambda)$ is a solution of the singular PDE problem (1.28),(1.29), (1.35). Furthermore, this optimal control problem will be referred to as *Problem $\mathscr{S}$*.

The main goals of the dissertation are

- to prove well-posedness of the optimal control problem for singular nonlinear parabolic PDEs

- to pursue discretization of the optimal control problem through method of finite differences, and to prove the convergence of the sequence of discrete optimal control problems to the optimal control problem for singular nonlinear parabolic PDEs both with respect to functional and control.

- to prove existence, uniqueness, and stability of singular nonlinear parabolic PDEs under minimal regularity assumptions on the data

In Chapter 2, optimal control *Problem* $\mathscr{S}$ with $d = 1$ is analyzed. In Chapter 3, optimal control *Problem* $\mathscr{I}$ is analyzed with arbitrary number of spatial variables. The following are the main results:

- Existence of a solution to the optimal control problem for singular nonlinear parabolic PDEs is proved.

- Approximation of the optimal control problems via method of finite differences is pursued. Convergence of the sequence of discrete optimal control problems to the continuous optimal control problem with respect to both functional and control is proved.

- Uniform L_∞ and $W_2^{1,1}$ energy estimates are established and the convergence of the method of finite differences for the singular PDE problem is proved.

- Existence, uniqueness, and stability of the singular nonlinear PDE problem is proved under minimal regularity assumptions on the coefficients and initial-boundary data.

Results of Chapter 2 are published in [9]. Section 2.1 formulates the optimal control problem for the singular nonlinear PDE under general Neumann boundary condition with one spatial variable. Section 2.2 describes the discretization of the optimal control problem through finite differences. Section 2.3 formulates the main results on the existence of the optimal control and on the convergence of the sequence of finite-dimensional discrete optimal control problems to the original optimal control problem both with respect to functional and control. Section 2.4 establishes some preliminary results essential for the prove of main results, including existence and uniqueness of the discrete solution, and uniqueness of the weak solution to the singular nonlinear PDE problem. Section 2.5 establishes the key discrete L_∞ and $W_2^{1,1}$ energy estimates, and completes the proof of the main results.

Chapter 3 describes the results on the optimal control of multidimensional singular PDE problem under Dirichlet boundary conditions. Section 3.1 formulates the definition of the weak solution of the singular PDE problem. Section 3.2 introduces discrete framework and pursues the discretization of the optimal control problem through method of finite differences. In Section 3.3 the main results on the existence of the optimal control, and the convergence of the sequence of discrete optimal control problems to problem $\mathscr{J}$ both with respect to functional and control are formulated. Section 3.4 proves essential preliminary results, including the existence and uniqueness of the discrete solution and uniqueness of the weak solution to the singular PDE problem. In Section 3.5 discrete L_∞ and $W_2^{1,1}$ energy estimates are proved for the singular PDE problem. The key approximation theorem on the convergence in the weak topology of Hilbert space $W_2^{1,1}(D)$ of the multilinear interpolations of solution to the discrete problem to the solution of the singular PDE problem is proved in Section 3.6. Proofs of the main results are completed in Section 3.6.

Chapter 2

Optimal Control of 1D Multiphase Free Boundary Problem for Nonlinear Parabolic Equations

The results of Chapter 2 are published in [9].

2.1 Optimal Control Problem

We first consider the following one-dimensional version of the open problem. Consider singular nonlinear PDE problem

$$\frac{\partial \beta(v)}{\partial t} - \mathscr{L}v - f(x,t) \ni 0, \qquad \text{in } D = \{(x,t) : 0 < x < \ell,\ 0 < t \leq T\}; \tag{2.1}$$

$$v(x,0) = \Phi(x), \qquad 0 < x < \ell \tag{2.2}$$

$$av_x + bv|_{x=0} = g(t), \qquad 0 < t \leq T, \tag{2.3}$$

$$av_x + bv|_{x=\ell} = p(t), \qquad 0 < t \leq T, \tag{2.4}$$

with β given as in (1.31) and $\mathscr{L}$ is an elliptic operator

$$\mathscr{L}v = (a(x,t)v_x + b(x,t)v)_x - c(x,t)v$$

with bounded measurable coefficients a, b, c and

$$a(x,t) \geq a_0 > 0, \quad \text{a.e. } (x,t) \in D = \{0 < x < \ell, 0 < t < T\}. \tag{2.5}$$

In the physical context, $v(x,t)$ is a temperature, $f(x,t)$ is a density of heat sources, $\Phi(x)$ is an initial temperature distribution, $g(t)$ and $p(t)$ are heat flux on fixed boundaries, v^j's are phase transition temperatures; $\beta_j'(v), v^j < v < v^{j+1}, j = 0, 1, ..., m$ characterize heat conductivities in each phase, and the positive jump constants $\nu^j, j = 1, ..., m$ are expressing latent heat of fusion during phase transition.

Let $\omega \in L_2(0,\ell)$ is given. Consider an optimal control problem on the minimization of the cost functional

$$\mathscr{J}(g) = \|v(x,T;g) - \omega(x)\|^2_{L_2(0,\ell)} \tag{2.6}$$

over the control set:

$$\mathscr{G}_R = \left\{g : g \in W_2^1(0,T),\ \|g\|_{W_2^1[0,T]} \leq R\right\},$$

where v is a solution of the singular nonlinear PDE problem (2.1)-(2.4). Described optimal control problem will be called a Problem $\mathscr{S}$. The aim of the Problem $\mathscr{S}$ is to achieve the desired temperature distribution $\omega(x)$ at the final moment by controlling the boundary flux $g(t)$ on the fixed boundary. Equivalently, it is a variational formulation of the inverse multiphase Stefan type free boundary problem on the identification of the boundary flux $g(t)$ through measurement $\omega(x)$ of the final moment temperature distribution.

Throughout the chapter we are going to use standard notation of Sobolev spaces ([63, 5]). We now formulate the notion of the weak solution of the nonlinear multiphase parabolic free boundary problem (2.1)-(2.4).

Definition 2.1.1. $v \in W_2^{1,1}(D) \cap L_\infty(D)$ is called a *weak solution of the problem* (2.1)-(2.4) if for some two functions B, B_0 of type $\mathscr{B}$, the following integral identity is satisfied

$$\int_0^T\int_0^\ell \Big[-B(x,t,v(x,t))\psi_t + a(x,t)v_x\psi_x + b(x,t)v\psi_x + c(x,t)v\psi - f\psi\Big]\,dxdt$$
$$-\int_0^\ell B_0(x,0,\Phi(x))\psi(x,0)\,dx - \int_0^T p(t)\psi(\ell,t)\,dt + \int_0^T g(t)\psi(0,t)\,dt = 0, \tag{2.7}$$

for all $\psi \in W_2^{1,1}(D)$ with $\psi(x,T) = 0$.

2.2 Discrete Optimal Control Problem

Let

$$\omega_\tau = \{t_k, k = \overline{1,n}\}, \tau = \frac{T}{n}, t_k = k\tau, \qquad \omega_h = \{x_i, i = \overline{1,m}\}, h = \frac{\ell}{m}, x_i = ih$$

be grids in the time and space domains, respectively, under the assumptions that $m \to \infty$ as $n \to \infty$ and

$$\frac{h}{\tau} \geq \frac{8\|b\|_{L_\infty}}{\bar{b}} \tag{2.8}$$

Define the Steklov averages

$$w_k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} w(t)\,dt, \qquad \Phi_i = \frac{1}{h} \int_{x_i}^{x_{i+1}} \Phi(x)\,dx, \ \ \Phi_m = \Phi(\ell), \tag{2.9}$$
$$q_{ik} = \frac{1}{\tau h} \int_{t_{k-1}}^{t_k} \int_{x_i}^{x_{i+1}} q(x,t)\,dxdt, \qquad k = \overline{1,n}, \quad i = \overline{0,m-1},$$

where w represents any of the functions p, Γ, g, or g^n, and q represents any of the functions a, b, c, and f. Introduce the discretized control set

$$\mathscr{G}_R^n = \{[g]_n \in \mathbb{R}^{n+1} : \|[g]_n\|_{w_2^1} \leq R\}$$

where $[g]_n = (g_0, g_1, \dots, g_n)$, and

$$\|[g]_n\|^2_{w_2^1} = \sum_{k=1}^{n} \tau g_k^2 + \sum_{k=1}^{n} \tau g_{k\bar{t}}^2$$

with $g_{k\bar{t}} = \frac{g_k - g_{k-1}}{\tau}$. Assume that every element $g \in W_2^1(0,T)$ is extended on the interval $[-\tau, 0]$ as a constant $g(0)$. Consider now the mappings between the discrete and continuous control sets, $\mathscr{Q}_n : W_2^1(0,T) \to \mathbb{R}^{n+1}$, $\quad \mathscr{P}_n : \mathbb{R}^{n+1} \to W_2^1(0,T)$ as

$$\mathscr{Q}_n(g) = [g]_n, \quad \text{for } g \in \mathscr{G}_R, \text{ where } g_k = \frac{1}{\tau} \int\limits_{t_{k-1}}^{t_k} g(t)\,dt, \quad k = \overline{0,n}, \tag{2.10}$$

$$\mathscr{P}_n([g]_n) = g^n, \text{ for } [g]_n \in \mathscr{G}_R^n;$$

$$g^n(t) = g_{k-1} + \frac{g_k - g_{k-1}}{\tau}(t - t_{k-1}), t \in [t_{k-1}, t_k),\ k = \overline{1,n}. \tag{2.11}$$

Approximate the function $\beta(v)$ by the infinitely differentiable sequence

$$b_n(v) = \int_{v-\frac{1}{n}}^{v+\frac{1}{n}} \beta(y)\omega_n(v-y)dy, \tag{2.12}$$

where ω_n is a standard mollifier defined as

$$\omega_n(v) = \begin{cases} \mathscr{C} n e^{-\frac{1}{1-n^2v^2}}, & |v| \le \frac{1}{n} \\ 0, & |v| > \frac{1}{n} \end{cases} \tag{2.13}$$

and the constant $\mathscr{C}$ is chosen so that $\int\limits_{\mathbb{R}} \omega_1(u)\,du = 1$. Since $\beta'(v)$ is piecewise-continuous, we also have

$$b_n'(v) = \int_{v-\frac{1}{n}}^{v+\frac{1}{n}} \beta'(y)\omega_n(v-y)dy. \tag{2.14}$$

This implies b_n is also strict monotonically increasing and by (1.32) we have

$$b_n'(v) \geq \bar{b} > 0 \tag{2.15}$$

We now define a solution to the problem (2.1)-(2.4) in the discrete sense:

Discrete State Vector. Given $[g]_n$, the vector function $[v([g]_n)]_n$ $= \big(v(0), v(1), \dots, v(n)\big); \quad v(k) \in \mathbb{R}^{m+1}, \quad k = 0, \dots, n$ is called a *discrete state vector* if

1. $v_i(0) = \Phi_i, \quad i = \overline{0,m},$

2. For arbitrary $k = 1, \dots, n$, the vector $v(k) \in \mathbb{R}^{m+1}$ satisfies

$$\sum_{i=0}^{m-1} h\Big[\big(b_n(v_i(k))\big)_{\bar{t}}\eta_i + a_{ik}v_{ix}(k)\eta_{ix} + b_{ik}v_i(k)\eta_{ix} + c_{ik}v_i(k)\eta_i - f_{ik}\eta_i\Big] \tag{2.16}$$
$$-p_k\eta_m + g_k^n\eta_0 = 0, \forall \eta = (\eta_i) \in \mathbb{R}^{m+1}.$$

Given $[g]_n \in \mathscr{G}_R^n$, the discrete cost functional $\mathscr{I}_n$ is defined as

$$\mathscr{I}_n([g]_n) = \sum_{i=1}^{m} h\Big(v_i(n) - w_i\Big)^2 \tag{2.17}$$

where $v_i(k)$ are components of the discrete state vector $[v([g]_n)]_n$. Finite-dimensional optimal control problem on the minimization of $\mathscr{I}_n([g]_n)$ on a control set $\mathscr{G}_R^n$ will be called Problem $\mathscr{S}_n$. We define

$$\mathscr{I}_{n_*} := \inf_{[g]_n \in \mathscr{G}_R^n} \mathscr{I}_n([g]_n).$$

Furthermore, the following interpolations will be considered:

$$\begin{aligned}
\tilde{v}(x,t) &= v_i(k), \qquad x\in[x_i,x_{i+1}], \quad t\in[t_{k-1},t_k], \qquad i=\overline{0,m-1}, \quad k=\overline{0,n},\\
\hat{v}(x;k) &= v_i(k)+v_{ix}(k)(x-x_i), \qquad x\in[x_i,x_{i+1}], \quad i=\overline{0,m-1},\\
v^{\tau}(x,t) &= \hat{v}(x;k), \qquad t\in[t_{k-1},t_k],\\
\hat{v}^{\tau}(x,t) &= \hat{v}(x;k-1)+\hat{v}_{\bar{t}}(x;k)(t-t_{k-1}), \qquad t\in[t_{k-1},t_k], \quad k=\overline{1,n}. \qquad (2.18)
\end{aligned}$$

2.3 Formulation of the Main Results

Unless stated otherwise, throughout the chapter we assume the following conditions are satisfied by the data:

$$\begin{aligned}
&f\in L_\infty(D), \quad p\in W_2^1(0,T), \quad \Phi\in W_2^1(0,\ell), \qquad (2.19)\\
&a,b\in W_\infty^{1,0}(D), \quad \frac{\partial a}{\partial t},\frac{\partial b}{\partial t}\in L_{\infty,1}(D), \quad c\in L_\infty(D)
\end{aligned}$$

where a satisfies (2.5); and $\Phi(x)=v^j$, $j=1,\dots,m$ on a set of measure 0 in the interval $(0,\ell)$.

Theorem 2.3.1. *The optimal control problem $\mathscr{S}$ has a solution, that is, the set*

$$\mathscr{G}_* = \left\{ g\in\mathscr{G}_R \middle| \mathscr{J}(g)=\mathscr{J}_* := \inf_{g\in\mathscr{G}_R}\mathscr{J}(g)\right\}$$

is not empty.

Theorem 2.3.2. *The sequence of discrete optimal control problems $\mathscr{S}_n$ approximates* *the optimal control problem $\mathscr{S}$ with respect to functional, that is,*

$$\lim_{n\to+\infty} \mathscr{I}_{n_*} = \mathscr{J}_*, \tag{2.20}$$

where

$$\mathscr{I}_{n_*} = \inf_{\mathscr{G}_R^n} \mathscr{I}_n([g]_n),\ n = 1,2,\dots\ .$$

If $[g]_{n_\varepsilon} \in \mathscr{G}_R^n$ is chosen such that

$$\mathscr{I}_{n_*} \le \mathscr{I}_n([g]_{n_\varepsilon}) \le \mathscr{I}_{n_*} + \varepsilon_n,\ \varepsilon_n \downarrow 0,$$

then the sequence $g^n = \mathscr{P}_n([g]_{n_\varepsilon})$ has a subsequence convergent to some element $g_* \in \mathscr{G}_*$ weakly in $W_2^1(0,T)$ and strongly in $L_2(0,T)$. Moreover, the piecewise linear interpolations $\hat{v}^\tau$ of the corresponding discrete state vectors $[v([g]_{n_\varepsilon})]_n$ converge to the weak solution $v(x,t;g_*) \in W_2^{1,1}(D) \cap L_\infty(D)$ of the singular PDE problem (2.1)-(2.4) weakly in $W_2^{1,1}(D)$, strongly in $L_2(D)$, and almost everywhere on D.

2.4 Preliminary Results

Lemma 2.4.1. *Given any $[g]_n \in \mathscr{G}^n$, and any h, τ, a discrete state vector exists uniquely.*

Proof. First we prove uniqueness by induction. For a given $[g]_n$, suppose v and $\tilde{v}$ both are discrete state vectors. Due to definition of how a discrete state vector is constructed, we have that $v(0) = \tilde{v}(0)$. Now suppose that $v(k-1) = \tilde{v}(k-1)$ for some fixed $k \ge 1$. Since v and $\tilde{v}$ both satisfy (2.16), subtract the identities for both v and $\tilde{v}$, choosing $\eta = v(k) - \tilde{v}(k)$

to get:

$$\sum_{i=0}^{m-1}\Big[\big(b_n(v_i(k))_{\bar t}-b_n(\tilde v_i(k))_{\bar t}\big)\big(v_i(k)-\tilde v_i(k)\big)+a_{ik}\big(v_{ix}(k)-\tilde v_{ix}(k)\big)^2$$
$$+b_{ik}\big(v_{ix}(k)-\tilde v_{ix}(k)\big)\big(v_i(k)-\tilde v_i(k)\big)+c_{ik}\big(v_i(k)-\tilde v_i(k)\big)^2\Big]=0. \tag{2.21}$$

By using Cauchy inequality with $\varepsilon = a_0$ we have

$$\sum_{i=0}^{m-1}\Big[\frac{1}{\tau}\big(b_n(v_i(k))-b_n(\tilde v_i(k))\big)\big(v_i(k)-\tilde v_i(k)\big)+a_0\big(v_{ix}(k)-\tilde v_{ix}(k)\big)^2$$
$$+c_{ik}\big(v_i(k)-\tilde v_i(k)\big)^2\Big]$$
$$\le \sum_{i=0}^{m-1}-b_{ik}\big(v_{ix}(k)-\tilde v_{ix}(k)\big)\big(v_i(k)-\tilde v_i(k)\big)$$
$$\le \sum_{i=0}^{m-1}\frac{a_0}{2}\big(v_{ix}(k)-\tilde v_{ix}(k)\big)^2+\sum_{i=0}^{m-1}\frac{\|b\|^2_{L^\infty}}{2a_0}\big(v_i(k)-\tilde v_i(k)\big)^2 \tag{2.22}$$

Absorbing to left hand side, and by using (2.15), we get:

$$\sum_{i=0}^{m-1}\Big[\Big(\frac{\bar b}{\tau}-\|c\|_{L^\infty}-\frac{\|b\|^2_{L^\infty}}{2a_0}\Big)\big(v_i(k)-\tilde v_i(k)\big)^2+\frac{a_0}{2}\big(v_{ix}(k)-\tilde v_{ix}(k)\big)^2\Big]\le 0 \tag{2.23}$$

The whole summand is non-negative for sufficiently small τ. Therefore, it is equal to 0, which implies that $v_i(k) = \tilde v_i(k), \quad \forall i = \overline{0,m}$. Hence, by induction, $v = \tilde v$.

Now we seek to prove existence through induction. Construct $v(0)$ through definition of a Discrete State Vector. Note that $v(0)$ is bounded since $\|v(0)\| \le \|\Phi\|_{L_\infty[0,\ell]}$. Fix $k \ge 1$, and assume that $v(k-1)$ has been constructed so that (2.16) is satisfied for all $K < k$. Moreover, assume that each element of $v(k-1)$ is bounded. Through manipulation, the summation identity (2.16) is equivalent to solving the following system of non-linear equations:

$$[a_{0k} - hb_{0k} + h^2 c_{0k}]v_0(k) + \frac{h^2}{\tau} b_n(v_0(k)) - a_{0k}v_1(k)$$
$$= \frac{h^2}{\tau} b_n(v_0(k-1)) + h^2 f_{0k} - hg_0^n$$

$$\frac{h^2}{\tau} b_n(v_i(k)) + [-a_{i-1,k} + hb_{i-1,k}]v_{i-1}(k) + [a_{i-1,k} + a_{ik} - hb_{ik} + h^2 c_{ik}]v_i(k)$$
$$-a_{ik}v_{i+1}(k) = \frac{h^2}{\tau} b_n(v_i(k-1)) + h^2 f_{ik}, \quad i = \overline{1, m-1}$$

$$[-a_{m-1,k} + hb_{m-1,k}]v_{m-1}(k) + a_{m-1,k}v_m(k) = hp_k \tag{2.24}$$

We will construct $v(k)$ by the method of successive approximations. Fix h and τ, and choose $v^0 = v(k-1)$. Having obtained v^N, we search v^{N+1} as a solution of the following:

$$[a_{0k} - hb_{0k} + h^2 c_{0k}] v_0^{N+1}(k) + \frac{h^2}{\tau} b_n(v_0^{N+1}(k)) - a_{0k} v_1^N(k)$$
$$= \frac{h^2}{\tau} b_n(v_0(k-1)) + h^2 f_{0k} - hg_0^n$$

$$\frac{h^2}{\tau} b_n(v_i^{N+1}(k)) + [-a_{i-1,k} + hb_{i-1,k}] v_{i-1}^N(k) + [a_{i-1,k} + a_{ik} - hb_{ik}$$
$$+ h^2 c_{ik}] v_i^{N+1}(k) - a_{ik} v_{i+1}^N(k)$$
$$= \frac{h^2}{\tau} b_n(v_i(k-1)) + h^2 f_{ik}, \quad i = \overline{1, m-1}$$

$$[-a_{m-1,k} + hb_{m-1,k}] v_{m-1}^{N+1}(k) + a_{m-1,k} v_m^{N+1}(k) = hp_k \tag{2.25}$$

We now proceed to prove that the sequence $\{v^N\}$ converges to the unique solution of (2.24). Subtract (2.25) for N and $N-1$ to get

$$[a_{0k} - hb_{0k} + h^2 c_{0k}] \big(v_0^{N+1}(k) - v_0^N(k)\big) + \frac{h^2}{\tau} \big(b_n(v_0^{N+1}(k)) - b_n(v_0^N(k))\big) \tag{2.26}$$
$$= a_{0k} \big(v_1^N(k) - v_1^{N-1}(k)\big)$$

$$[a_{ik} + a_{i-1,k} - hb_{ik} + h^2 c_{ik}] \big(v_i^{N+1}(k) - v_i^N(k)\big) + \frac{h^2}{\tau} \big(b_n(v_i^{N+1}(k)) - b_n(v_i^N(k))\big)$$
$$= a_{ik} \big(v_{i+1}^N(k) - v_{i+1}^{N-1}(k)\big) + [a_{i-1,k} - hb_{i-1,k}] \big(v_{i-1}^N(k) - v_{i-1}^{N-1}(k)\big), \quad i = \overline{1, m-1}$$

$$a_{m-1,k} \Big(v_m^{N+1}(k) - v_m^N(k)\Big) = [a_{m-1,k} - hb_{m-1,k}] \Big(v_{m-1}^{N+1}(k) - v_{m-1}^N(k)\Big) \tag{2.27}$$

which can be transformed to

$$\begin{cases} v_0^{N+1}(k)-v_0^N(k) &= \left(\frac{a_{0k}}{a_{0k}-hb_{0k}+h^2c_{0k}+\frac{h^2}{\tau}\zeta_{n,N}^0}\right)\left(v_1^N(k)-v_1^{N-1}(k)\right) \\ v_i^{N+1}(k)-v_i^N(k) &= \left(\frac{a_{ik}}{a_{i-1,k}+a_{ik}-hb_{ik}+h^2c_{ik}+\frac{h^2}{\tau}\zeta_{n,N}^i}\right)(v_{i+1}^N(k)-v_{i+1}^{N-1}(k)) \\ & \quad +\left(\frac{a_{i-1,k}-hb_{i-1,k}}{a_{i-1,k}+a_{ik}-hb_{ik}+h^2c_{ik}+\frac{h^2}{\tau}\zeta_{n,N}^i}\right)(v_{i-1}^N(k)-v_{i-1}^{N-1}(k)) \\ v_m^{N+1}(k)-v_m^N(k) &= \frac{a_{m-1,k}-hb_{m-1,k}}{a_{m-1,k}}\left(v_{m-1}^{N+1}(k)-v_{m-1}^N(k)\right) \end{cases} \tag{2.28}$$

where

$$\zeta_{n,N}^i := \int_0^1 b_n'(\theta v_i^{N+1}(k)+(1-\theta)v_i^N(k))d\theta, \quad i=\overline{0,m-1}.$$

Due to (2.15), we have $\zeta_{n,N}^i \geq \bar{b}, \ i=\overline{0,m-1}$. Let

$$A_N := \max_{0\leq i\leq m}\left|v_i^{N+1}(k)-v_i^N(k)\right|.$$

From (2.28), taking the first equation into consideration, we have:

$$\begin{aligned} |v_0^{N+1}(k)-v_0^N(k)| &\leq \left|\frac{a_{0k}}{a_{0k}-hb_{0k}+h^2c_{0k}+\frac{h^2}{\tau}\zeta_{n,N}^0}\right|A_{N-1} \\ &= |\delta_0^{-1}|A_{N-1},\ \delta_0 = 1+\frac{h(-b_{0k}+hc_{0k}+\frac{h}{\tau}\zeta_{n,N}^0)}{a_{0k}} \end{aligned} \tag{2.29}$$

We have

$$0 < a_0 \leq a_{0k} \leq \|a\|_{L_\infty}$$

and

$$-b_{0k}+hc_{0k}+\frac{h}{\tau}\zeta_{n,N}^0 \geq -\|b\|_{L_\infty}-h\|c\|_{L_\infty}+\frac{h}{\tau}\bar{b} > 0$$

by (2.8) and for sufficiently small h and τ. Thus, $0 < \delta_0^{-1} < 1$. Similarly,

$$|v_i^{N+1}(k) - v_i^N(k)| \leq \Big|\frac{a_{i-1,k} + a_{ik} - hb_{ik}}{a_{i-1,k} + a_{ik} - hb_{ik} + h^2 c_{ik} + \frac{h^2}{\tau}\zeta_{n,N}^i}\Big| A_{N-1}$$

$$= |\delta_i^{-1}| A_{N-1},\ \delta_i = 1 + \frac{h(hc_{ik} + \frac{h}{\tau}\zeta_{n,N}^i)}{a_{i-1,k} + a_{ik} - hb_{ik}}, i = \overline{1, m-1} \tag{2.30}$$

Through similar argument as with δ_0, we derive that $0 < \delta_i^{-1} < 1$ for $i = \overline{1, m-1}$ for $0 < h << 1$. For $i = m$, we get

$$|v_m^{N+1}(k) - v_m^N(k)| \leq |\delta_m^{-1}| A_{N-1},$$

$$\delta_m = \Big(1 - \frac{hb_{m-1,k}}{a_{m-1,k}}\Big)^{-1}\Big(1 + \frac{h^2 c_{m-1,k} + \frac{h^2}{\tau}\zeta_{n,N}^{m-1}}{a_{m-2,k} + a_{m-1,k} - hb_{m-1,k}}\Big)$$

For $0 < h << 1$, we can see that the term in left brackets will be close to 1, and due to (2.8), as before we derive that $0 < \delta_m^{-1} < 1$ for sufficiently small h and τ. Let $\delta = \max_{i=\overline{1,m}} \delta_i^{-1}$. We thus have $\delta < 1$, and

$$A_N \leq \delta A_{N-1} \leq \cdots \leq A_0 \delta^N. \tag{2.31}$$

Following the proof given in [5] (Lemma 1, Section 2) it follows that there exist finite limits

$$v_i(k) = \lim_{N \to +\infty} v_i^N(k), \qquad i = 0, 1, \ldots, m. \tag{2.32}$$

Passing to limit as $N \to +\infty$ in (2.25), we derive that $v_i(k), i = \overline{1, m}$ is a unique solution of (2.24). □

Given the existence and uniqueness of the discrete state vector for fixed n, we can uniquely define for each $k = 1, \ldots, n$ the vector ζ_k whose m components ζ_k^i are given by

$$\zeta_k^i = \int_0^1 b_n'(\theta v_i(k) + (1-\theta) v_i(k-1)) d\theta, \quad i = \overline{0, m-1}. \tag{2.33}$$

The following is a well known necessary and sufficient condition for the convergence of the discrete optimal control problems to continuous optimal control problem.

Lemma 2.4.2. *[80] The sequence of discrete optimal control problems approximates the continuous optimal control problem if and only if the following conditions are satisfied:*

- *for arbitrary sufficiently small $\varepsilon > 0$ there exists $M_1 = M_1(\varepsilon)$ such that $\mathscr{Q}_M(g) \in \mathscr{G}_R^M$ for all $g \in \mathscr{G}_{R-\varepsilon}$ and $M \geq M_1$; and for any fixed $\varepsilon > 0$ and for all $g \in \mathscr{G}_{R-\varepsilon}$ the following inequality is satisfied:*

$$\limsup_{M\to\infty} \Big(\mathscr{I}_M(\mathscr{Q}_M(g)) - \mathscr{J}(g) \Big) \leq 0. \tag{2.34}$$

- *for arbitrary sufficiently small $\varepsilon > 0$ there exists $M_2 = M_2(\varepsilon)$ such that $\mathscr{P}_M([g]_M) \in \mathscr{G}_{R+\varepsilon}$ for all $[g]_M \in \mathscr{G}_R^M$ and $M \geq M_2$; and for all $[g]_M \in \mathscr{G}_R^M$, $M \geq 1$ the following inequality is satisfied:*

$$\limsup_{M\to\infty} \Big(\mathscr{J}(\mathscr{P}_M([g]_M)) - \mathscr{I}_M([g]_M) \Big) \leq 0. \tag{2.35}$$

- *the following inequalities are satisfied:*

$$\limsup_{\varepsilon\to 0} \mathscr{J}_*(\varepsilon) \geq \mathscr{J}_*, \;\; \liminf_{\varepsilon\to 0} \mathscr{J}_*(-\varepsilon) \leq \mathscr{J}_*, \tag{2.36}$$

where $\mathscr{J}_*(\pm\varepsilon) = \inf_{\mathscr{G}_{R\pm\varepsilon}} \mathscr{J}(g)$.

Lemma 2.4.3. *[5] The mappings $\mathscr{P}_n, \mathscr{Q}_n$ satisfy the conditions of Lemma 2.4.2.*

Lemma 2.4.4. *There is at most one solution to the multiphase free boundary problem (2.1)-(2.4) in the sense of (2.7).*

Proof. The uniqueness of the weak solution is proved in Section 9 of Chapter V of [63] for the classical multiphase Stefan Problem ($\mathscr{L} = \Delta$) under zero Dirichlet boundary conditions on the fixed boundary. Lemma 4 of [5] generalized the result to the case of non-homogeneous Neumann boundary condition. We generalize the result to the case of multiphase free boundary problem with general elliptic operator $\mathscr{L}$. Uniqueness is proved over a wider class of solutions than given in (2.7). Suppose that $v \in L_\infty(D)$ only, not necessarily in the Sobolev space $W_2^{1,1}(D)$, and that for any two functions B, B_0 of type $\mathscr{B}$ it satisfies the identity

$$\int_0^T \int_0^\ell \Big[B(x,t,v)\psi_t + v(a\psi_x)_x - bv\psi_x - cv\psi + f\psi\Big]\,dxdt$$
$$+\int_0^\ell B_0(x,0,\Phi(x))\psi(x,0)\,dx + \int_0^T p(t)\psi(\ell,t)\,dt - \int_0^T g(t)\psi(0,t)\,dt = 0,$$
$$\forall \psi \in W_2^{2,1}(D), \psi(x,T) = 0,$$
$$a(0,t)\psi_x(0,t) = a(\ell,t)\psi_x(\ell,t) = 0. \tag{2.37}$$

Any function satisfying (2.1.1) will also satisfy the above definition. Suppose v and $\tilde{v}$ are two solutions in the sense of (2.37), and subtract (2.37) with solution $\tilde{v}$ from that of v. Due to Φ taking on phase transition temperatures on sets of measure 0, the B_0 term will vanish and we are left with the following:

$$\int_0^T \int_0^\ell \big(B(x,t,v) - \tilde{B}(x,t,\tilde{v})\big)\Big(\psi_t + z(x,t)\Big((a\psi_x)_x - b\psi_x - c\psi\Big)\Big)\,dxdt = 0 \tag{2.38}$$

where $z(x,t) = \frac{v-\tilde{v}}{B(x,t,v)-\tilde{B}(x,t,\tilde{v})}$. For $(x,t) \in D$ such that $v(x,t) = \tilde{v}(x,t)$, we have $z(x,t) = 0$. Otherwise, since B and $\tilde{B}$ are strictly increasing on v a.e. $(x,t) \in D$, we have that z is non-negative for a.e. (x,t). Moreover, we have:

$$|z(x,t)| = \left| \frac{v-\tilde{v}}{\int\limits_{\tilde{v}(x,t)}^{v(x,t)} \beta'(w)\,dw + \sum\limits_{i:v^i \in (\tilde{v}(x,t),v(x,t))} (\beta(v^i)^+ - \beta(v^i)^-)} \right| \leq \left| \frac{v-\tilde{v}}{\int\limits_{\tilde{v}}^{v} \bar{b}\,dv} \right| = \frac{1}{\bar{b}},$$

so that z is essentially bounded. Fix $\varepsilon > 0$, and take as $\psi(x,t)$ the solution of the following Neumann problem

$$\psi_t + (z(x,t)+\varepsilon)\Big((a\psi_x)_x - b\psi_x - c\psi\Big) = F(x,t), \tag{2.39}$$

$$a(0,t)\psi_x(0,t) = a(\ell,t)\psi_x(\ell,t) = 0. \tag{2.40}$$

$$\psi(x,T) = 0, \tag{2.41}$$

where the ε is added to ensure the conjugate diffusion coefficient is strictly positive, and F is an arbitrary smooth bounded function in D. Note that (2.39) is the conjugate parabolic equation. From [63], there exists a unique solution $\psi^\varepsilon \in W_2^{2,1}(D)$ of the problem (2.39)-(2.41). We will use the arbitrariness of F to obtain that $B - \tilde{B} = 0$ a.e. $(x,t) \in D$. Note that through (2.39), we can rewrite (2.38):

$$\int\limits_0^T \int\limits_0^\ell \big(B(x,t,v) - \tilde{B}(x,t,\tilde{v})\big)\Big(F - \varepsilon\Big((a\psi_x)_x - b\psi_x - c\psi\Big)\Big)\,dx\,dt = 0. \tag{2.42}$$

Thus our goal will be attained if we have an energy estimate on $\mathscr{L}\psi$ for solutions of (2.39). For simplicity, we obtain energy estimates through the second order parabolic equation, which will give analogous estimates for the conjugate parabolic equation by reversing the time variable. Let $z^{\varepsilon}(x,t) = z(x,t) + \varepsilon$, and for simplicity we will omit the superscript. Multiply the non-conjugate version of (2.39) by ψ_{xx} and integrate it over $D_t := (0,\ell)\times(0,t)$ to get

$$-\int_0^t\int_0^\ell (\psi_\tau - za\psi_{xx} - z(a_x - b)\psi_x + zc\psi)\psi_{xx}\,dx\,d\tau = -\int_0^t\int_0^\ell F\psi_{xx}\,dx\,d\tau$$

$$= \int_0^t\int_0^\ell F_x\psi_x\,dx\,d\tau - \int_0^t F\psi_x\Big|_0^\ell d\tau, \tag{2.43}$$

Due to (2.40), the second integral on the right hand side disappears. We can transform various terms on the right hand side as follows:

$$-\int_0^t\int_0^\ell \psi_\tau\psi_{xx}\,dx\,d\tau = \int_0^t\int_0^\ell (\psi_\tau)_x\psi_x\,dx\,d\tau - \int_0^t \psi_\tau\psi_x\Big|_0^\ell d\tau = \frac{1}{2}\int_0^\ell \psi_x^2(x,\ell)\,dx,$$

$$-\int_0^t\int_0^\ell (-za\psi_{xx})\psi_{xx}\,dx\,d\tau \geq a_0\int_0^t\int_0^\ell z\psi_{xx}^2\,dx\,d\tau$$

Using the above, and returning to (2.43), we get that:

$$\frac{1}{2}\int_0^\ell \psi_x^2(x,\ell)\,dx + a_0\int_0^t\int_0^\ell z\psi_{xx}^2\,dx\,d\tau \leq \int_0^t\int_0^\ell (-za_x\psi_x\psi_{xx})\,dx\,d\tau$$

$$+\int_0^t\int_0^\ell zb\psi_x\psi_{xx}\,dx\,d\tau + \int_0^t\int_0^\ell zc\psi\psi_{xx}\,dx\,d\tau + \int_0^t\int_0^\ell F_x\psi_x\,dx\,d\tau \tag{2.44}$$

We now estimate the terms on the right hand side using Cauchy inequality with $\varepsilon > 0$

and properties of given functions, and absorbing terms to the left hand side, we have:

$$\frac{1}{2}\int_0^\ell \psi_x^2(x,\ell)\,dx + \frac{a_0}{4}\int_0^t\int_0^\ell z\psi_{xx}^2\,dx\,d\tau \le \frac{2\|c\|^2_{L_\infty(D)}}{\bar{b}a_0}\int_0^t\int_0^\ell \psi^2\,dx\,d\tau +$$
$$\Big(\frac{2(\|a_x\|^2_{L_\infty(D)}+\|b\|^2_{L_\infty(D)})}{\bar{b}a_0}+\frac{1}{2}\Big)\int_0^t\int_0^\ell \psi_x^2\,dx\,d\tau + \frac{1}{2}\int_0^t\int_0^\ell F_x^2\,dx\,d\tau. \tag{2.45}$$

From Theorem 2.3. Chapter 1 of [63] it follows that ψ will have a uniform bound in $L_\infty(D)$, which we will denote as $\bar{C}$. Letting now $y(t) = \int_0^t\int_0^\ell \psi_x^2\,dx\,d\tau$ from (2.45) we deduce

$$y'(t) \le Cy(t) + \Big(\int_0^t\int_0^\ell F_x^2\,dx\,d\tau + \frac{4\|c\|^2_{L_\infty(D)}\bar{C}^2\ell T}{\bar{b}a_0}\Big).$$

where

$$C = 2\Big(\frac{2(\|a_x\|^2_{L_\infty(D)}+\|b\|^2_{L_\infty(D)})}{\bar{b}a_0}+\frac{1}{2}\Big)$$

By Gronwall's Inequality (e.g. Lemma 5.5, Chapter 2, [63]), we deduce from the above differential inequality that

$$\int_0^t\int_0^\ell \psi_x^2(x,\tau)\,dx\,d\tau \le \Big[\frac{e^{Ct}-1}{C}\Big]\Big[\int_0^t\int_0^\ell F_x^2\,dx\,d\tau + \frac{4\|c\|^2_{L_\infty(D)}\bar{C}^2\ell T}{\bar{b}a_0}\Big], \quad \forall t \in (0,T]$$

so that by (2.45),

$$\int_0^\ell \psi_x^2(x,t)\,dx + \frac{a_0}{2}\int_0^t\int_0^\ell z\psi_{xx}^2\,dx\,d\tau \le (e^{Ct}-1))\Big[\int_0^t\int_0^\ell F_x^2\,dx\,d\tau$$
$$+\frac{4\|c\|^2_{L_\infty(D)}\bar{C}^2\ell T}{\bar{b}a_0}\Big] + \frac{1}{2}\int_0^t\int_0^\ell F_x^2\,dx\,d\tau + \frac{4\|c\|^2_{L_\infty(D)}\bar{C}^2\ell T}{\bar{b}a_0} \qquad \forall t \in (0,T].$$

The first of the above inequalities implies that

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_0^\ell \psi_x^2(x,t)\,dx \le C_1\int_0^T\int_0^\ell F_x^2\,dx\,d\tau + C_2.$$

Now, since $\psi_t = az\psi_{xx} + z(a_x-b)\psi_x + cz\psi + F$, we have

$$\begin{aligned}
\|\psi_t\|^2_{L_2(D_t)} &= \|az\psi_{xx} + z(a_x-b)\psi_x + cz\psi + F\|^2_{L_2(D_t)}\\
&\le 4\Big(\|az\psi_{xx}\|^2_{L_2(D_t)} + \|(a_x-b)z\psi_x\|^2_{L_2(D_t)} + \|cz\psi\|^2_{L_2(D_t)} + \|F\|^2_{L_2(D_t)}\Big)\\
&\le C_3\Big[\|F_x\|^2_{L_2(D_t)} + \|F\|^2_{L_2(D_t)}\Big] + C_4,
\end{aligned}$$

where the constants C_3 and C_4 depend on $\bar{b}, a_0, T, \ell$ and L_∞-norms of a, a_x, b, c. Combining all the estimations we have the following desired energy estimate for $\psi \in W_2^{2,1}(D)$:

$$\begin{aligned}
&\int_0^T\int_0^\ell (\psi_t^2 + z(x,t)\psi_{xx}^2)\,dx\,dt + \operatorname*{ess\,sup}_{0\le t\le T}\int_0^\ell \psi_x^2(x,t)\,dx\\
&\qquad\le C_5\Big[\|F_x\|^2_{L_2(D_t)} + \|F\|^2_{L_2(D_t)}\Big] + C_6,
\end{aligned} \tag{2.46}$$

where the constants C_5 and C_6 are independent of ε, and depend on $\bar{b}, a_0, T, \ell$ and L_∞-norms of a, a_x, b, c.

For the rest of the proof, any constant depending on the bounded data, domain, F, or the uniform bound on ψ will be referred to as C^*.We can now observe that

$$\begin{aligned}
&\left|\int_0^T\int_0^\ell (B-\tilde{B})\varepsilon\Big((a\psi_x^\varepsilon)_x - b\psi_x^\varepsilon - c\psi^\varepsilon\Big)\,dx\,dt\right| \\
&\leq \int_0^T\int_0^\ell \left|B-\tilde{B}\right|\varepsilon\Big(\,|a\psi_{xx}^\varepsilon| + |(a_x-b)\,\psi_x^\varepsilon| + |c\psi^\varepsilon|\,\Big)\,dx\,dt \\
&\leq 2\text{esssup}\,\beta(v)\left[\int_0^T\int_0^\ell \frac{\varepsilon(z+\varepsilon)^{\frac{1}{2}}}{(z+\varepsilon)^{\frac{1}{2}}}\,|a\psi_{xx}^\varepsilon|\,dx\,dt\right. \\
&\left.+\int_0^T\int_0^\ell \varepsilon\,|a_x-b|\,|\psi_x^\varepsilon|\,dx\,dt + \int_0^T\int_0^\ell \varepsilon\,|c|\,|\psi^\varepsilon|\,dx\,dt\right] \\
&\leq 2\text{esssup}\,\beta(v)\left[\Big(\int_0^T\int_0^\ell \frac{\varepsilon^2}{z+\varepsilon}\,dx\,dt\Big)^{\frac{1}{2}}\Big(a^2(z+\varepsilon)(\psi_{xx}^\varepsilon)^2\,dx\,dt\Big)^{\frac{1}{2}}\right. \\
&\left.+\Big(\int_0^T\int_0^\ell \varepsilon^2(a_x-b)^2\,dx\,dt\Big)^{\frac{1}{2}}\Big(\int_0^T\int_0^\ell (\psi_x^\varepsilon)^2\,dx\,dt\Big)^{\frac{1}{2}} + \varepsilon C^*\right] \\
&\leq 2\sqrt{\varepsilon}\;\text{esssup}\,\beta(v)\left[C^*\Big(\int_0^T\int_0^\ell \frac{\varepsilon}{z+\varepsilon}\,dx\,dt\Big)^{\frac{1}{2}} + C^*\sqrt{\varepsilon}\right] \\
&\leq 2C^*\sqrt{\varepsilon}\;\text{esssup}\,\beta(v)[(T\ell)^{\frac{1}{2}} + \sqrt{\varepsilon}] \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Therefore, (2.42) now implies

$$\int_0^T\int_0^\ell \Big(B(x,t,v(x,t)) - \tilde{B}(x,t,\tilde{v}(x,t))\Big)F\,dx\,dt = 0.$$

Since F is arbitrary, the above equality gives that $B(x,t,v(x,t)) = \tilde{B}(x,t,\tilde{v}(x,t))$ a.e. $(x,t)\in D$. This implies $\beta(v(x,t)) = \beta(\tilde{v}(x,t))$, a.e. $(x,t)\in D$ s.t. $v(x,t)\neq v^j$, $j=1,\ldots,m$. Due to the fact that β is strictly increasing, we have $v(x,t) = \tilde{v}(x,t)$ a.e. (x,t). Thus v and $\tilde{v}$ are the same solution to (2.37) □

Corollary 2.4.5. *If v is weak solution, the sets $\{(x,t) \in D | v = v^j\}, j = 1,...,m$ have 2-dimensional measure 0.*

Indeed, due to uniqueness of the weak solution, for any two representatives B_1, B_2 of the class $\mathscr{B}$ we have

$$B_1(x,t,v(x,t)) = B_2(x,t,v(x,t)), \quad \text{a.e. } (x,t) \in D$$

and by the Definition 1.1.1 this will be a contradiction if any of the v^j-level sets of the weak solution would have a positive 2-dimensional measure.

2.5 Proof of Main Results

2.5.1 $L_\infty(D)$ Estimate for the Discrete Multiphase Free Boundary Problem

In this section we prove $L_\infty(D)$ bound for the discrete PDE problem under the following reduced assumptions:

$$p \in L_\infty(0,T),\ \Phi \in L_\infty(0,\ell),\ f,a,b,c \in L_\infty(D)$$

and a satisfies (2.5).

Theorem 2.5.1. *For $[g]_n \in \mathscr{G}_R^n$ and n,m large enough, the discrete state vector $[v([g]_n)]_n$*

satisfies the following estimate:

$$\|[v]_n\|_{\ell_\infty} := \max_{0\le k\le n}\left(\max_{0\le i\le m}|v_i(k)|\right)$$
$$\le C_\infty\Big(\|f\|_{L_\infty(D)}+\|p\|_{L_\infty(0,T)}+\|g^n\|_{L_\infty(0,T)}+\|\Phi\|_{L_\infty(0,\ell)}\Big) \tag{2.47}$$

where C_∞ is a constant independent of n and m.

Proof. Fix n arbitrarily large. Note $\max|v_i(0)| \le \|\Phi\|_{L_\infty(0,\ell)}$. Consider a positive function $\gamma(x)\in C^2[0,\ell]$ satisfying

$$\gamma(0)=\frac{1}{2},\ \gamma(\ell)=\frac{1}{2},\ \gamma'(0)=\frac{4(\|b\|_{L_\infty(D)}+\|c\|_{L_\infty(D)})}{a_0}+1,$$
$$\gamma'(\ell)=-1,\qquad \frac{1}{4}\le\gamma(x)\le 1,\ x\in[0,\ell]. \tag{2.48}$$

Define $\gamma_i=\gamma(x_i),\ i=\overline{0,m}$, and denote as x^i the value in $[x_i,x_{i+1}]$ that satisfies (by mean value theorem (MVT)) $\gamma(x_{i+1})-\gamma(x_i)=\gamma'(x^i)h$. Transform the discrete state vector as

$$w_i(k)=v_i(k)\gamma_i,\qquad i=\overline{0,m},\qquad k=\overline{0,n}.$$

System (2.24) can be rewritten as:

$$\begin{cases} h\zeta_k^0 v_{0\bar t}(k)-a_{0k}v_{0x}(k)+[hc_{0k}-b_{0k}]v_0(k)=hf_{0k}-g_k^n & \\ \zeta_k^i v_{i\bar t}(k)-[a_{i-1,k}-hb_{ik}]v_{ix\bar x}(k)+\frac{1}{h}[a_{i-1,k}-hb_{ik}-a_{ik}]v_{ix}(k) & \\ \qquad +c_{ik}v_i(k)=f_{ik} & ,\ i=\overline{1,m-1} \\ a_{m-1,k}v_{m-1,x}(k)+b_{m-1,k}v_{m-1}(k)=p_k & \end{cases}. \tag{2.49}$$

We note

$$v_i(k) = \frac{1}{\gamma_i} w_i(k), \qquad v_{i\bar{t}}(k) = \frac{1}{\gamma_i} w_{i\bar{t}}(k),$$

$$v_{ix}(k) = \frac{1}{\gamma_{i+1}} w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_x w_i(k) = \frac{1}{\gamma_i} w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_x w_{i+1}(k),$$

$$\begin{aligned} v_{ix\bar{x}}(k) &= \frac{1}{\gamma_{i-1}} w_{ix\bar{x}}(k) + \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right] w_{ix}(k) + \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} w_i(k) \\ &= \frac{1}{\gamma_{i+1}} w_{ix\bar{x}}(k) + \left[\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right] w_{i\bar{x}}(k) + \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} w_i(k), \end{aligned}$$

$$\left(\frac{1}{\gamma_i}\right)_x = -\frac{1}{\gamma_i\gamma_{i+1}}\gamma_{ix}, \qquad \left(\frac{1}{\gamma_i}\right)_{x\bar{x}} = -\frac{1}{\gamma_i\gamma_{i+1}}\gamma_{ix\bar{x}} + \frac{\gamma_{ix}+\gamma_{i\bar{x}}}{\gamma_{i-1}\gamma_i\gamma_{i+1}}\gamma_{i\bar{x}}.$$

Thus $w_i(0) = \gamma_i\Phi_i, \quad i = \overline{0,m}$, and for $k = \overline{1,n}$,

$$\begin{cases} \frac{h}{\gamma_0}\zeta_k^0 w_{0\bar{t}}(k) - \frac{a_{0k}}{\gamma_1} w_{0x}(k) - \left[a_{0k}\left(\frac{1}{\gamma_0}\right)_x + \frac{b_{0k}-hc_{0k}}{\gamma_0}\right] w_0(k) = hf_{0k} - g_k^n \\ \\ \frac{1}{\gamma_i}\zeta_k^i w_{i\bar{t}}(k) - \frac{a_{i-1,k}-hb_{ik}}{\gamma_{i-1}} w_{ix\bar{x}}(k) \\ -\left[(a_{i-1,k}-hb_{ik})\left(\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\right) + \frac{1}{\gamma_{i+1}h}[-a_{i-1,k}+hb_{ik}+a_{ik}]\right] w_{ix}(k) \\ -\left[(a_{i-1,k}-hb_{ik})\left(\frac{1}{\gamma_i}\right)_{x\bar{x}} + \frac{1}{h}(-a_{i-1,k}+hb_{ik}+a_{ik})\left(\frac{1}{\gamma_i}\right)_x + \frac{c_{ik}}{\gamma_i}\right] w_i(k) \\ = f_{ik}, \quad i = \overline{1,m-1} \\ \\ \frac{a_{m-1,k}-b_{m-1,k}h}{\gamma_{m-1}} w_{m-1,x}(k) + \left[a_{m-1,k}\left(\frac{1}{\gamma_{m-1}}\right)_x + \frac{b_{m-1,k}h}{\gamma_{m-1}}\right] w_m(k) = p_k \end{cases} \quad .. \tag{2.50}$$

Furthermore, transform $w_i(k)$ as:

$$u_i(k) = w_i(k)e^{-\lambda t_k}, \qquad i = \overline{0,m}, \quad k = \overline{0,n} \tag{2.51}$$

where λ satisfies

$$\bar{b}(\lambda-1)=(\|a\|_{L_\infty(D)}+\|b\|_{L_\infty(D)})(32\|\gamma''\|_{C[0,\ell]}+356\|\gamma'\|^2_{C[0,\ell]})$$
$$+32(\|a_x\|_{L_\infty(D)}+\|b\|_{L_\infty(D)})\|\gamma'\|_{C[0,\ell]}+8\|c\|_{L_\infty(D)} \tag{2.52}$$

and if $t^k\in[t_{k-1},t_k]$ satisfies through the MVT that $e^{\lambda t_k}-e^{\lambda t_{k-1}}=\lambda e^{\lambda t^k}\tau$, then

$$w_{i\bar{t}}(k)=e^{\lambda t_{k-1}}u_{i\bar{t}}(k)+\lambda e^{\lambda t^k}u_i(k).$$

So $u_i(0)=w_i(0)=\gamma_i\Phi_i,\quad i=\overline{0,m}$, and for $k=\overline{1,n}$, the vector $u(k)$ satisfies the system

$$\frac{h}{\gamma_0}\zeta_k^0 e^{-\lambda\tau}u_{0\bar{t}}(k)-\frac{a_{0k}}{\gamma_1}u_{0x}(k)+\Big[-a_{0k}\left(\frac{1}{\gamma_0}\right)_x-\frac{b_{0k}-hc_{0k}}{\gamma_0}$$
$$+\frac{h\lambda}{\gamma_0}\zeta_k^0 e^{-\lambda(t_k-t^k)}\Big]u_0(k)=e^{-\lambda t_k}(hf_{0k}-g_k^n),$$

$$\frac{1}{\gamma_i}\zeta_k^i e^{-\lambda\tau}u_{i\bar{t}}(k)-\frac{a_{i-1,k}-hb_{ik}}{\gamma_{i-1}}u_{ix\bar{x}}(k)$$
$$-\left[(a_{i-1,k}-hb_{ik})\Big(\left(\frac{1}{\gamma_i}\right)_{\bar{x}}+\left(\frac{1}{\gamma_i}\right)_x\Big)+\frac{1}{\gamma_{i+1}h}[-a_{i-1,k}+hb_{ik}+a_{ik}]\right]u_{ix}(k)$$
$$-\left[(a_{i-1,k}-hb_{ik})\left(\frac{1}{\gamma_i}\right)_{x\bar{x}}+\frac{1}{h}(-a_{i-1,k}+hb_{ik}+a_{ik})\left(\frac{1}{\gamma_i}\right)_x\right. \tag{2.53}$$
$$\left.+\frac{c_{ik}}{\gamma_i}-\frac{\zeta_k^i\lambda e^{-\lambda(t_k-t^k)}}{\gamma_i}\right]u_i(k)=e^{-\lambda t_k}f_{ik},\quad i=\overline{1,m-1}$$

$$\frac{a_{m-1,k}-b_{m-1,k}h}{\gamma_{m-1}}u_{m-1,x}(k)+\left[a_{m-1,k}\left(\frac{1}{\gamma_{m-1}}\right)_x+\frac{b_{m-1,k}h}{\gamma_{m-1}}\right]u_m(k)=e^{-\lambda t_k}p_k$$

Now fix $k_1 \le n$, and define the following sets of indexes for convenience:

$$\mathscr{M}_{k_1} = \{(i,k)|i = 0,\dots,m, \quad k = 0,\dots,k_1\},$$

$$\mathscr{N} = \{(i,k)|i = 1,\dots,m-1, \quad k = 1,\dots,k_1\},$$

$$\mathscr{T}_0 = \{(i,k)|i = 0, k = 1,\dots,k_1\},$$

$$\mathscr{T}_m = \{(i,k)|i = m, k = 1,\dots,k_1\},$$

$$\mathscr{X}_0 = \{(i,k)|i = 0,\dots,m, \quad k = 0\}.$$

Unless confusion may arise, we omit the subscript to $\mathscr{M}_{k_1}$. It is clear that

$$\mathscr{M} = \mathscr{N} \cup \mathscr{T}_0 \cup \mathscr{T}_m \cup \mathscr{X}_0.$$

If $u_i(k) \le 0$ in $\mathscr{M}$, then $\max\limits_{\mathscr{M}} u_i(k) \le 0$. Suppose that there exists (i,k) such that $u_i(k) > 0$. Then $\max\limits_{\mathscr{M}} u_i(k) > 0$. Let $(i^*,k^*) \in \mathscr{M}$ be such that $u_{i^*}(k^*) = \max\limits_{\mathscr{M}} u_i(k)$.

If $(i^*,k^*) \in \mathscr{X}_0$, then $u_{i^*}(k^*) = \max\limits_i \gamma_i \Phi_i \le \max\limits_i \Phi_i \le \max\limits_{[0,\ell]} \Phi(x)$.

If $(i^*,k^*) \in \mathscr{T}_m$, then $i^* = m$, $u_{m-1,x}(k^*) \ge 0$ and we can choose h small enough that $\gamma_{m-1,x} = \gamma'(x^{m-1}) \in (-\frac{3}{2}, -\frac{1}{2})$ so that

$$\left(-\frac{a_{m-1,k^*}\gamma'(x^{m-1})}{\gamma_m \gamma_{m-1}} + \frac{b_{m-1,k^*}h}{\gamma_{m-1}} \right) u_m(k^*) \le e^{-\lambda t_{k^*}} p_{k^*}$$

We can see that:

$$-\frac{a_{m-1,k^*}\gamma'(x^{m-1})}{\gamma_m \gamma_{m-1}} + \frac{b_{m-1,k^*}h}{\gamma_{m-1}} \ge \frac{a_0}{2} - 4\|b\|_{L_\infty(D)} h \ge \frac{a_0}{4}$$

for sufficiently small h. Thus we have:

$$u_m(k^*) \le \frac{4}{a_0} e^{-\lambda t_{k^*}} p_{k^*}$$

If $(i^*, k^*) \in \mathscr{T}_0$, then $i^* = 0, u_{0\bar{t}}(k^*) \ge 0$, $u_{0x}(k^*) \le 0$. Notice that $\left(\frac{1}{\gamma_0}\right)_x = -\frac{1}{\gamma_0 \gamma_1} \gamma_{0x}$. Note $\gamma_{0x} = \gamma'(x^0)$, so for h small enough, we can ascertain $\gamma_{0x} = \gamma'(x^0) \in (\frac{4(\|b\|_{L_\infty(D)} + \|c\|_{L_\infty(D)})}{a_0} + \frac{1}{2}, \frac{4(\|b\|_{L_\infty(D)} + \|c\|_{L_\infty(D)})}{a_0} + \frac{3}{2})$. It follows

$$\left[a_{0k^*} \frac{\gamma'(x^0)}{\gamma_0 \gamma_1} - \frac{b_{0k^*} - hc_{0k^*}}{\gamma_0} + \frac{h\lambda}{\gamma_0} \zeta^0_{k^*} e^{-\lambda(t_{k^*} - t^{k^*})}\right] u_0(k^*) = e^{-\lambda t_{k^*}} (hf_{0k^*} - g^n_{k^*})$$

Since the third term in the parenthesis on the left hand side is positive, we only consider first two terms. We can see that:

$$a_{0k^*} \frac{\gamma'(x^0)}{\gamma_0 \gamma_1} - \frac{b_{0k^*} - hc_{0k^*}}{\gamma_0} \ge a_0 \gamma'(x^0) - 4(\|b\|_{L_\infty(D)} + h\|c\|_{L_\infty(D)}) \ge \frac{a_0}{2}$$

Thus we have:

$$u_0(k^*) \le \frac{2}{a_0} e^{-\lambda t_{k^*}} (hf_{0k^*} - g^n_{k^*})$$

If $(i^*, k^*) \in \mathscr{N}$, then $u_{i^*\bar{t}}(k^*) \ge 0$, $u_{i^*x\bar{x}}(k^*) = \frac{1}{h^2}\left(u_{i^*+1}(k^*) - 2u_{i^*}(k^*) + u_{i^*-1}(k^*)\right) \le$

0. For $(i,k) \in \mathscr{N}$, the corresponding equation in (2.53) is equivalent to

$$\frac{1}{\gamma_{i^*}}\zeta_{k^*}^{i^*}e^{-\lambda\tau}u_{i^*\bar{t}}(k^*) - \left(\frac{a_{i^*-1,k^*} - hb_{i^*k^*}}{\gamma_{i^*-1}} + \frac{-a_{i^*-1,k^*} + hb_{i^*k^*} + a_{i^*k^*}}{\gamma_{i^*+1}}\right)u_{i^*x\bar{x}}(k^*)$$
$$-\left[(a_{i^*-1,k^*} - hb_{i^*k^*})\Big(\left(\frac{1}{\gamma_{i^*}}\right)_{\bar{x}} + \left(\frac{1}{\gamma_{i^*}}\right)_x\Big)\right.$$
$$\left.+\frac{1}{\gamma_{i^*+1}h}\Big[-a_{i^*-1,k^*} + hb_{i^*k^*} + a_{i^*k^*}\Big]\right]u_{i^*\bar{x}}(k^*)$$
$$-\left[(a_{i^*-1,k^*} - hb_{i^*k^*})\left(\frac{1}{\gamma_{i^*}}\right)_{x\bar{x}} + \frac{1}{h}(-a_{i^*-1,k^*} + hb_{i^*k^*} + a_{i^*k^*})\left(\frac{1}{\gamma_{i^*}}\right)_x\right.$$
$$\left.\frac{c_{i^*k^*}}{\gamma_{i^*}} - \frac{\zeta_{k^*}^{i^*}\lambda e^{-\lambda(t_{k^*}-t^{k^*})}}{\gamma_{i^*}}\right]u_{i^*}(k^*) = e^{-\lambda t_{k^*}}f_{i^*k^*} \tag{2.54}$$

Define the sets

$$\mathscr{N}_+ = \left\{(i,k) \in \mathscr{N}\,\middle|\right.$$
$$\left.\left[(a_{i-1,k} - hb_{ik})\Big(\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\Big) + \frac{1}{\gamma_{i+1}h}\Big[-a_{i-1,k} + hb_{ik} + a_{ik}\Big]\right] \geq 0\right\}$$
$$\mathscr{N}_- = \left\{(i,k) \in \mathscr{N}\,\middle|\right.$$
$$\left.\left[(a_{i-1,k} - hb_{ik})\Big(\left(\frac{1}{\gamma_i}\right)_{\bar{x}} + \left(\frac{1}{\gamma_i}\right)_x\Big) + \frac{1}{\gamma_{i+1}h}\Big[-a_{i-1,k} + hb_{ik} + a_{ik}\Big]\right] < 0\right\}.$$

And it's clear $\mathscr{N} = \mathscr{N}_+ \cup \mathscr{N}_-$. Suppose $(i^*,k^*) \in \mathscr{N}_+$. Then owing to (2.53) since $u_{i^*x}(k^*) \leq 0$, for sufficiently small h we can write

$$
-\left[(a_{i^*-1,k^*}-hb_{i^*k^*})\left(\frac{1}{\gamma_{i^*}}\right)_{x\bar{x}}+\frac{1}{h}(-a_{i^*-1,k^*}+hb_{i^*k^*}+a_{i^*k^*})\left(\frac{1}{\gamma_{i^*}}\right)_{x}+\frac{c_{i^*k^*}}{\gamma_{i^*}}\right.
$$
$$
\left.-\frac{\zeta_{k^*}^{i^*}\lambda e^{-\lambda(t_{k^*}-t^{k^*})}}{\gamma_{i^*}}\right]u_{i^*}(k^*)\leq e^{-\lambda t_{k^*}}f_{i^*k^*} \tag{2.55}
$$

If instead $(i^*,k^*)\in\mathcal{N}_-$, then we can use (2.54), the fact that $u_{i^*\bar{x}}(k^*)\geq 0$ and that

$$
\begin{aligned}
&-\left(\frac{a_{i-1,k}-hb_{ik}}{\gamma_{i-1}}+\frac{-a_{i-1,k}+hb_{ik}+a_{ik}}{\gamma_{i+1}}\right)\\
&=-\frac{a_{ik}}{\gamma_{i+1}}-\frac{(\gamma_{i+1}-\gamma_{i-1})a_{i-1,k}+h(\gamma_{i-1}-\gamma_{i+1})b_{ik}}{\gamma_{i+1}\gamma_{i-1}}\\
&\leq -a_0-\frac{2h\gamma'(\tilde{x})a_{i-1,k}+h(\gamma_{i-1}-\gamma_{i+1})b_{ik}}{\gamma_{i+1}\gamma_{i-1}}\leq -\frac{a_0}{2}
\end{aligned}
$$

for sufficiently small h, where $\tilde{x}$ is the value that satisfies the mean value theorem to achieve again (2.55). Therefore, (2.55) is achieved in any case. We can choose τ so small that $e^{-\lambda(t_k-t^k)}>\frac{1}{2},\quad\forall k$. The coefficient in front of $u_{i^*}(k^*)$ in (2.55) can be estimated as follows:

$$
\begin{aligned}
&\frac{\zeta_{k^*}^{i^*}\lambda e^{-\lambda(t_{k^*}-t^{k^*})}}{\gamma_{i^*}}-(a_{i^*-1,k^*}-hb_{i^*k^*})\left(\frac{-1}{\gamma_{i^*}\gamma_{i^*+1}}\gamma_{i^*x\bar{x}}+\frac{\gamma_{i^*x}+\gamma_{i^*\bar{x}}}{\gamma_{i^*-1}\gamma_{i^*}\gamma_{i^*+1}}\gamma_{i^*\bar{x}}\right)\\
&\qquad+(a_{i^*k^*,\bar{x}}+b_{i^*k^*})\frac{\gamma_{i^*x}}{\gamma_{i^*}\gamma_{i^*+1}}-\frac{c_{i^*k^*}}{\gamma_{i^*}}\\
&\geq\frac{\bar{b}\lambda}{2}-(\|a\|_{L_\infty(D)}+\|b\|_{L_\infty(D)})(16\|\gamma''\|_{C[0,\ell]}+128\|\gamma'\|^2_{C[0,\ell]})\\
&\qquad-16(\|a_x\|_{L_\infty(D)}+\|b\|_{L_\infty(D)})\|\gamma'\|_{C[0,\ell]}-4\|c\|_{L_\infty(D)}\geq\frac{\bar{b}}{2}
\end{aligned}
$$

due to definitions of λ and $\gamma(x)$. Then by (3.51), it is the case that the coefficient of

$u_{i^*}(k^*)$ is positive independently of i^*, k^*. Therefore,

$$u_{i^*}(k^*) \leq \frac{2}{\bar{b}} f_{i^* k^*} e^{-\lambda t_{k^*}}$$

We can put together the obtained estimations to deduce that for $(i,k) \in \mathscr{M}_{k_1}$,

$$u_i(k) \leq \max_{\mathscr{M}} u_i(k) \leq A \max \Big\{ 0,\ \|\Phi\|_{L_\infty(0,\ell)},\ \|p\|_{L_\infty(0,T)}, \|g^n\|_{L_\infty(0,T)},\ \|f\|_{L_\infty(D)} \Big\},$$

with $A = \max\{1,\ 4a_0^{-1},\ 2\bar{b}^{-1}\}$. But because $u_i(k) = \gamma_i e^{-\lambda t_k} v_i(k)$, we have the following uniform upper bound for the discrete state vector:

$$v_i(k) \leq 4Ae^{\lambda T} \max \Big\{ 0,\ \|\Phi\|_{L_\infty(0,\ell)},\ \|p\|_{L_\infty(0,T)},\ \|g^n\|_{L_\infty(0,T)},\ \|f\|_{L_\infty(D)} \Big\},$$

for $\forall (i,k) \in \mathscr{M}_{k_1}$. In a fully analogous manner, we arrive at a uniform lower bound for the discrete state vector:

$$v_i(k) \geq 4Ae^{\lambda T} \min \Big\{ 0, -\|\Phi\|_{L_\infty(0,\ell)}, -\|p\|_{L_\infty(0,T)}, -\|g^n\|_{L_\infty(0,T)}, -\|f\|_{L_\infty(D)} \Big\},$$

for $\forall (i,k) \in \mathscr{M}_{k_1}$. Combining the uniform upper and lower bounds imply (2.5.1) up to k_1. But k_1 was arbitrary in $1, \ldots, n$. Theorem is proved. $\square$

2.5.2 $W_2^{1,1}(D)$ Estimate for the Discrete Multiphase Free Boundary Problem

Theorem 2.5.2. *For $[g]_n \in \mathscr{G}_R^n$ and n,m large enough, the discrete state vector $[v([g]_n)]_n$ satisfies the following estimate:*

$$\begin{aligned}\|[v]_n\|_{\mathscr{E}}^2 := \sum_{k=1}^{n}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \max_{1\le k\le n}\left(\sum_{i=0}^{m-1} hv_{ix}^2(k)\right) + \sum_{k=1}^{n}\tau^2\sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \\ \le\ \tilde{C}_\infty\Big(\|\Phi\|^2_{W_2^1(0,\ell)} + \|f\|^2_{L_\infty(D)} + \|p\|^2_{W_2^1(0,T)} + \|g^n\|^2_{W_2^1(0,T)}\Big)\end{aligned} \tag{2.56}$$

where $\tilde{C}_\infty$ is a constant independent of n and m.

Proof. Consider n and m large enough that Theorem 2.5.1 is satisfied. In (2.16), choose $\eta = 2\tau v_{\bar{t}}(k)$. Using (2.33), write $(b_n(v_i(k)))_{\bar{t}} = \zeta_k^i v_{i\bar{t}}(k)$. Also, use the fact that

$$\begin{aligned}&2\tau a_{ik} v_{ix}(k)(v_{i\bar{t}}(k))_x \\ &= a_{ik}v_{ix}^2(k) - a_{i,k-1}v_{ix}^2(k-1) - \tau a_{ik\bar{t}}v_{ix}^2(k-1) + \tau^2 a_{ik}v_{ix\bar{t}}^2(k)\end{aligned}$$

Using the above equality, and the lower bound for $a(x,t)$, we thus have

$$\begin{aligned}2\tau\sum_{i=0}^{m-1} h\zeta_k^i v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} ha_{ik}v_{ix}^2(k) - \sum_{i=0}^{m-1} ha_{i,k-1}v_{ix}^2(k-1) + a_0\tau^2\sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \\ \le \tau\sum_{i=0}^{m-1} ha_{ik\bar{t}}v_{ix}^2(k-1) - 2\tau\sum_{i=0}^{m-1} hb_{ik}v_i(k)v_{ix\bar{t}}(k) - 2\tau\sum_{i=0}^{m-1} hc_{ik}v_i(k)v_{i\bar{t}}(k) \\ +2\tau\sum_{i=0}^{m-1} hf_{ik}v_{i\bar{t}}(k) + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k).\end{aligned} \tag{2.57}$$

We will now look to estimate the three summation terms on the right hand side of

(2.57). By using summation by parts and Cauchy inequality with $\varepsilon > 0$ we get:

$$-2\tau\sum_{i=0}^{m-1} hb_{ik}v_i(k)v_{ix\bar{t}}(k) = 2\tau\sum_{i=1}^{m-1} hb_{ik}v_{i-1,x}(k)v_{i\bar{t}}(k)$$

$$+2\tau\sum_{i=i}^{m-1} hb_{i-1,k,x}v_{i-1}(k)v_{i\bar{t}}(k) - 2\tau b_{m-1,k}v_{m-1}(k)v_{m\bar{t}}(k) + 2\tau b_{0k}v_0(k)v_{0\bar{t}}(k)$$

$$\leq \frac{\bar{b}}{4}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{4\|b\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_{ix}^2(k) + \frac{\bar{b}}{4}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k)$$

$$+\frac{4\|b_x\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_i^2(k) - 2\tau b_{m-1,k}v_{m-1}(k)v_{m,\bar{t}}(k) + 2\tau b_{0k}v_0(k)v_{0\bar{t}}(k)$$

We will also use the fact that

$$2\tau b_{0k}v_0(k)v_{0\bar{t}}(k) = \tau^2 b_{0k}v_{0\bar{t}}^2(k) - b_{0k}v_0^2(k-1) + b_{0k}v_0^2(k),$$

$$-2\tau b_{m-1,k}v_{m-1}(k)v_{m\bar{t}}(k)$$

$$= -2\tau b_{m-1,k}v_m(k)v_{m\bar{t}}(k) + 2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k)$$

$$= -\tau^2 b_{m-1,k}v_{m\bar{t}}^2(k) + b_{m-1,k}v_m^2(k-1) - b_{m-1,k}v_m^2(k)$$

$$+2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k)$$

Estimating the other two summation terms on the right-hand side of (2.57) via

Cauchy Inequality with $\varepsilon > 0$ and by recalling (2.15), we have:

$$
\begin{aligned}
&2\tau\bar{b}\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} ha_{ik}v_{ix}^2(k) - \sum_{i=0}^{m-1} ha_{i,k-1}v_{ix}^2(k-1) + a_0\tau^2\sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \\
&\leq \tau\sum_{i=0}^{m-1} ha_{ik\bar{t}}v_{ix}^2(k-1) + \frac{\bar{b}}{4}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{4\|b\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_{ix}^2(k) \\
&+\frac{\bar{b}}{4}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{4\|b_x\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_i^2(k) + \frac{\tau}{2}\bar{b}\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{2}{\bar{b}}\tau\sum_{i=0}^{m-1} hc_{ik}^2v_i^2(k) \\
&+\frac{\tau}{2}\bar{b}\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \frac{2}{\bar{b}}\tau\sum_{i=0}^{m-1} hf_{ik}^2 + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k) + \tau^2 b_{0k}v_{0\bar{t}}^2(k) \\
&-b_{0k}v_0^2(k-1) + b_{0k}v_0^2(k) - \tau^2 b_{m-1,k}v_{m\bar{t}}^2(k) + b_{m-1,k}v_m^2(k-1) - b_{m-1,k}v_m^2(k) \\
&+2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k).
\end{aligned} \tag{2.58}
$$

By absorbing several terms on the right hand side of (2.58) to the left-hand side, and further bounding the right hand side we get

$$
\begin{aligned}
&\frac{\bar{b}}{2}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} ha_{ik}v_{ix}^2(k) - \sum_{i=0}^{m-1} ha_{i,k-1}v_{ix}^2(k-1) + a_0\tau^2\sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k) \\
&\leq \tau\sum_{i=0}^{m-1} ha_{ik\bar{t}}v_{ix}^2(k-1) + \frac{4\|b_x\|_{L_\infty(D)}^2 + 2\|c\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_i^2(k) + \frac{2}{\bar{b}}\tau\sum_{i=0}^{m-1} hf_{ik}^2 \\
&+\frac{4\|b\|_{L_\infty(D)}^2}{\bar{b}}\tau\sum_{i=0}^{m-1} hv_{ix}^2(k) + 2\tau p_k v_{m\bar{t}}(k) - 2\tau g_k^n v_{0\bar{t}}(k) + \tau^2 b_{0k}v_{0\bar{t}}^2(k) \\
&-b_{0k}v_0^2(k-1) + b_{0k}v_0^2(k) - \tau^2 b_{m-1,k}v_{m\bar{t}}^2(k) + b_{m-1,k}v_m^2(k-1) - b_{m-1,k}v_m^2(k) \\
&+2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k),.
\end{aligned} \tag{2.59}
$$

$\forall k = \overline{1,n}$. Perform summation of (2.59) for k from 1 to q, $2 \leq q \leq n$. The second and third terms on the left-hand side telescope, and we obtain:

$$\frac{\bar{b}}{2}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)+\sum_{i=0}^{m-1}ha_{iq}v_{ix}^2(q)+a_0\sum_{k=1}^{q}\tau^2\sum_{i=0}^{m-1}hv_{ix\bar{t}}^2(k)$$
$$\leq\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}ha_{ik\bar{t}}v_{ix}^2(k-1)+\sum_{i=0}^{m-1}ha_{i0}v_{ix}^2(0)$$
$$+\frac{4\|b_x\|_{L_\infty(D)}^2+2\|c\|_{L_\infty(D)}^2}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_i^2(k)+\frac{2}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hf_{ik}^2$$
$$+\frac{4\|b\|_{L_\infty(D)}^2}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{ix}^2(k)+2\sum_{k=1}^{q}\tau p_k v_{m\bar{t}}(k)-2\sum_{k=1}^{q}\tau g_k^n v_{0\bar{t}}(k)$$
$$+\sum_{k=1}^{q}\tau^2 b_{0k}v_{0\bar{t}}^2(k)-\sum_{k=1}^{q}\tau^2 b_{m-1,k}v_{m\bar{t}}^2(k)+\sum_{k=1}^{q}\left(b_{0k}v_0^2(k)-b_{0k}v_0^2(k-1)\right)$$
$$-\sum_{k=1}^{q}\left(b_{m-1,k}v_m^2(k)-b_{m-1,k}v_m^2(k-1)\right)$$
$$+\sum_{k=1}^{q}2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k) \tag{2.60}$$

We can estimate the right hand side further and use (2.8) to get

$$\sum_{k=1}^{q}\tau^2 b_{0k}v_{0\bar{t}}^2(k)\leq\tau\|b\|_{L_\infty(D)}\sum_{k=1}^{q}\tau v_{0\bar{t}}^2(k)\leq\frac{\bar{b}}{8}h\sum_{k=1}^{q}\tau v_{0\bar{t}}^2(k)\leq\frac{\bar{b}}{8}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)$$

Similarly,

$$-\sum_{k=1}^{q}\tau^2 b_{m-1,k}v_{m\bar{t}}^2(k)\leq\frac{\bar{b}}{8}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)$$

We also have:

$$\sum_{k=1}^{q}(b_{0k}v_0^2(k)-b_{0k}v_0^2(k-1))=b_{0q}v_0^2(q)-b_{00}v_0^2(0)-\sum_{k=1}^{q}\tau b_{0k\bar{t}}v_0^2(k-1). \tag{2.61}$$

Since

$$\Big|\sum_{k=1}^{q}\tau b_{0k\bar{t}}v_0^2(k-1)\Big| \leq \|[v]_n\|_{\ell_\infty}^2\frac{1}{h\tau}\int_0^h\int_0^{t_q}\int_{t-\tau}^{t}|b_t(x,\theta)|\,d\theta\,dt\,dx$$
$$\leq \|[v]_n\|_{\ell_\infty}^2\int_{-\tau}^{T}\operatorname*{ess\,sup}_{x\in[0,\ell]}|b_t(x,t)|\,dt$$

from (2.61) it follows that

$$\Big|\sum_{k=1}^{q}(b_{0k}v_0^2(k)-b_{0k}v_0^2(k-1))\Big| \leq \|[v]_n\|_{\ell_\infty}^2\Big(2\|b\|_{L_\infty(D)}+\int_{-\tau}^{T}\operatorname*{ess\,sup}_{x\in[0,\ell]}|b_t(x,t)|\,dt\Big)$$

Similarly, we have:

$$\Big|\sum_{k=1}^{q}(b_{m-1,k}v_m^2(k)-b_{m-1,k}v_m^2(k-1))\Big|$$
$$\leq \|[v]_n\|_{\ell_\infty}^2\Big(2\|b\|_{L_\infty(D)}+\int_{-\tau}^{T}\operatorname*{ess\,sup}_{x\in[0,\ell]}|b_t(x,t)|\,dt\Big)$$

From [5], in a similar fashion to above, we have:

$$\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}ha_{ik\bar{t}}v_{ix}^2(k-1)\leq 2\int_0^T\operatorname*{ess\,sup}_{x\in[0,\ell]}|a_t(x,t)|\,dt\max_{1\leq k\leq n}\sum_{i=0}^{m-1}hv_{ix}^2(k)+C\sum_{i=0}^{m-1}h\Phi_{ix}^2$$

where C is a constant independent of n. Using Cauchy Inequality with $\varepsilon=\frac{\bar{b}}{8}$, we have:

$$\sum_{k=1}^{q}2\tau hb_{m-1,k}v_{m-1,x}(k)v_{m\bar{t}}(k)$$
$$\leq\frac{\bar{b}}{8}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1}hv_{i\bar{t}}^2(k)+\frac{8\|b\|_{L_\infty(D)}^2t_q}{\bar{b}}\max_{1\leq k\leq q}\sum_{i=0}^{m-1}hv_{ix}^2(k)$$

Use the summation by parts technique on the p and g sums:

$$\sum_{k=1}^{q}\tau p_k v_{m\bar{t}}(k) = -\sum_{k=1}^{q-1}\tau p_{kt} v_m(k) + p_q v_m(q) - p_1 v_m(0),$$
$$\sum_{k=1}^{q}\tau g_k^n v_{0\bar{t}}(k) = -\sum_{k=1}^{q-1}\tau g_{kt}^n v_0(k) + g_q^n v_0(q) - g_1^n v_0(0). \tag{2.62}$$

In view of (2.62) and the above estimates, (2.60) yields

$$\begin{aligned}
&\frac{\bar{b}}{8}\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1} h v_{i\bar{t}}^2(k) + \sum_{i=0}^{m-1} h a_{iq} v_{ix}^2(q) + a_0\sum_{k=1}^{q}\tau^2\sum_{i=0}^{m-1} h v_{ix\bar{t}}^2(k) \\
&\le 2\int_0^T \operatorname*{ess\,sup}_{x\in[0,\ell]} |a_t(x,t)|\,dt \max_{1\le k\le n}\sum_{i=0}^{m-1} h v_{ix}^2(k) + (C + \|a\|_{L_\infty(D)})\sum_{i=0}^{m-1} h\Phi_{ix}^2 \\
&\quad + \frac{(4\|b_x\|_{L_\infty(D)}^2 + 2\|c\|_{L_\infty(D)}^2)T\ell}{\bar{b}}\|[v]_n\|_{\ell_\infty}^2 + \frac{2}{\bar{b}}\sum_{k=1}^{n}\tau\sum_{i=0}^{m-1} h f_{ik}^2 \\
&\frac{4\|b\|_{L_\infty(D)}^2 t_q}{\bar{b}}\max_{1\le k\le n}\sum_{i=0}^{m-1} h v_{ix}^2(k) - 2\sum_{k=1}^{q-1}\tau p_{kt} v_m(k) + 2p_q v_m(q) - 2p_1 v_m(0) \\
&+2\sum_{k=1}^{q-1}\tau g_{kt}^n v_0(k) - 2g_q^n v_0(q) + 2g_1^n v_0(0) + \frac{8\|b\|_{L_\infty(D)}^2 t_q}{\bar{b}}\max_{1\le k\le n}\sum_{i=0}^{m-1} h v_{ix}^2(k) \\
&\quad + \Big(4\|b\|_{L_\infty(D)} + 2\int_{-\tau}^{T}\operatorname*{ess\,sup}_{x\in[0,\ell]} |b_t(x,t)|\,dt\Big)\|[v]_n\|_{\ell_\infty}^2
\end{aligned} \tag{2.63}$$

Applying Cauchy-Schwartz inequality to Steklov averages, by emloying Morrey's inequality ([63]), for sufficiently small h we have the following estimations:

$$\sum_{i=0}^{m-1} h\Phi_{ix}^2 \le \|\Phi'\|^2_{L_2(0,\ell)} + \|\Phi'\|^2_{L_2(\ell-h,\ell)} \le \|\Phi\|^2_{W_2^1(0,\ell)},$$

$$\|g^n\|_{L_\infty(0,T)} \le C\|g^n\|_{W_2^1(0,T)},$$

$$\sum_{k=1}^{q}\tau\sum_{i=0}^{m-1} hf_{ik}^2 \le \|f\|^2_{L_2(D)},$$

$$2\sum_{k=1}^{q-1}\tau g_{kt}^n v_0(k) \le \sum_{k=1}^{q-1}\tau\left(g_{kt}^n\right)^2 + \sum_{k=1}^{q-1}\tau v_0^2(k) \le \|(g^n)'\|^2_{L_2(0,T)} + T\|[v]_n\|^2_{\ell_\infty},$$

$$2p_q v_m(q) \le \|p\|^2_{L_\infty(0,T)} + v_m^2(q) \le C\|p\|^2_{W_2^1(0,T)} + \|[v]_n\|^2_{\ell_\infty},. \tag{2.64}$$

with last two estimations being extended to similar terms

$$2\sum_{k=1}^{q-1}\tau p_{kt} v_m(k),\ 2p_1 v_m(0),\ 2g_q^n v_0(q),\ 2g_1^n v_0(0)$$

Applying the results in (2.64), along with L_∞-estimate (2.5.1) to (2.63), and taking into account that $q=\overline{1,n}$ is arbitrary we derive

$$\begin{aligned}
&\frac{\bar{b}}{8}\sum_{k=1}^{n}\tau\sum_{i=0}^{m-1} hv_{i\bar{t}}^2(k) + a_0\max_{1\le k\le n}\sum_{i=0}^{m-1} hv_{ix}^2(k) + a_0\sum_{k=1}^{n}\tau^2\sum_{i=0}^{m-1} hv_{ix\bar{t}}^2(k)\\
&\le \tilde{C}_\infty\Big(\|\Phi\|^2_{W_2^1(0,\ell)} + \|f\|^2_{L_\infty(D)} + \|p\|^2_{W_2^1(0,T)} + \|g^n\|^2_{W_2^1(0,T)}\Big)\\
&+\left(2\int_0^T \operatorname*{ess\,sup}_{x\in[0,\ell]}|a_t(x,t)|\,dt + \frac{12\|b\|^2_{L_\infty(D)}T}{\bar{b}}\right)\max_{1\le k\le n}\sum_{i=0}^{m-1} hv_{ix}^2(k)
\end{aligned} \tag{2.65}$$

where $\tilde{C}_\infty$ is a constant independent of n,m. If

$$2\int_0^T \operatorname*{ess\,sup}_{x\in[0,\ell]}|a_t(x,t)|\,dt + \frac{12\|b\|^2_{L_\infty(D)}T}{\bar{b}} < a_0 \tag{2.66}$$

then we absorb the extra term to the left-hand side and (2.56) follows. If not, we partition $[0,T]$ into finitely many intervals such that in each interval I, (2.66) is satisfied with integral along I, and T in the second term replaced with the interval length $|I|$. Therefore, energy estimate (2.56) holds in each interval segment, and by adding finitely many inequalities, (2.56) in the whole segment follows. □

2.5.3 Existence of the Optimal Control and Convergence of the Discrete Optimal Control Problems

Theorem 2.5.3. *Let $\{[g]_n\}$ be a sequence in $\mathscr{G}_R^n$ such that the sequence of interpolations $\{\mathscr{P}_n([g]_n)\}$ converges weakly to $g \in W_2^1[0,T]$. Then the whole sequence of interpolations $\{\hat{v}^\tau\}$ of the associated discrete state vectors converges weakly in $W_2^{1,1}(D)$ to the unique weak solution $v = v(x,t;g)$ of the multiphase parabolic free boundary problem (2.1)-(2.4).*

Proof. Having estimates (2.5.1),(2.5.2), the proof is pursued similar to the proof of Theorem 5 in [5]. By the definitions of the interpolations in (2.18) we have that there is a subsequence of $\{\hat{v}^\tau\}$ that converges weakly in $W_2^{1,1}(D)$ to some function $v \in W_2^{1,1}(D) \cap L_\infty(D)$, strongly in $L_2(D)$, and a further subsequence that converges to v pointwise almost everywhere in D. We also have equivalence of $\{v^\tau\}$ and $\{\hat{v}^\tau\}$ in $W_2^{1,0}(D)$, and equivalence of $\{v^\tau\}$ and $\{\tilde{v}\}$ in $L_2(D)$. Accordingly, $v^\tau \to v$ weakly in $W_2^{1,0}(D)$ and $\tilde{v} \to v$ strongly in $L_2(D)$ and pointwise a.e. on D along a subsequence. Fix arbitrary $\psi \in W_2^{1,1}(D)$ with $\psi|_{t=T} = 0$. Due to density of $C^1(\bar{D})$ in $W_2^{1,1}(D)$, without loss of generality we can consider $\psi \in C^1(\overline{D})$ and $\psi|_{t=T} = 0$. Define $\psi_i(k) = \psi(x_i,t_k)$, and consider the interpolations:

$$\psi^\tau(x,t) := \psi_i(k), \qquad \psi_x^\tau(x,t) := \psi_{ix}(k) \qquad \psi_t^\tau(x,t) := \psi_{it}(k),$$
$$x_i \le x < x_{i+1}, \quad t_{k-1} < t \le t_k, \qquad i = \overline{0,m},\ k = \overline{0,n}. \tag{2.67}$$

It is readily checked that $\psi^\tau, \psi_x^\tau, \psi_t^\tau$ converge uniformly on $\overline{D}$ as $n,m \to \infty$ to the functions ψ, ψ_x, ψ_t respectively. For each k in (2.16) as satisfied by the discrete state vector $[v([g]_n)]_n$, choose $\eta_i = \tau\psi_i(k),\ i = 0,...,m$ and sum all equalities (2.16) over $k = 1,\dots,n$. The resulting expression is as follows:

$$\sum_{k=1}^{n} \tau \sum_{i=0}^{m-1} h\Big[\big(b_n(v_i(k))\big)_{\bar{t}}\psi_i(k) + a_{ik}v_{ix}(k)\psi_{ix}(k) + b_{ik}v_i(k)\psi_{ix}(k)$$
$$+c_{ik}v_i(k)\psi_i(k) - f_{ik}\psi_i(k)\Big] - \sum_{k=1}^{n} \tau p_k \psi_m(k) + \sum_{k=1}^{n} \tau g_k^n \psi_0(k) = 0 \tag{2.68}$$

The first term is transformed through summation by parts as in [5], and using the interpolations, (2.68) becomes the following integral identity:

$$\int_0^T \int_0^\ell \Big[-b_n(\tilde{v})\psi_t^\tau + av_x^\tau\psi_x^\tau + b\tilde{v}\psi_x^\tau + c\tilde{v}\psi^\tau - f\psi^\tau\Big]\,dx\,dt - \int_0^\ell b_n(\tilde{\Phi})\psi^\tau(x,\tau)\,dx$$
$$-\int_0^T p(t)\psi^\tau(\ell,t)\,dt + \int_0^T g^n(t)\psi^\tau(0,t)\,dt + \int_{T-\tau}^T \int_0^\ell b_n(\tilde{v})\psi_t^\tau\,dx\,dt = 0. \tag{2.69}$$

From [5], we have that $b_n(\tilde{v})$ has a subsequence such that both $b_n(\tilde{v})$ and $b_n(\tilde{\Phi})$ converge weakly in $L_2(D)$ and $L_2[0,\ell]$, respectfully, to functions of type $\mathscr{B}$, which we will denote as $\tilde{b}(x,t)$ and $\tilde{b}_0(x)$. We also have that the last integral tends to 0 due to absolute continuity of the integral. Using the convergence properties of the interpolations, due to weak convergence of $\{\hat{v}^\tau\}$, equivalence of $\{\hat{v}^\tau\}$ and $\{v^\tau\}$, and uniform convergence of

$\{\psi_x^\tau\}$, passing to the limit as $n \to +\infty$ we get:

$$\int_0^T \int_0^\ell \Big[-\tilde{b}(x,t)\psi_t + av_x\psi_x + bv\psi_x + cv\psi - f\psi \Big]\,dx\,dt - \int_0^\ell \tilde{b}_0(x)\psi(x,0)\,dx$$

$$- \int_0^T p(t)\psi(\ell,t)\,dt + \int_0^T g(t)\psi(0,t)\,dt = 0. \tag{2.70}$$

Since $\tilde{b}(x,t)$ and $\tilde{b}_0(x)$ are both of type $\mathscr{B}$, and by use of Mazur's lemma, we deduce as in [5] that $\tilde{b}(x,t) = B(x,t,v(x,t))$ $\tilde{b}_0(x) = B(x,0,\Phi(x))$ a.e on D and $(0,\ell)$ respectfully. This implies that v is a weak solution in the sense of Definition 2.1.1. By Lemma 2.4.4, this implies v is the only solution of the problem, and hence the only limit point of the sequence $\{\hat{v}^\tau\}$. $\square$

Having estimates (2.5.1),(2.5.2) and compactness Theorem 2.5.3, the completion of the proof of Theorems 2.3.1 and 2.3.2 coincides with the proof given in [5], through compactness arguments and proving weak continuity of cost functional $\mathscr{J}$, and verification of the conditions of Theorem 2.4.2 (see Lemmas A, B and C in [5]).

Chapter 3

Optimal Control of Singular Parabolic PDE Modeling Multidimensional and Multiphase Stefan-type Free Boundary Problems

The results of this chapter have been submitted for publication [10].

3.1 Optimal Control Problem for Singular PDE

Consider the formulation of the open problem given in Section 1.3. We now define the notion of the weak solution of the singular PDE problem (1.28)-(1.30).

Given f, a weak solution $v = v(x,t;f)$ of the problem (1.28)-(1.30) is defined as follows:

Definition 3.1.1. $v \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$ is called a *weak solution of the singular PDE problem* (1.28)-(1.30) if for some functions B, B_0 of type $\mathscr{B}$, the integral identity

$$\int_D \Big[-B(x,t,v(x,t))\psi_t + \sum_{i=1}^d [a_i(x,t)v_{x_i} + b_i(x,t)v]\psi_{x_i} + \sum_{i=1}^d c_i(x,t)v_{x_i}\psi + r(x,t)v\psi - f\psi \Big] dxdt$$
$$- \int_\Omega B_0(x,0,\Phi(x))\psi(x,0)\,dx = 0 \tag{3.1}$$

is satisfied for arbitrary $\psi \in \overset{\circ}{W}{}_2^{1,1}(D)$ with $\psi|_{\Omega\times\{t=T\}} = 0$.

3.2 Discrete Optimal Control Problem for Singular PDE

We employ a discrete framework introduced in [7] to pursue a discretization of the *Problem* $\mathscr{I}$. Let $n \in \mathbb{N}, \tau := \frac{T}{n}, h > 0$, and slice $\mathbb{R}^d \times \mathbb{R}$ by the planes

$$x_i = k_i h,\ i = 1,\ldots,d, \quad t = k_0\tau, \qquad \forall k_\ell \in \mathbb{Z},\ \ell = 0,1,\ldots,d,$$

so that $\mathbb{R}^d \times \mathbb{R}$ is split into cells of length h in every x_i direction, and of length τ in the t direction. We assume the following relation between h and τ;

$$\frac{h}{\tau} \geq \frac{1 + 2\sum_{i=1}^d \|b_i\|_{L_\infty(D)}}{\bar{b}} \tag{3.2}$$

where $\bar{b}$ is defined in (1.32). We will use Δ as a notation to represent a discretization with steps (τ,h). We consider a partial ordering on the set of discretizations: we say $\Delta_1 \leq \Delta_2$ if $\tau_1 \leq \tau_2$ and $h_1 \leq h_2$. We will denote $t_\ell = \tau\ell$ for $\ell = \overline{1,n}$. We will consider two multi-indexes, $\alpha = (k_1,k_2,\ldots,k_d,k_0)$ and $\gamma = (k_1,k_2,\ldots,k_d)$. We can also denote

$\alpha = (\gamma, k_0)$, and let α_i be the $i-$th component of α, provided $i \in \{1, 2, \ldots, d\}$ and α_0 is the $d+1-$st component of α, and γ_i is the $i-$th component of γ. We can thus represent each elementary cell C_Δ^α the rectangular prism R_Δ^γ uniquely as

$$C_\Delta^\alpha = \Big\{(x,t) \in \mathbb{R}^d \times \mathbb{R} \,\Big|\, k_i h \le x_i \le (k_i+1)h, \; i = 1, \ldots, d; \;\; (k_0 - 1)\tau \le t \le k_0 \tau \Big\}.$$

$$R_\Delta^\gamma = \Big\{x \in \mathbb{R}^d \,\Big|\, k_i h \le x_i \le (k_i+1)h, \; i = 1, \ldots, d\}.$$

and a superscript k represents the projection of R_Δ^γ onto the hyper-plane $t = k\tau$ of R^{d+1}:

$$R_\Delta^{\gamma,k} = \Big\{(x,t) \in \mathbb{R}^d \times \mathbb{R} \,\Big|\, k_i h \le x_i \le (k_i+1)h, \; i = 1, \ldots, d; \;\; t = k\tau\}.$$

Selecting the collection of prisms and cells contained in $\overline{D}$ and $\overline{\Omega}$ respectively,

$$\mathscr{C}_\Delta^D = \Big\{C_\Delta^\alpha \mid C_\Delta^\alpha \subset \overline{D}, \; \alpha \in \mathbb{Z}^{d+1}\Big\}, \;\; \mathscr{R}_\Delta^\Omega = \Big\{R_\Delta^\gamma \mid R_\Delta^\gamma \subset \overline{\Omega}, \; \gamma \in \mathbb{Z}^d\Big\},$$

discretized domains are defined as follows:

$$\Omega_\Delta = \bigcup_{R_\Delta^\gamma \in \mathscr{R}_\Delta^\Omega} R_\Delta^\gamma \subset \overline{\Omega}, \qquad D_\Delta = \bigcup_{C_\Delta^\alpha \in \mathscr{C}_\Delta^D} C_\Delta^\alpha \subset \overline{D}.$$

We define a *natural corner* of a prism R_Δ^γ as the vertex with the relatively smallest coordinates in respect to the other vertices. We define the natural corner of the cell $C_\Delta^{(\gamma,k)}$ as the vertex whose spatial coordinates coincide with those of the natural corner of R_Δ^γ and whose time coordinate is $k\tau$. Furthermore, each cell and prism will be identified by its natural corner. The lateral boundary of D_Δ is denoted by S_Δ and interior sets are defined as $D'_\Delta = (D_\Delta \backslash \partial D_\Delta) \cup (\Omega_\Delta \times \{t = T\})$ and $\Omega'_\Delta = \Omega_\Delta \backslash \partial \Omega_\Delta$. Similar notation will be used for cells C_Δ^α and prisms R_Δ^γ. Next, we introduce the lattice of points

$$\mathscr{L}_T = \Big\{(x,t) \in \mathbb{R}^d \times \mathbb{R} \mid \exists \alpha \in \mathbb{Z}^{d+1} \text{ s.t. } x_i = k_i h,\ i = 1,\dots,d,\ \ t = k_0 \tau\Big\},$$

$$\mathscr{L} = \Big\{x \in \mathbb{R}^d \mid \exists \gamma \in \mathbb{Z}^d \text{ s.t. } x_i = k_i h,\ i = 1,\dots,d\Big\}.$$

We will denote for notational purposes $y = (x,t)$, $y_\alpha = (k_1 h, k_2 h, \dots, k_d h, k_0 \tau), x_\gamma = (k_1 h, k_2 h, \dots, k_d h)$. Bijections $\alpha \mapsto y_\alpha$ and $\gamma \mapsto x_\gamma$ will be referred as natural bijections.

Let X be a set with a natural bijection to a collection of multi-indices, γ (or α). We will denote $\mathscr{A}(X)$ as the set of multi-indices corresponding to X. For $X \subset \mathbb{R}^d$ (or $X \subset \mathbb{R}^{d+1}$), we define $\mathscr{L}(X) := \mathscr{L} \cap X$. For ease of notation, we will write $\mathscr{A}(Y)$ in lieu of $\mathscr{A}(\mathscr{L}(Y))$ or $\mathscr{A}(\mathscr{L}_T(Y))$ and denote $\mathscr{A} := \mathscr{A}(\mathscr{R}^\Omega_\Delta)$. The latter means the set of all indices γ which are in natural bijection with the natural corners of the prisms in Ω_Δ. In contrast, $\mathscr{A}(\Omega'_\Delta)$ (or $\mathscr{A}(\Omega_\Delta)$) is the set of indices in natural bijection to the lattice points in the interior of Ω_Δ (or in Ω_Δ). It is obvious that $\mathscr{A}(\Omega'_\Delta)$ is a subset of $\mathscr{A}$. For ease of notation, in Σ and other operations requiring subscripts, expressions like $\gamma \in \mathscr{A}(X)$ will be replaced simply with $\mathscr{A}(X)$.

Given our data in the appropriate Sobolev or Lebesgue spaces of measurable functions, we define corresponding discrete grid functions via Steklov averages. Whenever it is necessary, we extend all functions to slightly wider region with preservation of the norm. In order to construct discrete version of $\Phi \in W_2^1(\Omega)$ defined on $\mathscr{A}(\Omega_\Delta)$, we construct an extension of Φ to $\Omega + B_1(0)$, so that the extension is in $W_2^1(\Omega + B_1(0))$. Such an extension is possible, since $\partial\Omega$ is Lipschitz [63]. Let

$$\phi_\gamma = \frac{1}{h^d} \int_{x_1}^{x_1+h} \int_{x_2}^{x_2+h} \cdots \int_{x_d}^{x_d+h} \phi(x)\, dx, \qquad \gamma \in \mathscr{A}(\Omega_\Delta), \tag{3.3}$$

where ϕ stands for functions Φ, Γ, or initial traces of functions such as $a_i(\cdot,0)$. Let

$$g_\alpha = \frac{1}{\tau h^d} \int\limits_{t_{k-1}}^{t_k} \int\limits_{x_1}^{x_1+h} \int\limits_{x_2}^{x_2+h} \cdots \int\limits_{x_d}^{x_d+h} g(x,t)\,dx\,dt, \qquad \alpha = (\gamma,k) \in \mathscr{A}(\mathscr{C}_\Delta^D),\ k \geq 1 \tag{3.4}$$

where g stands for any of the functions $f, r, a_i, b_i, c_i, i = 1, ..., d$.

Consider approximation of $\beta(v)$ by the sequence of infinitely differentiable functions

$$b_n(v) = \int_{v-\frac{1}{n}}^{v+\frac{1}{n}} \beta(y)\omega_n(v-y)dy, \tag{3.5}$$

where ω_n is a standard mollifier defined as

$$\omega_n(v) = \begin{cases} \mathscr{C} n e^{-\frac{1}{1-n^2v^2}}, & |v| \leq \frac{1}{n} \\ 0, & |v| > \frac{1}{n} \end{cases} \tag{3.6}$$

and the constant $\mathscr{C}$ is chosen so that $\int\limits_{\mathbb{R}} \omega_1(u)\,du = 1$. Since $\beta'(v)$ is piecewise-continuous, we also have

$$b_n'(v) = \int_{v-\frac{1}{n}}^{v+\frac{1}{n}} \beta'(y)\omega_n(v-y)dy. \tag{3.7}$$

This implies b_n is also strict monotonically increasing and by (1.32) we have

$$b_n'(v) \geq \bar{b} > 0 \tag{3.8}$$

For a given discretization Δ, we define a finite-dimensional discrete control vector

$$[f]_\Delta := \{f_\alpha : \ f_\alpha \in \mathbb{R}, \alpha \in \mathscr{A}(\mathscr{C}_\Delta^D)\}$$

and corresponding discrete norms:

$$\|[f]_\Delta\|_{\ell_\infty} := \max_{\mathscr{A}(\mathscr{C}^D_\Delta)} |f_\alpha|, \quad \|[f]_\Delta\|_{\ell_2} := \Big(\sum_{\mathscr{A}(\mathscr{C}^D_\Delta)} \tau h^d f_\alpha^2 \Big)^{\frac{1}{2}}.$$

For any collection $\{v_\alpha\}$, with $\alpha = (\gamma, k_0)$, we utilize the notation

$$\gamma \pm e_i := (k_1, \ldots, k_i \pm 1, \ldots, k_d), \quad \alpha \pm e_i := (k_1, \ldots, k_i \pm 1, \ldots, k_d, k_0)$$

for suitable i. We employ the standard notation for the first order backward and forward space and time differences:

$$v_{\alpha\bar{t}} = \frac{v_{(\gamma,k_0)} - v_{(\gamma,k_0-1)}}{\tau}, \ v_{\alpha t} = \frac{v_{(\gamma,k_0+1)} - v_{(\gamma,k_0)}}{\tau}, \ v_{\alpha x_i} = \frac{v_{\alpha+e_i} - v_\alpha}{h}, \ v_{\alpha \bar{x}_i} = \frac{v_\alpha - v_{\alpha-e_i}}{h}.$$

For a given $R > 0$, let

$$\mathscr{F}^R_\Delta := \Big\{ [f]_\Delta \ \Big| \ \|[f]_\Delta\|_{\ell_\infty} \le R \Big\}$$

be a *discrete control set*. We define

$$\mathscr{P}_\Delta : \bigcup_R \mathscr{F}^R_\Delta \longrightarrow \bigcup_R \mathscr{F}^R, \qquad \mathscr{P}_\Delta([f]_\Delta) = f^\Delta$$

to be the interpolating map from the discrete control set to the continuous control set, where

$$f^\Delta \Big|_{C^\alpha_\Delta} = f_\alpha, \ \ \alpha \in \mathscr{A}(\mathscr{C}^D_\Delta), \qquad f^\Delta \equiv 0 \ \text{ elsewhere on } D.$$

Similarly, we define

$$\mathscr{Q}_\Delta : \bigcup_R \mathscr{F}^R \longrightarrow \bigcup_R \mathscr{F}_\Delta^R, \qquad \mathscr{Q}_\Delta(f) = [f]_\Delta$$

to be the discretizing map from the continuous control set to the discrete control set, where f_α is given by (3.4) for each $\alpha \in \mathscr{A}(\mathscr{C}_\Delta^D)$.

Next, we are going to define a solution of the discrete singular PDE problem.

Definition 3.2.1. Given $[f]_\Delta$, the vector function $[v([f]_\Delta)]_\Delta = (v(0), v(1), \ldots, v(n))$, where $v(k)$ is a collection of real numbers $\{v_\gamma(k)\}, \gamma \in \mathscr{A}(\Omega_\Delta),\ k = 0, 1, 2 \ldots, n$, is called a *discrete state vector* if

1. $v_\gamma(0) = \Phi_\gamma,\ \gamma \in \mathscr{A}(\Omega'_\Delta)$,
2. For each fixed $k = 1, \ldots, n$, the collection $v(k)$ satisfies

$$\sum_{\mathscr{A}} h^d \Bigg[\Big(b_n(v_\gamma(k)) \Big)_{\bar{t}} \eta_\gamma + \sum_{i=1}^{d} [(a_i)_\alpha v_{\gamma x_i}(k) + (b_i)_\alpha v_\gamma(k)] \eta_{\gamma x_i} + \sum_{i=1}^{d} (c_i)_\alpha v_{\gamma x_i}(k) \eta_\gamma$$
$$+ r_\alpha v_\gamma(k) \eta_\gamma - f^\Delta_{(\gamma,k)} \eta_\gamma \Bigg] = 0 \qquad (3.9)$$

   for arbitrary $\{\eta_\gamma\},\ \gamma \in \mathscr{A}(\Omega_\Delta)$ such that $\eta_\gamma = 0$ for $\gamma \in \mathscr{A}(\partial\Omega_\Delta)$.

3. For each $k = 0, 1, \ldots, n$, we have $v_\gamma(k) = 0$ for $\gamma \in \mathscr{A}(\partial\Omega_\Delta)$.

It should be mentioned that the collection $\{f_\alpha^\Delta\}$ in (3.9) coincides with $\mathscr{Q}_\Delta\Big(\mathscr{P}_\Delta([f]_\alpha)\Big)$. It will be proved in Lemma 3.4.2, Section 3.4 that for any $[f]_\Delta \in \mathscr{F}_\Delta^R$ there exists a unique discrete state vector. We can thus define the discrete cost functional $\mathscr{I}_\Delta : \cup_R \mathscr{F}_\Delta^R \to$

$[0,+\infty)$ by

$$\mathscr{I}_\Delta([f]_\Delta) = \sum_{\mathscr{A}} h^d |v_\gamma(n) - \Gamma_\gamma|^2 \tag{3.10}$$

where $v_\gamma(n)$ represents the nth component of the discrete state vector $[v([f]_\Delta)]_\Delta$. We will refer to the discrete optimal control problem on the minimization of the functional (3.10) on a discrete control set $\mathscr{F}_\Delta^R$ as a *Problem* $\mathscr{I}_\Delta$.

Next, we introduce various interpolations of the discrete state vector $[v]_\Delta$. First, define a piecewise constant interpolation $\tilde{V}_\Delta : D \to \mathbb{R}$, which assigns the value of $[v]_\Delta$ on the natural corner to the interior and top face of each cell, i.e.

$$\tilde{V}_\Delta\Big|_{C_\Delta^{\alpha'} \cup R_\Delta^{\gamma,k}} = v_\gamma(k), \qquad \forall \alpha = (\gamma, k) \in \mathscr{A}(\mathscr{C}_\Delta^D), \tag{3.11}$$

with $\tilde{V}_\Delta = 0$ everywhere else in D. We also define a piecewise constant interpolation of the discrete x_i-derivative denoted as $\tilde{V}_\Delta^i : D \to \mathbb{R}$, which assigns the value of the forward x_i-difference of $[v]_\Delta$ at the natural corner to interior and top face of each cell, i.e.

$$\tilde{V}_\Delta^i\Big|_{C_\Delta^{\alpha'} \cup R_\Delta^{\gamma,k}} = v_{\gamma x_i}(k), \qquad \forall \alpha = (\gamma, k) \in \mathscr{A}(\mathscr{C}_\Delta^D) \tag{3.12}$$

with $\tilde{V}_\Delta^i = 0$ everywhere else in D.

For fixed $k = \overline{0,n}$, we define a multilinear interpolation $V_\Delta^k : \Omega \to \mathbb{R}$ as a function that takes the value $v_\gamma(k)$ at corresponding lattice points of Ω_Δ, is linear with respect to every variable, when all other variables are fixed, and vanishes in $\Omega \backslash \Omega_\Delta$. Note that $V_\Delta^k \in C(\bar{\Omega}) \cap W_2^1(\Omega)$. Next, we define $V_\Delta : D \to \mathbb{R}$ as a piecewise constant interpolation of V_Δ^k onto $[0,T]$:

$$V_\Delta(x,t) = V_\Delta^k(x), \qquad t \in (t_{k-1}, t_k],\ k = 1, 2, \ldots, n \tag{3.13}$$

with $V_\Delta(x,0) = V_\Delta^0(x)$. We have $V_\Delta \in W_2^{1,0}(D)$. Finally, we define multilinear interpolation $V'_\Delta \in W_2^{1,1}(D) \cap C(\bar{D})$:

$$V'_\Delta(x,t) = V_\Delta^{k-1}(x) + \big(V_\Delta^k(x)\big)_{\bar{t}}(t - t_{k-1}), \qquad t \in [t_{k-1}, t_k],\ k = 1,2,\dots,n. \tag{3.14}$$

3.3 Formulation of Main Results

Throughout the chapter we assume the following assumptions are satisfied:

$$\Phi \in W_2^1(\Omega) \cap L_\infty(\Omega), \quad \Gamma \in L_2(\Omega) \tag{3.15}$$

$$a_i, b_i, c_i, \in W_\infty^{1,0}(D), \quad \frac{\partial a_i}{\partial t} \in L_{\infty,1}(D),\ i = \overline{1,d}, \quad r \in L_\infty(D), \tag{3.16}$$

$$\{x \in \Omega \,|\, \Phi(x) = v^j\},\ j = \overline{1,m} \text{ has d-dimensional Lebesgue measure } 0, \tag{3.17}$$

$a_i, i = 1,...,d$ satisfy (1.34); β is a maximal monotone graph satisfying (1.31),(1.32).

The following are the main results of this chapter:

Theorem 3.3.1. *There exists an optimal control in problem $\mathscr{I}$, i.e.*

$$\mathscr{F}_* := \left\{ f \in \mathscr{F}^R \;\middle|\; \mathscr{J}(f) = \mathscr{J}_* := \inf_{f \in \mathscr{F}^R} \mathscr{J}(f) \right\} \neq \emptyset$$

Theorem 3.3.2. *The sequence of discrete optimal control problems $\mathscr{I}_n$ approximates the optimal control problem $\mathscr{I}$ with respect to the functional and control, i.e.*

$$\lim_{\Delta \to 0} \mathscr{I}_{\Delta_*} = \mathscr{J}_* \tag{3.18}$$

where

$$\mathscr{I}_{\Delta_*} = \inf_{\mathscr{F}_\Delta^R} \mathscr{I}([f]_\Delta);$$

if $[f]_{\Delta,\varepsilon} \in \mathscr{F}_\Delta^R$ *is chosen such that*

$$\mathscr{I}_{\Delta_*} \le \mathscr{I}_\Delta([f]_{\Delta,\varepsilon}) \le \mathscr{I}_{\Delta_*} + \varepsilon_\Delta, \ \ \varepsilon_\Delta \downarrow 0, \tag{3.19}$$

then we have

$$\lim_{\Delta \to 0} \mathscr{J}(\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})) = \mathscr{J}_*, \tag{3.20}$$

the sequence $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ *is weakly precompact in* $L_2(D)$*, and all of its weak limit points lie in* $\mathscr{F}_*$*. Moreover, if* f_* *is such a weak limit point, then there is a subsequence* Δ' *such that the multilinear interpolations* $V'_{\Delta'}$ *of the discrete state vectors* $[v([f]_{\Delta',\varepsilon})]_{\Delta'}$ *converge to the weak solution* $v = v(x,t;f_*) \in W_2^{1,1}(D) \cap L_\infty(D)$ *of the singular PDE problem* (1.28)-(1.30)*, weakly in* $W_2^{1,1}(D)$*, strongly in* $L_2(D)$*, and almost everywhere on* D*.*

3.4 Preliminary Results

We define

$$\zeta_\Delta^{\gamma,k} := \int_0^1 b_n'\Big(\theta v_\gamma(k) + (1-\theta) v_\gamma(k-1)\Big)\, d\theta, \tag{3.21}$$

for each $(\gamma,k) \in \mathscr{A}(D_\Delta), k \neq 0$. Note that for every (γ,k), we have

$$\Big(b_n(v_\gamma(k))\Big)_{\bar{t}} = \zeta_\Delta^{\gamma,k} v_{\gamma\bar{t}}(k). \tag{3.22}$$

and

$$\zeta_\Delta^{\gamma,k} \geq \inf_{x\in\mathbb{R}} b_n'(x) \geq \bar{b}, \tag{3.23}$$

independently of $\gamma, k, \Delta, v_\gamma(k)$.

Lemma 3.4.1. *For a fixed discretization Δ and given discrete control $[f]_\Delta$, a vector function $[v([f]_\Delta)]_\Delta$ is a discrete state vector in the sense of Definition 3.2.1 if and only if it satisfies conditions (i), (ii)', and (iii), where*

(ii)' $\forall k = \overline{1,n}$ *and $\gamma \in \mathscr{A}(\Omega'_\Delta)$, we have*

$$\Big(b_n(v_\gamma(k))\Big)_{\bar{t}} - \sum_{i=1}^{d}\Big([(a_i)_\alpha v_{\gamma x_i}(k) + (b_i)_\alpha v_\gamma(k)]\Big)_{\bar{x}_i} + \sum_{i=1}^{d}(c_i)_\alpha v_{\gamma x_i}(k) + r_\alpha v_\gamma(k) = f^\Delta_{(\gamma,k)}. \tag{3.24}$$

Proof. Assume $[v([f]_\Delta)]_\Delta$ satisfy (i),(ii)' and (iii), and $k \in \{1,\dots,n\}$ is fixed. Take $\{\eta_\gamma \in \mathbb{R} : \gamma \in \mathscr{A}(\Omega_\Delta)\}$ with $\eta_\gamma = 0$ for $\gamma \in \mathscr{A}(\partial\Omega_\Delta)$. Multiplying (3.24) by $h^d\eta_\gamma$, and performing summation with respect to $\gamma \in \mathscr{A}(\Omega'_\Delta)$ we have

$$\sum_{\mathscr{A}(\Omega'_\Delta)} h^d\Bigg[\Big(b_n(v_\gamma(k))\Big)_{\bar{t}}\eta_\gamma - \sum_{i=1}^{d}\Big([(a_i)_\alpha v_{\gamma x_i}(k) + (b_i)_\alpha v_\gamma(k)]\Big)_{\bar{x}_i}\eta_\gamma + \sum_{i=1}^{d}(c_i)_\alpha v_{\gamma x_i}(k)\eta_\gamma + r_\alpha v_\gamma(k)\eta_\gamma - f^\Delta_{(\gamma,k)}\eta_\gamma\Bigg] = 0. \tag{3.25}$$

For any fixed i, we can rewrite

$$
\begin{aligned}
&-\sum_{\mathscr{A}(\Omega'_\Delta)}\Big([(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]\Big)_{\bar{x}_i}\eta_\gamma \\
&=-\sum_{\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\eta_\gamma+\sum_{\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_{\alpha-e_i}v_{(\gamma-e_i)x_i}(k)+(b_i)_{\alpha-e_i}v_{\gamma-e_i}(k)]}{h}\eta_\gamma, \\
&=-\sum_{\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\eta_\gamma+\sum_{\gamma \text{ s.t. } \gamma+e_i\in\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\eta_{\gamma+e_i} \\
&=\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\Omega'_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\Omega'_\Delta)}[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]\eta_{\gamma x_i} \\
&\quad-\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\Omega'_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\partial\Omega_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\eta_\gamma \\
&\quad+\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\partial\Omega_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\eta_{\alpha+e_i} \\
&=\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\Omega'_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\Omega'_\Delta)}[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]\eta_{\gamma x_i}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\Omega'_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\partial\Omega_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\big(-\eta_\gamma+\eta_{\gamma+e_i}\big) \\
&+\sum_{\gamma \text{ s.t. } \gamma\in\mathscr{A}(\partial\Omega_\Delta) \text{ and } \gamma+e_i\in\mathscr{A}(\Omega'_\Delta)}\frac{[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]}{h}\big(\eta_{\gamma+e_i}-\eta_\gamma\big) \\
&=\sum_{\mathscr{A}}[(a_i)_\alpha v_{\gamma x_i}(k)+(b_i)_\alpha v_\gamma(k)]\eta_{\gamma x_i}.
\end{aligned}
$$

Taking into account this transformation in (3.25), (ii) easily follows. Now conversely, suppose (i), (ii) and (iii) are satisfied. We fix any $k\in\overline{1,n}$, and any $\gamma'\in\mathscr{A}(\Omega'_\Delta)$, and choose the collection $\{\eta_\gamma\}$ as follows: $\eta_{\gamma'}=1$, and $\eta_\gamma=0,\forall\gamma\neq\gamma'$. Then from (3.9) we get

$$\Big(b_n(v_{\gamma'}(k))\Big)_{\bar t}+\sum_{i=1}^{d}\Bigg(-\frac{[(a_i)_{\alpha'}v_{\gamma' x_i}(k)+(b_i)_{\alpha'}v_{\gamma'}(k)]}{h}\Bigg)$$

$$\sum_{i=1}^{d}\frac{[(a_i)_{\alpha'-e_i}v_{(\gamma'-e_i)x_i}(k)+(b_i)_{\alpha'-e_i}v_{\gamma'-e_i}(k)]}{h}+\sum_{i=1}^{d}(c_i)_{\alpha'}v_{\gamma' x_i}(k)\ +r_{\alpha'}v_{\gamma'}(k)-f^{\Delta}_{(\gamma',k)}=0$$

which yields (3.24) for γ'. Since $\gamma'\in\mathscr{A}(\Omega'_\Delta)$ is arbitrary, statement (ii)' follows. □

Lemma 3.4.2. *For any discretization Δ with sufficiently small h and τ satisfying* (3.2)*, and for any $[f]_\Delta\in\mathscr{F}^R_\Delta$, there exists a unique discrete state vector $[v([f]_\Delta)]_\Delta$.*

Proof. To prove uniqueness, assume that $[v([f_\Delta])]_\Delta,[\tilde v([f_\Delta])]_\Delta$ are two discrete state vectors. We use induction on k. We have $v(0)=\tilde v(0)$, due to conditions (i) and (iii). Fix any $k,\ 1\le k\le n$ and assume $v(k-1)=\tilde v(k-1)$. By selecting $\eta=v(k)-\tilde v(k)$, and by subtracting (3.9) for both $v(k)$ and $\tilde v(k)$, we derive

$$\sum_{\mathscr{A}}h^d\Bigg[\Big((b_n(v_\gamma(k)))_{\bar t}-(b_n(\tilde v_\gamma(k)))_{\bar t}\Big)(v_\gamma(k)-\tilde v_\gamma(k))+\sum_{i=1}^{d}\Big[(a_i)_\alpha(v_{\gamma x_i}(k)-\tilde v_{\gamma x_i}(k))^2\Big]$$
$$+\sum_{i=1}^{d}\Big[\Big((b_i)_\alpha+(c_i)_\alpha)(v_\gamma(k)-\tilde v(k))(v_{\gamma x_i}(k)-\tilde v_{\gamma x_i}(k))\Big]+r_\alpha(v_\gamma(k)-\tilde v_\gamma(k))^2\Bigg]=0,$$

We can rewrite the following:

$$(b_n(v_\gamma(k)))_{\bar t}-(b_n(\tilde v_\gamma(k)))_{\bar t}=\frac{b_n(v_\gamma(k))-b_n(\tilde v_\gamma(k))}{\tau}$$

Using (1.34), (1.32), the integral mean value theorem as in (3.22),(3.23), and Cauchy inequality with $\varepsilon > 0$, we get

$$\sum_{\mathscr{A}} h^d \left[\frac{\bar{b}}{\tau}(v_\gamma(k) - \tilde{v}_\gamma(k))^2 + a_0 \sum_{i=1}^{d} (v_{\gamma x_i}(k) - \tilde{v}_{\gamma x_i}(k))^2 + r_\alpha (v_\gamma(k) - \tilde{v}_\gamma(k))^2 \right] \leq$$
$$\frac{a_0}{2} \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} \left[(v_{\gamma x_i}(k) - \tilde{v}_{\gamma x_i}(k))^2 \right] + \frac{1}{a_0} \sum_{\mathscr{A}} h^d \left[\sum_{i=1}^{d} (\|b_i\|^2_{L_\infty} + \|c_i\|^2_{L_\infty})(v_\gamma(k) - \tilde{v}_\gamma(k))^2 \right],$$

and therefore,

$$\sum_{\mathscr{A}} h^d \left[\left(\frac{\bar{b}}{\tau} - \frac{1}{a_0} \sum_{i=1}^{d} (\|b_i\|^2_{L_\infty} + \|c_i\|^2_{L_\infty}) - \|r\|_{L_\infty} \right) (v_\gamma(k) - \tilde{v}_\gamma(k))^2 \right.$$
$$\left. + \frac{a_0}{2} \sum_{i=1}^{d} (v_{\gamma x_i}(k) - \tilde{v}_{\gamma x_i}(k))^2 \right] \leq 0$$

By taking τ sufficiently small, all the terms on the left hand side become non-negative, and therefore, each term is equal to 0. This implies that $v_\gamma(k) = \tilde{v}_\gamma(k)$ for $\gamma \in \mathscr{A}(\Omega'_\Delta)$. By (iii) and induction argument, we get $v = \tilde{v}$, and uniqueness follows.

Now we prove existence, again through induction on k. Let discretization Δ and $[f]_\Delta$ are fixed. For $k = 0$, $v(0)$ is given by (i) and (iii) of Definition 3.2.1. Assuming that $v(0), v(1), \ldots, v(k-1)$ exist, we prove the existence of $v(k)$ by the method of successive approximations. By (iii), $v(k)$ taken to be 0 for any lattice point on the boundary of Ω_Δ. For the interior lattice points, we rewrite (3.24) as

$$\frac{h^2}{\tau}\left[b_n(v_\gamma(k)) - b_n(v_\gamma(k-1))\right] + \left[\sum_{i=1}^{d} \Big((a_i)_\alpha + (a_i)_{\alpha - e_i} - h(b_i)_\alpha - h(c_i)_\alpha \Big) + h^2 r_\alpha \right] v_\gamma(k)$$
$$- \sum_{i=1}^{d} \left[\Big((a_i)_\alpha - h(c_i)_\alpha \Big) v_{\gamma + e_i}(k) + \Big((a_i)_{\alpha - e_i} - h(b_i)_{\alpha - e_i} \Big) v_{\gamma - e_i}(k) \right] = h^2 f^\Delta_{(\gamma,k)}. \quad (3.26)$$

We set $v^0 = v(k-1)$, and having calculated v^N, v^{N+1} is found as a solution of the system

$$\frac{h^2}{\tau}b_n(v_\gamma^{N+1}) + \left[\sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_\alpha - h(c_i)_\alpha\Big) + h^2 r_\alpha\right]v_\gamma^{N+1} = \frac{h^2}{\tau}b_n(v_\gamma(k-1))$$
$$+\sum_{i=1}^{d}\left[\Big((a_i)_\alpha - h(c_i)_\alpha\Big)v_{\gamma+e_i}^N + \Big((a_i)_{\alpha-e_i} - h(b_i)_{\alpha-e_i}\Big)v_{\gamma-e_i}^N\right] + h^2 f_{(\gamma,k)}^\Delta. \qquad (3.27)$$

Since the left hand side of (3.27) is monotonically increasing with respect to v^{N+1} for sufficiently small h, and has a range $\mathbb{R}$, there exists a unique solution v^{N+1}. This implies the sequence $\{v^N\}$ is well-defined. Subtracting (3.27) for N and $N-1$ we have

$$\frac{h^2}{\tau}\Big(b_n(v_\gamma^{N+1}) - b_n(v_\gamma^N)\Big) + \left[\sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_\alpha - h(c_i)_\alpha\Big) + h^2 r_\alpha\right]\Big(v_\gamma^{N+1} - v_\gamma^N\Big)$$
$$= \sum_{i=1}^{d}\left[\Big((a_i)_\alpha - h(c_i)_\alpha\Big)(v_{\gamma+e_i}^N - v_{\gamma+e_i}^{N-1}) + \Big((a_i)_{\alpha-e_i} - h(b_i)_{\alpha-e_i}\Big)(v_{\gamma-e_i}^N - v_{\gamma-e_i}^{N-1})\right] \qquad (3.28)$$

Similar to (3.22),(3.21), we have

$$b_n(v_\gamma^{N+1}) - b_n(v_\gamma^N) = \zeta_{\Delta,N}^{\gamma,k}\Big(v_\gamma^{N+1} - v_\gamma^N\Big),\ \zeta_{\Delta,N}^{\gamma,k} := \int_0^1 b_n'\Big(\theta v_\gamma^{N+1} + (1-\theta)v_\gamma^N\Big)\,d\theta, \quad (3.29)$$

where $\zeta_{\Delta,N}^{\gamma,k}$ satisfies (3.23) uniformly with respect to Δ,γ,k,N. From (3.28), (3.29) it follows that

$$v_\gamma^{N+1} - v_\gamma^N = \frac{1}{\frac{h^2}{\tau}\zeta_{\Delta,N}^{\gamma,k} + \sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_\alpha - h(c_i)_\alpha\Big) + h^2 r_\alpha} \times$$
$$\left(\sum_{i=1}^{d}\left[\Big((a_i)_\alpha - h(c_i)_\alpha\Big)(v_{\gamma+e_i}^N - v_{\gamma+e_i}^{N-1}) + \Big((a_i)_{\alpha-e_i} - h(b_i)_{\alpha-e_i}\Big)(v_{\gamma-e_i}^N - v_{\gamma-e_i}^{N-1})\right]\right) \tag{3.30}$$

Due to (3.23),(1.34),(3.16), for sufficiently small h we have

$$\begin{aligned}0 &< \frac{1}{\frac{h^2}{\tau}\zeta_{\Delta,N}^{\gamma} + \sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_\alpha - h(c_i)_\alpha\Big) + h^2 r_\alpha} \\ &\leq \frac{1}{\frac{h^2}{\tau}\bar{b} + \sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_\alpha - h(c_i)_\alpha\Big) + h^2 r_\alpha}\end{aligned}$$

Let

$$A_N := \max_\gamma |v_\gamma^{N+1} - v_\gamma^N|.$$

From (3.30) we deduce that for sufficiently small h and for every γ

$$|v_\gamma^{N+1} - v_\gamma^N| \leq \delta A_{N-1}, \tag{3.31}$$

where

$$\delta := \left(1 + \frac{\frac{h^2}{\tau}\bar{b} + h\sum_{i=1}^{d}(b_i)_{\alpha-e_i} - h\sum_{i=1}^{d}(b_i)_\alpha + h^2 r_\alpha}{\sum_{i=1}^{d}\Big((a_i)_\alpha + (a_i)_{\alpha-e_i} - h(b_i)_{\alpha-e_i} - h(c_i)_\alpha\Big)}\right)^{-1}$$

From (1.34),(3.16),(3.2) it follows that for sufficiently small h

$$\sum_{i=1}^{d}\Big((a_i)_\alpha+(a_i)_{\alpha-e_i}-h(b_i)_{\alpha-e_i}-h(c_i)_\alpha\Big)\geq 2da_0-h\sum_{i=1}^{d}\Big(\|b_i\|_{L_\infty(D)}+(\|c_i\|_{L_\infty(D)}\Big)>0,$$

$$\frac{h}{\tau}\bar{b}+\sum_{i=1}^{d}(b_i)_{\alpha-e_i}-\sum_{i=1}^{d}(b_i)_\alpha+hr_\alpha\geq\frac{h}{\tau}\bar{b}-2\sum_{i=1}^{d}\|b_i\|_{L_\infty(D)}-h\|r\|_{L_\infty(D)}\geq 1-h\|r\|_{L_\infty(D)}>0.$$

Hence, $\delta\in(0,1)$, and by taking maximum with respect to γ from (3.31) we derive inductive chain of inequalities

$$A_N\leq\delta A_{N-1}\leq\delta^2A_{N-2}\leq\cdots\leq\delta^NA_0. \tag{3.32}$$

Following the proof of the Lemma 7, [7], from (3.32) it follows that there exists a limit

$$v_\gamma(k)=\lim_{N\to\infty}v_\gamma^N,\quad \gamma\in\mathscr{A}(\Omega'_\Delta). \tag{3.33}$$

and $v(k)$, given by (3.33) satisfies (3.9). Thus, the existence of the discrete state vector is proved. □

3.4.1 Proof of Uniqueness of the Weak Solution to the Singular Parabolic PDE

The uniqueness of the weak solution of the multiphase Stefan problem, or singular PDE problem (1.28)-(1.30) with $\mathscr{L}=\Delta$, in the sense of Definition 3.1.1, is proved in [63]. Next proposition formulates uniqueness of the weak solution of the singular PDE problem (1.28)-(1.30).

Proposition 3.4.3. *There exists at most one solution $v\in\overset{\circ}{W}{}_2^{1,1}(D)\cap L_\infty(D)$ of the singular PDE problem* (1.28)-(1.30).

Proof. We prove uniqueness in a broader class of solutions $v \in L_\infty(D)$, which satisfy the following integral identity instead of (3.1):

$$\int_D \Big[B(x,t,v)\psi_t + v\mathscr{L}^*\psi + f\psi\Big]\,dxdt + \int_\Omega B_0(x,0,\Phi(x))\,\psi(x,0)\,dx = 0, \tag{3.34}$$

$\forall \psi \in W_2^{2,1}(D)$ such that $\psi(x,T)|_{x\in\Omega} = 0, \psi|_{\partial\Omega\times(0,T]} = 0$, where

$$\mathscr{L}^*\psi = \sum_{i=1}^d (a_i\psi_{x_i})_{x_i} - \sum_{i=1}^d b_i\psi_{x_i} + \sum_{i=1}^d (c_i\psi)_{x_i} - r\psi.$$

Subtracting any solutions $v, \tilde{v} \in L_\infty(D)$ of (3.34), and by taking into account (3.17), we get

$$\int_D \big(B(x,t,v) - \tilde{B}(x,t,\tilde{v})\big)\,(\psi_t + z(x,t)\mathscr{L}^*\psi)\,dxdt = 0, \tag{3.35}$$

where

$$z(x,t) = \begin{cases} \frac{v-\tilde{v}}{B(x,t,v)-\tilde{B}(x,t,\tilde{v})}, \text{ if } v(x,t) \neq \tilde{v}(x,t), \\ 0, \text{ if } v(x,t) = \tilde{v}(x,t). \end{cases}$$

Since $B, \tilde{B} \in \mathscr{B}$, we have

$$0 \le z \le \frac{1}{\bar{b}}, \text{ a.e. } (x,t) \in D. \tag{3.36}$$

Fix $\varepsilon > 0$, and take $\psi(x,t)$ to be the solution of the Dirichlet problem for the backward parabolic PDE:

$$\psi_t + z^\varepsilon(x,t)\mathscr{L}^*\psi = F(x,t), \text{ in } \Omega\times[0,T), \tag{3.37}$$

$$\psi(x,T)|_{x\in\Omega} = 0, \ \ \psi|_{\partial\Omega\times(0,T]} = 0, \tag{3.38}$$

where $z^\varepsilon(x,t) = z(x,t) + \varepsilon$, F is an arbitrary compactly-supported, smooth function in D. From [63] it follows there exists a unique solution $\psi^\varepsilon \in W_2^{2,1}(D)$. By using (3.35),(3.37), we can write

$$\int_D \hat{B}(x,t)\,(F - \varepsilon \mathscr{L}^* \psi)\,dx\,dt = 0. \tag{3.39}$$

where $\hat{B}(x,t) = B(x,t,v(x,t)) - \tilde{B}(x,t,\tilde{v}(x,t))$. Our goal is to eliminate the ε-term by passing to limit as $\varepsilon \downarrow 0$, and use the arbitrariness of F to derive that $\hat{B} = 0$ a.e. on D. To do that we need to attain energy estimate for the solution of (3.37),(3.38). For simplicity, we will derive the required energy estimate for the parabolic PDE by assuming that time variable t is replaced with $T-t$ in (3.37),(3.38). Let us multiply the parabolic version of (3.37) by $\sum\limits_{i=1}^{d}(a_i\psi_{x_i})_{x_i}$, integrate it over $D_t := \Omega \times (0,t)$, to get

$$-\int_{D_t} (\psi_\tau - z^\varepsilon \mathscr{L}^* \psi) \sum_{i=1}^{d} (a_i \psi_{x_i})_{x_i}\,dx\,d\tau = \int_{D_t} \sum_{i=1}^{d} a_i \psi_{x_i} F_{x_i}\,dx\,d\tau, \tag{3.40}$$

Transforming the first term on the left hand side as

$$-\int_{D_t} \psi_\tau \sum_{i=1}^{d} (a_i \psi_{x_i})_{x_i}\,dx\,d\tau = \frac{1}{2}\int_\Omega \sum_{i=1}^{d} a_i(x,t)\psi_{x_i}^2(x,t)\,dx - \frac{1}{2}\int_{D_t} \sum_{i=1}^{d} (a_i)_\tau \psi_{x_i}^2\,dx\,d\tau,$$

and using (1.34), from (3.40), we derive

$$\begin{aligned} &\frac{a_0}{2}\int_\Omega |D\psi(x,t)|^2\,dx + \int_{D_t} z^\varepsilon \Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2 dx\,d\tau \\ &\leq \int_{D_t} z^\varepsilon \Big(\sum_{i=1}^{d}(b_i - c_i)\psi_{x_i}\Big)\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)\,dx\,d\tau + \int_{D_t} z^\varepsilon \Big((r - \sum_{i=1}^{d} c_{i,x_i})\psi\Big)\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)\,dx\,d\tau \\ &\quad + \int_{D_t}\sum_{i=1}^{d} a_i\psi_{x_i}F_{x_i}\,dx\,d\tau + \frac{1}{2}\int_{D_t}\sum_{i=1}^{d}(a_i)_\tau \psi_{x_i}^2\,dx\,d\tau, \end{aligned} \tag{3.41}$$

where $D\psi$ denotes the spatial gradient of ψ. Using Cauchy inequality with appropriately chosen small parameter, and (3.16), we get the following estimations:

$$\int_{D_t} z^{\varepsilon}\Big(\sum_{i=1}^{d}(b_i - c_i)\psi_{x_i}\Big)\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)\,dx\,d\tau \le \frac{1}{4}\int_{D_t} z^{\varepsilon}\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2\,dx\,d\tau$$
$$+2d\bar{b}^{-1}\Big(\max_i \|b_i\|^2_{L_\infty(D)} + \max_i \|c_i\|^2_{L_\infty(D)}\Big)t\,\|D\psi\|^2_{L_{2,\infty}(D_t)},$$

$$\int_{D_t} z^{\varepsilon}\Big((r - \sum_{i=1}^{d} c_{i,x_i})\psi\Big)\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)\,dx\,d\tau \le \frac{1}{4}\int_{D_t} z^{\varepsilon}\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2\,dx\,d\tau$$
$$+2\bar{b}^{-1}\Big(\|r\|^2_{L_\infty(D)} + d\sum_{i=1}^{d}\|c_{i,x_i}\|^2_{L_\infty(D)}\Big)\|\psi\|^2_{L_2(D_t)},$$

$$\int_{D_t}\sum_{i=1}^{d} a_i\psi_{x_i}F_{x_i}\,dx\,d\tau \le t\|D\psi\|^2_{L_{2,\infty}(D_t)} + \frac{1}{4}\max_i \|a_i\|^2_{L_\infty(D)}\|DF\|^2_{L_2(D_t)},$$

$$\frac{1}{2}\int_{D_t}\sum_{i=1}^{d}(a_i)_\tau\psi_{x_i}^2\,dx\,d\tau \le \frac{1}{2}\max_i \Big\|\frac{\partial a_i}{\partial t}\Big\|_{L_{\infty,1}(D_t)}\|D\psi\|^2_{L_{2,\infty}(D_t)}.$$

Plugging these estimates in (3.41), absorbing similar terms to the left hand side, and by taking ess sup with respect to τ in $0 \le \tau \le t$ in the first term, we have

$$\frac{a_0}{2}\|D\psi\|^2_{L_{2,\infty}(D_t)} + \frac{1}{2}\int\limits_{D_t} z^{\varepsilon}\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2 dx\,d\tau \le$$

$$\bigg(\Big(2d\bar{b}^{-1}\big(\max_i \|b_i\|^2_{L_\infty(D)} + \max_i \|c_i\|^2_{L_\infty(D)}\big) + 1\Big)t + \frac{1}{2}\max_i \Big\|\frac{\partial a_i}{\partial t}\Big\|_{L_{\infty,1}(D_t)}\bigg)\|D\psi\|^2_{L_{2,\infty}(D_t)}$$

$$+2\bar{b}^{-1}\Big(\|r\|^2_{L_\infty(D)} + d\sum_{i=1}^{d}\|c_{i,x_i}\|^2_{L_\infty(D)}\Big)\|\psi\|^2_{L_2(D_t)} + \frac{1}{4}\max_i \|a_i\|^2_{L_\infty(D)}\|DF\|^2_{L_2(D_t)}. \quad (3.42)$$

By choosing $t > 0$ sufficiently small such that

$$\Big(2d\bar{b}^{-1}\big(\max_i \|b_i\|^2_{L_\infty(D)} + \max_i \|c_i\|^2_{L_\infty(D)}\big) + 1\Big)t + \frac{1}{2}\max_i \Big\|\frac{\partial a_i}{\partial t}\Big\|_{L_{\infty,1}(D_t)} < \frac{a_0}{4}, \quad (3.43)$$

and by absorbing the first term on the right hand side, we derive

$$\frac{a_0}{4}\|D\psi\|^2_{L_{2,\infty}(D_t)} + \frac{1}{2}\int\limits_{D_t} z^{\varepsilon}\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2 dx\,d\tau \le$$

$$2\bar{b}^{-1}\Big(\|r\|^2_{L_\infty(D)} + d\sum_{i=1}^{d}\|c_{i,x_i}\|^2_{L_\infty(D)}\Big)\|\psi\|^2_{L_2(D_t)} + \frac{1}{4}\max_i \|a_i\|^2_{L_\infty(D)}\|DF\|^2_{L_2(D_t)}. \quad (3.44)$$

From the maximum principle (e.g. Theorem 2.1, Chapter 1 of [63]) it follows that ψ is essentially bounded, and $\|\psi\|_{L_\infty(D)}$ depends on $\bar{b}$ and L_∞-norms of $a_i, (a_i)_{x_i}, b_i, c_i, (c_i)_{x_i}, r$ and F. Hence $\|\psi\|_{L_2(D)}$, and therefore the right hand side of (3.44) is bounded uniformly with respect to ε. If (3.43) is not satisfied in the whole time interval $[0,T]$, it can be divided into finitely many intervals that satisfy (3.43), and summing up respective inequalities (3.44) we arrive at the estimation:

$$\|D\psi\|^2_{L_{2,\infty}(D)} + \int\limits_{D} z^{\varepsilon}\Big(\sum_{i=1}^{d}(a_i\psi_{x_i})_{x_i}\Big)^2 dx\,d\tau \le C \quad (3.45)$$

where C is independent of ε. Furthermore, any constant independent of ε will be denoted by C. Although (3.45) is satisfactory for our purpose, it is worth mentioning that since $\psi_t = F - z^\varepsilon \mathscr{L}^* \psi$, from (3.16),(3.36),(3.45) and L_∞ bounds of ψ and F it follows that $\|\psi_t\|_{L_2(D)}$ is uniformly bounded. Therefore, complete energy estimate for ψ reads

$$\|\psi\|_{L_\infty(D)} + \|\psi_t\|^2_{L_2(D)} + \|D\psi\|^2_{L_{2,\infty}(D)} + \int_D z^\varepsilon \Big(\sum_{i=1}^d (a_i \psi_{x_i})_{x_i} \Big)^2 dx\,d\tau \le C \tag{3.46}$$

Having (3.46), we can estimate ε-term in (3.39) as follows:

$$\begin{aligned} \left| \int_D \hat{B} \varepsilon \mathscr{L}^* \psi\, dx\,dt \right| &\le \operatorname*{ess\,sup}_D |\hat{B}| \left[\Big(\int_D \frac{\varepsilon^2}{z^\varepsilon} dx\,dt \int_D z^\varepsilon \Big(\sum_{i=1}^d (a_i \psi_{x_i})_{x_i} \Big)^2 dx\,dt \Big)^{\frac{1}{2}} + C\varepsilon \right] \\ &\le C \operatorname*{ess\,sup}_D |\hat{B}| \Big(\varepsilon^{\frac{1}{2}} |D|^{\frac{1}{2}} + \varepsilon \Big) \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Therefore, (3.39) implies

$$\int_D \hat{B} F\, dx\,dt = 0.$$

Since the choice of F is arbitrary, it follows that

$$B(x,t,v(x,t)) = \tilde{B}(x,t,\tilde{v}(x,t)), \text{ a.e. } (x,t) \in D.$$

Since, $B, \tilde{B} \in \mathscr{B}$, clearly we have

$$\{(x,t) \in D : B(x,t,v(x,t)) = \tilde{B}(x,t,\tilde{v}(x,t))\} \subset \{(x,t) \in D : v(x,t) = \tilde{v}(x,t)\}$$

and therefore, $v(x,t) = \tilde{v}(x,t)$ for a.e. $(x,t) \in D$. Uniqueness is proved. □

The following lemma recalls the criteria for the convergence of the discrete optimal control problems.

Lemma 3.4.4. *[81] The sequence of discrete optimal control problems $\mathscr{I}_n$ approximates the continuous optimal control problem $\mathscr{I}$ with respect to the functional, i.e.* (3.18) *holds, if and only if the following conditions are satisfied:*

(i) For any $f \in \mathscr{F}^R$, we have $\mathscr{Q}_\Delta(f) \in \mathscr{F}_\Delta^R$, and

$$\limsup_{\Delta \to 0} \Big(\mathscr{I}_\Delta(\mathscr{Q}_\Delta(f)) - \mathscr{J}(f) \Big) \leq 0. \tag{3.47}$$

(ii) For any $[f]_\Delta \in \mathscr{F}_\Delta^R$, we have $\mathscr{P}_\Delta([f]_\Delta) \in \mathscr{F}^R$, and

$$\limsup_{\Delta \to 0} \Big(\mathscr{J}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{I}_\Delta([f]_\Delta) \Big) \leq 0. \tag{3.48}$$

Finally, we recall two lemmas proved in [7].

Lemma 3.4.5. *[7] The maps $\mathscr{P}_\Delta$ and $\mathscr{Q}_\Delta$ satisfy the conditions of Lemma 3.4.4.*

Lemma 3.4.6. *[7] For any fixed $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{\mathscr{A}} h^d \sum_{i=1}^{d} |\Phi_{\gamma x_i}|^2 \leq (1+\varepsilon)\|D\Phi\|^2_{L_2(\Omega)} \tag{3.49}$$

whenever $h < \delta$.

3.5 Discrete Energy Estimates

3.5.1 Discrete $L_\infty(D)$ Estimate

Theorem 3.5.1. *(Discrete Maximum Principle) For any $R > 0$, $[f]_\Delta \in \mathscr{F}^R_\Delta$, and Δ, the discrete state vector $[v([f]_\Delta)]_\Delta$ given in Definition 3.2.1 satisfies*

$$\|[v]_\Delta\|_{\ell_\infty} \leq e^{\lambda T} \max\left\{\|[f]_\Delta\|_{\ell_\infty}\,,\ \|\Phi\|_{L_\infty(\Omega)}\right\} \tag{3.50}$$

where

$$\lambda = \frac{2}{\bar{b}}\Big(1 + \sum_{i=1}^{d} \|b_{i,x_i}\|_{L_\infty(D)} + \|r\|_{L_\infty(D)}\Big) \tag{3.51}$$

Proof. Fix a discretization $\Delta = (\tau, h)$ and $[f]_\Delta \in \mathscr{F}^R_\Delta$. By Lemma 3.4.2, there exists a unique discrete state vector, $[v([f]_\Delta)]_\Delta$. We transform it using (3.51) as

$$u_\gamma(k) := v_\gamma(k) e^{-\lambda t_k}, \quad \forall (\gamma, k) \in \mathscr{A}(D_\Delta). \tag{3.52}$$

Then by (3.22), we get

$$\frac{b_n(v_\gamma(k)) - b_n(v_\gamma(k-1))}{\tau} = \zeta^{\gamma,k}_\Delta e^{\lambda t_{k-1}} u_{\gamma\bar{t}}(k) + \zeta^{\gamma,k}_\Delta \lambda e^{\lambda t^k} u_\gamma(k)$$

where $t^k \in [t_{k-1}, t_k]$ represents the value resulting from the mean value theorem:

$$e^{\lambda t_k} - e^{\lambda t_{k-1}} = \lambda e^{\lambda t^k} \tau.$$

Substituting in (3.24), we get

$$\zeta_\Delta^{\gamma,k}\lambda e^{\lambda t^k}u_\gamma(k)+\zeta_\Delta^{\gamma,k}u_{\gamma\bar{t}}(k)\,e^{\lambda t_{k-1}}-e^{\lambda t_k}\sum_{i=1}^{d}\Big([(a_i)_\alpha u_{\gamma x_i}(k)+(b_i)_\alpha u_\gamma(k)]\Big)_{\bar{x}_i}$$
$$+e^{\lambda t_k}\sum_{i=1}^{d}(c_i)_\alpha u_{\gamma x_i}(k)\ +e^{\lambda t_k}r_\alpha u_\gamma(k)=f_{\gamma,k}^{\Delta}$$

Splitting up the third term and gathering similar terms, we have

$$\zeta_\Delta^{\gamma,k}\lambda e^{\lambda t^k}u_\gamma(k)+\zeta_\Delta^{\gamma,k}e^{\lambda t_{k-1}}u_{\gamma\bar{t}}(k)\,e^{\lambda t_{k-1}}$$
$$-e^{\lambda t_k}\sum_{i=1}^{d}\frac{1}{h}\Big(((a_i)_\alpha-h(c_i)_\alpha)u_{\gamma x_i}(k)-((a_i)_{\alpha-e_i}-h(b_i)_{\alpha-e_i})u_{\gamma\bar{x}_i}(k)\Big)$$
$$+e^{\lambda t_k}\Big(r-\sum_{i=1}^{d}(b_i)_{\alpha,\bar{x}_i}\Big)u_\gamma(k)=f_{\gamma,k}^{\Delta} \tag{3.53}$$

If $u_\gamma(k)\le 0$ for every $\alpha\in\mathscr{A}(D_\Delta)$, then it is clear that $\max\limits_{\mathscr{A}(D_\Delta)}u_\gamma(k)\le 0$. We now suppose that for some $\alpha=(\gamma,k)\in\mathscr{A}(D_\Delta)$, we have $u_\gamma(k)>0$. This implies that $\max\limits_{\mathscr{A}(D_\Delta)}u_\gamma(k)>0$. Assume that maximum occurs as $\alpha^*=(\gamma^*,k^*)$, i.e.

$$u_{\gamma^*}(k^*)=\max_{\mathscr{A}(D_\Delta)}u_\gamma(k).$$

Due to (iii) in Definition 3.2.1, $\alpha^*\notin\mathscr{A}(S_\Delta)$. If $\alpha^*=(\gamma^*,0)$, $\gamma^*\in\mathscr{A}(\Omega'_\Delta)$, this would imply

$$u_{\gamma^*}(k^*)=\max_{\mathscr{A}(\Omega_\Delta)}\Phi_\gamma\le\|\Phi\|_{L_\infty(\Omega)}.$$

The only other possibility is $\alpha^*\in\mathscr{A}(D'_\Delta)$, i.e. (3.53) is true for α^*, and moreover,

$$u_{\gamma^*\bar{t}}(k^*)\ge 0,\qquad u_{\gamma^*x_i}(k^*)\le 0\ \forall i,\qquad u_{\gamma^*\bar{x}_i}(k^*)\ge 0\ \forall i$$

since maximum occurs at α^*. By using this properties and (3.23) in (3.53), we have

$$e^{\lambda t_{k^*}}\left(\lambda\bar{b}e^{-\lambda(t_{k^*}-t^{k^*})}-\sum_{i=1}^{d}(b_i)_{\alpha^*,\bar{x}_i}+r_{\alpha^*}\right)u_{\gamma^*}(k^*)$$
$$-e^{\lambda t_{k^*}}\sum_{i=1}^{d}\frac{1}{h}\Big(((a_i)_{\alpha^*}-h(c_i)_{\alpha^*})u_{\gamma^* x_i}(k)-((a_i)_{\alpha^*-e_i}-h(b_i)_{\alpha^*-e_i})u_{\gamma^*\bar{x}_i}(k)\Big)\leq f^{\Delta}_{\gamma^*,k^*} \tag{3.54}$$

Assume that $\forall i=1,...,d$, we have $(a_i)_{\alpha^*}-h(c_i)_{\alpha^*}-(a_i)_{\alpha^*-e_i}+h(b_i)_{\alpha^*-e_i}\geq 0$. Then we can rewrite (3.54) as

$$e^{\lambda t_{k^*}}\left(\lambda\bar{b}e^{-\lambda(t_{k^*}-t^{k^*})}-\sum_{i=1}^{d}(b_i)_{\alpha^*,\bar{x}_i}+r^*_{\alpha}\right)u_{\gamma^*}(k^*)$$
$$-e^{\lambda t_{k^*}}\sum_{i=1}^{d}\frac{1}{h}\Big((a_i)_{\alpha^*}-h(c_i)_{\alpha^*}-(a_i)_{\alpha^*-e_i}+h(b_i)_{\alpha^*-e_i}\Big)u_{\gamma^* x_i}(k)$$
$$-e^{\lambda t_{k^*}}\sum_{i=1}^{d}\Big((a_i)_{\alpha^*-e_i}-h(b_i)_{\alpha^*-e_i}\Big)u_{\gamma^* x_i\bar{x}_i}(k^*)\leq f^{\Delta}_{\gamma^*,k^*} \tag{3.55}$$

Since $u_{\gamma^* x_i(k^*)}\leq 0$, $u_{\gamma^* x_i\bar{x}_i(k^*)}\leq 0$ and since for small enough h, we have $(a_i)_{\alpha^*-e_i}-h(b_i)_{\alpha^*-e_i}\geq 0$, from (3.55) we deduce

$$e^{\lambda t_{k^*}}\left(\lambda\bar{b}e^{-\lambda(t_{k^*}-t^{k^*})}-\sum_{i=1}^{d}(b_i)_{\alpha^*,\bar{x}_i}+r^*_{\alpha}\right)u_{\gamma^*}(k^*)\leq f^{\Delta}_{\gamma^*,k^*} \tag{3.56}$$

If for some $i=\overline{1,d}$ we have that $(a_i)_{\alpha^*}-h(c_i)_{\alpha^*}-(a_i)_{\alpha^*-e_i}+h(b_i)_{\alpha^*-e_i}<0$, then we rewrite that specific term in (3.54) as

$$\frac{1}{h}\left(((a_i)_{\alpha^*}-h(c_i)_{\alpha^*})u_{\gamma^* x_i}(k)-((a_i)_{\alpha^*-e_i}-h(b_i)_{\alpha^*-e_i})u_{\gamma^*\bar{x}_i}(k)\right)$$
$$=\frac{1}{h}\Big((a_i)_{\alpha^*}-h(c_i)_{\alpha^*}-(a_i)_{\alpha^*-e_i}+h(b_i)_{\alpha^*-e_i}\Big)u_{\gamma^*\bar{x}_i}(k)+((a_i)_{\alpha^*}-h(c_i)_{\alpha^*})u_{\gamma^* x_i\bar{x}_i}(k^*)$$

Since $u_{\gamma^* \bar{x}_i(k^*)} \geq 0$, $u_{\gamma^* x_i \bar{x}_i(k^*)} \leq 0$ and since for small enough h, we have $(a_i)_{\alpha^*} - h(c_i)_{\alpha^* - e_i} \geq 0$, we get the same estimate as (3.56). By choosing τ sufficiently small such that $e^{-\lambda(t_k - t^k)} > \frac{1}{2}$ $\quad, \forall k$, due to (3.51) we get

$$u_{\gamma^*}(k^*) \leq f^{\Delta}_{\alpha^*} e^{-\lambda t^k} \leq \|[f]_\Delta\|_{\ell_\infty}.$$

These estimations result in

$$\max_{\mathscr{A}(D_\Delta)} v_\gamma(k) \leq e^{\lambda T} \max\left\{\|[f]_\Delta\|_{\ell_\infty}\ ,\ \|\Phi\|_{L_\infty(\Omega)}\right\}.$$

We can similarly derive a uniform lower bound

$$\min_{\mathscr{A}(D_\Delta)} v_\gamma(k) \geq e^{\lambda T} \min\left\{-\|[f]_\Delta\|_{\ell_\infty}\ ,\ -\|\Phi\|_{L_\infty(\Omega)}\right\},$$

giving (3.50), and thus proving the theorem. □

3.5.2 Discrete $W_2^{1,1}(D)$ estimate

Theorem 3.5.2. *(Discrete $W_2^{1,1}$ Energy Estimate) For any $R > 0$, Δ and $[f]_\Delta \in \mathscr{F}_\Delta^R$, the discrete state vector $[v([f]_\Delta)]_\Delta$ satisfies*

$$\sum_{k=1}^{n} \tau \sum_{\mathscr{A}} h^d (v_{\gamma\bar{t}}(k))^2 + \max_{1\leq k\leq n} \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} (v_{\gamma x_i}(k))^2 +$$

$$+ \sum_{k=1}^{n} \tau^2 \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} \left(v_{\gamma x_i \bar{t}}(k)\right)^2 \leq C\left(\|\Phi\|^2_{L_\infty(\Omega)} + \|D\Phi\|^2_{L_2(\Omega)} + \|f^\Delta\|^2_{L_\infty(D)}\right) \qquad (3.57)$$

where C is a constant independent of Δ, R.

Proof. Choose $\eta_\gamma = 2\tau v_{\gamma\bar{t}}(k)$ in (3.9) with $k = 1, \ldots, n$. Using (3.22) and the identity

$$2\tau(a_i)_\alpha v_{\gamma x_i}(k)\big(v_\gamma(k)_{\bar{t}}\big)_{x_i}$$
$$= (a_i)_\alpha (v_{\gamma x_i}(k))^2 - (a_i)_{\alpha-e_k}(v_{\gamma x_i}(k-1))^2 + \tau^2 (a_i)_\alpha \big(v_{\gamma x_i \bar{t}}(k)\big)^2 - \tau(a_i)_{\alpha\bar{t}}(v_{\gamma x_i}(k-1))^2$$

we have

$$\sum_{\mathscr{A}} h^d \Bigg[2\tau\zeta_\Delta^{\gamma,k}(v_{\gamma\bar{t}}(k))^2 + \sum_{i=1}^d (a_i)_\alpha (v_{\gamma x_i}(k))^2 - \sum_{i=1}^d (a_i)_{\alpha-e_k}(v_{\gamma x_i}(k-1))^2$$
$$+ \sum_{i=1}^d \tau^2 (a_i)_\alpha \big(v_{\gamma x_i\bar{t}}(k)\big)^2 - \sum_{i=1}^d \tau(a_i)_{\alpha\bar{t}}(v_{\gamma x_i}(k-1))^2\Big) + \sum_{i=1}^d 2\tau(b_i)_\alpha v_\gamma(k) v_{\gamma x_i \bar{t}}(k)$$
$$\sum_{i=1}^d 2\tau(c_i)_\alpha v_{\gamma x_i}(k) v_{\gamma\bar{t}}(k) + 2\tau\, r_\alpha v_\gamma(k) v_{\gamma\bar{t}}(k) - 2\tau f_{\gamma,k}^\Delta v_{\gamma\bar{t}}(k)\Bigg] = 0. \tag{3.58}$$

Since $v_\gamma(k) = 0$ for $\gamma \in \mathscr{A}(\partial\Omega_\Delta)$ through summation by parts we deduce

$$\sum_{\mathscr{A}} h^d \sum_{i=1}^d 2\tau(b_i)_\alpha v_\gamma(k) v_{\gamma x_i\bar{t}}(k) = -\sum_{\mathscr{A}} h^d \sum_{i=1}^d 2\tau((b_i)_\alpha v_\gamma(k))_{\bar{x}_i} v_{\gamma\bar{t}}(k)$$
$$= -\sum_{\mathscr{A}} h^d \sum_{i=1}^d 2\tau(b_i)_{\alpha\bar{x}_i} v_\gamma(k) v_{\gamma\bar{t}}(k) - \sum_{\mathscr{A}} h^d \sum_{i=1}^d 2\tau(b_i)_{\alpha-e_i} v_{\gamma-e_i,x_i}(k) v_{\gamma\bar{t}}(k). \tag{3.59}$$

Due to (3.23) and (3.59), from (3.58) it follows

$$\sum_{\mathscr{A}} h^d \Big[2\tau\bar{b}(v_{\gamma\bar{t}}(k))^2 + \sum_{i=1}^d (a_i)_\alpha (v_{\gamma x_i}(k))^2 - \sum_{i=1}^d (a_i)_{\alpha-e_k}(v_{\gamma x_i}(k-1))^2$$
$$+ \sum_{i=1}^d \tau^2 (a_i)_\alpha \big(v_{\gamma x_i\bar{t}}(k)\big)^2\Big] \le \sum_{\mathscr{A}} h^d \Big[\sum_{i=1}^d \tau(a_i)_{\alpha\bar{t}}(v_{\gamma x_i}(k-1))^2 + \sum_{i=1}^d 2\tau(b_i)_{\alpha\bar{x}_i} v_\gamma(k) v_{\gamma\bar{t}}(k)$$
$$+ \sum_{i=1}^d 2\tau(b_i)_{\alpha-e_i} v_{\gamma-e_i,x_i}(k) v_{\gamma\bar{t}}(k) - \sum_{i=1}^d 2\tau(c_i)_\alpha v_{\gamma x_i}(k) v_{\gamma\bar{t}}(k)$$
$$-2\tau(d)_\alpha v_\gamma(k) v_{\gamma\bar{t}}(k) + 2\tau f_{\gamma,k}^\Delta v_{\gamma\bar{t}}(k)\Big] \tag{3.60}$$

Applying Cauchy inequality with appropriately chosen small parameter we estimate various terms in (3.60) as follows:

$$\sum_{i=1}^{d} 2\tau (b_i)_{\alpha - e_i} v_{\gamma - e_i, x_i}(k) v_{\gamma\bar{t}}(k) \leq \tau \frac{\bar{b}}{5} (v_{\gamma\bar{t}}(k))^2 + \frac{5d}{\bar{b}} \max_i \|b_i\|^2_{L_\infty(D)} \tau \sum_{i=1}^{d} (v_{\gamma - e_i, x_i}(k))^2,$$

$$\sum_{i=1}^{d} 2\tau (b_i)_{\alpha\bar{x}_i} v_{\gamma}(k) v_{\gamma\bar{t}}(k) \leq \tau \frac{\bar{b}}{5} (v_{\gamma\bar{t}}(k))^2 + \frac{5d^2}{\bar{b}} \max_i \|(b_i)_{x_i}\|^2_{L_\infty(D)} \tau (v_{\gamma}(k))^2,$$

$$-\sum_{i=1}^{d} 2\tau (c_i)_{\alpha} v_{\gamma x_i}(k) v_{\gamma\bar{t}}(k) \leq \tau \frac{\bar{b}}{5} (v_{\gamma\bar{t}}(k))^2 + \frac{5d}{\bar{b}} \max_i \|(c_i)\|^2_{L_\infty(D)} \tau \sum_{i=1}^{d} (v_{\gamma x_i}(k))^2,$$

$$-2\tau r_{\alpha} v_{\gamma}(k) v_{\gamma\bar{t}}(k) \leq \frac{\bar{b}}{5} (v_{\gamma\bar{t}}(k))^2 + \frac{5}{\bar{b}} \|r\|^2_{L_\infty(D)} (v_{\gamma}(k))^2,$$

$$2\tau f^{\Delta}_{\gamma,k} v_{\gamma\bar{t}}(k) \leq \frac{\bar{b}}{5} (v_{\gamma\bar{t}}(k))^2 + \frac{5}{\bar{b}} \tau (f^{\Delta}_{\gamma,k})^2$$

Implementing these estimates in (3.60), and absorbing similar terms into the left hand side, we derive

$$\sum_{\mathscr{A}} h^d \Bigg[\tau \bar{b} (v_{\gamma\bar{t}}(k))^2 + \sum_{i=1}^{d} (a_i)_\alpha (v_{\gamma x_i}(k))^2 - \sum_{i=1}^{d} (a_i)_{\alpha - e_k} (v_{\gamma x_i}(k-1))^2$$
$$+ \sum_{i=1}^{d} \tau^2 (a_i)_\alpha \big(v_{\gamma x_i \bar{t}}(k)\big)^2 \Bigg] \le \sum_{\mathscr{A}} h^d \Bigg[\sum_{i=1}^{d} \tau (a_i)_{\alpha\bar{t}} (v_{\gamma x_i}(k-1))^2$$
$$+ \frac{5d}{\bar{b}} (\max_i \|b_i\|^2_{L_\infty(D)} + \max_i \|c_i\|^2_{L_\infty(D)}) \tau \sum_{i=1}^{d} (v_{\gamma x_i}(k))^2$$
$$+ \frac{5d^2}{\bar{b}} \max_i \|(b_i)_{x_i}\|^2_{L_\infty(D)} + \frac{5}{\bar{b}} \|r\|^2_{L_\infty(D)} \tau (v_\gamma(k))^2 + \frac{5}{\bar{b}} \tau (f^\Delta_{\gamma,k})^2 \Bigg]. \tag{3.61}$$

Pursuing summation over all $k = \overline{1,q}$, where $q \le n$, and using (1.34), from (3.61) we have

$$\sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d \bar{b} (v_{\gamma\bar{t}}(k))^2 + a_0 \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} (v_{\gamma x_i}(q))^2$$
$$+ a_0 \sum_{k=1}^{q} \tau^2 \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} \big(v_{\gamma x_i \bar{t}}(k)\big)^2 \le \sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} (a_i)_{\alpha\bar{t}} (v_{\gamma x_i}(k-1))^2$$
$$+ \frac{5d}{\bar{b}} \Big(\max_i \|b_i\|^2_{L_\infty(D)} + \max_i \|c_i\|^2_{L_\infty(D)} \Big) \sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} (v_{\gamma x_i}(k))^2$$
$$+ \frac{5}{\bar{b}} \Big(d^2 \max_i \|(b_i)_{x_i}\|^2_{L_\infty(D)} + \|r\|^2_{L_\infty(D)} \Big) \sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d (v_\gamma(k))^2$$
$$+ \frac{5}{\bar{b}} \sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d (f^\Delta_{\gamma,k})^2 + \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} (a_i)_{(\gamma,0)} (v_{\gamma x_i}(0))^2 \tag{3.62}$$

We estimate the first term on the right hand side as follows:

$$\sum_{k=1}^{q}\tau\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(a_i)_{\alpha\bar{t}}(v_{\gamma x_i}(k-1))^2=\sum_{k=0}^{q-1}\tau\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(a_i)_{\alpha t}(v_{\gamma x_i}(k))^2$$

$$=\sum_{k=1}^{q-1}\sum_{\mathscr{A}}\sum_{i=1}^{d}\left(\frac{1}{\tau}\int_{R_\Delta^\gamma}\int_{t_k}^{t_{k+1}}\int_{t-\tau}^{t}\frac{\partial a(x,\xi)}{\partial\xi}\,d\xi\,dt\,dx\right)(v_{\gamma x_i}(k))^2$$

$$+\sum_{\mathscr{A}}\sum_{i=1}^{d}\left(\frac{1}{\tau}\int_{R_\Delta^\gamma}\int_{0}^{\tau}\int_{0}^{t}\frac{\partial a(x,\xi)}{\partial\xi}\,d\xi\,dt\,dx\right)(v_{\gamma x_i}(0))^2\leq$$

$$\leq 2\max_{1\leq i\leq d}\left\|\frac{\partial a_i}{\partial t}\right\|_{L_{\infty,1}(D_{t_q})}\max_{1\leq k\leq q}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2$$

$$+\max_{1\leq i\leq d}\left\|\frac{\partial a_i}{\partial t}\right\|_{L_{\infty,1}(D_\tau)}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(\Phi_{\gamma x_i})^2. \tag{3.63}$$

We also have

$$\sum_{k=1}^{q}\tau\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2\leq t_q\max_{1\leq k\leq q}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2. \tag{3.64}$$

Applying (3.63) and (3.64) in (3.62), and by noting that the index q in the second term on the left hand side of (3.62) can be replaced with any $1\leq k\leq q$, we derive

$$\bar{b}\sum_{k=1}^{q}\tau\sum_{\mathscr{A}}h^d(v_{\gamma\bar{t}}(k))^2+a_0\max_{1\leq k\leq q}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2$$

$$+a_0\sum_{k=1}^{q}\tau^2\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}\left(v_{\gamma x_i\bar{t}}(k)\right)^2\leq 2\max_{1\leq i\leq d}\left\|\frac{\partial a_i}{\partial t}\right\|_{L_{\infty,1}(D_{t_q})}\max_{1\leq k\leq q}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2$$

$$+\frac{5d}{\bar{b}}\left(\max_i\|b_i\|_{L_\infty(D)}^2+\max_i\|c_i\|_{L_\infty(D)}^2\right)t_q\max_{1\leq k\leq q}\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(v_{\gamma x_i}(k))^2$$

$$+\frac{5}{\bar{b}}\left(d^2\max_i\|(b_i)_{x_i}\|_{L_\infty(D)}^2+\|r\|_{L_\infty(D)}^2\right)T|\Omega|\;\|[v]_\Delta\|_{\ell_\infty}^2+\frac{5}{\bar{b}}\sum_{k=1}^{q}\tau\sum_{\mathscr{A}}h^d(f_{\gamma,k}^{\Delta})^2$$

$$+\left(\max_{1\leq i\leq d}\|a_i\|_{L_\infty(D)}+\max_{1\leq i\leq d}\left\|\frac{\partial a_i}{\partial t}\right\|_{L_{\infty,1}(D)}\right)\sum_{\mathscr{A}}h^d\sum_{i=1}^{d}(\Phi_{\gamma x_i})^2 \tag{3.65}$$

Note that by the Cauchy-Schwartz inequality, we have

$$\sum_{k=1}^{q} \tau \sum_{\mathscr{A}} h^d (f_{\gamma,k}^{\Delta})^2 \le \int_{D_\Delta} (f^\Delta)^2 \, dx\, dt \le \|f^\Delta\|_{L_2(D)}^2. \tag{3.66}$$

If the length of the time interval T is small enough to guarantee

$$2 \max_{1\le i\le d} \left\| \frac{\partial a_i}{\partial t} \right\|_{L_{\infty,1}(D)} + \frac{5d}{\bar{b}} \Big(\max_i \|b_i\|_{L_\infty(D)}^2 + \max_i \|c_i\|_{L_\infty(D)}^2 \Big) T \le \frac{a_0}{2}, \tag{3.67}$$

then by choosing $q = n$, and by absorbing first two terms on the right hand side of (3.65) into the second term on the left hand side, and by using (3.50),(3.49),(3.66), from (3.65), the energy estimate (3.57) follows. If (3.67) is not satisfied, then we can partition $[0,T]$ into finitely many subsegments which obey (3.67), pursue the energy estimation in each subsegment as before, and through summation achieve the same for (3.65) in general. Theorem is proved. □

Theorems 3.5.1 and 3.5.2 imply the following corollary:

Corollary 3.5.3. *Let $\{[f]_\Delta\}$ be a sequence of discrete control vectors such that there exists $R > 0$ for which $[f]_\Delta \in \mathscr{F}_\Delta^R$ for each Δ. The following statements hold:*

1. *The sequences $\{\tilde{V}_\Delta\}, \{V_\Delta\}, \{V'_\Delta\}$ are uniformly bounded in $L_\infty(D)$.*

2. *For each $i \in \{1, \dots, d\}$, the sequences $\{\tilde{V}_\Delta^i\}, \{\partial V_\Delta / \partial x_i\}, \{\partial V'_\Delta / \partial x_i\}$ are uniformly bounded in $L_2(D)$. Moreover, the sequence $\{\partial V'_\Delta / \partial t\}$ is uniformly bounded in $L_2(D)$.*

3. *The sequence $\{V_\Delta - V'_\Delta\}$ converges strongly to 0 in $L_2(D)$ as $\tau \to 0$.*

4. *For each $k = 1, \dots, n$, the sequence $\{V_\Delta^k - \tilde{V}_\Delta(\cdot, t_k)\}$ converges strongly to 0 in $L_2(\Omega)$ as $h \to 0$. Furthermore, the sequence $\{\tilde{V}_\Delta - V_\Delta\}$ converges strongly to 0 in*

$L_2(D)$ *as* $h \to 0$.

5. *For each* $i \in \{1,2,\ldots,d\}$, *the sequence* $\{\partial V_\Delta/\partial x_i - \partial V'_\Delta/\partial x_i\}$ *converges strongly to* 0 *in* $L_2(D)$ *as* $\tau \to 0$.

6. *For each* $i \in \{1,2,\ldots,d\}$, *the sequence* $\{\tilde{V}^i_\Delta - \partial V_\Delta/\partial x_i\}$ *converges weakly to* 0 *in* $L_2(D)$ *as* $\Delta \to 0$.

Having estimates (3.50),(3.57), the proof of the corollary coincides with the proof of corresponding result of [7] (Theorem 14, pp. 22-30).

3.6 Proof of Main Results

The key to complete the proof of main results is the following approximation theorem.

Theorem 3.6.1. *Let* $R > 0$ *is fixed, and for the sequence of discrete control vectors* $[f]_\Delta \in \mathscr{F}^R_\Delta$, *corresponding sequence of interpolations* $\{\mathscr{P}_\Delta([f]_\Delta)\}$ *converges weakly to* f *in* $L_2(D)$. *Then the sequence of multilinear interpolations* $\{V'_\Delta\}$ *of associated discrete state vectors converges weakly in* $W_2^{1,1}(D)$ *to weak solution* $v = v(x,t;f) \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$ *of the singular PDE problem* (1.28)-(1.30).

Proof. From Theorems 3.5.1, 3.5.2 and Corollary 3.5.3 it follows that $\{V'_\Delta\}$ is a uniformly bounded sequence in $W_2^{1,1}(D) \cap L_\infty(D)$, and hence it is weakly precompact in $W_2^{1,1}(D)$. Let v be its weak limit point. By the Rellich-Kondrachev compact embedding [68], there is a subsequence that converges strongly in $L_2(D)$, and hence further subsequence can be chosen which converges pointwise almost everywhere on D. Since $\{V'_\Delta\}$ is a uniformly bounded in $L_\infty(D)$, and subspace $\overset{\circ}{W}{}_2^{1,1}(D)$ is closed in the weak topology of $W_2^{1,1}(D)$, it follows $v \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$. Next, we prove that v is a weak solution of the singular PDE problem (1.28)-(1.30).

Without loss of generality assume that the whole sequence $\{V'_\Delta\}$ converges to v, weakly in $W_2^{1,1}(D)$ and pointwise a.e. on D. Let $\psi \in \overset{\bullet}{C}{}^1(D)$ is a continuously differentiable function on $\bar{D}$, whose support is positive distance away from S and $\Omega \times \{t = T\}$. Due to construction of D_Δ, there exists a discretization Δ^*, such that $\overline{\text{supp } \psi} \subset D_\Delta$ for all $\Delta \le \Delta^*$. For $\Delta \le \Delta^*$. We define a discrete vector

$$[\psi]_\Delta = \{\psi_\gamma^k : \ \psi_\gamma^k = \psi(x_\gamma, t_k), \ \alpha = (\gamma, k) \in \mathscr{A}(D_\Delta)\}$$

Note that $\psi_\gamma^n = 0$, for all $\gamma \in \mathscr{A}(\Omega_\Delta)$. Plugging $\eta_\gamma := \tau\psi_\gamma^k$ into (3.9), and pursuing summation over $k = \overline{1,n}$, we get

$$\sum_{k=1}^{n} \tau \sum_{\mathscr{A}} h^d \Bigg[\big(b_n(v_\gamma(k))\big)_{\bar{t}} \psi_\gamma^k + \sum_{i=1}^{d} \Big((a_i)_\alpha v_{\gamma x_i}(k) + (b_i)_\alpha v_\gamma(k)\Big) \psi_{\gamma x_i}^k$$
$$+ \sum_{i=1}^{d} (c_i)_\alpha v_{\gamma x_i}(k) \psi_\gamma^k + r_\alpha v_\gamma(k) \psi_\gamma^k - f_{(\gamma,k)}^\Delta \psi_\gamma^k \Bigg] = 0. \qquad (3.68)$$

Since

$$\sum_{k=1}^{n} \tau \sum_{\mathscr{A}} h^d \big(b_n(v_\gamma(k))\big)_{\bar{t}} \psi_\gamma^k = -\sum_{k=1}^{n-1} \tau \sum_{\mathscr{A}} h^d b_n(v_\gamma(k)) \psi_{\gamma t}^k - \sum_{\mathscr{A}} h^d b_n(\Phi_\gamma) \psi_\gamma^1, \qquad (3.69)$$

from (3.68)) we have

$$-\sum_{k=1}^{n-1} \tau \sum_{\mathscr{A}} h^d b_n(v_\gamma(k)) \psi_{\gamma t}^k + \sum_{k=1}^{n} \tau \sum_{\mathscr{A}} h^d \Bigg[\sum_{i=1}^{d} \Big((a_i)_\alpha v_{\gamma x_i}(k) + (b_i)_\alpha v_\gamma(k)\Big) \psi_{\gamma x_i}^k$$
$$+ \sum_{i=1}^{d} (c_i)_\alpha v_{\gamma x_i}(k) \psi_\gamma^k + r_\alpha v_\gamma(k) \psi_\gamma^k - f_{(\gamma,k)}^\Delta \psi_\gamma^k \Bigg] - \sum_{\mathscr{A}} h^d b_n(\Phi_\gamma) \psi_\gamma^1 = 0. \qquad (3.70)$$

We define the following interpolations

$$\overline{\Phi}_\Delta\Big|_{R_\Delta^\gamma} = \Phi_\gamma, \ \ \gamma \in \mathscr{A}, \qquad \overline{\Phi}_\Delta \equiv 0 \text{ elsewhere on } \Omega,$$

$$\overline{\psi}_\Delta\Big|_{C_\Delta^\alpha} = \psi_\gamma^k, \ \ \alpha \in \mathscr{A}(\mathscr{C}_\Delta^D), \qquad \overline{\psi}_\Delta \equiv 0 \text{ elsewhere on } D,$$

$$\overline{\psi}_\Delta^t\Big|_{C_\Delta^\alpha} = \psi_{\gamma t}^k, \ \ \alpha \in \mathscr{A}(\mathscr{C}_\Delta^D \backslash \mathscr{R}_\Delta^{\gamma,n}), \qquad \overline{\psi}_\Delta^t \equiv 0 \text{ elsewhere on } D,$$

$$\overline{\psi}_\Delta^i\Big|_{C_\Delta^\alpha} = \psi_{\gamma x_i}^k, \ \ \alpha \in \mathscr{A}(\mathscr{C}_\Delta^D), \ \ k = 1,\ldots,n, \qquad \overline{\psi}_\Delta^i \equiv 0 \text{ elsewhere on } D.$$

and rewrite (3.70) in integral form:

$$-\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \sum_{\mathscr{A}} \int_{R_\Delta^\gamma} b_n(\tilde{V}_\Delta)\overline{\psi}_\Delta^t \, dx\, dt + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \sum_{\mathscr{A}} \int_{R_\Delta^\gamma} \Big[\sum_{i=1}^{d} \Big(a_i(x,t)\tilde{V}_\Delta^i + b_i(x,t)\tilde{V}_\Delta \Big) \overline{\psi}_\Delta^i$$
$$+ \sum_{i=1}^{d} c_i(x,t)\tilde{V}_\Delta^i \overline{\psi}_\Delta + r(x,t)\tilde{V}_\Delta \overline{\psi}_\Delta - f^\Delta \overline{\psi}_\Delta \Big] \, dx\, dt - \sum_{\mathscr{A}} \int_{R_\Delta^\gamma} b_n(\overline{\Phi}_\Delta)\overline{\psi}_\Delta(x,\tau)\, dx = 0, \quad (3.71)$$

which then implies

$$\int_D \Big[- b_n(\tilde{V}_\Delta)\overline{\psi}_\Delta^t + \sum_{i=1}^{d} \Big(a_i(x,t)\tilde{V}_\Delta^i + b_i(x,t)\tilde{V}_\Delta \Big) \overline{\psi}_\Delta^i$$
$$+ \sum_{i=1}^{d} c_i(x,t)\tilde{V}_\Delta^i \overline{\psi}_\Delta + r(x,t)\tilde{V}_\Delta \overline{\psi}_\Delta - f^\Delta \overline{\psi}_\Delta \Big] \, dx\, dt - \int_\Omega b_n(\overline{\Phi}_\Delta)\overline{\psi}_\Delta(x,\tau)\, dx = 0, \quad (3.72)$$

due to $\overline{\psi}^t_\Delta \equiv 0$ on $\Omega \times (T-\tau, T]$. We transform (3.72) as follows:

$$\int\limits_D \Big[-b_n(\tilde{V}_\Delta)\frac{\partial \psi}{\partial t} + \sum_{i=1}^d \Big(a_i(x,t)\tilde{V}^i_\Delta + b_i(x,t)\tilde{V}_\Delta \Big) \frac{\partial \psi}{\partial x_i}$$
$$+ \sum_{i=1}^d c_i(x,t)\tilde{V}^i_\Delta \psi + r(x,t)\tilde{V}_\Delta \psi - f^\Delta \psi \Big] \, dx\,dt - \int\limits_\Omega b_n(\overline{\Phi}_\Delta)\psi(x,0)\,dx + I = 0, \quad (3.73)$$

where

$$I = \int\limits_D \Big[-b_n(\tilde{V}_\Delta)\Big(\overline{\psi}^t_\Delta - \frac{\partial \psi}{\partial t}\Big) + \sum_{i=1}^d \Big(a_i(x,t)\tilde{V}^i_\Delta + b_i(x,t)\tilde{V}_\Delta \Big) \Big(\overline{\psi}^i_\Delta - \frac{\partial \psi}{\partial x_i}\Big)$$
$$+ \sum_{i=1}^d c_i(x,t)\tilde{V}^i_\Delta \Big(\overline{\psi}_\Delta - \psi\Big) + r(x,t)\tilde{V}_\Delta(\overline{\psi}_\Delta - \psi) - f^\Delta(\overline{\psi}_\Delta - \psi)\Big] \, dx\,dt$$
$$- \int\limits_\Omega b_n(\overline{\Phi}_\Delta)\Big(\overline{\psi}_\Delta(x,\tau) - \psi(x,0)\Big)\,dx. \quad (3.74)$$

Since sequences $b_n(\tilde{V}_\Delta)$ and $b_n(\overline{\Phi}_\Delta)$ are uniformly bounded, and the sequences $\overline{\psi}_\Delta, \overline{\psi}^t_\Delta, \overline{\psi}^i_\Delta$ converge uniformly on $\overline{D}$ to the functions $\psi, \partial\psi/\partial t, \partial\psi/\partial x_i$ respectively as $\Delta \to 0$, it easily follows that $I \to 0$ as $\Delta \to 0$. In [7], it is proved that $b_n(\tilde{V}_\Delta)$, and $b_n(\overline{\Phi}_\Delta)$ are weakly convergent sequences in $L_2(D)$ and $L_2(\Omega)$ respectively, and their weak limits are functions of type $\mathscr{B}$. Precisely, it is proved that

$$b_n(\tilde{V}_\Delta) \rightharpoonup \tilde{b}(x,t) \text{ in } L_2(D); \;\; \tilde{b}(x,t) = B(x,t,v(x,t)), \text{ a.e. in } D, \quad (3.75)$$

$$b_n(\overline{\Phi}_\Delta) \rightharpoonup \tilde{b}_0(x) \text{ in } L_2(\Omega); \;\; \tilde{b}_0(x) = B_0(x,\Phi(x)), \text{ a.e. in } \Omega, \quad (3.76)$$

where B and B_0 are some functions of class $\mathscr{B}$. Passing to limit as $\Delta \to 0$, from

(3.73),(3.75),(3.76) it follows that

$$\int_0^T \int_\Omega \Big[-\tilde{b}(x,t)\frac{\partial \psi}{\partial t} + \sum_{i=1}^d \Big(a_i(x,t)\frac{\partial v}{\partial x_i} + b_i(x,t)v \Big) \frac{\partial \psi}{\partial x_i} + \sum_{i=1}^d c_i(x,t)\frac{\partial v}{\partial x_i}\psi + r(x,t)v\psi - f\psi \Big]\,dx\,dt - \int_\Omega b_0(x)\psi(x,0)\,dx = 0. \tag{3.77}$$

Since $\overset{\bullet}{C}{}^1(D)$ is dense in the set of admissible test functions ψ, and by using (3.75),(3.76) again, it follows that v is a weak solution of the singular PDE problem (1.28)-(1.30). □

Theorem 3.6.1 and Proposition 3.4.3 together with energy estimates of Section 3.5 imply the general existence, uniqueness and stability result for the singular PDE problem (1.28)-(1.30), when the data satisfy assumptions formulated in Section 3.3 and $f \in L_\infty(D)$.

Corollary 3.6.2. *There exists a unique weak solution $v \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$ of the singular PDE problem* (1.28)-(1.30) *and the following estimates are satisfied:*

$$\|v\|_{L_\infty(D)} \le e^{\lambda T} \max\left\{ \|f\|_{L_\infty(D)}\,,\ \|\Phi\|_{L_\infty(\Omega)} \right\}, \tag{3.78}$$

$$\|D_x v\|^2_{L_2(D)} + \|v_t\|^2_{L_2(D)} \le C\Big[\ \|f\|^2_{L_\infty(D)} + \|\Phi\|^2_{L_\infty(\Omega)} + \|D\Phi\|^2_{L_2(\Omega)}\Big] \tag{3.79}$$

where C is a constant depending on d, $\bar{b}$, a_0 and norms of coefficients a_i, b_i, c_i, r in respective spaces given in (3.16).

Proof. The uniqueness is proved in Proposition 3.4.3. The existence of the weak solution is a direct consequence of Theorem 3.6.1. Indeed, given $f \in L_\infty(D)$, consider the sequence of discrete vectors $[f]_\Delta := \mathscr{Q}_\Delta(f)$. Corresponding sequence of interpolations $\mathscr{P}_\Delta([f]_\Delta)$ converge strongly to f in $L_2(D)$, and Theorem 3.6.1 implies the existence of the weak solution $v(x,t;f) \in \overset{\circ}{W}{}_2^{1,1}(D) \cap L_\infty(D)$. There is a sequence of multilinear interpolations $\{V'_\Delta\}$ of the solution to the discrete PDE problem, which converge to v weakly in $W_2^{1,1}(D)$, strongly in $L_2(D)$, and pointwise a.e. on D. From the discrete maximum estimate (3.50) of Theorem 3.5.1 it follows that $\|V'_\Delta\|_{L_\infty(D)}$ is bounded above by the right-hand side of (3.50). Noting that,

$$\|[f]_\Delta\|_{\ell_\infty} = \|f^\Delta\|_{L_\infty(D)} \leq \|f\|_{L_\infty(D)}, \tag{3.80}$$

from (3.50), (3.78) follows. To prove the energy estimate (3.79) we use the following two estimates proved in [7] ((4.17),(4.18)):

$$\|D_x V'_\Delta\|^2_{L_2(D)} \leq 2^{d+1} T \max_{0\leq k\leq n} \sum_{\mathscr{A}} h^d \sum_{i=1}^{d} |v_{\gamma x_i}(k)|^2. \tag{3.81}$$

$$\left\| \frac{\partial}{\partial t} V'_\Delta \right\|^2_{L_2(D)} \leq 2^d \sum_{k=1}^{n} \tau \sum_{\mathscr{A}} h^d |v_{\gamma \bar{t}}(k)|^2 \, dx. \tag{3.82}$$

Weak convergence in $W_2^{1,1}(D)$ implies that

$$\|D_x v\|_{L_2(D)} \leq \liminf_{\Delta\to 0} \|D_x V'_\Delta\|_{L_2(D)}, \quad \|v_t\|_{L_2(D)} \leq \liminf_{\Delta\to 0} \left\| \frac{\partial}{\partial t} V'_\Delta \right\|_{L_2(D)}. \tag{3.83}$$

From (3.81),(3.82), (3.83), (3.49), (3.80) and (3.57), (3.79) follows. □

Having estimates (3.50),(3.57), and approximation Theorem 3.6.1, the completion of the proofs of Theorems 3.3.1 and 3.3.2 coincides with the proofs given in [7]. Theorem 3.6.1 implies that the cost functional $\mathscr{J}(f)$ is continuous on $\mathscr{F}^R$ in a weak topology of $L_2(D)$. Therefore, existence of the optimal control is a consequence of the Weierstrass theorem in a weak topology due to weak compactness of the control set $\mathscr{F}^R$ [47]. Proof of the convergence with respect to functional, or claim(3.18) of Theorem 3.3.2 is pursued by proving claims (i) and (ii) of the Lemma 3.4.4. Claim of Theorem 3.3.2 on the convergence with respect to control is a direct consequence of Theorem 3.6.1.

Chapter 4

Conclusions and Future Research

4.1 Conclusions

Dissertation analyzes optimal control of systems with distributed parameters described by singular nonlinear partial differential equations (PDE) modeling multiphase Stefan type second order parabolic free boundary problems. This type of free boundary problems arise in various applications, such as biomedical engineering problem on the laser ablation of biological tissues, aerospace engineering problem on the ice accretion in aircrafts mid-flight, biomedical problem on the growth of cancerous tumor in the body, and many other phase transition processes in thermophysics and fluid mechanics. Motivation for the dissertation research on the optimal control of distributed free boundary systems is twofold:

- Identification of functional parameters of the mathematical model via solving inverse Stefan type free boundary problems.
- Optimizing the performance of free boundary systems with distributed parameters

via optimal choice of control parameters.

Ill-posed nature of inverse free boundary problems, formation of singularities by free boundaries, and irregularity of solutions are major difficulties in modeling and controlling distributed free boundary systems. Dissertation develops a new method introduced in *U.G. Abdulla & B. Poggi, Calculus of Variations & PDEs, 59:61, 2020*, which is based on the transformation of the multiphase multidimensional Stefan problem to singular PDE problem with discontinuous coefficient in a fixed domain. Optimal control of second order singular parabolic PDE with principal part in divergence form with bounded measurable coefficients is analyzed. Chapter 2 of the dissertation analyzes optimal control of the Neumann problem for the singular PDE problem in one-dimensional setting, where control parameter is a boundary heat flux and cost functional is a norm difference of the trace of the solution from the available temperature measurement at the final moment. Chapter 3 pursues analysis of the optimal control of the multidimensional singular PDE modeling multiphase Stefan type free boundary problem under the Dirichlet boundary condition, where the control parameter is a density of the heat sources, and the cost functional is an L_2-nom difference of the trace of the solution to singular PDE and the given data function at the final moment.

- Existence of the optimal control is proved. Proofs are based on $L_\infty(D)$ and $W_2^{1,1}(D)$ energy estimates of the solution to the singular PDE problem, weak continuity of the functional and Weierstrass theorem in a weak topology of the Banach space.

- Discretization of the optimal control problems via finite differences is pursued. Convergence of the sequence of the finite-dimensional discrete optimal control problems to the original optimal control problem is proved both with respect to functional and control. Precisely, it is proved that the sequence of multilinear interpolations of the discrete minimizers converge to the optimal solution of the singular PDE problem in a weak topology of the Hilbert space of weakly differentiable functions.

- Uniform discrete L_∞ and $W_2^{1,1}$ energy estimates are established and the convergence of the method of finite differences for the singular PDE problem is proved.
- Existence, uniqueness, and stability of the weak solution of the singular nonlinear PDE problem is proved under minimal regularity assumptions on the coefficients and initial-boundary data in terms of anisotropic Sobolev spaces.

4.2 Publications and Conference Presentations

The results of the dissertation are published in:

- U.G. Abdulla and **E. Cosgrove**, Optimal Control of Multiphase Free Boundary Problems for Nonlinear Parabolic Equations, *Applied Mathematics and Optimization, (2020). https://doi.org/10.1007/s00245-020-09655-6*
- U.G. Abdulla and **E. Cosgrove**, Optimal Control of Singular Parabolic PDEs Modeling Multiphase Stefan-type Free Boundary Problems, 2020, submitted. http://arxiv.org/abs/2006.07426.

The following are conference presentations given throughout the completion of the doctoral degree:

- SEARCDE, Florida Gulf Coast University, Fort Myers, FL, November 2016. *On the Frechet differentiability in optimal control of coefficients in parabolic free boundary problems*.

- JMM, Atlanta, GA, January 2017. *On the Frechet differentiability in optimal control of coefficients in parabolic free boundary problems*.

- AMS Fall Southeastern Sectional Meeting, University of Central Florida, Orlando, FL, September 2017. *On the Frechet differentiability in optimal control of coefficients in parabolic free boundary problems*.

- JMM, San Diego, CA, January 2018. *On the Frechet differentiability in optimal control of coefficients in parabolic free boundary problems*.

- AMS Fall Eastern Sectional Meeting, University of Delaware, Newark, DE, September 2018. *On the Optimal Control of the Multiphase Free Boundary Problems for the Nonlinear Parabolic Equations*.

- JMM, Baltimore, MD, January 2019. *On the Optimal Control of the Multiphase Free Boundary Problems for the Nonlinear Parabolic Equations*.

The following is a poster presentation given during the completion of the doctoral degree:

- Young Mathematicians Conference, Ohio State University, Columbus, OH, August 2016. *Frechet Differentiability in Optimal Control of the Stefan Problem*

4.3 Future Research

The results of the dissertation open a perspective to apply the developed methods for the solution of the several open problems in the field.

- It is a challenging open problem to extend the results of Chapter 3 to optimal control of singular PDE problem (1.28), (1.29), (1.35), which is the distributional formulation of the multidimensional and multiphase Stefan problem under the Neumann boundary condition on the fixed boundary. Optimal control problem is to minimize the cost functional (1.37) on a control set (1.38), with control parameters chosen as a boundary flux $\Lambda(x,t)$, or any of the coefficients of the PDE (1.28), such as a_i, b_i, c_i, r. This problem is very important in applications outlined in Chapter 1.

- Another open problem consists in proving Frechet differentiability and optimality conditions in optimal control problems for singular PDEs. Solution of this problem will open perspective to develop gradient type iterative numerical methods [3, 4].

- Optimal control of singular PDE (1.28), where β is a non-differentiable function of type

$$\beta(s) = |s|^{\frac{1}{m}}\ sign\ s,\ m > 1$$

  is an outstanding open problem. This class of PDEs form a porous medium type nonlinear degenerate parabolic PDEs arising in flow of a gas or Newtonian fluid in a porous media, and in many other applications [85]. It is a very challenging open problem to analyze optimal control of the free boundary problems for the nonlinear degenerate parabolic PDEs by using general theory in non-cylindrical domains [11, 12, 13, 14]), and properties of the interfaces of nonlinear degenerate parabolic PDEs [15, 16].

- Another open problem would be analysis of optimal control of singular PDEs in unbounded domains. Generalization of the methods to this class of optimal control problems would require the application of the delicate results on the well posedness of the elliptic and parabolic PDEs in domains with non-compact boundaries [19, 20, 17, 18, 19, 20, 21].

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