11-2018

On the Qualitative Theory of the Nonlinear Parabolic p-Laplacian Type Reaction-Diffusion Equations

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by

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A dissertation
submitted to Florida Institute of Technology
in partial fulfillment of the requirements
for the degree of

Doctorate of Philosophy
in
Applied Mathematics

Melbourne, Florida
November, 2018

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ABSTRACT

Title:

On the Qualitative Theory of the Nonlinear Parabolic p-Laplacian Type

Reaction-Diffusion Equations

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This dissertation presents full classification of the evolution of the interfaces and asymptotics of the local solution near the interfaces and at infinity for the nonlinear second order parabolic p-Laplacian type reaction-diffusion equation of non-Newtonian elastic filtration

$$u_t - \left(|u_x|^{p-2}u_x\right)_x + bu^\beta = 0,\ p > 1, \beta > 0. \tag{1}$$

Nonlinear partial differential equation (1) is a key model example expressing competition between nonlinear diffusion with gradient dependent diffusivity in either slow ($p > 2$) or fast ($1 < p < 2$) regime and nonlinear state dependent reaction ($b > 0$) or absorption ($b < 0$) forces. If interface is finite, it may expand, shrink, or remain stationary as a result of the competition of the diffusion and reaction terms near the interface, expressed in terms of the parameters $p, \beta, sign\ b$, and asymptotics of the initial function near its support. In the fast diffusion regime strong domination of the diffusion causes infinite speed of propagation and interfaces are absent. In all cases with finite interfaces we prove the explicit formula for the interface and the local solution with accuracy up to constant coefficients. We prove explicit asymptotics of the local solution at infinity in all cases with infinite speed of propagation. The methods of the proof are general-

ization of the methods developed in *U.G. Abdulla & J. King, SIAM J. Math. Anal., 32, 2(2000), 235-260; U.G. Abdulla, Nonlinear Analysis, 50, 4(2002), 541-560* and based on rescaling laws for the nonlinear PDE and blow-up techniques for the identification of the asymptotics of the solution near the interfaces, construction of barriers using special comparison theorems in irregular domains with characteristic boundary curves.

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Acknowledgements

First of all, I would like to thanks and express my deepest appreciation to my degree advisor Professor Dr. Ugur G. Abdulla, Ph.D., Dr. Sci., who has the attitude and the substance of genius: he taught me well and has been kindly advising me until I became a mathematician. I am so lucky to have him as a supervisor, without his guidance and persistent help this dissertation would not have been possible.

I would like to extend my sincere thanks to the committee members Professor Dr. Kanishka Perera, Ph.D., Professor Dr. Jay Kovats, Ph.D. and Professor Dr. Ming Zhang, Ph.D. for their valuable guidance. I am grateful for their support.

I would also like to thanks all faculty members and staff in FIT who taught and helped me.

Dedication

I am very thankful to my parents; whose love and guidance are with me in whatever I pursue. They are the ultimate role models. I wish to thank my supportive husband, and my three wonderful children, Samar, Ali and Sama, who provide unending inspiration.

Chapter 1

Introduction

1.1 Physical Motivation

Consider the one-dimensional, turbulent, poly-tropic flow of a gas in a porous medium [60].This flow can be mathematically described by the following laws.

- A poly-tropic equation of state

$$P = c\gamma^n \tag{1.1}$$

  where γ is a density of the gas, P is the pressure.

- The continuity equation

$$k\gamma_t + (\gamma V)_x = 0 \tag{1.2}$$

  where V is the velocity of the gas at the space point x at the time instant t.

- The flux under turbulent condition

$$\gamma V = -M|\Phi_x|^{p-2}\Phi_x \tag{1.3}$$

where c, k, M are positive physical constants. $n \geq 1,\ p \geq 3/2$ and

$$\Phi = P^{(n+1)/n}. \tag{1.4}$$

Combining (1.1)-(1.4), we get

$$k\gamma_t = Mc^{(p-1)(n+1)/n} \frac{\partial}{\partial x}\Big(\Big|\frac{\partial(\gamma^{n+1})}{\partial x}\Big|^{p-2} \frac{\partial(\gamma^{n+1})}{\partial x}\Big). \tag{1.5}$$

Scaling the constants in (1.5) we obtain the nonlinear diffusion equation

$$u_t = \frac{\partial}{\partial x}\Big(\Big|\frac{\partial u^m}{\partial x}\Big|^{p-2} \frac{\partial u^m}{\partial x}\Big)$$

where $m = n + 1,\ m(p-1) > 1$. If $m = 1$, then we have non-Newtonian elastic filtration equation

$$u_t = \frac{\partial}{\partial x}\Big(\Big|\frac{\partial u}{\partial x}\Big|^{p-2} \frac{\partial u}{\partial x}\Big)$$

where $p > 1$. The case $p > 2$ is called slow diffusion case and the case $1 < p < 2$ is called fast diffusion case [85]. In the case $p = 2$ we have a classical linear heat equation.

1.1.1 Instantaneous Point-Source Solution

The prelude of the mathematical theory of the nonlinear degenerate parabolic equations is the papers [42, 105](see also [43]), where instantaneous point source type particular solutions were constructed and analyzed. The property of finite speed of propagation and the existence of compactly supported nonclassical solutions and interfaces became a motivating force of the general theory. Consider the instantaneous point-source problem for the nonlinear p-Laplacian equation

$$\begin{cases} u_t = (|u_x|^{p-2} u_x)_x, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \delta(x), & x \in \mathbb{R} \\ \int_{\mathbb{R}} u(x,t)dx = 1, & t \geq 0 \\ u(x,t) \geq 0 \end{cases} \tag{1.6}$$

where $\delta(\cdot)$ is Dirac's point mass with support at the origin. The solution of this problem (1.6) is given by

$$u_*(x,t) = t^{\frac{-1}{2(p-1)}} \Big\{ C - k(p) \Big(\frac{|x|}{t^{1/2(p-1)}} \Big)^{p/(p-1)} \Big\}^{\frac{p-1}{p-2}}$$

where $k(p) = \frac{p-2}{p}(2(p-1))^{-1/(p-1)}$. In the slow diffusion case ($p > 2$) [42, 43]. In the fast diffusion case ($1 < p < 2$), solution of this problem (1.6) has an infinite speed of propagation. Meaning that the solution is instantaneously positive everywhere in the space. Where $(\mathcal{X})_+ = \{\mathcal{X},\ if\ \mathcal{X} > 0; 0,\ if\ \mathcal{X} \leq 0\}$. The profile of the solution in different moments of time is depicted in Figure1.1and Figure1.2. Two key features of the Barenbaltt's solution became vital both for application of the nonlinear diffusion type degenerate parabolic PDEs, and the fascinating mathematical theory.

- **Finite speed of propagation:** Support of the solution is compact

$$\text{spt}(u) = \overline{\{(x,t) : u(x,t) > 0\}} = \{|x| \leq \eta_0 t^{1/2(p-1)}\}.$$

Hence, the solution of the nonlinear degenerate parabolic PDE demonstrate finite speed of propagation property like hyperbolic equations, which is in contrast to infinite speed of propagation property of the linear heat equation. This property

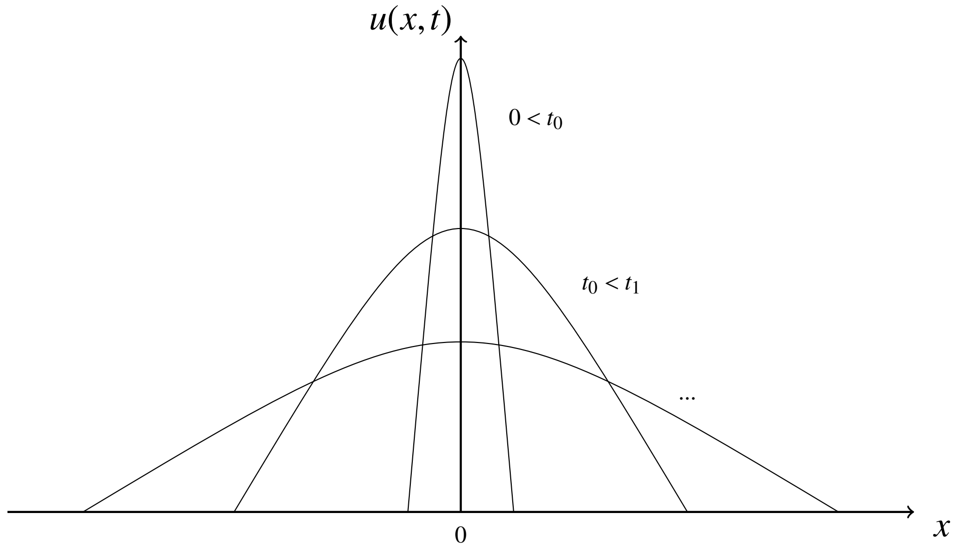

Figure 1.1: Barenblatt solution when $p > 2$

suggest that nonlinear degenerate parabolic PDEs are more relevant for real world applications than their linear predecessors.

- **The Barenblatt solution is not a classical solution:** Despite being physically relevant, Barenblatt's solution doesn't solve the PDE in classical sense, second derivative with respect to x and first derivative with respect to t are discontinuous along the boundary surfaces of the support, called interfaces or free boundaries.

1.2 Historical Review

Mathematical theory of nonlinear degenerate parabolic equations began with paper [93] on the porous medium equation. To explain the notion of the weak solution, consider a

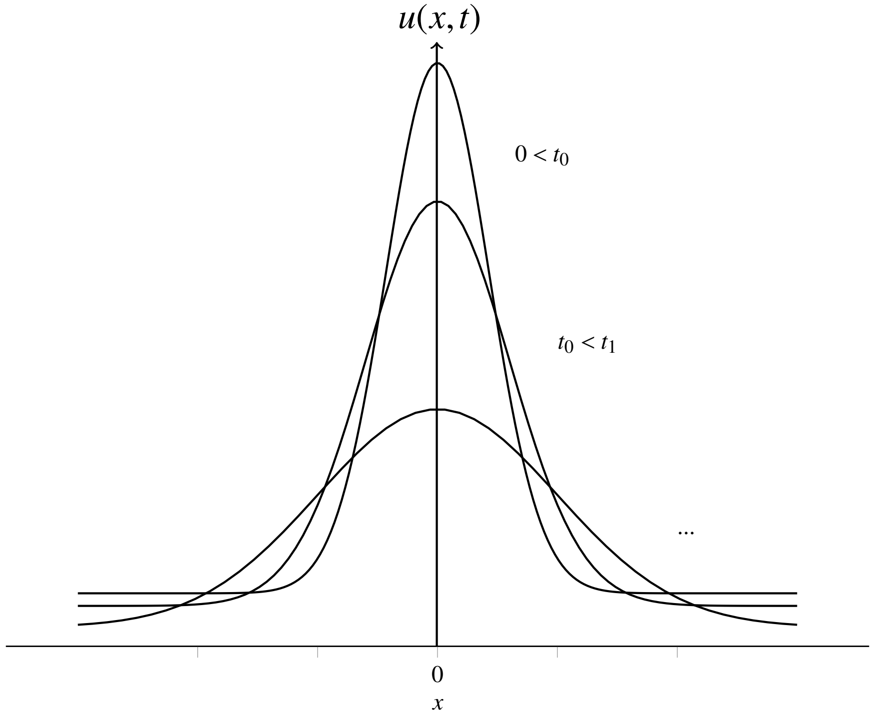

Figure 1.2: Barenblatt solution when $1 < p < 2$

Dirichlet problem for PDE

$$u_t = \triangle u^m \qquad \text{in } \ Q = D \times (0,T] \tag{1.7}$$

where $D \subset \mathbb{R}^N$ be open domain, under the initial-boundary conditions

$$u(x,0) = u_0(x), \quad x \in Q \tag{1.8}$$

$$u(x,t) = 0, \quad (x,t) \in S = \partial D \times (0,T) \tag{1.9}$$

Definition 1.2.1. We say that a non-negative function $u = u(x,t)$ is a weak solution of the Dirichlet problem(1.7)-(1.9) if

- $u^m \in L_2(0,T;H_0^1(D))$

- u satisfies the integral identity

$$\iint_{Q_T} (\nabla u^m \cdot \nabla \phi - u\phi_t)dxdt = \int_D u_0(x)\phi(x,0)dx$$

for any $\phi \in C^1(\overline{Q_T})$ satisfying $\phi(x,T) = \phi|_{S_T} = 0$, where,

$$L_2(0,T;H_0^1(D)) = \{u = u(t) : [0,T] \to H_0^1(D)\}$$

is a Hilbert space with the norm

$$||u||_{L_2\left(0,T;H_0^1(D)\right)} = (\int_0^T ||u||^2_{H_0^1(D)}dt)^{1/2} = (\int_0^T \int_D (|u|^2 + |\nabla u|^2)dxdt)^{1/2}$$

In fact, instantaneous point-source solution is a weak solution in the sense of the Definition 1.2.1. Currently there is a well established general theory of the nonlinear degenerate parabolic equations (see [104, 58]). The questions of existence, uniqueness of solutions to Cauchy problem and other initial-boundary value problems, comparison theorems, regularity of weak solutions are analyzed in [19]-[105]. The general theory of nonlinear degenerate second order parabolic PDEs in non-cylinrical non-smooth domains was developed in [3, 14, 6, 12, 4]

1.3 Formulation of the open problems

We consider the Cauchy problem(CP) for the nonlinear degenerate parabolic equation:

$$Lu \equiv u_t - \left(|u_x|^{p-2}u_x\right)_x + bu^\beta = 0,\ x \in \mathbb{R}, 0 < t < T, \tag{1.10}$$

with

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.11}$$

where $p > 1$, $b \in \mathbb{R}$, $\beta > 0$, $0 < T \leq +\infty$, and u_0 is non-negative and continuous. We assume that $b > 0$ if $\beta < 1$, and b is arbitrary if $\beta \geq 1$ (see Remark 1.1). Equation (1.10) arises in many applications, such as the filtration of non-Newtonian fluids in porous media [42] or non-linear heat conduction [43] in the presence of the reaction term expressing additional release ($b > 0$) or absorption ($b < 0$) of energy. Due to the property of the finite speed of propagation the problem develops interfaces or free boundaries separating the region where $u > 0$ from the region where $u = 0$. The aim of the dissertation is to present full classification of the short-time evolution of interfaces and local structure of solutions near the interfaces. Due to invariance of (1.10) with respect to translation, without loss of generality, we will investigate the case when $\eta(0) = 0$, where

$$\eta(t) = \sup\{x : u(x,t) > 0\},$$

and precisely, we are interested in the short-time behavior of the interface function $\eta(t)$ and local solution near the interface. We shall assume that

$$u_0 \sim C(-x)_+^{\alpha} \text{ as } x \to 0- \quad \text{for some} \quad C > 0,\ \alpha > 0. \tag{1.12}$$

The direction of the movement of the interface and its asymptotics is an outcome of the competition between the diffusion and reaction terms and depends on the parameters p, b, β, C, and α. Since the main results are local in nature, without loss of generality we may suppose that u_0 either is bounded or satisfies some restriction on its growth rate as $x \to -\infty$ which is suitable for existence, uniqueness, and comparison results (see

Section 2.2). The special global case

$$u_0(x) = C(-x)_+^{\alpha}, \quad x \in \mathbb{R}, \tag{1.13}$$

will be considered when the solution to the problem (1.10), (1.13) is of self-similar form. Our estimations are global in time in these special cases.

Initial development of interfaces and structure of local solution near the interfaces is very well understood in the case of the reaction-diffusion equations with porous medium type diffusion term:

$$u_t - (u^m)_{xx} + bu^{\beta} = 0 \ x \in \mathbb{R}, 0 < t < T. \tag{1.14}$$

Full classification of the evolution of interfaces and the local behavior of solutions near the interfaces in CP (1.14), (1.11), (1.12) was presented in [22] for the case of slow diffusion ($m > 1$) case, and in [7] for the fast diffusion ($0 < m < 1$) case.The major obstacle in solving the interface development problem for non-linear degenerate parabolic equations is a problem of non-uniform asymptotics in the sense of singular perturbations theory, namely that the dominant balance as $t \to 0+$ between the terms in (1.10), (1.14) on curves that approach the boundary of the support on the initial line depending on how they do so. The general theory, including existence, boundary regularity, uniqueness and comparison theorems, for the reaction-diffusion equations of type (1.14) in general non-cylindrical and non-smooth domains is developed in [3] in the one-dimensional case, and in [6, 12, 14] in the multi-dimensional case. Comparison theorems proved in [3] were essential tools in developing the rigorous proof method in [22, 7] for solving interface problem for the reaction-diffusion equation (1.14). The rigorous proof method developed in [22, 7] is based on a barrier technique using special comparison theorems

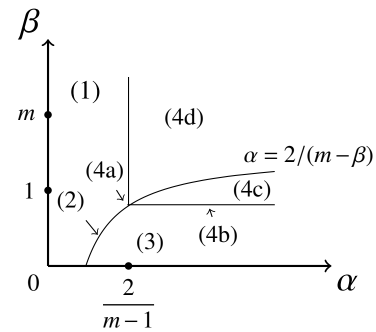

Figure 1.3: Classification of different cases in the (α,β) plane for interface development in problem (1.14), (1.11), (1.12) which is presented in [22] .

in irregular domains with characteristic boundary curves. Evolution of interfaces on local solutions for reaction-diffusion (1.14) in the slow diffusion case was solved in [22] and Figure 1.3 demonstrates the following classification:

Region one when $\alpha < 2/(m-\min\{1,\beta\})$; diffusion dominates and interface expands. Region two when $\alpha = 2/(m-\beta), 0 < \beta < 1$; diffusion and absorption are in balance in this borderline case. There is a critical constant C_* such that interface expands for $C > C_*$, and shrinks for $C < C_*$. Region three when $\alpha > 2/(m-\beta), 0 < \beta < 1$; absorption term dominates and interface shrinks. Region four when $\alpha \geq 2/(m-1), \beta \geq 1$; interface has initial 'waiting time'.

In the case of fast diffusion $0 < m < 1$, interface development for reaction-diffusion equation was solved in [7] and Figure 1.4 demonstrates the classification as the following:

Region one when $0 < \beta < m$, $0 < \alpha < 2/(m-\beta)$; diffusion dominates and interface expands. Region two when $\alpha = 2/(m-\beta)$, $0 < \beta < m$; diffusion and absorption are in balance in this borderline case. There is a critical constant C_* such that interface expands for $C > C_*$, and shrinks for $C < C_*$. Region three when $\alpha > 2/(m-\beta)$, $0 < \beta < m$; absorption term dominates and interface shrinks. Region four when $\alpha > 0$, $0 < m = \beta < 1$; there

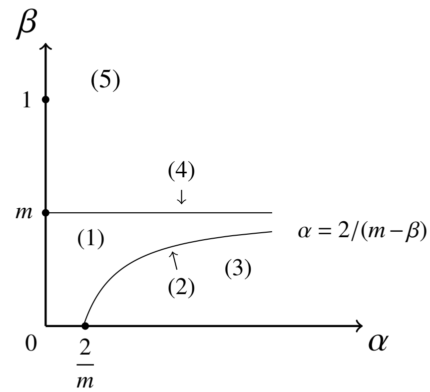

Figure 1.4: Classification of different cases in the (α,β) plane for interface development in problem (1.14), (1.11), (1.12) which is presented in [7] .

is an infinite speed of propagation. Region five when $\beta > m$; there is an infinite speed of propagation. In all cases asymptotic formula for the interface and local solution was proved [22] and [7].

The goal of this dissertation is to solve the open problem both for slow diffusion case ($p > 2$) and fast diffusion case ($1 < p < 2$) and present full classification of the evolution of interfaces and asymptotics of the local solutions near the interfaces for p-Laplacian type reaction-diffusion equation (1.10). Although in the absence of reaction term if $b = 0$ in the fast diffusion case there is no interface there is an infinite speed of propagation and adding absorption term there is possible that they will be finite speed of propagation. The direction of the interface in Cauchy problem for p-Laplacian type reaction-diffusion equation (1.10) in slow diffusion case ($p > 2$) was considered in [91] and one sided rough estimations for interfaces was proved in [91]. The aim of the dissertation is to develop and applied the methods of papers [22, 7, 3] to prove sharp estimations of the interfaces and local solutions near the interfaces in full parameter scale both for slow diffusion case ($p > 2$) and fast diffusion case ($1 < p < 2$).

Chapter 2

Evolution of Interface for the Nonlinear p-Laplacian type Reaction-Diffusion Equations with Slow Diffusion

In this chapter we present full classification of the evolution of interfaces and local structure of solution near the interfaces of the problem (1.10) -(1.13) in the slow diffusion case ($p > 2$). The results of Chapter 2 are published in [20]

2.1 Description of Main Results

In Figure 2.1 we present classification diagram in (α,β)-plane for the initial interface development in CP (1.10) -(1.12) if $b > 0$.

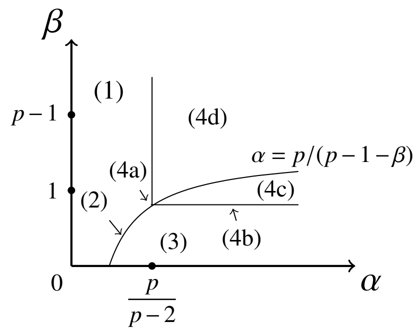

Figure 2.1: Classification of different cases in the (α,β) plane for interface development in problem (1.10)-(1.13) (when $p > 2$).

- **Region (1):** $\alpha < p/(p-1-\min\{1,\beta\})$; diffusion dominates and interface expands.
- **Region (2):** $\alpha = p/(p-1-\beta), 0 < \beta < 1$; diffusion and absorption are in balance in this borderline case. There is a critical constant C_* such that interface expands for $C > C_*$, and shrinks for $C < C_*$.
- **Region (3):** $\alpha > p/(p-1-\beta), 0 < \beta < 1$; absorption term dominates and interface shrinks.
- **Region (4):** $\alpha \geq p/(p-2), \beta \geq 1$; interface has initial 'waiting time'.

To describe the asymptotic properties of the interface and local solution near the interface, we divide the results into the two different subcases:

(I) $b \neq 0$ (either $b > 0, \beta > 0$ or $b < 0, \beta \geq 1$) and $p > 2$; and **(II)** $b = 0$.

(I) In this case there are four different subcases, as shown in Figure 2.1 and itemized above. (In view of our assumptions, the case $b < 0$ relates to the part of the (α,β) plane with $\beta \geq 1$.)

Region (1)

Theorem 2.1.1. *Let* u_0 *satisfies* (1.12) *with* $\alpha < \frac{p}{p-1-min\{1,\beta\}}$. *Then, interface initially expands and*

$$\eta(t) \sim \xi_* t^{1/(p-\alpha(p-2))} \quad as\ t \to 0+, \tag{2.1}$$

where

$$\xi_* = C^{\frac{p-2}{p-\alpha(p-2)}} \xi'_* \tag{2.2}$$

and $\xi'_* > 0$ *depends on* p *and* α *only (see Lemma 2.3.1). For arbitrary* $\rho < \xi_*$ *there exists* $f(\rho) > 0$ *depending on* $C, p,$ *and* α *such that*

$$u(x,t) \sim f(\rho) t^{(\alpha/p-\alpha(p-2))} \quad as \quad t \to 0+ \tag{2.3}$$

along the curve $x = \xi_\rho(t) = \rho t^{1/(p-\alpha(p-2))}$.

A function f is a shape function of the self-similar solution of (1.10),(1.13) with $b = 0$ (see Lemma 2.3.1):

$$u_*(x,t) = t^{\frac{\alpha}{p-\alpha(p-2)}} f(\xi), \qquad \xi = x t^{-\frac{1}{p-\alpha(p-2)}}, \tag{2.4}$$

In fact, f is a unique solution of the following nonlinear ODE problem:

$$\begin{cases} (|f'(\xi)|^{p-2} f'(\xi))' + \frac{1}{p-\alpha(p-2)} \xi f'(\xi) - \frac{\alpha}{p-\alpha(p-2)} f(\xi) = 0, \ -\infty < \xi < \xi_* \\ f(-\infty) \sim C(-\xi)^\alpha, f(\xi_*) = 0, f(\xi) \equiv 0, \ \xi \geq \xi_*. \end{cases} \tag{2.5}$$

Its dependence on C is given through the following relation:

$$f(\rho) = C^{p/(p-\alpha(p-2))} f_0\Big(C^{(p-2)/(\alpha(p-2)-p)} \rho\Big), \tag{2.6a}$$

$$f_0(\rho) = w(\rho, 1), \qquad \xi'_* = \sup\{\rho : f_0(\rho) > 0\} > 0, \tag{2.6b}$$

where w is a solution of (1.10), (1.13) with $b = 0, C = 1$. Lower and upper estimations for f are given in (2.31). Moreover,

$$\xi'_* = A_0^{\frac{p-2}{p}} \Big[\frac{(p-1)^{p-1}(p-\alpha(p-2))}{(p-2)^{p-1}}\Big]^{\frac{1}{p}} \xi''_*, \tag{2.7}$$

where $A_0 = w(0,1)$ and ξ''_* is some number in $[\xi_1, \xi_2]$, where

$$\xi_1 = (p-1)^{\frac{1}{p}}\Big(\alpha(p-2)\Big)^{-\frac{1}{p}}, \quad \xi_2 = 1 \qquad \text{if} \quad (p-1)(p-2)^{-1} \le \alpha < p(p-2)^{-1},$$

$$\xi_1 = 1, \qquad \xi_2 = (p-1)^{\frac{1}{p}}\Big(\alpha(p-2)\Big)^{-\frac{1}{p}}, \qquad \text{if} \quad 0 < \alpha \le (p-1)(p-2)^{-1}. \tag{2.8}$$

In particular, if $\alpha = (p-1)(p-2)^{-1}$and $p > 1 + (\min\{1,\beta\})^{-1}$, then the explicit solution of the problem (1.10), (1.13) with $b = 0$ is given by (2.29), and we have

$$\xi_1 = \xi_2, \quad \xi'_* = (p-1)^{p-1}(p-2)^{1-p}, \quad f_0(x) = (\xi'_* - x)_+^{(p-1)/(p-2)}. \tag{2.9}$$

The explicit formulae (2.1) and (2.3) mean that the local behavior of the interface and solution along $x = \xi_\rho(t)$ coincide with that of the problem (1.10), (1.13) with $b = 0$.

Region (2)

Theorem 2.1.2. *Let* $b > 0, 0 < \beta < 1, p \geq 2, \alpha = p/(p-1-\beta)$ *and*

$$C_* = \Big[\frac{|b||p-1-\beta|^p}{(1+\beta)p^{p-1}(p-1)}\Big]^{\frac{1}{p-1-\beta}}. \tag{2.10}$$

If u_0 *satisfies* (1.12)*, then interface expands or shrinks according as* $C > C_*$ *or* $C < C_*$ *and*

$$\eta(t) \sim \zeta_* t^{\frac{p-1-\beta}{p(1-\beta)}} \quad as \;\; t \to 0+, \tag{2.11}$$

where $\zeta_* \lesseqgtr 0$ *if* $C \lesseqgtr C_*$*, and for arbitrary* $\rho < \zeta_*$ *there exists* $f_1(\rho) > 0$ *such that*

$$u(x,t) \sim f_1(\rho)t^{1/(1-\beta)} \quad for \quad x = \zeta_\rho(t) = \rho t^{\frac{p-1-\beta}{p(1-\beta)}}, \quad t \to 0+. \tag{2.12}$$

Assume that u_0 is defined by (1.13). If $\beta(p-1) = 1$, then the explicit solution to (1.10), (1.13) is

$$u(x,t) = C(\zeta_* t - x)_+^{\frac{1}{1-\beta}}, \quad \zeta_* = b(1-\beta)C^{\beta-1}((C/C_*)^{p-1-\beta} - 1). \tag{2.13}$$

It has an expanding interface if $C > C_*$, a shrinking interface if $0 < C < C_*$, and is a stationary solution if $C = C_*$.

Let $\beta(p-1) \neq 1$. If $C = C_*$, then u_0 is a stationary solution to (1.10), (1.13). If $C \neq C_*$, then the solution to (1.10), (1.13) is of the self similar form:

$$u(x,t) = t^{1/(1-\beta)} f_1(\zeta), \qquad \zeta = x t^{-\frac{p-1-\beta}{p(1-\beta)}}, \tag{2.14}$$

$$\eta(t) = \zeta_* t^{\frac{p-1-\beta}{p(1-\beta)}}, \qquad 0 \leq t < +\infty. \tag{2.15}$$

If $C > C_*$ then the interface expands, $f_1(0) = A_1 > 0$ (see Lemma 2.3.3), and

$$C_1 t^{\frac{1}{1-\beta}}\left(\zeta_1 - \zeta\right)_+^{\mu} \leq u \leq C_2 t^{\frac{1}{1-\beta}}\left(\zeta_2 - \zeta\right)_+^{\frac{p}{p-1-\beta}}, \quad 0 \leq x < +\infty, \quad 0 < t < +\infty, \tag{2.16}$$

where

$$\mu = (p-1)(p-2)^{-1} \text{ if } \beta(p-1) > 1; \ \mu = p(p-1-\beta)^{-1} \text{ if } \beta(p-1) < 1$$

which implies

$$\zeta_1 \leq \zeta_* \leq \zeta_2. \tag{2.17}$$

The right-hand side of (2.16) ((2.17), respectively) may be replaced by $\bar{C}_2 t^{\frac{1}{1-\beta}}(\bar{\zeta}_2 - \zeta)_+^{\frac{p-1}{p-2}}$ ($\bar{\zeta}_2$, respectively); see the Appendix Part A for the description of all the relevant constants. Let $\beta(p-1) \neq 1$ and $0 < C < C_*$. Then, the interface shrinks and if $\beta(p-1) > 1$, then

$$[C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}} - b(1-\beta)t]_+^{\frac{1}{1-\beta}} \leq u$$

$$\leq [C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}} - b(1-\beta)(1-\left(\frac{C}{C_*}\right)^{p-1-\beta})t]_+^{\frac{1}{1-\beta}}, \ x \in \mathbb{R}, \ 0 \leq t < +\infty, \tag{2.18}$$

which again implies (2.17), where ζ_1(ζ_2, respectively) is replaced with

$$-C^{-\frac{p-1-\beta}{p}}(b(1-\beta))^{\frac{p-1-\beta}{p(1-\beta)}}$$

$$\left(\text{respectively}, -C^{-\frac{p-1-\beta}{p}}(b(1-\beta)(1-(C/C_*)^{p-1-\beta})^{\frac{p-1-\beta}{p(1-\beta)}}\right).$$

However, if $\beta(p-1)<1$, then

$$C_*\Big(-\zeta_3 t^{\frac{p-1-\beta}{p(1-\beta)}}-x\Big)_+^{\frac{p}{p-1-\beta}} \le u \le C_3(-\zeta_4 t^{\frac{p-1-\beta}{p(1-\beta)}}-x)_+^{\frac{p}{p-1-\beta}},\ 0\le t<+\infty, \tag{2.19}$$

where the left-hand side is valid for $x \ge -\ell_0 t^{\frac{p-1-\beta}{p(1-\beta)}}$, whereas the right-hand side is valid for $x \ge -\ell_1 t^{\frac{p-1-\beta}{p(1-\beta)}}$. From (2.19),(2.17) follows if we replace ζ_1 and ζ_2 with $-\zeta_3$ and $-\zeta_4$, respectively.

If $\beta(p-1)\neq 1$, in general, the precise value ζ_* can be found only by solving the ODE $\mathcal{L}^0 f_1=0$ (see (2.77b)) below) and calculating $\zeta_*=\sup\{\zeta : f_1(\zeta)>0\}$.

The right-hand side of (2.12) ((2.11), respectively) relates to the self-similar solution (2.14) (to its interface, as in (2.15), respectively). If $\beta(p-1)=1$, we then have explicit values of ζ_* and $f_1(\rho)$ via (2.13), whereas in general we have lower and upper bounds via (2.16)-(2.19). If u_0 satisfies (1.12) with $\alpha=p/(p-1-\beta), C=C_*$, then the small-time behavior of the interface and local solution depend on the terms smaller than $C_*(-x)^{p/(p-1-\beta)}$ in the expansion of u_0 as $x\to 0-$.

Region (3)

Theorem 2.1.3. *Let $b>0, 0<\beta<1, p\ge 2, \alpha>p/(p-1-\beta)$. If u_0 satisfies* (1.12)*, then interface shrinks and*

$$\eta(t)\sim -\ell_* t^{1/\alpha(1-\beta)} \text{ as } t\to 0+, \tag{2.20}$$

where $\ell_*=C^{-1/\alpha}(b(1-\beta))^{1/\alpha(1-\beta)}$. For arbitrary $\ell>\ell_*$, we have

$$u(x,t)\sim [C^{1-\beta}(-x)_+^{\alpha(1-\beta)}-b(1-\beta)t]^{1/(1-\beta)} \text{ as } t\to 0+ \tag{2.21}$$

along the curve $x = \eta_l(t) = -lt^{1/\alpha(1-\beta)}$.

Hence, the interface initially coincides with that of the solution

$$\bar{u}(x,t) = [C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t]_+^{1/(1-\beta)} \tag{2.22}$$

to the problem

$$\bar{u}_t + b\bar{u}^{\beta} = 0, \qquad \bar{u}(x,0) = C(-x)_+^{\alpha}. \tag{2.23}$$

Respective lower and upper estimations are given in Section 2.4 (see (2.89) and (2.92)).

Region (4)

In this case, the interface initially has a waiting time. We divide the results into four different subcases (see Figure 2.1).

(4a) Let $\beta = 1, \alpha = p/(p-2)$. This case reduces to the case $b = 0$ by a simple transformation (see Section 2.2). If u_0 is defined by (1.13), then the unique solution to (1.10), (1.13) is

$$u_C(x,t) = C(-x)_+^{p/(p-2)}\exp(-bt)[1-(C/\bar{C})^{p-2}b^{-1}(1-\exp(-b(p-2)t))]^{1/(p-2)}, \tag{2.24}$$

for $x \in \mathbb{R}$, $t \in [0,T)$,where

$$T = +\infty \quad \text{if } b \geq (C/\bar{C})^{p-2},$$

$$T = (b(2-p))^{-1}\ln[1-b(\bar{C}/C)^{p-2}], \qquad \text{if } -\infty < b < (C/\bar{C})^{p-2},$$

$$\bar{C} = \left[(p-2)^p/(2(p-1)p^{p-1})\right]^{1/(p-2)}.$$

If u_0 satisfies (1.12), then lower and upper estimations are given by $u_{C\pm\epsilon}$.

(4b) Let $\beta = 1, \alpha > p/(p-2)$. Then, for arbitrary $\epsilon > 0$ there exists $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^\alpha \exp(-bt) \le u(x,t) \le (C+\epsilon)(-x)_+^\alpha \exp(-bt) \tag{2.25}$$

$$\times\left[1-\epsilon b^{-1}(p-2)^{-p}(1-\exp(-b(p-2)t))\right]^{1/2-p}, \qquad x > x_\epsilon,\ \ 0 \le t \le \delta_\epsilon.$$

(4c) Let $1 < \beta < p-1,\ \alpha \ge p/(p-1-\beta)$. Then, for $\forall\epsilon > 0\ \exists x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$g_{-\epsilon}(x,t) \le u(x,t) \le g_\epsilon(x,t),\ \ x \ge x_\epsilon,\ \ \ 0 \le t \le \delta_\epsilon, \tag{2.26}$$

where

$$g_\epsilon(x,t) = \begin{cases} \left[(C+\epsilon)^{1-\beta}|x|^{\alpha(1-\beta)} + b(\beta-1)(1-d_\epsilon)t\right]^{1/(1-\beta)}, & x_\epsilon \le x < 0, \\ 0, & x \ge 0, \end{cases}$$

$$d_\epsilon = \begin{cases} \epsilon \text{ sign } b & \text{if } \alpha > p/(p-1-\beta), \\ \left(((C+\epsilon)/C_*)^{p-1-\beta} + \epsilon\right) \text{sign } b & \text{if } \alpha = p/(p-1-\beta), \end{cases}$$

and the constant C_* is defined in Region (2) of (I).

(4d) Let either $1 < \beta < p-1,\ p/(p-2) \le \alpha < p/(p-1-\beta)$, or $\beta \ge p-1,\ \alpha \ge p/(p-2)$.

If $\alpha = p/(p-2)$ then for arbitrary $\epsilon > 0$ there exists $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^{p/(p-2)}(1-\gamma_{-\epsilon}t)^{1/(2-p)} \leq u \leq (C+\epsilon)(-x)_+^{p/(p-2)}(1-\gamma_\epsilon t)^{1/(2-p)}, \qquad (2.27)$$

where

$$\gamma_\epsilon = [2(p-1)p^{p-1}(C+\epsilon)^{p-2}/(p-2)^{1-p}] + \epsilon.$$

However, if $\alpha > p/(p-2)$, then for arbitrary $\epsilon > 0$ there exists $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^{\alpha} \leq u \leq (C+\epsilon)(-x)_+^{\alpha}(1-\epsilon t)^{1/2-p)}, \qquad x \geq x_\epsilon,\ 0 \leq t \leq \delta_\epsilon. \qquad (2.28)$$

(II) $b = 0$. We divide this case into three subcases.

(1) Let $p > 2,\ 0 < \alpha < p/(p-2)$. In this case, the interface expands. First, assume that u_0 is defined by (1.13). Then, if $\alpha = (p-1)/(p-2)$, the explicit solution to the problem (1.10), (1.13) is

$$u(x,t) = C(\xi_* t - x)_+^{(p-1)/(p-2)}, \qquad \xi_* = C^{p-2}\Big(\frac{p-1}{p-2}\Big)^{p-1}. \qquad (2.29)$$

If $0 < \alpha < p/(p-2)$, then the solution to (1.10), (1.13) has the self-similar form (2.4)

$$\eta(t) = \xi_* t^{\frac{1}{p-\alpha(p-2)}}, \qquad 0 \leq t < +\infty, \qquad (2.30)$$

where ξ_* and f satisfy (2.2), (2.5)-(2.8). Moreover, we have

$$C_4 t^{\frac{\alpha}{p-\alpha(p-2)}}(\xi_3 - \xi)_+^{\frac{p-1}{p-2}} \leq u \leq C_5 t^{\frac{\alpha}{p-\alpha(p-2)}}(\xi_4 - \xi)_+^{\frac{p-1}{p-2}}, \qquad (2.31)$$

$$0 \le x < +\infty, \quad 0 < t < +\infty,$$

where ξ_3 (ξ_4, respectively) is defined by the right-hand side of (2.7), where we replace ξ_*'' with $C^{\frac{p-2}{p-\alpha(p-2)}}\xi_1$ (with $C^{\frac{p-2}{p-\alpha(p-2)}}\xi_2$, respectively) and

$$C_4 = C^{p/(p-\alpha(p-2))}A_0\xi_3^{(p-1)/(2-p)}, \qquad C_5 = C^{p/(p-\alpha(p-2))}A_0\xi_4^{-(p-1)/(p-2)}.$$

In the particular case $\alpha = (p-1)(p-2)^{-1}$, when an explicit solution is given by (2.29), we have $\xi_3 = \xi_4 = \xi_*$ and both lower and upper estimations in (2.31) lead to the explicit solution (2.29). In general, when $\alpha \neq (p-1)(p-2)^{-1}$ the precise value ξ_* relates to the similarity ODE for $f(\xi)$ from (2.5), namely, $\xi_* = \sup\{\xi : f(\xi) > 0\}$. If u_0 satsfies (1.12) with $(0 < \alpha < p/(p-2))$, then (2.1) and (2.3) are valid. Lower and upper bounds for $f(\rho)$ follow from (2.31).

(2) Let $p > 2, \alpha = p/(p-2)$. In this case, the interface initially has a waiting time. If u_0 is defined by (1.13), then the explicit solution to (1.10), (1.13) is

$$u_C(x,t) = C(-x)_+^{\alpha}[1-(C/\bar{C})^{p-2}(p-2)t]^{1/(2-p)} \quad x \in \mathbb{R}, \; 0 \le t < T, \tag{2.32}$$

where

$$T = (\bar{C}/C)^{p-2}(p-2)^{-1}$$

and the constant $\bar{C}$ is defined in Region (4) of (I).

If u_0 satisfies (1.12) with $\alpha = p/(p-2)$, then lower and upper estimations are given by $u_{C\pm\epsilon}$.

(3) Let $p > 2, \alpha > p/(p-2)$. In this case also the interface initially remains stationary and for arbitrary $\epsilon > 0$ there exists $x_\epsilon < 0$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(-x)_+^\alpha \le u \le (C+\epsilon)(-x)_+^\alpha(1-\epsilon t)^{1/2-p)}, \qquad x_\epsilon \le x,\ \ 0 \le t \le \delta_\epsilon. \tag{2.33}$$

Remark 1.1. We are not interested in the special case $p = 2$ of semi-linear heat equation. This case was completed in [74, 75] (see also [22]). However, we will mention when our results extend to the limit case $p = 2$. In general, the case $p = 2$ is in some sense a singular limit. For example, if $b > 0, 0 < \beta < 1, p-1 > \beta, \alpha < \frac{p}{p-1-\beta}$, then the interface initially expands and if $p > 2$, then we prove in this chapter that

$$\eta(t) \sim C_1 t^{1/(p-\alpha(p-2))} \ \text{ as } t \to 0+,$$

while if $1 < p < 2$, we prove in Chapter 3 that

$$\eta(t) \sim C_2 t^{(p-1-\beta)/p(1-\beta)} \ \text{ as } t \to 0+\,.$$

Formally, as $p \to 2$ both estimates yield a false result, and from [75] it follows that if $p = 2$, then

$$\eta(t) \sim C_3(t\ \log 1/t)^{\frac{1}{2}}$$

($C_i, i = \overline{1,3}$ are positive constants).

2.2 Preliminary results

The mathematical theory of non-linear p-Laplacian type degenerate parabolic equations is well developed. We shall follow the definition of weak solutions and of supersolutions (or subsolutions) of the equation (1.10) in the following sense:

Definition 2.2.1. A measurable function $u \geq 0$ is a local weak solution (respectively sub- or supersolution) of (1.10) in $\mathbb{R} \times (0,T]$ if

- $u \in C_{loc}(0,T; L^2_{loc}(\mathbb{R}) \cap L^p_{loc}(0,T; W^{1,p}_{loc}(\mathbb{R}) \cap L^{1+\beta}_{loc}(\mathbb{R}));$

- For $\forall$ subinterval $[t_0,t_1] \subset (0,T]$ and for $\forall \mu_i \in C^1[t_0,t_1],\ i = 1,2$ such that $\mu_1(t) < \mu_2(t)$ for $t \in [t_0,t_1]$

$$\int_{\mu_1(t)}^{\mu_2(t)} u\phi dx \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\mu_1(t)}^{\mu_2(t)} (-u\phi_t + |u_x|^{p-2} u_x \phi_x + bu^\beta \phi) dxdt = 0\ (resp. \leq or \geq\ 0), \tag{2.34}$$

  where $\phi \in C^{2,1}_{x,t}(\overline{D})$ is an arbitrary function that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1, i = 1,2$, and

$$D = \{(x,t) : \mu_1(t) < x < \mu_2(t), t_0 < t < t_1\}.$$

The questions of existence and uniqueness of initial boundary value problems for (1.10), comparison theorems, and regularity of weak solutions are known due to [58, 57, 59, 53, 60, 82, 83, 99] etc. Qualitative properties of free boundaries for the quasilinear degenerate parabolic equations were studied via energy methods in [38]. The proof of the following typical comparison result is standard.

Lemma 2.2.2. *Let g be a non-negative and continuous function in $\overline{Q}$, where*

$$Q = \{(x,t) : \eta_0(t) < x < +\infty, 0 < t < T \leq +\infty\},$$

f is in $C^{2,1}_{x,t}$ in Q outside a finite number of curves $x = \eta_j(t)$, which divide Q into a finite number of subdomains Q^j, where $\eta_j \in C[0,T]$; for arbitrary $\delta > 0$ and finite $\Delta \in (\delta,T]$

the function η_j is absolutely continuous in $[\delta, \Delta]$. Let g satisfy the inequality

$$Lg \equiv g_t - \left(|g_x|^{p-2} g_x\right)_x + b g^\beta \geq 0, \ (\leq 0)$$

at the points of Q, where $g \in C^{2,1}_{x,t}$. Assume also that the function $|g_x|^{p-2} g_x$ is continuous in Q and $g \in L^\infty(Q \cap (t \leq T_1))$ for any finite $T_1 \in (0, T]$. Then, g is a supersolution (subsolution) of (1.10)*. If, in addition we have*

$$g\Big|_{x=\eta_0(t)} \geq (\leq)\ u\Big|_{x=\eta_0(t)},\ g\Big|_{t=0} \geq (\leq)\ u\Big|_{t=0},$$

then

$$g \geq (\leq)\ u, \quad in\ \ \overline{Q}.$$

Suppose that $b \geq 0$ and that u_0 may have unbounded growth as $|x| \to +\infty$. It is well known that in this case some restriction must be imposed on the growth rate for existence, uniqueness and comparison results in the CP (1.10), (1.11). Optimal growth condition for the equation (1.10) with $b = 0, p > 2$ was derived in [57, 59]. If initial data may be majorised by power law function (1.13), then there exists a unique solution (with $T = +\infty$) and a comparison principle is valid if $0 < \alpha < p/(p-2)$. If $\alpha = p/(p-2)$, then existence, uniqueness, and comparison results are valid only locally in time. In particular, from [57, 59] it follows that the unique explicit solution to (1.10), (1.13) with $b = 0, \alpha = p/(p-2), T = (\bar{C}/C)^{p-2}(p-2)^{-1}$ is $u_C(x,t)$ from (2.32).

If the function $u(x,t)$ is a solution to CP (1.10), (1.13) with $b = 0$, then the function

$$\bar{u}(x,t) = \exp(-bt) u(x, (b(2-p))^{-1}(\exp(b(2-p)t) - 1))$$

is a solution to (1.10) with $b \neq 0, \beta = 1$. Hence, from the above mentioned result it follows that the unique solution to CP (1.10), (1.13) with $p > 2, b \neq 0, \beta = 1, \alpha = p/(p-2)$ is the function $\bar{u}_C(x,t)$ from (2.24).

It is proved in [53] that existence, uniqueness and comparison theorems are valid for the CP (1.10),(1.11) with $b = 0$, $1 < p < 2$ without any growth condition on the initial function u_0 at infinity. In particular, $\alpha > 0$ is arbitrary in (1.13).

We are not interested in necessary and sufficient conditions on the growth rate at infinity for existence, uniqueness, and comparison results for the CP (1.10), (1.11) with $b > 0, p > 2, \beta > 0$; for our purposes, it is enough to mention that if u_0 may be majorised by the function (1.13) with α satisfying $0 < \alpha < p/(p-2)$, then the CP (1.10), (1.11) with $b > 0, p > 2, \beta > 0, T = +\infty$ has a unique solution and for this class of initial data a comparison principle is valid. This easily follows from the fact that the solution of the CP (1.10), (1.11) with $b = 0$ is a supersolution of the CP with $b > 0$, and hence it becomes a global locally bounded uniform upper bound for the increasing sequence of approximating bounded solutions of the CP with $b > 0$. Similarly, if $b > 0, 1 < p < 2$ due to above mentioned result of [53], existence, uniqueness and comparison theorems are valid for the CP (1.10),(1.11), and for the respective boundary value problems without any growth condition on the continuous initial function at infinity. In particular, $\alpha > 0$ could be arbitrary in (1.13).

2.2.1 Traveling wave solutions

Traveling wave solution for equation(1.10) is investigated in [91]. We consider the parabolic p-Laplacian equation with strong absorption (1.10) with $p > 2$, $0 < \beta < 1$, $b > 0$, $x \in \mathbb{R}$, $t \geq 0$ and equation (1.11), where $u_0(x)$ is a continuous and non-negative function with compact support. We seek solutions of the form:

$$u(x,t) = \phi((kt - x)),$$

such that $0 \neq k \in \mathbb{R}$, $\phi(\eta) \geq 0$, $\phi \not\equiv 0$, and $\phi \to 0$ as $\eta \to -\infty$. In the case $\phi(\eta) = 0$ for $\eta \leq \eta_0 \in \mathbb{R}$. Without loss of generality we can assume that $\eta_0 = 0$. Plugging $u(x,t) = \phi((kt - x))$ into (1.10), we have

$$\begin{cases} (|\phi'|^{p-2}\phi')' - k\phi' - b\phi^\beta = 0, \quad \phi = \phi(\eta), \ \eta > 0, \\ \phi(0) = \phi'(0) = 0. \end{cases} \tag{2.35}$$

If we can find a positive solution, ϕ, to the problem above in $\mathbb{R}^+$, then (1.10) admits a finite traveling-wave solution. Note that the solution to the problem (2.35) is understood in the weak sense. In [91] it is proven that there exists a unique positive and monotonically increasing solution $\phi(\eta) : \mathbb{R}^+ \to \mathbb{R}^+$ of the problem (2.35) with $k \neq 0, p > 2, b > 0, 0 < \beta < 1$. By introducing the variables

$$X = \phi, \ \ Y = (\phi')^{p-1}$$

the problem can be reformulated as finding the nontrivial trajectories of the dynamical system

$$X' = \phi' = Y^{1/(p-1)}, \ \ Y' = ((\phi')^{p-1})' = (|\phi'|^{p-2}\phi')' = k\phi' + b\phi^\beta = kY^{1/(p-1)} + bX^\beta$$

which start from $(0,0)$ at $\eta = 0$, and stay in the first quadrant $\Omega_1 = \{(X,Y) : X > 0, \ Y > 0\}$ for $0 < \eta < +\infty$. We write the system as an O.D.E problem

$$\begin{cases} \frac{dY}{dX} = f(X,Y) = k + bX^\beta Y^{-\frac{1}{p-1}}, \\ Y(0) = 0. \end{cases} \tag{2.36}$$

The problem (2.36) has a unique global solution [91]. Having global solution Y, consider the problem

$$\begin{cases} \frac{d\phi}{d\eta} = Y^{1/(p-1)}(\phi(\eta)), \\ \phi(0) = 0. \end{cases} \tag{2.37}$$

There exists a unique maximal solution defined in $(-\infty,\beta')$ such that

$$\lim_{\eta\to\beta'-} \phi(\eta) = +\infty$$

if β' is finite. In fact, it easily follows from (2.37) that the same is true if $\beta' = +\infty$ [91]. It follows that the solution of (2.37) defined in $(-\infty,\beta')$ and satisfies

$$\begin{cases} (|\phi'|^{p-2}\phi')' - k\phi' - \lambda\phi^q = 0 \quad \text{in } (-\infty,\beta'), \\ \phi(0) = 0, \quad \phi'(0) = 0. \end{cases} \tag{2.38}$$

It remains to prove that $\beta' = +\infty$. In [91] it is demonstrated that this follows from the following lemma on the asymptotic properties of the solution to the problem (2.36).

Lemma 2.2.3. *[91] Let Y be a solution of* (2.36). *Then we have*

(i) $Y(X) \sim [\frac{bp}{(p-1)(\beta+1)}]^{(p-1)/p} X^{\frac{(p-1)(\beta+1)}{p}}$ *as* $X \to +\infty$ *if* $\beta(p-1) > 1$;

(ii) $Y(X) \sim [\frac{p}{(p-1)(\beta+1)}]^{(p-1)/p} X^{\frac{(p-1)(\beta+1)}{p}}$ *as* $X \to 0$ *if* $\beta(p-1) < 1$;

(iii) $Y(X) \sim kX$ *as* $X \to +\infty$ *if* $k > 0$, $\beta(p-1) < 1$;

(iv) $Y(X) \sim kX$ *as* $X \to 0$ *if* $k > 0$, $(p-1)\beta > 1$;

(v) $Y(X) \sim (-\frac{k}{b})^{1-p} X^{\beta(p-1)}$ *as* $X \to +\infty$ *if* $k < 0$, $\beta(p-1) < 1$;

(vi) $Y(X) \sim (-\frac{k}{b})^{1-p} X^{\beta(p-1)}$ *as* $X \to 0$ *if* $k < 0$, $\beta(p-1) > 1$.

In particular, Lemma 2.2.3 is equivalent to the following key lemma on the asymptotic properties of the traveling wave solutions of the PDE (1.10).

Lemma 2.2.4. *[91] The equation* (1.10) *admits a finite traveling-wave solution* $u(x,t) = \phi(kt-x)$ *with* $\phi(0) = 0$ *if* $k \neq 0$. *Moreover,*

$(i)\lim_{\eta\to+\infty}\eta^{-\frac{p}{p-1-\beta}}\phi(\eta) = [\frac{b(p-1-\beta)^p}{p^{p-1}(p-1)(\beta+1)}]^{1/(p-1-\beta)}$ *if* $\beta(p-1) > 1$;
$(ii)\lim_{\eta\to 0}\eta^{-\frac{p}{p-1-\beta}}\phi(\eta) = [\frac{b(p-1-\beta)^p}{p^{p-1}(p-1)(\beta+1)}]^{1/(p-1-\beta)}$ *if* $\beta(p-1) < 1$;
$(iii)\lim_{\eta\to+\infty}\eta^{-\frac{p-1}{p-2}}\phi(\eta) = (\frac{p-2}{p-1})^{\frac{p-1}{p-2}}k^{1/(p-2)}$ *if* $k > 0$, $\beta(p-1) < 1$;
$(iv)\lim_{\eta\to 0}\eta^{-\frac{p-1}{p-2}}\phi(\eta) = (\frac{p-2}{p-1})^{\frac{p-1}{p-2}}k^{1/(p-2)}$ *if* $k > 0$, $\beta(p-1) > 1$;
$(v)\lim_{\eta\to+\infty}\eta^{-\frac{1}{1-\beta}}\phi(\eta) = [(1-\beta)(-\frac{b}{k})]^{\frac{1}{1-\beta}}$ *if* $k < 0$, $\beta(p-1) < 1$;
$(vi)\lim_{\eta\to 0}\eta^{-\frac{1}{1-\beta}}\phi(\eta) = [(1-\beta)(-\frac{b}{k})]^{\frac{1}{1-\beta}}$ *if* $k < 0$, $\beta(p-1) > 1$.

Both Lemma 2.2.3 and Lemma 2.2.4 were proved in [91]. Below we sketch simple proof of the Lemma 2.2.3 based on the scaling method.

***Proof of Lemma 2.2.3*.**

- proof of (i): Consider equation (2.36). Rescaled solution

$$Y_m(X) = mY(m^{\gamma}X), \quad Y(X) = m^{-1}Y_m(m^{-\gamma}X) \tag{2.39}$$

with $\gamma = \frac{-p}{(p-1)(\beta+1)}$ satisfies

$$\begin{cases} \frac{dY_m}{dX} = m^{\frac{\beta(p-1)-1}{(p-1)(\beta+1)}}k + bX^{\beta}Y_m^{-\frac{1}{p-1}}, \ 0 < X < +\infty, \\ Y_m(0) = 0. \end{cases} \tag{2.40}$$

One can easily prove that the sequence $\{Y_m\}$ is uniformly bounded, and monotonically decreasing if $k > 0$, and monotonically increasing if $k < 0$. Therefore,

$$\lim_{m\to 0} Y_m(X) = Y(X) \tag{2.41}$$

exists. Since $\beta(p-1) > 1$, passing to the limit as $m \to 0$ from (2.40) it easily follows that Y satisfies the ODE problem

$$\frac{dY}{dX} = bX^{\beta}Y^{-\frac{1}{p-1}}, 0 < X < +\infty;\ Y(0) = 0,$$

that is to say

$$Y(X) = \Big[\frac{bp}{(\beta+1)(p-1)}\Big]^{\frac{p-1}{p}} X^{\frac{(p-1)(\beta+1)}{p}}.$$

Changing variable in (2.41) as $Z = m^{\gamma}X$, we easily deduce claim (i) in Lemma 2.2.3.

- proof of (ii): Since $\beta(p-1) < 1$, the exponent of m in (2.40) is negative. In a similar way one can prove that the limit

$$\lim_{m\to+\infty} Y_m(X) = \Big[\frac{bp}{(\beta+1)(p-1)}\Big]^{\frac{p-1}{p}} X^{\frac{(p-1)(\beta+1)}{p}} \tag{2.42}$$

exists. By changing variable in (2.42), claim (ii) follows.

- proof of (iii): Rescaled solution of (2.36)

$$Y_m = mY(m^{-1}X)$$

satisfies

$$\begin{cases} \frac{dY_m}{dX} = k + m^{\frac{1-\beta(p-1)}{p-1}} bX^{\beta}Y_m^{-\frac{1}{p-1}},\ \ 0 < X < +\infty, \\ Y_m(0) = 0. \end{cases} \tag{2.43}$$

Standard comparison lemma implies that

$$Y_m(X) \geq kX,\ 0 < X < +\infty. \tag{2.44}$$

Similarly, as in the case (i), it is proved that the sequence $\{Y_m\}$ is uniformly bounded and equicontinuous on compact subsets of $(0,+\infty)$ as $m \to 0$. Arzela-Ascoli theorem implies the existence of the convergent subsequence of $\{Y_m\}$ on every compact subset of $\mathbb{R}^+$. By selecting expanding sequence of compact subsets and Cantor's diagonalization one can deduce the existence of the limit

$$\lim_{m' \to 0} Y_{m'}(X) = kX,\ 0 < X < +\infty,$$

for some subsequence m', and convergence being uniform on compact subsets. By changing the variable under the limit sign, the claim (iii) follows. The proof of (iv) is similar as the proof of (iii).

- proof of (v) and (vi) can be pursued similarly by rescaling solution of (2.36) as

$$Y_m = mY(m^{-\frac{1}{\beta(p-1)}}X)$$

which solves the problem

$$\begin{cases} m^{\frac{1-\beta(p-1)}{\beta(p-1)}} \frac{dY_m}{dX} = k + bX^{\beta} Y_m^{-\frac{1}{p-1}},\ 0 < X < +\infty, \\ Y_m(0) = 0. \end{cases} \tag{2.45}$$

Similar analysis as in the case (iii) implies the limit relation

$$\lim_{m \to 0} Y_m = \Big(\frac{-k}{b}\Big)^{1-p} X^{\beta(p-1)}, 0 < X < +\infty$$

By changing variable under the limit sign, claim (v) follows. The proof of (vi) is similar.

□

2.3 Asymptotic Properties of solutions based on scaling laws

In the next four lemmas, we apply rescaling to establish some preliminary estimations of the solution to CP.

Lemma 2.3.1. *If* $b = 0$ *and* $p > 2, 0 < \alpha < p/(p-2)$, *then the solution* u *of the CP* (1.10), (1.13) *has a self-similar form* (2.4)*, where the self-similarity function* f *satisfies* (2.6). *If* u_0 *satisfies* (1.12)*, then the solution to CP* (1.10), (1.11) *satisfies* (2.1)-(2.3)

Lemma 2.3.2. *Let* u *be a solution to the CP*(1.10)*,* (1.11) *and* u_0 *satisfy* (1.12)*. Let one of the following conditions be valid:*

(a) $b > 0, \quad 0 < \beta < 1 < p, \quad 0 < \alpha < p/(p-1-\beta)$;

(b) $b \neq 0, \quad \beta \geq 1, \quad p > 2, \quad 0 < \alpha < p/(p-2)$.

Then, u *satisfies* (2.3).

Lemma 2.3.3. *Let* u *be a solution to the CP*(1.10)*,* (1.13) *with* $b > 0, \ 0 < \beta < 1, \ p > 2, \ \alpha = p/(p-1-\beta)$. *Then, the solution* u *has the self-similar form* (2.14)*. If* $C > C_*$*, then* $f_1(0) = A_1$*, where* A_1 *is a positive number depending on* $p, \beta, C,$ *and* b*. If* u_0 *satisfies* (1.12) *with* $\alpha = p/(p-1-\beta), C > C_*$*, then* u *satisfies*

$$u(0,t) \sim A_1 t^{1/(1-\beta)} \text{ as } t \to 0+. \tag{2.46}$$

Lemma 2.3.4. *Let* u *be a solution to the CP* (1.10)-(1.12) *with* $b > 0, 0 < \beta < 1, \alpha > p/(p-1-\beta)$*. Then, for arbitrary* $\ell > \ell_*$ *(see* (2.20)*) the asymptotic formula* (2.21) *is valid with* $x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}$.

2.3.1 Proof of Lemma 2.3.1 & Lemma 2.3.2: Diffusion dominates over the reaction

Proof of Lemma 2.3.1. If we consider a function

$$u_k(x,t) = ku(k^{-\frac{1}{\alpha}}x, k^{\frac{\alpha(p-2)-p}{\alpha}}t) \qquad k > 0,$$

it may easily be checked that this satisfies (1.10), (1.13). From [57, 59], it follows that under the condition of the lemma there exists a unique global solution to (1.10), (1.13). Therefore, we have

$$u(x,t) = ku(k^{-1/\alpha}x, k^{(\alpha(p-2)-p)/\alpha}), \qquad k > 0. \tag{2.47}$$

If we choose $k = t^{\alpha/(p-\alpha(p-2))}$, then (2.47) implies (2.4) for u with $f(\xi) = u(\xi, 1)$. In fact, f is a unique non-negative and differentiable weak solution of the boundary value problem (2.5) and there exists an $\xi_* > 0$ such that f satisfies (2.5): it is positive and smooth for $\xi < \xi_*$ and $f = 0$ for $\xi \geq \xi_*$([42]). Thus, (2.30) is valid. To find the dependence of f on C, we can again use scaling. Let

$$v(x,t) = C^{-1}u(x,t).$$

Plugging it into (1.10) with $b = 0$ such that

$$v_t(x,t) = C^{-1}u_t(x,t)$$

$$\Big(|v_x(x,t)|^{p-2}v_x(x,t)\Big)_x = C^{1-p}\Big(|u_x(x,t)|^{p-2}u_x(x,t)\Big)_x.$$

We have

$$\begin{cases} C^{2-p}v_t(x,t) = \Big(|v_x(x,t)|^{p-2}v_x(x,t)\Big)_x; \quad x \in \mathbb{R}, t > 0 \\ v(x,0) = C^{-1}u(x,0) = C^{-1}C(-x)_+^{\alpha} = (-x)_+^{\alpha}; \quad x \in \mathbb{R}. \end{cases}$$

Choose time variable $\tau = C^{p-2}t$, we have

$$w(x,\tau) = v(x, c^{2-p}\tau)$$

which solves CP (1.10), (1.13) with $C = 1$. Then, it may be easily checked that for arbitrary $k > 0$

$$u(x,t) = kw(C^{1/\alpha}k^{-1/\alpha}x, C^{p/\alpha}k^{(\alpha(p-2)-p)/\alpha}t).$$

By choosing $k = (C^{p/\alpha}t)^{\alpha/(p-\alpha(p-2))}$, we then have

$$u(x,t) = C^{\frac{p}{p-\alpha(p-2)}}w(C^{\frac{p-2}{\alpha(p-2)-p}}\xi, 1)t^{\alpha/(p-\alpha(p-2))}. \tag{2.48}$$

Formulae (2.6) and (2.2) follow from (2.48) and (2.4).

Now assume that u_0 satisfies (1.12). Then, for arbitrary sufficiently small $\epsilon > 0$ there exists $x_\epsilon < 0$, such that

$$(C-\epsilon/2)(-x)_+^{\alpha} \leq u_0(x) \leq (C+\epsilon/2)(-x)_+^{\alpha}, \;\; x \geq x_\epsilon. \tag{2.49}$$

Let $u_\epsilon(x,t)$ ($u_{-\epsilon}(x,t)$, respectively) be a solution to the CP (1.10), (1.11) with initial data $(C+\epsilon)(-x)_+^{\alpha}$ ($(C-\epsilon)(-x)_+^{\alpha}$, respectively). Since the solution to the CP (1.10), (1.11) is continuous, there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$u_\epsilon(x_\epsilon,t) \geq u(x_\epsilon,t), \;\; u_{-\epsilon}(x_\epsilon,t) \leq u(x_\epsilon,t) \text{ for } 0 \leq t \leq \delta. \tag{2.50}$$

From (2.49), (2.50), and a comparison principle, it follows that

$$u_{-\epsilon} \le u \le u_\epsilon \ \text{ for } \ x \ge x_\epsilon, \quad 0 \le t \le \delta. \tag{2.51}$$

Obviously

$$u_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho; C \pm \epsilon) t^{\alpha/(p-\alpha(p-2))}, \quad t \ge 0. \tag{2.52}$$

(Furthermore, we denote the right-hand side of(2.6a) by $f(\rho, C)$.) Now taking $x = \xi_\rho(t)$ in (2.51), after multiplying to $t^{-\alpha/(p-\alpha(p-2))}$ and passing to the limit, first as $t \to 0$ and then as $\epsilon \to 0$, we can easily derive (2.3). Similarly, from (2.51), (2.30), and (2.2), (2.1) easily follows. □

Proof of Lemma 2.3.2. As in the previous proof, (2.49)-(2.51) follow from (1.12). Let the conditions of one of the cases (a) or (b) with $b > 0$ be valid. Then, from the results mentioned earlier it follows that the existence, uniqueness, and comparison results of the CP (1.10), (1.11) with $u_0 = (C \pm \epsilon)(-x)_+^\alpha$, $T = +\infty$ hold. Now, if we rescale

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{-1/\alpha} x, k^{(\alpha(p-2)-p)/\alpha} t), \qquad k > 0, \tag{2.53}$$

then $u_k^{\pm\epsilon}(x,t)$ satisfies the following problem:

$$u_t - (|u_x|^{p-2} u_x)_x + b k^{(\alpha(p-1-\beta)-p)/\alpha} u^\beta = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.54a}$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad x \in \mathbb{R}. \tag{2.54b}$$

There exists a unique solution to CP (2.54), which also obeys a comparison principle. Since $\alpha(p-1-\beta) - p < 0$, by using a comparison principle in Lemma 2.3.1 it follows that

$$u_{k_1}^{\pm\epsilon}(x,t) \le u_{k_2}^{\pm\epsilon}(x,t) \le \cdots \le v_{\pm}(x,t), \quad x \in \mathbb{R}, \quad t \ge 0; \quad \text{if } k_1 < k_2, \tag{2.55}$$

where $v_{\pm\epsilon}$ is a solution to CP (1.10), (1.11) with $b = 0$, $u_0 = (C \pm \epsilon)(-x)_+^{\alpha}$, $T = +\infty$. From the results of [57, 99], it follows that the sequence of non-negative and locally bounded solutions $\{u_k^{\pm\epsilon}\}$ is locally uniformly Hölder continuous, and weakly pre-compact in $W_{loc}^{1,p}(\mathbb{R} \times (0,T))$. Since $\alpha(p-1-\beta) - p < 0$, passing to limit as $k \to +\infty$, from (2.34) it follows that the limit function is a solution of the CP (1.10), (1.11) with $b = 0$, $u_0 = (C \pm \epsilon)(-x)_+^{\alpha}$, $T = +\infty$. Due to uniqueness we have

$$\lim_{k \to +\infty} u_k^{\pm\epsilon}(x,t) = v_{\pm}(x,t), \quad x \in \mathbb{R}, \quad t \ge 0. \tag{2.56}$$

Hence, $v_{\pm\epsilon}$ satisfies (2.52). If we now take $x = \xi_\rho(t)$, where ρ is an arbitrary fixed number satisfying $\rho < \xi_*$, then from (2.56) it follows that

$$\lim_{k \to +\infty} k u_{\pm\epsilon}(k^{-1/\alpha}\xi_\rho(t), k^{(\alpha(p-2)-p)/\alpha} t) = f(\rho; C \pm \epsilon) t^{\alpha(p-\alpha(p-2))}, \quad t > 0. \tag{2.57}$$

If we take $\tau = k^{(\alpha(p-2)-p)/\alpha} t$, then (2.57) implies

$$u_{\pm\epsilon}(\xi_\rho(\tau), \tau) \sim f(\rho; C \pm \epsilon) \tau^{\alpha(p-\alpha(p-2))}, \text{ as } \tau \to 0+. \tag{2.58}$$

As before, (2.3) follows from (2.51), (2.58).

Now consider the case (b) with $b < 0$. Suppose that $u_{\pm\epsilon}$ is a solution of the Dirichlet problem

$$u_t - (|u_x|^{p-2} u_x)_x + b u^\beta = 0, \quad |x| < |x_\epsilon|, \ 0 < t < \delta, \tag{2.59a}$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \quad |x| \le |x_\epsilon|, \tag{2.59b}$$

$$u(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)^\alpha, \quad u(-x_\epsilon, t) = 0, \;\; 0 \le t \le \delta. \tag{2.59c}$$

The function $u_k^{\pm\epsilon}$ is defined as in (2.53), satisfies the Dirichlet problem:

$$u_t - (|u_x|^{p-2}u_x)_x + bk^{(\alpha(p-1-\beta)-p)/\alpha}u^\beta = 0 \text{ in } D_\epsilon^k, \tag{2.60a}$$

$$u(k^{1/\alpha}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)^\alpha, \quad u(-k^{1/\alpha}x_\epsilon, t) = 0, \qquad 0 \le t \le k^{(p-\alpha(p-2))/\alpha}\delta \tag{2.60b}$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad |x| \le k^{1/\alpha}|x_\epsilon|, \tag{2.60c}$$

where

$$D_\epsilon^k = \{(x,t) : |x| < k^{1/\alpha}|x_\epsilon|, \quad 0 < t \le k^{(p-\alpha(p-2))/\alpha}\delta\}.$$

There exists a number $\delta > 0$ (which does not depend on k) such that both (2.59a)-(2.59c) and (2.60a)-(2.60c) have a unique solution (see discussion preceding Lemma 2.3.1). In view of finite speed of propagation, $\delta = \delta(\epsilon) > 0$ may be chosen such that

$$u(-x_\epsilon, t) = 0, \quad 0 \le t \le \delta. \tag{2.61}$$

Applying the comparison theorem, from (2.49), (2.50) and (2.61),(2.51) follows for $|x| \le |x_\epsilon|$, $0 \le t \le \delta$.

To prove the convergence of the sequences $\{u_k^{\pm\epsilon}\}$ as $k \to +\infty$, we need to prove uniform boundedness. Consider a function

$$g(x,t) = (C+1)(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{1}{2-p}},\ x \in \mathbb{R},\ 0 \le t \le t_0 = \frac{\nu^{-1}}{2},$$

where

$$\nu = h_* + 1,\ h_* = h_*(\alpha;p) = \max_{x\in\mathbb{R}} h(x), \tag{2.62}$$

$$h(x) = (p-2)\alpha^{p-1}(C+1)^{p-2}(1+x^2)^{\frac{(\alpha-2)(p-1)-2-\alpha}{2}} x^2|x|^{p-2}$$
$$\times\Big(\frac{1+x^2}{x^2} + (p-2)\frac{1+x^2}{|x|^2} + (\alpha-2)(p-1)\Big).$$

Then, we have

$$L_k g \equiv g_t - (|g_x|^{p-2}g_x)_x + bk^{\frac{\alpha(p-\beta-1)-p}{\alpha}} g^\beta = (C+1)(p-2)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{p-1}{2-p}} S \ \text{ in } D_\epsilon^k,$$

$$S = \nu - h(x) + b(p-2)(C+1)^{\beta-1} k^{\frac{\alpha(p-\beta-1)-p}{\alpha}}(1+x^2)^{\frac{\alpha(\beta-1)}{2}}(1-\nu t)^{\frac{\beta+1-p}{2-p}},$$

and hence

$$S \ge 1 + R \text{ in } D_{0\epsilon}^k = D_\epsilon^k \cap \{0 < t \le t_0\}, \tag{2.63}$$

where

$$R = O\Big(k^{p-2-p/\alpha}\Big) \ \text{ uniformly for } (x,t) \in D_{0\epsilon}^k \text{ as } k \to +\infty.$$

Moreover, we have for $0 < \epsilon \ll 1$

$$g(x,0) \qquad \ge u_k^{\pm\epsilon}(x,0) \text{ for } |x| \le k^{1/\alpha}|x_\epsilon|, \tag{2.64a}$$

$$g(\pm k^{1/\alpha}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{1/\alpha}x_\epsilon, t) \text{ for } 0 \leq t \leq t_0. \tag{2.64b}$$

Hence, $\exists\, k_0 = k_0(\alpha; p)$ such that for $\forall k \geq k_0$ the comparison theorem implies

$$0 \leq u_k^{\pm\epsilon}(x,t) \leq g(x,t) \text{ in } \bar{D}_{0\epsilon}^k. \tag{2.65}$$

Let G be an arbitrary fixed compact subset of

$$P = \{(x,t) : x \in \mathbb{R}, \quad 0 < t \leq t_0\}.$$

We take k_0 so large that $G \subset D_{0\epsilon}^k$ for $k \geq k_0$. From (2.65), it follows that the sequences $\{u_k^{\pm\epsilon}\}$, $k \geq k_0$, are uniformly bounded in G. As before, from the results of [57, 99] it follows that the sequence of non-negative and locally bounded solutions $\{u_k^{\pm\epsilon}\}$ is locally uniformly Hölder continuous, and weakly pre-compact in $W_{loc}^{1,p}(\mathbb{R} \times (0,T))$. It follows that for some subsequence k'

$$\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \qquad (x,t) \in P. \tag{2.66}$$

Since $\alpha(p-1-\beta) - p < 0$, passing to limit as $k' \to +\infty$, from (2.34) for $u_{k'}^{\pm\epsilon}$ it follows that $v_{\pm\epsilon}$ is a solution to the CP (1.10), (1.11) with $b = 0, T = t_0, u_0 = (C \pm \epsilon)(-x)_+^\alpha$. As before, from (2.52), (2.57), (2.58) and (2.51), the required estimation (2.3) follows. □

2.3.2 Proof of Lemma 2.3.3 : Diffusion & Reaction are in balance

Proof of Lemma 2.3.3. If we consider a function

$$u_k(x,t) = ku(k^{-\frac{p-1-\beta}{p}}x, k^{\beta-1}t), \qquad k > 0, \tag{2.67}$$

it may easily be checked that this satisfies (1.10), (1.13). From [57, 59], it follows that under the condition of the lemma there exists a unique global solution to (1.10), (1.13). In [57, 59] growth rate is necessarily and sufficient condition for existing and unique solution to diffusion equation (1.10) with $b = 0$ and sufficient condition for reaction-diffusion equation (1.10) since $b > 0$. Also from [91] we have

$$u(x,t) = ku(k^{-\frac{p-1-\beta}{p}}x, k^{\beta-1}t), \qquad k > 0, \tag{2.68}$$

If we choose $k = t^{1/(1-\beta)}$, then (2.68) implies (2.14) for u with $f_1(\zeta) = u(\zeta, 1)$. In fact, f is a unique non-negative and differentiable weak solution of the boundary value problem:

$$\begin{cases} (|f'(\zeta)|^{p-2}f'(\zeta))' + f^{\beta} + \frac{1}{\beta-1}f(\zeta) + \frac{p-1-\beta}{p(1-\beta)}\zeta f'(\zeta) = 0, \ \zeta \in \mathbb{R} \\ f(\zeta) \sim C(-\zeta)_+^{\frac{p}{p-1-\beta}} \text{ as } \zeta \downarrow -\infty, \ f(\zeta) \sim o(\zeta^{\frac{p}{p-1-\beta}}) \text{ as } \zeta \uparrow +\infty \end{cases} \tag{2.69}$$

and there exists a $\zeta_* > 0$ such that f is positive and smooth for $\zeta < \zeta_*$ and $f = 0$ for $\zeta \geq \zeta_*$([42]). Thus, (2.15) is valid. If $C > C*, \alpha = p/(p-1-\beta)$, then [91] and Lemma 2.2.3 implies that $f_1(0) = A_1 > 0$. Therefore we have $u(0,t) = A_1 t^{\frac{1}{1-\beta}}$. If u_0 satisfies (1.12), then it implies (2.49). Let $u_{+\epsilon}(0,t)$ ($u_{-\epsilon}(0,t)$, respectively) be a solution to the CP (1.10), (1.11) with initial data $(C+\epsilon)(-x)_+^{\alpha}$ ($(C-\epsilon)(-x)_+^{\alpha}$, respectively). From (2.49), there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$u_{-\epsilon}(0,t) \le u(0,t) \le u_{+\epsilon}(0,t) \quad \text{for } \ 0 \le t \le \delta. \tag{2.70}$$

Obviously

$$u_{\pm\epsilon}(0,t) = (A_1 \pm \epsilon)t^{1/(1-\beta)}, \quad t \ge 0. \tag{2.71}$$

After multiplying to $t^{-1/(1-\beta))}$ and passing to the limit, first as $t \to 0+$ and then as $\epsilon \to 0$, we can easily derive (2.46). □

2.3.3 Proof of Lemma 2.3.4 : Absorption dominates over the diffusion

Proof of Lemma 2.3.4. Asymptotic behavior (1.12) imply (2.49) and (2.50). Assume that that $v_{\pm\epsilon}$ solves the problem:

$$v_t - (|v_x|^{p-2}v_x)_x + bv^\beta = 0, \qquad |x| < |x_\epsilon|, \ 0 < t \le \delta,$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad |x| \le |x_\epsilon|,$$

$$v(x_\epsilon,t) = (C \pm \epsilon)(-x_\epsilon)_+^\alpha, \ v(-x_\epsilon,t) = u(-x_\epsilon,t), \quad 0 \le t \le \delta.$$

According to comparison result from (2.49) and (2.50), (2.51) follows for $|x| \le |x_\epsilon|, \ 0 \le t \le \delta$. If we rescale

$$u_k^{\pm\epsilon}(x,t) = ku_{\pm\epsilon}(k^{-\frac{1}{\alpha}}x, k^{\beta-1}t), \quad k > 0,$$

then $u_k^{\pm\epsilon}$ satisfies the Dirichlet problem

$$v_t - k^{\frac{p-\alpha(p-1-\beta)}{\alpha}}(|v_x|^{p-2}v_x)_x + bv^\beta = 0 \text{ in } E_\epsilon^k,$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \ \ |x| \le k^{\frac{1}{\alpha}}|x_\epsilon|,$$

$$v(k^{\frac{1}{\alpha}}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)_+^\alpha, \quad v(-k^{\frac{1}{\alpha}}x_\epsilon, t) = ku(-x_\epsilon, k^{\beta-1}t), \ 0 \le t \le k^{1-\beta}\delta,$$

where

$$E_\epsilon^k = \{|x| < k^{\frac{1}{\alpha}}|x_\epsilon|, \ 0 < t \le k^{1-\beta}\delta\}.$$

The goal is to in prove the convergence of the sequence $\{u_k^{\pm\epsilon}\}$ as $k \to +\infty$. To establish uniform bound consider $g(x,t) = (C+1)(1+x^2)^{\alpha/2} \exp t$. We have

$$\tilde{\mathrm{L}}_k g \equiv g_t - k^{\frac{p-\alpha(p-1-\beta)}{\alpha}}(|g_x|^{p-2}g_x)_x + bg^\beta \ge g\Big[1 - k^{\frac{p-\alpha(p--1-\beta)}{\alpha}}\alpha^{p-1}(C+1)^{p-2}e^{t(p-2)} \tag{2.72}$$

$$\times(1+x^2)^{\frac{(\alpha-2)(p-1)-2-\alpha}{2}}x^2|x|^{p-2}\Big(\frac{1+x^2}{x^2} + (p-2)\frac{1+x^2}{|x|^2} + (\alpha-2)(p-1)\Big)\Big] \text{ in } E_\epsilon^k.$$

Let $t_0 > 0$ be fixed and let $E_{0\epsilon}^k = E_\epsilon^k \cap \{(x,t): \ \ 0 < t \le t_0\}$. From (2.72), it follows that

$$\tilde{L}_k g \ge (1+R) \quad \text{in} \ \ E_0^k,$$

where

$$R = O(k^\theta) \text{ uniformly for } (x,t) \in E_{0\epsilon}^k \ \text{ as } \ k \to +\infty$$

$$\theta = (p - \alpha(p-1-\beta)/\alpha) \ \ \text{if} \ \ \alpha < p/(p-2),$$

$$\theta = \beta - 1, \qquad\qquad \text{if } \alpha \ge p/(p-2).$$

We have for $0 < \epsilon \ll 1$ that

$$g(x,0) = u_k^{\pm\epsilon}(x,0), \quad \text{for} \ \ |x| \le k^{1/\alpha}|x_\epsilon|,$$

and

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = o(k),\ 0 \le t \le t_0 \text{ as } k \to \infty,$$

$$g(\pm k^{\frac{1}{\alpha}}x_\epsilon, t) \ge u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}}x_\epsilon, t), \quad \text{for} \quad 0 \le t \le t_0,$$

if k is chosen large enough. Therefore, the comparison principle implies (2.65) in $\bar{E}^k_{0\epsilon}$, where the respective functions $u_k^{\pm\epsilon}$ and g apply in the context of the this proof. As before, from the interior regularity results [57, 99], it follows that the sequence of non-negative and locally bounded solutions $\{u_k^{\pm\epsilon}\}$ is locally uniformly Hölder continuous, and weakly pre-compact in $W^{1,p}_{loc}(\mathbb{R}\times(0,T))$. It follows that for some subsequence k' , (2.66) is valid. Since $\alpha > p/(p-1-\beta)$, it follows that the limit functions $v_{\pm\epsilon}$ are solutions to the problem:

$$v_t + bv^\beta = 0,\ x \in \mathbb{R},\ 0 < t \le t_0; \quad v(x,0) = (C\pm\epsilon)(-x)_+^\alpha,\ x \in \mathbb{R},$$

i.e.,

$$v_{\pm\epsilon}(x,t) = \Big[(C\pm\epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\Big]_+^{\frac{1}{1-\beta}}.$$

Let $l > l_*$ be an arbitrary number and $\epsilon > 0$ be chosen such that

$$(C-\epsilon)^{1-\beta}\ell^{\alpha(1-\beta)} > b(1-\beta).$$

If we now take $x = \eta_\ell(t)$ and $\tau = k^{\beta-1}t$, it follows from (2.66) that

$$u_{\pm\epsilon}(\eta_\ell(\tau),\tau) \sim \Big[(C\pm\epsilon)^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)\Big]^{\frac{1}{1-\beta}}\tau^{\frac{1}{1-\beta}}f \text{ as } \tau \to 0+. \qquad (2.73)$$

Since $\epsilon > 0$ is arbitrary, from (2.51) and (2.73), (2.21) follows.

□

2.4 Proofs of the main results

In this section, we prove the main results for slow diffusion case.

(I) $b \neq 0$ and $p > 2$.

2.4.1 Domination by diffusion: Interface expands

Region (1)

Proof of Theorem 2.1.1. Assume $\alpha < p/(p-1-\min\{1,\beta\})$. The formula (2.3) follows from Lemma 2.3.1. Since ρ is arbitrary, we take $\rho = \xi_* - \epsilon$ for $\epsilon > 0$,

$$\lim_{t\downarrow 0} \frac{u((\xi_* - \epsilon)t^{\frac{1}{p-\alpha(p-2)}}, t)}{t^{\frac{\alpha}{p-\alpha(p-2)}}} = f(\xi_* - \epsilon) > 0$$

$\exists \delta > 0 \quad \forall t \in (0, \delta]$ such that

$$\liminf_{t\downarrow 0} \eta(t) t^{\frac{1}{\alpha(p-2)-p}} \geq (\xi_* - \epsilon).$$

Let $\epsilon \downarrow 0 \Rightarrow$

$$\liminf_{t\downarrow 0} \eta(t) t^{\frac{1}{\alpha(p-2)-p}} \geq \xi_* \tag{2.74}$$

Take an arbitrary sufficiently small number $\epsilon > 0$. Let u_ϵ be a solution of the(1.10), (1.13) with $b = 0$ and with C replaced by $C + \epsilon$. As before, the second inequality of (2.49) and the first inequality of (2.50) follow from (1.12). Suppose that $b > 0$. In this case, u_ϵ is a supersolution of (1.10). From (2.49), (2.50), and a comparison principle, the second inequality of (2.51) follows. By Lemma 2.3.1, we then have

$$\eta(t) \leq (C + \epsilon)^{\frac{2-p}{\alpha(p-2)-p}} \xi'_* t^{1/(p-\alpha(p-2))}, \quad 0 \leq t \leq \delta,$$

and

$$\limsup_{t\downarrow 0} \eta(t)t^{\frac{1}{\alpha(p-2)-p}} \leq \xi_*. \tag{2.75}$$

Assume now that $b < 0$ and $\beta \geq 1$. The function

$$\bar{u}_\epsilon(x,t) = \exp(-bt)u_\epsilon\Big(x, \frac{1}{b(2-p)}[\exp(b(2-p)t) - 1]\Big)$$

is a solution to the (1.10), (1.13) with $\beta = 1$ and with C replaced by $C + \epsilon$. As before, from (1.12) the first inequality of (2.50) follows, where we replace u_ϵ with $\bar{u}_\epsilon$. Choose $|x_\epsilon|$ and δ so small that

$$\bar{u}_\epsilon < 1 \text{ in } B = \{(x,t) : x \geq x_\epsilon,\ 0 < t \leq \delta\}.$$

Obviously, $\bar{u}_\epsilon$ is a supersolution of (1.10) in B. From (2.49), (2.50), and a comparison principle, the second inequality of (2.51), with u_ϵ replaced by $\bar{u}_\epsilon$, follows. Thus, we have

$$\eta(t) \leq (C+\epsilon)^{\frac{2-p}{\alpha(p-2)-p}} \xi'_* \Big\{(b(2-p))^{-1}[\exp(b(2-p)t) - 1]\Big\}^{1/(p-\alpha(p-2))}, \quad 0 \leq t \leq \delta,$$

which again implies (2.75). From (2.74) and (2.75), (2.1) follows. Finally, (2.7), (2.8), (2.9) follow from (2.31), which will be proved later in this section. □

2.4.2 Borderline case: Diffusion & Reaction are in balance

Region (2)

Proof of Theorem 2.1.2. First, consider the global case of (1.13). The problem (1.10), (1.13) has a unique global solution and for this class of initial data a comparison principle is valid [57, 59].

If $\beta(p-1)=1$, it may be easily checked that the explicit solution to (1.10), (1.13) is given by (2.13).

Let $\beta(p-1)\neq 1$. The self-similar form (2.14) follows from Lemma 2.3.3. Let $C>C_*$. Consider a function

$$g(x,t)=t^{1/(1-\beta)}f_1(\zeta), \quad \zeta=xt^{-\frac{p-1-\beta}{p(1-\beta)}}. \tag{2.76}$$

We then have

$$\mathrm{L}g=t^{\frac{\beta}{1-\beta}}\mathcal{L}^0 f_1, \tag{2.77a}$$

$$\mathcal{L}^0 f_1=\frac{1}{1-\beta}f_1-(|f_1'|^{p-2}f_1')'-\frac{p-1-\beta}{p(1-\beta)}\zeta f_1'+bf_1^{\beta}. \tag{2.77b}$$

Choose as a function f_1

$$f_1(\zeta)=C_0(\zeta_0-\zeta)_+^{\gamma_0}, \;\; 0<\zeta<+\infty,$$

where C_0,ζ_0,γ_0 are some positive constants. Taking $\gamma_0=p/(p-1-\beta)$, from (2.77b) we have

$$\mathcal{L}^0 f_1=bC_0^{\beta}(\zeta_0-\zeta)_+^{\frac{p\beta}{p-1-\beta}}\Big\{1-\Big(\frac{C_0}{C_*}\Big)^{p-1-\beta}+\frac{C_0^{1-\beta}}{b(1-\beta)}\zeta_0(\zeta_0-\zeta)_+^{\frac{\beta(1-p)+1}{p-1-\beta}}\Big\}. \tag{2.78}$$

To prove an upper estimation, we take $C_0=C_2,\zeta_0=\zeta_2$ (see the Appendix Part A).

If $\beta(p-1) > 1$, then we have

$$\mathcal{L}^0 f_1 \geq bC_2^{\beta}(\zeta_2-\zeta)_+^{\frac{p\beta}{p-1-\beta}}\Big\{1-\Big(\frac{C_2}{C_*}\Big)^{p-1-\beta}+\frac{C_2^{1-\beta}}{b(1-\beta)}\zeta_2^{\frac{p(1-\beta)}{p-1-\beta}}\Big\}=0,\ \text{for } 0\leq\zeta\leq\zeta_2,$$

whereas if $\beta(p-1) < 1$, we have

$$\mathcal{L}^0 f_1 \geq bC_2^{\beta}(\zeta_2-\zeta)_+^{\frac{p\beta}{p-1-\beta}}\Big\{1-\Big(\frac{C_2}{C_*}\Big)^{p-1-\beta}\Big\}=0,\ \text{for } 0\leq\zeta\leq\zeta_2.$$

From (2.77a), it follows that

$$\mathrm{L}g \geq 0 \quad \text{for } 0 < x < \zeta_2 t^{\frac{p-1-\beta}{p(1-\beta)}},\ 0 < t < +\infty, \tag{2.79a}$$

$$\mathrm{L}g = 0 \quad \text{for } x > \zeta_2 t^{\frac{p-1-\beta}{p(1-\beta)}},\ 0 < t < +\infty. \tag{2.79b}$$

Lemma 2.2.2 implies that g is a supersolution of (1.10) in$\{(x,t) : x > 0, t > 0\}$. Since

$$g(x,0) = u(x,0) = 0 \quad \text{for } 0 \leq x < +\infty, \tag{2.80a}$$

$$g(0,t) = u(0,t) \quad \text{for } 0 \leq t < +\infty, \tag{2.80b}$$

the right-hand side of (2.16) follows. If $\beta(p-1) < 1$, then to prove the lower estimation we take $C_0 = C_1, \zeta_0 = \zeta_1,\ \gamma_0 = p/(p-1-\beta)$. Then, from (2.78) we derive

$$\mathcal{L}^0 f_1 \leq bC_1^{\beta}(\zeta_1-\zeta)^{\frac{p\beta}{p-1-\beta}}\Big\{1-\Big(\frac{C_1}{C_*}\Big)^{p-1-\beta}+\frac{C_1^{1-\beta}}{b(1-\beta)}\zeta_1^{\frac{p(1-\beta)}{p-1-\beta}}\Big\}=0 \text{ for } 0\leq\zeta\leq\zeta_1,$$

and from (2.77a) it follows that

$$\mathrm{L}g \le 0 \quad \text{for } 0 < x < \zeta_1 t^{\frac{p-1-\beta}{p(1-\beta)}},\ 0 < t < +\infty, \tag{2.81a}$$

$$\mathrm{L}g = 0 \quad \text{for } \ x > \zeta_1 t^{\frac{p-1-\beta}{p(1-\beta)}},\ 0 < t < +\infty. \tag{2.81b}$$

As before, from Lemma 2.2.2 and (2.80a),(2.80b), the left-hand side of (2.16) follows.

If $\beta(p-1) > 1$, then to prove the lower estimation we take $C_0 = C_1, \zeta_0 = \zeta_1,\ \gamma_0 = (p-1)/(p-2)$. Then, from (2.77b) we have

$$\mathcal{L}^0 f_1 = C_1(1-\beta)^{-1}(\zeta_1-\zeta)^{\frac{1}{p-2}}\Big\{\zeta_1 - \Big(\frac{\beta(p-1)-1}{p(p-2)}\Big)\zeta - (1-\beta)C_1^{p-2}\Big(\frac{p-1}{p-2}\Big)^p$$

$$+b(1-\beta)C_1^{\beta-1}(\zeta_1-\zeta)^{\frac{\beta(p-1)-1}{p-2}}\Big\} \le C_1(1-\beta)^{-1}(\zeta_1-\zeta)^{\frac{1}{p-2}}$$

$$\times\Big\{\zeta_1 - C_1^{p-2}\frac{(1-\beta)(p-1)^p}{(p-2)^p} + b(1-\beta)C_1^{\beta-1}\zeta_1^{\frac{\beta(p-1)-1}{p-2}}\Big\} = 0, \quad \text{for } \ 0 < \zeta < \zeta_1$$

which again implies (2.81a),(2.81b). From Lemma 2.2.2, the left-hand side of (2.16) follows.

By applying the same analysis, it may easily be checked that the alternative upper estimation is valid if $C_0 = \bar{C}_2, \zeta_0 = \bar{\zeta}_2, \gamma_0 = (p-1)/(p-2)$.

Let $\beta(p-1) > 1$ and $0 < C < C_*$. Consider a function

$$g(x,t) = [C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}} - b(1-\beta)(1-\gamma)t]_+^{\frac{1}{1-\beta}}, \quad x \in \mathbb{R},\ t > 0,$$

where $\gamma \in [0,1)$. Let us estimate Lg in

$$M = \{(x,t) : -\infty < x < \mu_\gamma(t),\ t > 0\}, \quad \mu_\gamma(t) = -[b(1-\beta)(1-\gamma)C^{\beta-1}t]^{\frac{p-1-\beta}{p(1-\beta)}}.$$

We have

$$Lg = bg^\beta S,$$

where

$$S = \gamma - p^{p-1}(\beta(1-p)+1)(p-1)b^{-1}(p-1-\beta)^{-p}C^{p-1-\beta}\Big[1 - \frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}}}\Big)\Big]^{\frac{\beta(p-2)}{1-\beta}}$$

$$-p^p\beta(p-1)b^{-1}(p-1-\beta)^{-p}C^{p-1-\beta}\Big[1 - \frac{b(1-\beta)(1-\gamma)t}{C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}}}\Big]^{\frac{\beta(p-1)-1}{1-\beta}}. \tag{2.82a}$$

Hence

$$S|_{t=0} = \gamma - \Big(\frac{C}{C_*}\Big)^{p-1-\beta}, \quad S|_{x=\mu_\gamma(t)} = \gamma. \tag{2.82b}$$

Moreover,

$$S_t = \frac{p^{p-1}(p-1)(1-\gamma)C^{p-2}}{(p-1-\beta)^p}(-x)_+^{\frac{p(\beta-1)}{p-1-\beta}}\Big[1 - C^{\beta-1}(-x)_+^{\frac{p(\beta-1)}{p-1-\beta}}b(1-\beta)(1-\gamma)t\Big]^{\frac{p\beta-2}{1-\beta}}$$

$$\times\Big[(\beta(p-1)-1)\beta(p-2)C^{\beta-1}b(1-\beta)(-x)_+^{-\frac{p(1-\beta)}{p-1-\beta}}(1-\gamma)t + (\beta(p-1)-1)(2\beta)\Big] \geq 0 \quad \text{in } M.$$

Thus,

$$\gamma - \Big(\frac{C}{C_*}\Big)^{p-1-\beta} \leq S \leq \gamma \quad \text{in } M.$$

If we take $\gamma = \left(\frac{C}{C_*}\right)^{p-1-\beta}$ ($\gamma = 0$, respectively), then we have

$$\mathrm{L}g \geq 0 (\text{respectively}, \mathrm{L}g \leq 0) \text{ in } M, \tag{2.83a}$$

$$\mathrm{L}g = 0 \text{ for } x > \mu_\gamma(t),\ t > 0, \tag{2.83b}$$

and the estimation (2.18) follows from the Lemma 2.2.2.

Let $\beta(p-1) < 1$ and $0 < C < C_*$. First, we can establish the following rough estimation:

$$\left[C^{1-\beta}(-x)_+^{\frac{p(1-\beta)}{p-1-\beta}} - b(1-\beta)(1-\left(\frac{C}{C_*}\right)^{p-1-\beta})t\right]_+^{\frac{1}{1-\beta}} \leq u(x,t) \leq C(-x)_+^{\frac{p}{p-1-\beta}}\ x \in \mathbb{R}, 0 \leq t < +\infty. \tag{2.84}$$

To prove the left-hand side, we consider the function g as in the case when $\beta(p-1) > 1$ with $\gamma = (C/C_*)^{p-1-\beta}$. As before, we then derive (2.82a) and, since

$$S_t = \frac{p^{p-1}(p-1)(1-\gamma)C^{p-2}(-x)_+^{-\frac{p(1-\beta)}{p-1-\beta}}}{(p-1-\beta)^p}\left[1-(\frac{-\mu_\gamma(t)}{(-x)_+})^{\frac{p(1-\beta)}{p-1-\beta}})\right]^{\frac{p\beta-2}{1-\beta}}\Big\{(\beta(p-1)-1)(2\beta)+$$

$$+(\beta(p-1)-1)\beta(p-2)b(1-\beta)(1-\gamma)C^{\beta-1}(-x)_+^{-\frac{p(1-\beta)}{p-1-\beta}}t\Big\} \leq 0 \text{ in } M,$$

we have $S \leq 0$ in M. Hence, (2.83a),(2.83b) are valid with reversed inequality. As before, from Lemma 2.2.2 the left-hand side of (2.84) follows. Since

$$\mathrm{L}u_0 = bu_0^\beta(1-\left(C/C_*\right)^{p-1-\beta}) \geq 0 \quad \text{for} \quad x \in \mathbb{R},\ t \geq 0,$$

the second inequality in (2.84) follows. Using (2.84), we can now establish a more accurate estimation (2.19). Consider a function

$$g(x,t) = C_0(-\zeta_0 t^{\frac{p-1-\beta}{p(1-\beta)}} - x)_+^{\frac{p}{p-1-\beta}} \text{ in } G_\ell,$$

$$G_\ell = \{(x,t) : \zeta(t) = -\ell t^{\frac{p-1-\beta}{p(1-\beta)}} < x < +\infty,\ 0 < t < +\infty\},$$

where, $C_0 > 0,\ \zeta_0 > 0,\ \ell > \zeta_0$ are some constants. Calculating Lg in

$$G_\ell^+ = \{(x,t); \zeta(t) < x < -\zeta_0 t^{\frac{p-1-\beta}{p(1-\beta)}},\ 0 < t < +\infty\},$$

we have

$$\mathrm{L}g = bg^\beta S, \quad S = 1 - \left(C_0/C_*\right)^{p-1-\beta} - (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 t^{\frac{\beta(p-1)-1}{p(1-\beta)}}$$

$$\times(-\zeta_0 t^{\frac{p-1-\beta}{p(1-\beta)}} - x)^{\frac{\beta(1-p)+1}{p-1-\beta}}. \tag{2.85}$$

Hence, if we take $C_0 = C_*$, then

$$\mathrm{L}g \le 0 \text{ in } G_\ell^+;\ \mathrm{L}g = 0 \text{ in } G_\ell \backslash \bar{G}_\ell^+. \tag{2.86}$$

To obtain a lower estimation, we now choose $\zeta_0 = \zeta_3,\ \ell = \ell_0$ (see the Appendix Part A). Using (2.84), we have

$$g(\zeta(t),t) = C_*(\ell_0 - \zeta_3)^{\frac{p}{p-1-\beta}} t^{\frac{1}{1-\beta}} = \left(b(1-\beta)\theta_* t\right)^{\frac{1}{1-\beta}}$$

$$= \left[C^{1-\beta} \ell_0^{\frac{p(1-\beta)}{p-1-\beta}} - b(1-\beta)(1-\left(C/C_*\right)^{p-1-\beta})\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} \le u(\zeta(t),t),\ t \ge 0, \tag{2.87a}$$

$$g(x,0) = u(x,0) = 0,\ 0 \le x \le x_0, \tag{2.87b}$$

$$g(x_0,t) = u(x_0,t) = 0,\ t \ge 0, \tag{2.87c}$$

where $x_0 > 0$ is an arbitrary fixed number. By using (2.86), (2.87a)-(2.87c), we can apply Lemma 2.2.2 in

$$G'_{\ell_0} = G_{\ell_0} \cap \Big\{x < x_0\Big\}.$$

Since $x_0 > 0$ is arbitrary number, the desired lower estimation from (2.19) follows.

Let us now prove the right-hand side of (2.19). Since

$$S_x = (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 t^{\frac{\beta(p-1)-1}{p(1-\beta)}} \Big(\frac{\beta(1-p)+1}{p-1-\beta}\Big)\Big(-\zeta_0 t^{\frac{p-1-\beta}{p(1-\beta)}} - x\Big)^{\frac{\beta(2-p)-p+2}{p-1-\beta}} \ge 0,$$

$$\text{for } \ \zeta(t) < x < -\zeta_0 t^{\frac{p-1-\beta}{p(1-\beta)}},\ t > 0,$$

from (2.85) it follows that

$$S \ge S|_{x=\zeta((t)} = 1 - \Big(C_0/C_*\Big)^{p-1-\beta} - (b(1-\beta))^{-1} C_0^{1-\beta} \zeta_0 (\ell - \zeta_0)^{\frac{\beta(1-p)+1}{p-1-\beta}}.$$

Taking now $C_0 = C_3,\ \zeta_0 = \zeta_4,\ \ell = \ell_1$ (see the Appendix Part A), we have

$$S|_{x=\zeta(t)} = 0;$$

hence (by using (2.84))

$$\mathrm{L}g \ge 0 \ \text{ in } \ G^+_{\ell_1},\ \mathrm{L}g = 0 \ \text{ in } \ G_{\ell_1} \backslash \bar{G}^+_{\ell_1},$$

$$u(\zeta(t),t) \le C\ell_1^{\frac{p}{p-1-\beta}} t^{\frac{1}{1-\beta}} = C_3(\ell_1 - \zeta_4)^{\frac{p}{p-1-\beta}} t^{\frac{1}{1-\beta}} = g(\zeta(t),t), \ t \ge 0,$$

and, for arbitrary $x_0 > 0$, (2.87b) and (2.87c) are valid. As before, applying Lemma 2.2.2 in G'_{ℓ_1}, we then derive the right-hand side of (2.19), since $x_0 > 0$ is arbitrary. From (2.16), (2.18), and (2.19), it follows that

$$\zeta_1 t^{\frac{p-1-\beta}{p(1-\beta)}} \le \eta(t) \le \zeta_2 t^{\frac{p-1-\beta}{p(1-\beta)}}, \quad 0 \le t < +\infty,$$

where the constants ζ_1 and ζ_2 are chosen according to relevant estimations for u. If u_0 satisfies (1.12) with $\alpha = p/(p-1-\beta)$ and with $C \neq C_*$, then the asymptotic formulae (2.11) and (2.12)may be proved as the similar estimations (2.1) and (2.3) were in Lemma 2.3.1.

□

2.4.3 Domination by absorption: Interface shrinks

Region (3)

Proof of Theorem 2.1.3. Take an arbitrary sufficiently small number $\epsilon > 0$. From (1.12), (2.49) follows. Then, consider a function

$$g_\epsilon(x,t) = \left[(C+\epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)t\right]_+^{1/(1-\beta)}. \tag{2.88}$$

We estimate Lg in

$$M_1 = \{(x,t) : x_\epsilon < x < \eta_\ell(t), \ \ 0 < t < \delta_1\},$$

$$\eta_\ell(t) = -\ell t^{1/(\alpha(1-\beta)}, \ \ \ell(\epsilon) = (C+\epsilon)^{-1/\alpha}\big[b(1-\beta)(1-\epsilon)\big]^{1/\alpha(1-\beta)},$$

where $\delta_1 > 0$ is chosen such that $\eta_{\ell(\epsilon)}(\delta_1) = x_\epsilon$. We have

$$Lg_\epsilon = bg_\epsilon^\beta\{\epsilon + S\},$$

$$S = -b^{-1}(p-1)\alpha^{p-1}(\alpha(1-\beta)-1)(C+\epsilon)^{p-1-\beta}(-x)_+^{\alpha(p-1-\beta)-p}\left\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\right\}^{\beta(p-2)}$$

$$-b^{-1}\beta(p-1)\alpha^p(C+\epsilon)^{p-1-\beta}(-x)_+^{\alpha(p-1-\beta)-p}\left\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\right\}^{\beta(p-1)-1}$$

$$= -b^{-1}\alpha^{p-1}(C+\epsilon)^{p-1-\beta}(-x)_+^{\alpha(p-1-\beta)-p}\left\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\right\}^{\beta(p-1)-1}S_1,$$

$$S_1 = \left\{(\alpha(1-\beta)-1)(p-1)\left[g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\right]^{1-\beta} + \alpha\beta(p-1)\right\}.$$

If $\beta(p-1) \geq 1$, then we can choose $x_\epsilon < 0$ such that (with sufficiently small $|x_\epsilon|$)

$$|S| < \frac{\epsilon}{2} \text{ in } M_1.$$

Thus ,we have

$$\mathrm{L}g_\epsilon > b(\epsilon/2)(g_\epsilon)^\beta \qquad (\mathrm{L}g_{-\epsilon} < -b(\epsilon/2)(g_{-\epsilon})^\beta, \text{ respectively}) \text{ in } M_1,$$

$$\mathrm{L}g_{\pm\epsilon} = 0 \quad \text{ for } x > \eta_{\ell(\pm\epsilon)}(t),\ 0 < t \leq \delta_1,$$

$$g_\epsilon(x,0) \geq u_0(x) \quad (g_{-\epsilon}(x,0) \leq u_0(x), \text{respectively}),\ \ x \geq x_\epsilon.$$

Since u and g are continuous functions, $\delta = \delta(\epsilon) \in (0, \delta_1]$ may be chosen such that

$$g_\epsilon(x_\epsilon,t) \geq u(x_\epsilon,t)\ \ (g_{-\epsilon}(x_\epsilon,t) \leq u(x_\epsilon,t), \text{respectively}),\ \ 0 \leq t \leq \delta.$$

From comparison Lemma 2.3.1, it follows that

$$g_{-\epsilon} \leq u \leq g_{\epsilon} \quad x \geq x_{\epsilon}, \;\; 0 \leq t \leq \delta, \tag{2.89a}$$

$$\eta_{\ell(-\epsilon)}(t) \leq \eta(t) \leq \eta_{\ell(\epsilon)}, \;\; 0 \leq t \leq \delta, \tag{2.89b}$$

which imply (2.20) and (2.21).

Let $\beta(p-1) < 1$. In this case the left-hand side of (2.89a), (2.89b) may be proved similarly. Moreover, we can replace $1+\epsilon$ with 1 in $g_{-\epsilon}$ and $\eta_{\ell}(-\epsilon)$.

To prove a relevant upper estimation, consider a function

$$g(x,t) = C_6\big(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\big)_+^{\alpha} \text{ in } G_{\ell,\delta},$$

$$G_{\ell,\delta} = \{(x,t) : \eta_{\ell}(t) < x < +\infty, \;\; 0 < t < \delta\},$$

where $\ell \in (\ell_*, +\infty)$ and

$$\zeta_5 = (\ell_*/\ell)^{\alpha(1-\beta)}(1-\epsilon)\ell,$$

$$C_6 = [1-(\ell_*/\ell)^{\alpha(1-\beta)}(1-\epsilon)]^{-\alpha}[C^{1-\beta} - \ell^{-\alpha(1-\beta)}b(1-\beta)(1-\epsilon))]^{1/(1-\beta)}.$$

From (2.21), it follows that for arbitrary $\ell > \ell_*$ and $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \ell) > 0$ such that

$$u(\eta_{\ell}(t), t) \leq [C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \;\; 0 \leq t \leq \delta. \tag{2.90}$$

Calculating Lg in

$$G^+_{\ell,\delta} = \{(x,t) : \eta_{\ell}(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}}, \;\; 0 < t < \delta\},$$

we have

$$\mathrm{L}g = bg^{\beta}S,$$

$$S = 1-(b(1-\beta))^{-1}\zeta_5 C_6^{1/\alpha}\{gt^{1/(\beta-1)}\}^{1-\beta-1/\alpha} - b^{-1}(\alpha-1)(p-1)\alpha^{p-1}C_6^{p/\alpha}g^{p-1-\beta-(p/\alpha)}.$$

Since

$$S_x = \alpha(1-\beta-\frac{1}{\alpha})(b(1-\beta))^{-1}\zeta_5 C_6^{\frac{1}{\alpha}} g^{-\beta-\frac{1}{\alpha}} t^{\frac{1-\alpha(1-\beta)}{\alpha(1-\beta)}} C_6(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x)_+^{\alpha-1} +$$

$$+\alpha b^{-1}(\alpha-1)(p-1)\alpha^{p-1}(p-1-\beta-\frac{p}{\alpha})C_6^{\frac{p}{\alpha}} g^{p-\beta-2-\frac{p}{\alpha}} C_6(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x)_+^{\alpha-1} \geq 0 \ \text{ in } \ G_{l,\delta}^{+},$$

$$S \geq S|_{x=\eta_\ell(t)} = 1-(b(1-\beta))^{-1}\zeta_5 C_6^{1-\beta}(\ell-\zeta_5)^{\alpha(1-\beta)-1}$$

$$-b^{-1}(\alpha-1)(p-1)\alpha^{p-1}C_6^{p-1-\beta}\{(\ell-\zeta_5)t^{1/\alpha(1-\beta)}\}^{\alpha(p-1-\beta)-p}.$$

Then, we have

$$S \geq \epsilon - b^{-1}C_6^{p-1-\beta}(\alpha-1)(p-1)\alpha^{p-1}\{(\ell-\zeta_5)t^{1/\alpha(1-\beta)}\}^{\alpha(p-1-\beta)-p} \text{ in } G_{\ell,\delta}^{+}.$$

Hence, we can choose $\delta = \delta(\epsilon) > 0$ so small that

$$\mathrm{L}g \ \geq \ b(\epsilon/2)g^{\beta} \ \text{ in } \ G_{\ell,\delta}^{+}. \tag{2.91a}$$

Using (2.90), we can apply Lemma 2.2.2 in $G'_{\ell,\delta} = G_{\ell,\delta} \cap \{x < x_0\}$, for $\forall x_0 > 0$. We have

$$\mathrm{L}g = 0 \ \text{ in } \ G'_{\ell,\delta}\backslash\bar{G}_{\ell,\delta}^{+}, \tag{2.91b}$$

$$u(\eta_\ell(t),t) \le [C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} = C_6(\ell-\zeta_5)^\alpha t^{\frac{1}{(1-\beta)}} = g(\eta_\ell(t),t),\ 0 \le t \le \delta. \tag{2.91c}$$

$$u(x_0,t) = g(x_0,t) = 0,\ 0 \le t \le \delta,\ \ u(x,0) = g(x,0) = 0,\ 0 \le x \le x_0. \tag{2.91d}$$

Since $x_0 > 0$ is arbitrary, from (2.91a)-(2.91d) and comparison principle it follows that for all $\ell > \ell_*$ and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, \ell) > 0$ such that

$$u(x,t) \le C_6(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x)_+^\alpha \ \text{ in } \ \bar{G}_{\ell,\delta}. \tag{2.92}$$

Since (2.21) is valid along $x = \eta_\ell(t)$, δ may be chosen so small that

$$-\ell t^{1/\alpha(1-\beta)} \le \eta(t) \le -\zeta_5 t^{1/\alpha(1-\beta)},\ 0 \le t \le \delta. \tag{2.93}$$

Since $\ell > \ell_*$ and $\epsilon > 0$ are arbitrary numbers, (2.20) follows from (2.93). □

2.4.4 Waiting time phenomena

Region (4)

(4a) This case is immediate.

(4b) Let $\beta = 1,\ \alpha > p/(p-2)$. As before, from (1.12), (2.49) follows. Then, consider a function

$$g(x,t) = (C-\epsilon)(-x)_+^\alpha \exp(-bt),$$

which satisfies

$$\text{L}g \leq 0 \ \text{ for } x_\epsilon < x < 0,\ t > 0; \quad Lg = 0 \ \text{ for } x > 0,\ t > 0.$$

We can choose $\delta = \delta(\epsilon) > 0$ such that

$$g(x_\epsilon, t) \leq u(x_\epsilon, t), \ \ 0 \leq t \leq \delta_\epsilon,$$

and from a comparison principle, the left-hand side of (2.25) follows. To prove the right-hand side, consider

$$g(x,t) = (C+\epsilon)(-x)_+^\alpha \exp(-bt)[1-\epsilon(b(p-2))^{-1}(1-\exp(-b(p-2)t))]^{1/2-p}.$$

We have

$$\text{L}g = (p-2)^{-1}(C+\epsilon)(-x)_+^\alpha \exp(-b(p-1)t)g^{p-1}$$

$$\times\{\epsilon-(p-2)\alpha^{p-1}(\alpha-1)(p-1)(C+\epsilon)^{p-2}(-x)_+^{\alpha(p-2)-p}\}, \ \ x < 0,\ t > 0,$$

and hence, if $|x_\epsilon|$ is small enough,

$$\text{L}g \geq 0 \ \text{ for } \ x_\epsilon < x < 0,\ t > 0; \quad \text{L}g = 0 \ \text{ for } \ x > 0,\ t > 0.$$

As before, a comparison principle implies the right-hand side of (2.25). The estimations (2.26)-(2.28) in the cases (4c) and (4d) may be proved similarly.

(II) $b = 0$.

(1) Let $p > 2,\ 0 < \alpha < p/(p-2)$.

First assume that u_0 is defined by (1.13). The self-similar form (2.4) and the formula(2.30) are well-known results (see Lemma 2.3.1). To prove (2.31), consider a function

$$g(x,t) = t^{\alpha/(p-\alpha(p-2))} f(\xi).$$

We have

$$\mathrm{L}g = t^{(\alpha(p-1)-p)/(p-\alpha(p-2))}\mathcal{L}_t f,$$

$$\mathcal{L}_t f = \frac{\alpha}{p-\alpha(p-2)} f - \frac{1}{p-\alpha(p-2)}\xi f' - (|f'|^{p-2} f')'.$$

Choose

$$f(\xi) = C_0(\xi_0 - \xi)_+^{(p-1)/(p-2)}, \ \ 0 < \xi < +\infty,$$

where C_0 and ξ_0 are some positive constants. Then, we have

$$\mathcal{L}_t f = (p-\alpha(p-2))^{-1}(p-1)(p-2)^{-1} C_0(\xi_0-\xi)^{1/p-2} R(\xi) \quad \text{for } 0 \le \xi \le \xi_0, \ \ t > 0$$

$$R(\xi) = \alpha(p-2)(p-1)^{-1}\xi_0 + (1-\alpha(p-2)(p-1)^{-1})\xi - (p-1)^{p-1}(p-2)^{-(p-1)}$$

$$\times (p-\alpha(p-2)) C_0^{p-2}$$

To prove an upper estimation, we take $C_0 = C_5, \ \ \xi_0 = \xi_4$. Then, we have

$$R(\xi) \ge \nu_\alpha \xi_4 - (p-1)^{p-1}(p-2)^{-(p-1)}(p-\alpha(p-2)) C_5^{p-2} = 0 \quad \text{for } \ 0 \le \xi \le \xi_4,$$

where

$$\nu_\alpha = \{1 \ \text{ if } \ \alpha \ge (p-1)(p-2)^{-1}; \ \alpha(p-2)(p-1)^{-1} \ \text{ if } \alpha < (p-1)(p-2)^{-1}\}.$$

Hence,

$$\mathrm{L}g \ \geq 0 \ \text{ for } \ 0 < x < \xi_4 t^{1/p-\alpha(p-2)}, \ \ t > 0,$$

$$\mathrm{L}g \ = 0 \ \text{ for } \ 0 > \xi_4 t^{1/p-\alpha(p-2)}, \ \ t > 0,$$

$$u(0,t) = g(0,t), \ t \geq 0; \ u(x,0) = g(x,0), \ x \geq 0,$$

and a comparison principle imply the right-hand side of (2.31). The left-hand side of (2.31) may be established similarly if we take $C_0 = C_4$, $\xi_0 = \xi_3$. Equations (2.2) and (2.6) follow from Lemma 2.3.1. Finally, (2.7)-(2.9) easily follow from (2.30) and (2.31). If u_0 satisfies (1.12) with $0 < \alpha < p/(p-2)$, then (2.1)-(2.3) follow from Lemma 2.3.1.

The cases (2) and (3) are immediate.

Chapter 3

Evolution of Interface for the Nonlinear p-Laplacian type Reaction-Diffusion Equations with Fast Diffusion

In this chapter we present full classification of the evolution of interfaces and local structure of solution near the interfaces and at infinity for the problem (1.10) -(1.13) in the fast diffusion case ($1 < p < 2$). The results of this chapter are contained in the paper [21]

3.1 Main Results

Throughout this section we assume that u is a unique weak solution of the CP (1.10)-(1.12). There are five different subcases, as shown in Fig. 3.1. The main results are

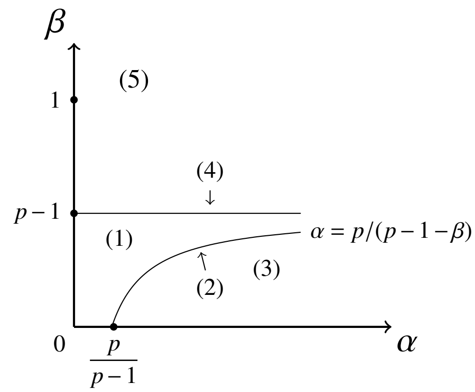

Figure 3.1: Classification of different cases in the (α,β) plane for interface development in problem (1.10)-(1.13) (when $1 < p < 2$).

outlined below in Theorems 3.1.1, 3.1.2, 3.1.3, 3.1.4 and 3.1.5 corresponding directly to the cases (1), (2), (3), (4) and (5) in Fig. 3.1.

Theorem 3.1.1. *Let $0 < \beta < p-1$, $0 < \alpha < p/(p-1-\beta)$. Then, interface initially expands and for some positive $\delta > 0$*

$$\zeta_1 t^{(p-1-\beta)/p(1-\beta)} \leq \eta(t) \leq \zeta_2 t^{(p-1-\beta)/p(1-\beta)}, \ 0 < t \leq \delta, \tag{3.1}$$

(see Appendix Part B for explicit values of ζ_1, ζ_2). Moreover, for arbitrary $\rho \in \mathbb{R}$, there exists a positive number $f(\rho)$ depending on C, p and α which satisfies (2.3) *along the curve $x = \xi_\rho(t) = \rho t^{1/(p+\alpha(2-p))}$.*

Theorem 3.1.2. *Let $0 < \beta < p-1$, $\alpha = p/(p-1-\beta)$ and C_* is defined as in* (2.10)*. Then the interface expands or shrinks accordingly as $C > C_*$ or $C < C_*$ and satisfies* (2.11) *where $\zeta_* \lessgtr 0$ if $C \lessgtr C_*$, and for arbitrary $\rho < \zeta_*$ there exists $f_1(\rho) > 0$ satisfies* (2.12).

Theorem 3.1.3. *Let $b > 0$, $0 < \beta < p-1$, $\alpha > p/(p-1-\beta)$. Then interface shrinks and satisfies* (2.20) *where $\ell_* = C^{-1/\alpha}(b(1-\beta))^{1/\alpha(1-\beta)}$. For arbitrary $\ell > \ell_*$, we have* (2.21) *along the curve $x = \eta_l(t) = -lt^{1/\alpha(1-\beta)}$.*

Theorem 3.1.4. *Let $b > 0,\ 0 < \beta = p-1 < 1,\ \alpha > 0$. Then there is an infinite speed of propagation and $\forall\ \epsilon > 0,\ \exists\ \delta = \delta(\epsilon) > 0$ such that*

$$t^{1/(2-p)}\phi(x) \le u(x,t) \le (t+\epsilon)^{1/(2-p)}\phi(x) \qquad \text{for } 0 < x < \infty,\ 0 \le t \le \delta_\epsilon, \tag{3.2}$$

where $\phi(x)$ solves ODE problem

$$(|\phi'(x)|^{p-2}\phi'(x))' = \frac{1}{2-p}\phi(x) + b\phi^{p-1}(x) \tag{3.3a}$$

$$\phi(0) = 1,\ \phi(\infty) = 0. \tag{3.3b}$$

Solution u satisfies asymptotic formula

$$\log u(x,t) \sim -\left(\frac{b}{p-1}\right)^{1/p} x \ \text{ as } x \to +\infty. \tag{3.4}$$

Theorem 3.1.5. *Let either $b > 0,\ \beta > p-1$ or $b < 0,\ \beta \ge 1$ and*

$$D = \Big(2(p-1)p^{p-1}(2-p)^{1-p}\Big)^{1/(2-p)}. \tag{3.5}$$

Then there is an infinite speed of propagation and (2.3) is valid. If either $b > 0,\ \beta \ge 2/p$ or $b < 0,\ \beta \ge 1$ then $\exists \delta > 0$ such that for $\forall$ fixed $t \in (0,\delta]$

$$u(x,t) \sim Dt^{1/(2-p)}x^{p/(p-2)} \qquad \text{as } \ x \to +\infty. \tag{3.6}$$

If $b > 0, 1 \leq \beta < 2/p$, *then*

$$\lim_{t\to 0+}\lim_{x\to +\infty} ut^{1/(p-2)}x^{\frac{p}{2-p}} = D. \tag{3.7}$$

If $b > 0,\ p-1 < \beta < 1$ *then* $\exists\delta > 0$ *such that for arbitrary fixed* $t \in (0,\delta]$

$$u(x,t) \sim C_* x^{p/(p-1-\beta)} \qquad as\ \ x \to +\infty. \tag{3.8}$$

3.2 Further Details of the Main Results

In this section we outline some essential details of the main results described in Theorems 3.1.1 - 3.1.5.

Further details of Theorem 3.1.1. Solution u satisfies the estimation

$$C_1 t^{1/(1-\beta)}(\zeta_1-\zeta)_+^{p/(p-1-\beta)} \leq u \leq C_* t^{1/(1-\beta)}(\zeta_2-\zeta)_+^{p/(p-1-\beta)},\ 0 < t \leq \delta, \tag{3.9}$$

where $\zeta = xt^{-(p-1-\beta)/p(1-\beta)}$ and the left-hand side of (3.9) is valid for $0 \leq x < +\infty$, while the right-hand side is valid for $x \geq \ell_0 t^{(p-1-\beta)/p(1-\beta)}$ and the constants C_*, C_1, ζ_1, ζ_2 and ℓ_0 are positive and depend only on p, β and b (see Appendix Part B).

A function f is a shape function of the self-similar solution of (1.10),(1.13) with $b = 0$ (see Lemma 3.3.1) and satisfies (2.6) where w is a solution of (1.10), (1.13) with $b = 0$, $C = 1$. Lower and upper estimations for f are given in (2.4), (3.23). If u_0 is defined as in (1.13), then the right-hand sides of (3.9), (3.1) are valid for $0 < t < +\infty$. The explicit formula (2.3) means that the local behavior of solution along the curves $x = \xi_\rho(t)$ approaching the origin coincides with that of the problem (1.10), (1.13) with $b = 0$. In other words, diffusion completely dominates in this region. However, domination of diffusion over the reaction fails along the curves $x = \zeta_\rho(t) = \rho t^{(p-1-\beta)/p(1-\beta)}$, $\rho > 0$ approaching the origin and the balance between diffusion and reaction in this region governs the interface, as expressed in estimations (3.9), (3.1). We stress the fact that the constants C_1, ζ_1, ζ_2 and ℓ_0 in (3.9), (3.1) do not depend on C and α.

Further details of Theorem 3.1.2. Assume that u_0 is defined by (1.13). If $C = C_*$ then u_0 is a stationary solution to (1.10),(1.13). If $C \neq C_*$ the solution to (1.10),(1.13) is of self-similar form as in (2.14) and (2.15). If $C > C_*$ then the interface expands, $f_1(0) = A_1 > 0$ (see Lemma 3.3.3) and

$$C'(\zeta' t^{(p-1-\beta)/p(1-\beta)} - x)_+^{p/(p-1-\beta)} \leq u(x,t) \leq C''(\zeta'' t^{(p-1-\beta)/p(1-\beta)} - x)_+^{p/(p-1-\beta)}, \quad (3.10a)$$

$$\zeta' \leq \zeta_* \leq \zeta'', \quad (3.10b)$$

where $0 \leq x < +\infty$, $0 < t < +\infty$ and $C' = C_2$, $C'' = C_*$, $\zeta' = \zeta_3$, $\zeta'' = \zeta_4$ (see Appendix Part B).

If $0 < C < C_*$ then the interface shrinks. There exists a constant $\ell_1 > 0$ such that for arbitrary $\ell \leq -\ell_1$, there exists a $\lambda > 0$ such that

$$u(\ell t^{\frac{p-1-\beta}{p(1-\beta)}}, t) = \lambda t^{1/(1-\beta)}, \; t \geq 0. \quad (3.11)$$

Moreover, u and ζ_* satisfy (3.10) with $C' = C_*$, $C'' = C_3$, $\zeta' = -\zeta_5 = -\ell_1 + (\lambda/C_*)^{\frac{p-1-\beta}{p}} < 0, \zeta'' = -\zeta_6$ and the left-hand side of (3.10a) is valid for $x \geq -\ell_1 t^{(p-1-\beta)/p(1-\beta)}$, while the right-hand side is valid for $x \geq -\ell_2 t^{(p-1-\beta)/p(1-\beta)}$ (see Appendix Part B, Lemma 3.3.3 and (3.24)).

In general the precise value ζ_* can be found only by solving the similarity ODE

$\mathcal{L}^0 f_1 = 0$ (see (2.77b)) and by calculating $\zeta_* = \sup\{\zeta : f_1(\zeta) > 0\}$.

The right-hand side of (2.11) (respectively (2.12)) relates to the self-similar solution (2.14), for which we have lower and upper bounds via (3.10). If u_0 satisfies (1.12) with $\alpha = p/(p-1-\beta), C = C_*$ then the small-time behavior of the interface and the local solution depends on the terms smaller than $C_*(-x)^{p/(p-1-\beta)}$ in the expansion of u_0 as $x \to 0-$.

It should be noted that if $C > C_*$ then the estimation (3.10) coincides with the estimation (2.19), proved for the case $\beta(p-1) < 1,\ p > 2$. If $0 < C < C_*$ then the right-hand side of estimation (3.10) coincides with (2.19) proved for the case $\beta(p-1) < 1,\ p > 2$, while the left-hand side of (3.10) is new. It should also be noted that the left-hand side of the estimation (2.19) proved above for the case $\beta(p-1) < 1,\ p > 2$, is still valid if $p \geq 2 - \beta$.

Further details of Theorem 3.1.3. The interface initially coincides with that of the solution (2.22) to the problem (2.23)

Further details of Theorem 3.1.4. The solution of (3.3) is

$$\phi(x) = F^{-1}(x), \quad 0 \leq x < +\infty, \tag{3.12}$$

where $F^{-1}(x)$ is an inverse function of

$$F(z) = \int_z^1 \frac{dy}{y\left[\frac{b}{p-1} + \frac{p}{2(p-1)(2-p)} y^{2-p}\right]^{1/p}},\ 0 < z \leq 1. \tag{3.13}$$

ϕ satisfies

$$\log \phi(x) \sim -\Big(\frac{b}{p-1}\Big)^{1/p} x \ \text{ as } x \to +\infty. \tag{3.14}$$

and the global estimation

$$0 < \phi(x) \leq e^{-\left(\frac{b}{p-1}\right)^{1/p} x}, \ \ 0 \leq x < +\infty. \tag{3.15}$$

Therefore, for any $\gamma > (\frac{b}{p-1})^{1/p}$ we have

$$\lim_{x \to +\infty} \frac{\phi(x)}{e^{-\gamma x}} = +\infty. \tag{3.16}$$

Respectively, the solution u satisfies

$$\lim_{t \to 0^+} \lim_{x \to +\infty} u(x,t) e^{\left(\frac{b}{p-1}\right)^{1/p} x} = 0, \tag{3.17}$$

and for any $\gamma > (\frac{b}{p-1})^{1/p}$

$$\lim_{x \to +\infty} \frac{u(x,t)}{e^{-\gamma x}} = +\infty, \ \ 0 < t \leq \delta. \tag{3.18}$$

Further details of Theorem 3.1.5. Let $\beta \geq 1$. Then for arbitrary sufficiently small $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$C_5 t^{\alpha/(p+\alpha(2-p))} (\xi_1 + \xi)^{\frac{p}{p-2}} \leq u \leq C_6 t^{\alpha/(p+\alpha(2-p))} (\xi_2 + \xi)^{\frac{p}{p-2}} \qquad x \geq 0, \ \ 0 \leq t \leq \delta, \tag{3.19}$$

where $\xi = xt^{-1/(p+\alpha(2-p))}$ (see Appendix Part B for the relevant constants). If $b > 0$, $\beta \geq 1$, then the following upper estimation is also valid

$$u(x,t) \leq D t^{1/(2-p)} x^{p/(p-2)} \qquad 0 < x < +\infty, \ \ 0 < t < +\infty, \tag{3.20}$$

Let $b < 0$, $\beta \geq 1$. Then for arbitrary sufficiently small $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$

such that

$$u(x,t) \le D(1-\epsilon)^{1/(2-p)} t^{1/(2-p)} x^{p/(p-2)} \qquad \text{for } \mu t^{1/(p+\alpha(2-p))} < x < +\infty, \ 0 < t \le \delta, \tag{3.21}$$

with

$$\mu = (D^{-1}(A_0+\epsilon))^{(p-2)/p}(1-\epsilon)^{-1/p}.$$

From (3.19) and (3.21), (3.6) again follows.

Let $b > 0$, $p-1 < \beta < 1$. Then there exists a number $\delta > 0$ such that

$$C_*(1-\epsilon)t^{1(1-\beta)}(\zeta_8+\zeta)_+^{p/(p-1-\beta)} \le u(x,t) \le C_* x^{p/(p-1-\beta)} \qquad 0 < x < +\infty,\ 0 < t \le \delta. \tag{3.22}$$

where $\epsilon > 0$ is an arbitrary sufficiently small number

As in the case I(1), the explicit formulae (2.3) expresses the domination of diffusion over the reaction. If $\beta \ge 1$, then from (3.19), (3.6), (3.7) it follows that domination of diffusion is the case for $x \gg 1$ as well, and the asymptotic behavior as $x \to +\infty$ coincides with that of the solution to problem (1.10), (1.13) with $b = 0$ (see the case II below). However, if $p-1 < \beta < 1$ then domination of the diffusion fails for $x \gg 1$ and there is solution of Eq. (1.10) on the right-hand side of (3.8).

(II) $b = 0,\ 1 < p < 2,\ \alpha > 0$.

In the case there is an infinite speed of propagation. First, assume that u_0 is defined by (1.13). Then the solution to (1.10), (1.13) has the self-similar form (2.4) where f satisfies (2.6). Moreover, we have

$$Dt^{\alpha/(p+\alpha(2-p))}(\xi_3+\xi)^{p/(p-2)} \le u \le C_7 t^{\alpha/(p+\alpha(2-p))}(\xi_4+\xi)^{p/(p-2)}, \quad 0 \le x,\ t < +\infty \tag{3.23}$$

(see Appendix Part B). The right-hand side of (3.23) is not in fact sharp enough as $x \to +\infty$ and the required upper estimation is provided by an explicit solution to (1.10), as in (3.20). From (3.23) and (3.20) it follows that, for arbitrary fixed $0 < t < +\infty$, the asymptotic result (3.6) is valid.

Now assume that u_0 satisfies (1.12) with $\alpha > 0$. Then (2.3) is valid and for arbitrary sufficiently small $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that the estimation (3.23) is valid for $0 < t \leq \delta$, except that in the left- hand side (respectively in the right-hand side) of (3.23) the constant A_0 should be replaced by $A_0 - \epsilon$ (respectively $A_0 + \epsilon$). Moreover, there exists a number $\delta > 0$ (which does not depend on ϵ) such that, for arbitrary $t \in (0, \delta]$, the asymptotic result (3.6) is valid.

3.3 Asymptotic Properties of solutions based on scaling laws

In the next two lemmas, we establish some preliminary estimations of the solution to CP, the proof of these estimations begin based on scale of variables.

Lemma 3.3.1. *If* $b = 0$ *and* $1 < p < 2$, $\alpha > 0$, *then the solution* u *of the CP* (1.10), (1.13) *has the self-similar form* (2.4), *where the self-similarity function* f *satisfies* (2.6). *If* u_0 *satisfies* (1.12) *then the solution to the CP* (1.10), (1.11) *satisfies* (2.3).

The proof of the lemma coincides with the proof of Lemma 2.3.1.

Lemma 3.3.2. *Let* u *be a solution of the* (1.10), (1.11) *and let* u_0 *satisfy* (1.12). *Let one of the following conditions be valid:*

(a) $b > 0, \ 0 < \beta < p - 1 < 1, \ 0 < \alpha < \frac{p}{p-1-\beta}$;

(b) $b > 0, \ 0 < p - 1 < 1, \ \beta \geq p - 1, \ \alpha > 0$;

(c) $b < 0,\ \beta \geq 1,\ 0 < p-1 < 1,\ \alpha > 0.$

Then u satisfies (2.3) *with the same function* f *as in Lemma 3.3.1.*

Lemma 3.3.3. *Let u be a solution to the CP* (1.10)*,* (1.13) *with* $b > 0,\ 0 < \beta < 1,\ p-1 > \beta,\ \alpha = p/(p-1-\beta),\ C > 0$. *Then the solution u has the self-similar form* (2.14)*. There is a constant* $\ell_1 > 0$ *such that for arbitrary* $\ell \in (-\infty, -\ell_1]$ *there exists* $\lambda > 0$ *such that* (3.11) *is valid. If* $0 < C < C_*$ *then*

$$0 < \lambda < C_*(-\ell)^{p/(p-1-\beta)}. \tag{3.24}$$

If $C > C_*$ *then* $f_1(0) = A_1 > 0$ *where* A_1 *depends on* p, β, C *and* b.

Lemma 3.3.4. *Let u be a solution to the CP* (1.10)*,* (1.12) *with* $b > 0,\ 0 < \beta < 1,\ p-1 > \beta,\ \alpha = p/(p-1-\beta),\ C > 0$. *Then for arbitrary* $\ell \in (-\infty, -\ell_1]$ *we have*

$$u(\ell t^{(p-1-\beta)/(p(1-\beta))}, t) \sim \lambda t^{1/(1-\beta)} \text{ as } t \to 0+, \tag{3.25}$$

where $\ell_1 > 0$, $\lambda > 0$ *are the same as in Lemma 3.3.3 and if* $0 < C < C_*$ *then* (3.24) *is also valid. If* $C > C_*$ *then u satisfies*

$$u(0,t) \sim A_1 t^{1/(1-\beta)} \text{ as } t \to 0+, \tag{3.26}$$

where $A_1 = f_1(0) > 0$ *(see Lemma 3.3.3).*

Lemma 3.3.5. *Let u be a solution to the CP* (1.10)*-*(1.12) *with* $b > 0,\ 0 < \beta < 1,\ p-1 > \beta,\ \alpha > p/(p-1-\beta),\ C > 0$. *Then for arbitrary* $\ell > \ell_*$ *(see* (2.20)*), the asymptotic formula* (2.21) *is valid, with* $x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}$.

3.3.1 Proof of Lemma 3.3.2: Diffusion dominates over the reaction

Proof of Lemma 3.3.2. The proof for cases (a) and (b) coincides with the proof for case (a) and (b) with $b > 0$ in Lemma 2.3.2. The proof for (c) coincides (with some modifications) with the proof for case (b) with $b < 0$ in the Lemma 2.3.2; namely, instead of zero boundary condition on the line $x = -x_\epsilon$ and $x = -k^{1/\alpha}x_\epsilon$ (see (2.59) and (2.60)). we take

$$u_{\pm\epsilon}(-x_\epsilon, t) = u(-x_\epsilon, t),\ 0 \le t \le \delta$$

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = ku(-x_\epsilon, k^{(\alpha(p-2)-p)/\alpha}t),\ \ 0 \le t \le k^{\frac{p-\alpha(p-2)}{\alpha}}\delta,$$

which are used to imply (2.52). Moreover, if $\beta > 1$ then to prove uniform boundedness of the sequence $\{u_k^{\pm\epsilon}\}$ we choose

$$g(x,t) = (C+1)(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{1}{1-\beta}},\ x \in \mathbb{R},\ 0 \le t \le t_0 = \frac{\nu^{-1}}{2},$$

where ν, h_* are choosen as in (2.62) and

$$h(x) = (\beta-1)\alpha^{p-1}(C+1)^{p-2}(1-\nu t)^{\frac{p-1-\beta}{1-\beta}}(1+x^2)^{\frac{(\alpha-2)(p-1)-2-\alpha}{2}}x^2|x|^{p-2}$$

$$\times\Big(\frac{1+x^2}{x^2} + (p-2)\frac{1+x^2}{|x|^2} + (\alpha-2)(p-1)\Big)$$

Then, we have

$$L_{kg} \equiv g_t - \big(|g_x|^{p-2}g_x\big)_x + bk^{\frac{\alpha(p-\beta-1)-p}{\alpha}}g^\beta = (C+1)(\beta-1)^{-1}(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{\beta}{1-\beta}}S,$$

where

$$S = \nu - h(x) + b(\beta-1)(C+1)^{\beta-1}k^{\frac{\alpha(p-\beta-1)-p}{\alpha}}(1+x^2)^{\frac{\alpha(\beta-1)}{2}}.$$

Let $R = b(\beta-1)(C+1)^{\beta-1}k^{\frac{\alpha(p-\beta-1)-p}{\alpha}}(1+x^2)^{\frac{\alpha(\beta-1)}{2}}$. Therefore it follows (2.63), where

$$R = O\Big(k^{p-2-p/\alpha}\Big) \text{ uniformly for } (x,t) \in D_{0\epsilon}^{k} \text{ as } k \to +\infty.$$

Now if $\beta = 1$ we take

$$g = (C+1)\exp(\nu t)(1+x^2)^{\frac{\alpha}{2}}$$

where

$$\nu = 1 + \max_{x \in \mathbb{R}} h^{\dagger}(x).$$

$$h^{\dagger}(x) = \alpha^{p-1}(C+1)^{p-2}\exp(\nu t(p-2))(1+x^2)^{\frac{(\alpha-2)(p-1)-2-\alpha}{2}} x^2|x|^{p-2}$$
$$\times\Big(\frac{1+x^2}{x^2} + (p-2)\frac{1+x^2}{|x|^2} + (\alpha-2)(p-1)\Big).$$

Then, we have

$$L_{kg} \equiv g_t - \left(|g_x|^{p-2}g_x\right)_x + bk^{\frac{\alpha(p-2)-p}{\alpha}} g = (C+1)(1+x^2)^{\frac{\alpha}{2}} \exp(\nu t)S,$$

where

$$S = \nu - h^{\dagger}(x) + bk^{\frac{\alpha(p-2)-p}{\alpha}}.$$

Let $R = bk^{\frac{\alpha(p-2)-p}{\alpha}}$. Since $\alpha(p-2)-p < 0$, then $R \to 0$ as $k \to +\infty$.

$$R = O\Big(k^{\frac{\alpha(p-2)-p}{\alpha}}\Big) \text{ uniformly for } (x,t) \in D_{0\epsilon}^{k} \text{ as } k \to +\infty.$$

Moreover, we have for $0 < \epsilon \ll 1$ which implies (2.64). Hence, $\exists\, k_0 = k_0(\alpha; p)$ such that for $\forall k \geq k_0$ the Comparison Theorem 2.4 of [3] implies (2.65). Let G be an arbitrary fixed compact subset of

$$P = \{(x,t) : x \in \mathbb{R}, \quad 0 < t \le t_0\}.$$

We take k_0 so large that $G \subset D^k_{0\epsilon}$ for $k \ge k_0$. From (2.65), it follows that the sequences $\{u_k^{\pm\epsilon}\}, k \ge k_0$, are uniformly bounded in G. As before, from the results of [57, 99] it follows that the sequence of non-negative and locally bounded solutions $\{u_k^{\pm\epsilon}\}$ is locally uniformly Hölder continuous and weakly pre-compact in $W^{1,p}_{loc}(\mathbb{R}\times(0,T))$. It follows for some subsequence k' (2.66). Since $\alpha(p-1-\beta)-p<0$, passing to limit as $k'\to+\infty$, from (2.34) for $u_{k'}^{\pm\epsilon}$ it follows that $v_{\pm\epsilon}$ is a solution to the CP (1.10), (1.11) with $b=0, T=t_0, u_0=(C\pm\epsilon)(-x)_+^{\alpha}$. From Lemma 3.3.1, the required estimation (2.3) follows. □

3.3.2 Proof of Lemma 3.3.3 & Proof of Lemma 3.3.4 : Diffusion & Reaction are in balance

Proof of Lemma 3.3.3. The first assertion of the lemma is known when $p-1\ge 1$ (see Lemma 2.3.3). The proof is similar if $\beta < p-1 < 1$. If we consider a function $u_k(x,t)$ defined as in (2.67). It may easily be checked that (2.67) satisfies (1.10), (1.13). Since under the conditions of the lemma there exists a unique global solution to (1.10), (1.13) we have equation (2.68). If we choose $k = t^{1/(1-\beta)}$ in (2.68), then (2.68) implies (2.14) with $f_1(\zeta) = u(\zeta,1)$.

To prove the second assertion of the lemma, Take an arbitrary $x_1 < 0$. Since u is continuous, there exists $\delta_1 > 0$ such that

$$(C/2)(-x_1)^{p/(p-1-\beta)} \le u(x_1,\delta) \ \text{ for } \delta \in [0,\delta_1] \tag{3.27a}$$

If $C \in (0, C_*)$ then we also choose $\delta_1 > 0$ such that

$$u(x_1, \delta) < C_*(-x_1)^{p/(p-1-\beta)} \quad \text{for } \delta \in [0, \delta_1] \tag{3.27b}$$

Choose $k = (t/\delta)^{1/(1-\beta)}$ in (2.68) and then taking

$$x = -\ell t^{(p-1-\beta)/p(1-\beta)}, \qquad \ell = \ell(\delta) = x_1 \delta^{-(p-1-\beta)/p(1-\beta)},\ \delta \in (0, \delta_1]$$

we obtain (3.11) with

$$\ell_1 = -x_1 \delta_1^{-(p-1-\beta)/p(1-\beta)}, \qquad \lambda = \lambda(\delta) = \delta^{1/(\beta-1)} u(x_1, \delta),\ \delta \in (0, \delta_1].$$

If $0 < C < C_*$, then (3.24) follows from (3.27b). Let $C > C_*$, to prove that $f_1(0) = A_1 > 0$ it is enough to prove that there exists a $t_0 > 0$ such that

$$u(0, t_0) > 0. \tag{3.28}$$

If $p \geq 2$, (3.28) is a known result (see Lemma 2.3.3). To prove (3.28) when $\beta < p-1 < 1$, Consider the function

$$g(x,t) = C_1(-x+t)_+^{p/(p-1-\beta)}$$

where $C_1 \in (C_*, C)$. If $x < t$ we have

$$Lg = bg^\beta S, \qquad S = 1 - \left(\frac{C_1}{C_*}\right)^{p-1-\beta} + \frac{p}{b(p-1-\beta)} C_1^{1-\beta}(-x+t)^{(\beta(1-p)+1)/(p-1-\beta)}.$$

we can choose $x_1 < 0$ and $t_1 > 0$ such that

$$S \le 0 \qquad \text{if } \ x_1 \le x \le t, \quad 0 \le t \le t_1.$$

Since u is continuous, we can also choose $t_1 > 0$ sufficiently small that

$$g(x_1,t) \le u(x_1,t) \ \text{ for } \ 0 \le t \le t_1.$$

Moreover

$$g(x,0) \le u_0(x) \ \text{ for } \ x \ge x_1$$

Applying Lemma 2.1 of [3] we have

$$u(x,t) \ge g(x,t) \qquad \text{for } \ x \ge x_1, \ 0 \le t \le t_1,$$

which implies (3.28). The lemma is proved. □

Lemma 3.3.4 may be proved by localization of the proof given in Lemma 3.3.3.The proof of Lemma 3.3.5 coincides with the proof of Lemma 2.3.4.

3.4 Proofs of the main results

In this section, we prove the main results for fast diffusion case.

(I) $b \ne 0$ and $p < 2$.

3.4.1 Domination by diffusion: Interface expands

Proof of Theorem 3.1.1. The asymptotic estimations(2.3) and(2.6) follow from Lemma 3.3.2. Take an arbitrary sufficiently small number $\epsilon > 0$; from (2.3) it follows that there exists a number $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$(A_0 - \epsilon)t^{\frac{\alpha}{p-\alpha(p-2)}} \le u(0,t) \le (A_0 + \epsilon)t^{\frac{\alpha}{p-\alpha(p-2)}}, \quad 0 \le t \le \delta_1, \tag{3.29}$$

where $A_0 = f(0) > 0$. Consider a function $g(x,t)$ as given in (2.76), then we obtain(2.77). For the function f_1, we take

$$f_1(\zeta) = C_0(\zeta_0 - \zeta)_+^{p/(p-1-\beta)},\ 0 < \zeta < +\infty$$

where C_0, ζ_0 are some positive constants. From (2.77b), we have (2.78). To prove a lower estimation, we take $C_0 = C_1, \zeta_0 = \zeta_1$ (see Appendix Part B). Then we have

$$\mathcal{L}^0 f_1 \le bC_1^{\beta}(\zeta_1 - \zeta)_+^{\frac{p\beta}{p-1-\beta}}\Big\{1 - \Big(\frac{C_1}{C_*}\Big)^{p-1-\beta} + \frac{C_1^{1-\beta}}{b(1-\beta)}\zeta_1^{\frac{p(1-\beta)}{p-1-\beta}}\Big\} = 0. \tag{3.30}$$

From (2.77), it follows that (2.81). Lemma 2.1 of [3] implies that g is a sub-solution of equation (1.10) in $\{(x,t) : x > 0, t > 0\}$. Since $1/(1-\beta) > \alpha/(p-\alpha(p-2))$, it follows from (3.29) that there exists a $\delta_2 > 0$, which does not depend on ϵ, such that

$$g(0,t) \le u(0,t) \quad \text{for } 0 \le t < \delta_2. \tag{3.31}$$

we also have (2.80a). Now we can fix a particular value of $\epsilon = \epsilon_0$ and take $\delta = \min(\delta_1, \delta_2)$. From (2.81), (3.31), (2.80a) and Lemma 2.1 of [3], the left-hand side of (3.9), (3.1) follow. To prove an upper estimation, first we use the rough estimation (3.20). The estimation (3.20) is obvious, since by Comparison Theorem 2.4 of [3] $u(x,t)$ may be upper estimated by the solution of equation (1.10) with $b = 0$. Using (3.20), we can now establish a more accurate estimation. For that, consider a function g with $C_0 = C_*$, $\zeta_0 = \zeta_2$ in $G_{\ell_0,\delta}$, where

$$G_{\ell_0,\delta} = \{(x,t) : \zeta_{\ell_0}(t) = \ell_0 t^{(p-1-\beta)/p(1-\beta)} < x < +\infty,\ 0 < t \leq \delta\}.$$

From (2.77),(2.78) it follows (2.79). Moreover, from (3.20) we have

$$u(\zeta_{\ell_0}(t),t) \leq D\ell_0^{p/(p-2)} t^{1/(1-\beta)} = C_*(\zeta_2 - \ell_0)_+^{p/(p-1-\beta)} t^{1/(1-\beta)} = g(\zeta_{\ell_0}(t),t) \quad \text{for } 0 \leq t \leq \delta. \tag{3.32}$$

By applying the Lemma 2.1 of [3] in $G_{\ell_0,\delta}$, the right hand side of (3.9), (3.1) follow from (2.79), (3.32) and (2.80a).

If u_0 is defined as in (1.13), then the CP (1.10), (1.13) has a global solution and from a Comparison Theorem 2.4 of [3] it follows that the solution may be globally upper estimated by the solution to the CP (1.10), (1.13) with $b = 0$. Hence (3.20), (3.32) and the right-hand side of (3.9) is valid for $0 < t < +\infty$. □

3.4.2 Borderline case: Diffusion & Reaction are in balance

Proof of Theorem 3.1.2. First, assume that u_0 is defined by (1.13). The self-similar form (2.14) follows from Lemma 3.3.3. The proof of the estimation (3.10a) when $C > C_*$ and the proof of the right-hand side of (3.10a) when $0 < C < C_*$ (and of the corresponding local ones when u_0 satisfies (1.12)) fully coincides with the proof given in Theorem 2.1.2 for the case $1 < (p-1) < \beta^{-1}$, $p > 2$ (see (2.16) and (2.19)). To prove the left-hand side of (3.10a), consider a function g from (2.76) with

$$f_1(\zeta) = C_*(-\zeta_5 - \zeta)_+^{p/(p-1-\beta)}, \qquad -\infty < \zeta < +\infty$$

From (2.77),(2.78) it follows that

$$Lg \leq 0 \quad \text{in } G_{-\ell_1,\infty} \tag{3.33}$$

Moreover, we have

$$u(-\ell_1 t^{(p-1-\beta)/(p(1-\beta))}, t) = \lambda t^{1/(1-\beta)} = C_*(\ell_1 - \zeta_5)_+^{p/(p-1-\beta)} t^{1/(1-\beta)}$$

$$= g(-\ell_1 t^{(p-1-\beta)/p(1-\beta)}, t), \qquad 0 \leq t < +\infty, \tag{3.34}$$

(2.87b) and (2.87c), where $x_0 > 0$ is an arbitrary fixed number. By using (3.33), (3.34), (2.87b) and (2.87c), we can apply Lemma 2.1 of [3] in

$$G'_{-\ell_1,\infty} = G_{-\ell_1,\infty} \cap \{x < x_0\}.$$

Since $x_0 > 0$ is an arbitrary number the desired lower estimation from (3.10a) follows .

Suppose that u_0 satisfies (1.12) with $\alpha = p/(p-1-\beta)$, $0 < C < C_*$. Then from (3.25) it follows that for arbitrary sufficiently small $\epsilon > 0$ there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$(\lambda - \epsilon) t^{1/(1-\beta)} \leq u(-\ell_1 t^{(p-1-\beta)/(p(1-\beta))}, t) \leq (\lambda + \epsilon) t^{1/(1-\beta)}, \qquad 0 \leq t \leq \delta.$$

Using this estimation, the left-hand side of (3.10a) may be established locally in time. The proof completely coincides with the proof given above for the global estimations, except that λ should be replaced by $\lambda - \epsilon$. (2.14) and (3.10a) easily imply (2.15) and (3.10b). □

3.4.3 Domination by absorption: Interface shrinks

Proof of Theorem 3.1.3. The asymptotic estimation (2.21) follows from Lemma 3.3.5. The proof of the asymptotic estimation (2.20) coincides with the proof given in Theorem 2.1.3. In particular, the estimation (2.92) and (2.93) are true in this case as well. □

3.4.4 Infinite speed propagation: Diffusion dominates weakly over the reaction

Proof of Theorem 3.1.4. The asymptotic estimations (2.3) and (2.6) follow from Lemma 3.3.2. From (2.3), (3.29) follows, where we fix a particular value of $\epsilon = \epsilon_0$. The function $g(x,t) = t^{1/(2-p)}\phi(x)$ is a solution of (1.10). Since $1/(2-p) > \alpha/(p + \alpha(2-p))$, there exists $\delta > 0$ such that

$$u(0,t) = A_0 t^{\frac{\alpha}{p+\alpha(2-p)}} \geq t^{\frac{1}{2-p}} = \phi(0)t^{\frac{1}{2-p}} = g(0,t), \qquad 0 \leq t \leq \delta.$$

$$u(x,0) = g(x,0) = 0, \qquad 0 \leq x < \infty$$

Therefore, from Lemma 2.1 of [3], the left-hand side of (3.2) follows. Let us prove the right-hand side of (3.2). As it was mentioned in Section 3.1, the right-hand side of (3.2) is valid for $0 < t < +\infty$ if the initial data u_0 from (1.11) vanishes for $x \geq 0$. For all $\epsilon > 0$

and consider a function

$$g_\epsilon(x,t) = (t+\epsilon)^{1/(2-p)}\phi(x),$$

$$g_\epsilon(0,t) = (t+\epsilon)^{1/(2-p)}\phi(0) = (t+\epsilon)^{1/(2-p)} \geq \epsilon^{1/(2-p)} \geq$$

$$\geq (A_0+\epsilon)t^{\frac{\alpha}{p+\alpha(2-p)}} = u(0,t), \text{ for } 0 \leq t \leq \delta_\epsilon = \left[(A_0+\epsilon)^{-1}\epsilon^{1/(2-p)}\right]^{\frac{p+\alpha(2-p)}{\alpha}},$$

Due to continuity of g_ϵ and u, $\exists\, \delta_{1\epsilon} > 0$ such that $g_\epsilon(0,t) \geq u(0,t)$. Since g_ϵ is a solution of (1.10), from the Lemma 2.1 of [3] it follows that

$$u(x,t) \leq g_\epsilon(x,t) = (t+\epsilon)^{1/(2-p)}\phi(x), \quad \text{for } 0 \leq x < +\infty,\ 0 \leq t \leq \delta_\epsilon. \tag{3.35}$$

Integration of (3.3) implies (3.12). Global estimation (3.15) (3.16) (3.17) (3.18). By rescaling $x \to \epsilon^{-1}x, \epsilon > 0$ from (3.12) we have

$$\frac{x}{\epsilon} = \int_{\phi(\frac{x}{\epsilon})}^{1} y^{-1}\left[\frac{b}{p-1} + \frac{p}{2(p-1)(2-p)}y^{2-p}\right]^{-1/p} dy.$$

Change of variable $z = -\epsilon \log y$ implies

$$x = \mathcal{F}[\Phi_\epsilon(x)], \tag{3.36}$$

where

$$\mathcal{F}(y) = \int_0^y \left[\frac{b}{p-1} + \frac{p}{2(p-1)(2-p)}e^{\frac{(p-2)}{\epsilon}z}\right]^{-1/p} dz,$$

$$\Phi_\epsilon(x) = -\epsilon \log \phi(\frac{x}{\epsilon}).$$

From (3.36) it follows that

$$\Phi_\epsilon(x) = \mathcal{F}^{-1}(x), \tag{3.37}$$

where $\mathcal{F}^{-1}$ is an inverse function of $\mathcal{F}$. Since $1 < p < 2$ it easily follows that

$$\lim_{\epsilon\to 0}\mathcal{F}(y) = \Big(\frac{b}{p-1}\Big)^{-1/p} y,\ \lim_{\epsilon\to 0}\mathcal{F}^{-1}(y) = \Big(\frac{b}{p-1}\Big)^{1/p} y, \tag{3.38}$$

for $y \geq 0$ and convergence is uniform in bounded subsets of $\mathbb{R}^+$. From (3.37), (3.38) it follows that

$$-\lim_{\epsilon\to 0}\epsilon \log \phi\Big(\frac{x}{\epsilon}\Big) = \Big(\frac{b}{p-1}\Big)^{1/p} x, \quad 0 < x < +\infty. \tag{3.39}$$

By letting $y = x/\epsilon$ from (3.39), (3.14) follows. Global estimation (3.15), and accordingly also (3.17) (3.18) easily follow from (3.12), (3.13). □

3.4.5 Infinite speed propagation: Diffusion dominates strongly over the reaction

Proof of Theorem 3.1.5. Let either $b > 0$, $\beta > p-1$ or $b < 0$, $\beta \geq 1$. The asymptotic estimations (2.3) and (2.6) follow from Lemma 3.3.2. Take an arbitrary sufficiently small number $\epsilon > 0$. From (2.3), it follows that there exists a number $\delta_1 = \delta_1(\epsilon) > 0$ such that (3.29) is valid. Let $\beta \geq 1$.

Consider a function

$$g(x,t) = t^{\alpha/(p+\alpha(2-p))} f(\xi), \quad \xi = xt^{-1/(p+\alpha(2-p))}. \tag{3.40}$$

We have

$$Lg = t^{(\alpha(p-1)-p)/(p+\alpha(2-p))}\mathcal{L}_t f \tag{3.41a}$$

$$\mathcal{L}_t f = \frac{\alpha}{p+\alpha(2-p)} f - \frac{1}{p+\alpha(2-p)} \xi f' - (|f'|^{p-2} f')' + bt^{(p-\alpha(p-1-\beta))/(p-\alpha(p-2))} f^{\beta}. \tag{3.41b}$$

As a function f we take

$$f(\xi) = C_0(\xi_0 + \xi)^{-\gamma_0}, \qquad 0 \le \xi < +\infty \tag{3.42}$$

where C_0, ξ_0, γ_0 are some positive constants. Taking $\gamma_0 = p/(2-p)$ from (3.41b) we have

$$\mathcal{L}_t f = (p + \alpha(2-p))^{-1} C_0 (\xi_0 + \xi)^{\frac{p}{p-2}}$$

$$\times \Big[R(\xi) + bt^{(p-\alpha(p-1-\beta))/(p-\alpha(p-2))} (p+\alpha(2-p)) C_0^{\beta-1} (\xi_0+\xi)^{\frac{p(1-\beta)}{2-p}} \Big] \tag{3.43a}$$

$$R(\xi) = [\alpha - 2(p-1)p^{p-1}(p+\alpha(2-p))(2-p)^{-p} C_0^{p-2} + p(2-p)^{-1}\xi(\xi_0+\xi)^{-1}]. \tag{3.43b}$$

To prove an upper estimation we take $C_0 = C_6$, $\xi_0 = \xi_2$ (see Appendix Part B). Then we have

$$R(\xi) \ge \alpha(\mu_b - 1)\mu_b^{-1} \tag{3.44}$$

From (3.43), (3.44) it follows that

$$\mathcal{L}_t f \ge 0 \quad \text{for } \xi \ge 0, \ \ 0 \le t \le \delta_2,$$

where

$$\delta_2 = \delta_1 \quad \text{if } b > 0, \qquad \delta_2 = \min(\delta_1, \delta_3) \quad \text{if } b < 0$$

and

$$\delta_3 = [\alpha\epsilon(A_0+\epsilon)^{1-\beta}(-b(p+\alpha(2-p))(1+\epsilon))^{-1}]^{(p+\alpha(2-p))/(p+\alpha(\beta+1-p))}$$

Hence , from (3.41) we have

$$Lg \geq 0 \qquad \text{for } \ 0 \leq x < +\infty, \ \ 0 < t \leq \delta_2. \tag{3.45}$$

From (3.29) and Lemma 2.1 of [3], the right-hand side of (3.19) follows with $\delta = \delta_2$. To prove a lower estimation in this case we take $C_0 = C_5$, $\xi_0 = \xi_1$. If $b > 0$ and $\beta < 2/p$ we derive from (3.43) that

$$R(\xi) \leq \alpha + p(2-p)^{-1} - 2(p-1)p^{p-1}(p+\alpha(2-p))(p-2)^{-p}C_5^{p-2}$$

$$= -(p+\alpha(2-p))((2-p)(1-\epsilon))^{-1}\epsilon \tag{3.46a}$$

$$\mathcal{L}_t f \leq 0 \qquad \text{for } \ \xi \geq 0, \ 0 \leq t \leq \delta_4, \tag{3.46b}$$

where $\delta_4 = \min(\delta_1, \delta_5)$ and

$$\delta_5 = \left[(A_0-\epsilon)^{1-\beta}(b(2-p)(1-\epsilon))^{-1}\epsilon\right]^{(p+\alpha(2-p))/(p+\alpha(\beta+1-p))}.$$

From (3.46) it follows that

$$Lg \leq 0 \qquad \text{for } 0 \leq x < +\infty,\ 0 < t \leq \delta_4. \tag{3.47}$$

If either $b > 0$, $\beta \geq 2/p$ or $b < 0$, $\beta \geq 1$, from (3.43) we have

$$\mathcal{L}_t f = (p + \alpha(2-p))^{-1} C_5 (\xi_1 + \xi)^{\frac{2}{p-2}}$$

$$\times \Big[R_1(\xi) + b t^{(p-\alpha(p-1-\beta))/(p-\alpha(p-2))} (p + \alpha(2-p)) C_5^{\beta-1} (\xi_1 + \xi)^{\frac{(2-p\beta)}{2-p}} \Big] \tag{3.48a}$$

$$R_1(\xi) = [\alpha - 2(p-1)p^{p-1}(p + \alpha(2-p))(2-p)^{-p} C_5^{p-2}](\xi_1 + \xi) + p(2-p)^{-1}\xi$$

$$= -p(2-p)^{-1}\xi_1, \tag{3.48b}$$

which again imply (3.46b), where $\delta_4 = \delta_1$ if $b < 0$, $\delta_4 = \min(\delta_1, \delta_5)$ if $b > 0$ and

$$\delta_5 = [p(b(p + \alpha(2-p))(2-p))^{-1}(A_0 - \epsilon)^{1-\beta}]^{(p+\alpha(2-p))/(p+\alpha(\beta+1-p))}.$$

As before (3.47) follows from (3.48b). From (3.29), and Lemma 2.1 of [3], the left-hand side of (3.19) follows with $\delta = \delta_4$. Thus we have proved (3.19) with $\delta = \min(\delta_2, \delta_4)$.

Let $b > 0$, $\beta \geq 1$. The upper estimation of (3.20) is an easy consequence of Lemma 2.1 of [3], since the right-hand side of it is a solution of (1.10) with $b = 0$. Let $b > 0$ and $\beta \geq 2/p$. Now we can fix a particular value of $\epsilon = \epsilon_0$ and take $\delta = \delta(\epsilon_0) > 0$ in

(3.19). Then from the left-hand side of (3.19) and (3.20), the asymptotic result (3.6) follows. However, if $b > 0,\ 1 \le \beta < 2/p$, from (3.19) and (3.20) it follows that for $\forall$ fixed $t \in (0, \delta(\epsilon)]$

$$D(1-\epsilon)^{1/(2-p)} \le \liminf_{x\to+\infty} ut^{1/(p-2)} x^{\frac{p}{2-p}} \le \limsup_{x\to+\infty} ut^{1/(p-2)} x^{\frac{p}{2-p}} \le D,$$

which easily implies (3.7) in view of arbitrariness of ϵ.

We now let $b < 0$, $\beta \ge 1$ and prove (3.21). Consider a function

$$\bar{g}(x,t) = D(1-\epsilon)^{1/(p-2)} t^{1/(2-p)} x^{p/(p-2)}$$

in $G = \{(x,t) : \mu t^{1/(p+\alpha(2-p))} < x < +\infty,\ \ 0 < t \le \delta\}$, where μ is defined as in (3.21). Let $g(x,t) = \bar{g}(x,t)$ for $(x,t) \in \bar{G}\backslash(0,0)$ and $g(0,0) = 0$. We have

$$Lg = D(2-p)^{-1}(1-\epsilon)^{(1/(p-2)} t^{(p-1)/(2-p)} x^{p/(p-2)} \mathcal{G} \quad \text{in } G$$

$$\mathcal{G} = \epsilon + b(2-p)D^{\beta-1}(1-\epsilon)^{(\beta-1)/(p-2)} t^{(\beta+1-p)/(2-p)} x^{p(\beta-1)/(p-2)}.$$

We then derive

$$\mathcal{G} \ge \epsilon + b(2-p)D^{\beta-1}(1-\epsilon)^{(\beta-1)/(p-2)} \mu^{p(\beta-1)/(p-2)} t^{(p+\alpha(\beta+1-p))/(p+\alpha(2-p))} \quad \text{in } G.$$

Hence,

$$\mathcal{G} \ge 0 \quad \text{in } G, \quad \text{for } \ \delta \in (0, \delta_0]$$

$$\delta_0 = \Big[\big(-b(2-p)\big)^{-1} D^{1-\beta}(1-\epsilon)^{(1-\beta)/(p-2)} \mu^{p(1-\beta)/(p-2)} \epsilon\Big]^{(p+\alpha(2-p))/(p+\alpha(\beta+1-p))},$$

which implies

$$Lg \geq 0 \quad \text{in } G. \tag{3.49a}$$

Moreover, we have

$$g|_{x=\mu t^{1/(p+\alpha(2-p))}} = (A_0+\epsilon)t^{\alpha/(p+\alpha(2-p)} \quad \text{for } \ 0 \leq t \leq \delta.$$

From(3.19), it follows that

$$u|_{x=\mu t^{1/(p+\alpha(2-p))}} \leq C_6(\xi_2+\mu)^{\frac{p}{p-2}} t^{\alpha/(p+\alpha(2-p))}$$

$$\leq (A_0+\epsilon)t^{\alpha/(p+\alpha(2-p)} \quad \text{for } \ 0 \leq t \leq \delta.$$

Therefore, we have

$$g \geq u \qquad \text{on } \ \bar{G} \setminus G, \tag{3.49b}$$

From (3.49), and Lemma 2.1 of [3], the desired estimation (3.21) follows. Since $\epsilon > 0$ is arbitrary, from the left-hand side of (3.19) and (3.21) the asymptotic result (3.6) follows as before.

Let $b > 0$, $p-1 < \beta < 1$. The left-hand side of (3.22) may be proved as the left-hand side of (3.9) was earlier. The only difference is that we take $f_1(\zeta) = C_*(1-\epsilon)(\zeta_8+\zeta)_+^{p/(p-1-\beta)}$ in (2.76), (2.77). The right-hand side of (3.22) is almost trivial, since $C_* x^{p/(p-1-\beta)}$ is a stationary solution of (1.10). The important point in (3.22) is that $\delta > 0$ does not depend on $\epsilon > 0$. This is clear from the analysis involved in the proof of the similar estimation (3.9). From (3.22), it follows that $\forall$ fixed $t \in (0,\delta]$, we have

$$C_*(1-\epsilon) \leq \liminf_{x\to+\infty} u x^{p/(\beta+1-p)} \leq \limsup_{x\to+\infty} u x^{p/(\beta+1-p)} \leq C_*.$$

Since $\epsilon > 0$ is arbitrary, (3.8) easily follows.

II. $b = 0$

First assume that u_0 is defined by (1.13). The self-similar form (2.4) and the formula (2.6) follow from Lemma 3.3.1. To prove (3.23), consider a function g from (3.40), which satisfies (3.41) with $b = 0$. As a function f we take (3.42) with $\gamma_0 = p/(2-p)$. Then we drive (3.43) with $b = 0$. To prove an upper estimation we take $C_0 = C_7$, $\xi_0 = \xi_4$ and from (3.43b) we have

$$R(\xi) \geq [\alpha - 2p^{p-1}(p-1)(p+\alpha(2-p))(2-p)^{-p}C_7^{p-2}] = 0,$$

which implies (3.45) with $\delta_2 = +\infty$. As before, from (3.45) and Lemma 2.1 of [3], the right-hand side of (3.23) follows. The left-hand side of (3.23) may be established similarly if we take $C_0 = D, \xi_0 = \xi_3$. To prove the estimation (3.20), consider

$$g_\mu(x,t) = D(t+\mu)^{1/(2-p)}(x+\mu)^{p/(p-2)}, \qquad \mu > 0,$$

which is a solution of (1.10) for $x > 0,\ t > 0$. Since

$$g_\mu(0,t) \geq D\mu^{(p-1)/(p-2)} \geq u(0,t) \quad \text{for } 0 \leq t \leq T(\mu) = [DA_0^{-1}\mu^{(p-1)/(p-2)}]^{(p+\alpha(2-p))/\alpha},$$

the Lemma 2.1 of [3] implies

$$u(x,t) \leq g_\mu(x,t) \qquad \text{for } 0 < x < +\infty,\ 0 \leq t \leq T(\mu).$$

In the limit as $\mu \to 0+$, we can easily derive (3.20). Finally, from (3.23) and (3.20) it easily follows that for arbitrary fixed $0 < t < +\infty$ the asymptotic formula (3.6) is valid. If

u_0 satisfies (1.12) with $\alpha > 0$, then (2.3) and (3.29) follow from Lemma 3.3.1. Similarly, we can then prove that for arbitrary sufficiently small $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that (3.23) is valid for $0 \leq t \leq \delta(\epsilon)$, except that in the left-hand side (respectively in the right-hand side) of (3.23) the constant A_0 is replaced by $A_0 - \epsilon$ (respectively by $A_0 + \epsilon$). Then we can fix a particular value of $\epsilon = \epsilon_0$ and let $\delta = \delta(\epsilon_0) > 0$. Obviously, from the local analog of (3.23) and (3.20) it follows that, for arbitrary fixed $t \in (0, \delta]$, the asymptotic formula (3.6) is valid. □

Chapter 4

Conclusions

The dissertation presents full classification of the short-time behavior of the interfaces and local solution near the interfaces or at infinity in the Cauchy problem for the non-linear parabolic p-Laplacian type reaction-diffusion equation of non-Newtonian elastic filtration in both slow and fast diffusion regimes:

$$u_t = \left(|u_x|^{p-2}u_x\right)_x - bu^\beta = 0,\ x \in \mathbb{R}, 0 < t < T,\ p > 1, b, \beta > 0;\ u(x,0) \sim C(-x)_+^\alpha,\ \text{as}\ x \to 0-.$$

The classification is based on the relative strength of the diffusion and absorption forces. The following are the main results:

- If $p > 2, \alpha < \frac{p}{p-1-\min\{1,\beta\}}$, then slow diffusion dominates over the absorption, and interface expands with asymptotics

$$\eta(t) \sim \xi_*(C, p, \alpha)t^{1/(p-\alpha(p-2))} \ \text{as}\ t \to 0+$$

- If $p \geq 2, 0 < \beta < 1, \alpha = p/(p-1-\beta)$ then diffusion and absorption are in balance, there is a critical value C_* such that the interface expands or shrinks accordingly as $C > C_*$ or $C < C_*$ and

$$\eta(t) \sim \zeta_*(C,p,\beta)t^{\frac{p-1-\beta}{p(1-\beta)}} \quad \text{as} \quad t \to 0+,$$

where $\zeta_* \lessgtr 0$ if $C \lessgtr C_*$.

- If $p \geq 2, 0 < \beta < 1, \alpha > p/(p-1-\beta)$, then absorption strongly dominates over the diffusion and interface shrinks with asymptotics

$$\eta(t) \sim -\ell_*(C,p,\alpha,\beta)t^{1/\alpha(1-\beta)} \text{ as } t \to 0+,$$

- If $p > 2, \alpha \geq p/(p-2), \beta \geq 1$, then slow difusion dominates over the absorption and interface has initial waiting time.

- If $1 < p < 2, 0 < \beta < p-1, 0 < \alpha < p/(p-1-\beta)$, then diffusion weakly dominates over the absorption and interface expands with asymptotics

$$\eta(t) \sim \gamma(C,p,\alpha)t^{(p-1-\beta)/p(1-\beta)} \quad \text{as } t \to 0+.$$

- If $1 < p < 2, 0 < \beta < p-1,\ \alpha = p/(p-1-\beta)$, then diffusion and absorption are in balance, there is a critical value C_* such that the interface expands or shrinks accordingly as $C > C_*$ or $C < C_*$ and

$$\eta(t) \sim \zeta_*(C,p,\beta)t^{(p-1-\beta)/p(1-\beta)}, \qquad \text{as } t \to 0+,$$

where $\zeta_* \lessgtr 0$ if $C \lessgtr C_*$.

- If $1 < p < 2, 0 < \beta < p-1, \alpha > p/(p-1-\beta)$, then absorption strongly dominates over the diffusion and interface shrinks with asymptotics

$$\eta(t) \sim -\ell_*(C, \alpha, p, \beta) t^{1/\alpha(1-\beta)} \qquad \text{as } t \to 0+,$$

- If $1 < p < 2, 0 < \beta = p-1 < 1, \alpha > 0$, then domination of the diffusion over absorption is moderate, there is an infinite speed of propagation and solution has exponential decay at infinity.

- If $1 < p < 2, \beta > p-1$, then diffusion strongly dominates over the absorption, and solution has power type decay at infinity.

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Appendix

Part A: We give here explicit values of the constants used in section 2.1 in the outline of the results for Case (I(2)) and later in section 2.4 during the proof of these results.

$\zeta_1 = A^{\frac{p-2}{p}}(1-\beta)^{\frac{1}{p}}(p-1)(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{p}}(p-2)^{-1}$,

$C_1 = A_1\zeta_1^{-\mu} \quad \text{if} \quad \beta(p-1) > 1$,

$\zeta_1 = A_1^{\frac{p-2}{p}}((1-\beta)(1+\beta)p^{p-1}(p-1))^{\frac{1}{p}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{p}}(p-1-\beta)^{-1}$,

$C_1 = A_1\zeta_1^{-\frac{p}{p-1-\beta}}, \quad \text{if} \quad \beta(p-1) < 1$,

$\zeta_2 = A_1^{\frac{p-2}{p}}((1-\beta)(1+\beta)p^{p-1}(p-1))^{\frac{1}{p}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{p}}(p-1-\beta)^{-1}$,

$C_2 = A_1\zeta_2^{-\frac{p}{p-1-\beta}}, \quad \text{if} \quad \beta(p-1) > 1$,

$\zeta_2 = \left(A_1/C_*\right)^{\frac{p-1-\beta}{p}}, \quad C_2 = C_*, \quad \text{if} \ \beta(p-1) < 1$,

$\bar{\zeta}_2 = A_1^{\frac{p-2}{p}}\left(\frac{p(p-1)^p(p-2)^{1-p}(1-\beta)}{p(p-2)-\beta(p-1)+1}\right)^{\frac{1}{p}}, \quad \bar{C}_2 = A_1\bar{\zeta}_2^{-\frac{(p-1)}{(p-2)}}, \quad \text{if} \quad \beta(p-1) > 1$,

$\bar{\zeta}_2 = A^{\frac{p-2}{p}}(1-\beta)^{\frac{1}{p}}(p-1)(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{p}}(p-2)^{-1}$,

$\bar{C}_2 = A_1\bar{\zeta}_2^{-\frac{(p-1)}{(p-2)}}, \quad \text{if} \quad \beta(p-1) < 1$,

$\ell_0 = C_*^{\frac{1+\beta-p}{p}}(C_*/C)^{\frac{(1-\beta)(p-1-\beta)}{1-\beta(p-1)}}(b(1-\beta)\theta_*)^{\frac{p-1-\beta}{p(1-\beta)}}$,

$\zeta_3 = C_*^{\frac{1+\beta-p}{p}}\left[(C_*/C)^{\frac{(1-\beta)(p-1-\beta)}{1-\beta(p-1)}} - 1\right](b(1-\beta)\theta_*)^{\frac{p-1-\beta}{p(1-\beta)}}$,

$\theta_* = \left[1-\left(C/C_*\right)^{p-1-\beta}\right]\left[\left(C_*/C\right)^{\frac{(1-\beta)(p-1-\beta)}{1-\beta(p-1)}} - 1\right]^{-1}$,

$\ell_1 = C^{\frac{1+\beta-p}{p}}\left[b(1-\beta)(\delta_*\Gamma)^{-1}\left((1-\delta_*\Gamma)-(1-\delta_*\Gamma)^{1-p}\left(C/C_*\right)^{p-1-\beta}\right)\right]^{\frac{p-1-\beta}{p(1-\beta)}}$,

$\zeta_4 = \delta_*\Gamma\ell_1, \quad \Gamma = 1-(C/C_*)^{\frac{p-1-\beta}{p}}, \quad C_3 = C(1-\delta_*\Gamma)^{\frac{p}{1+\beta-p}}$,

where $\delta_* \in (0,1)$ satisfies

$$g(\delta_*) = \max_{[0;1]} g(\delta), \qquad g(\delta) = \delta^{\frac{1+\beta(1-p)}{p(1-\beta)}}\Big[(1-\delta\Gamma) - \Big(C/C_*\Big)^{p-1-\beta}(1-\delta\Gamma)^{1-p}\Big)\Big],$$

$\ell_* = C^{-\frac{1}{\alpha}}(b(1-\beta))^{1/(\alpha(1-\beta))}$

$\zeta_5 = (\frac{\ell_*}{\ell})^{\alpha(1-\beta)}(1-\epsilon)\ell, \qquad \text{if } \beta(p-1) < 1,$

$C_6 = (1-(\frac{\ell_*}{\ell})^{\alpha(1-\beta)}(1-\epsilon))^{-\alpha}[C^{1-\beta} - \ell^{-\alpha(1-\beta)}b(1-\beta)(1-\epsilon)]^{\frac{1}{1-\beta}}.$

Part B: We given here explicit values of the constants used in Section 2 in the outline of the results and later in section 4 during the proof of these results.

(1) when $0 < \beta < p-1,\ 0 < \alpha < p/(p-1-\beta)$

$C_* = \Big[(b|p-1-\beta|^p)/((1+\beta)(p-1)p^{p-1})\Big]^{1/(p-1-\beta)}, \ C_1 = ((1-\beta)/(2-p))^{1/(p-1-\beta)}C_*$

$\zeta_1 = b^{\frac{p-2}{p(1-\beta)}}(p^{p-1}(p-1))^{1/p}(1+\beta)^{1/p}(p-1-\beta)^{\frac{\beta(p-1)-1}{p(1-\beta)}}((2-p)/(1-\beta))^{(2-p)/p(1-\beta)},$

$\zeta_2 = b^{(p-2)/p(1-\beta)}(p-1)^{1/p}p^{(p-1)/p}(1+\beta)^{(2-p)/p(1-\beta)}2^{(p-1-\beta)/p(1-\beta)}(2-p)^{\frac{\beta(p-1)-1}{p(1-\beta}}(1-\beta)(p-1-\beta)^{-1},$

$\ell_0 = \frac{p-1-\beta}{1-\beta}\zeta_2,$

(2) when $b > 0,\ 0 < \beta < 1,\ \beta < p-1 < \beta^{-1},\ \alpha = p(p-1-\beta)^{-1}$

$\zeta_3 = A_1^{\frac{p-2}{p}}((1-\beta)(1+\beta)p^{p-1}(p-1))^{\frac{1}{p}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{p}}(p-1-\beta)^{-1},$

$\zeta_4 = \Big(A_1/C_*\Big)^{\frac{p-1-\beta}{p}}, \qquad C_2 = A_1\zeta_3^{-\frac{p}{p-1-\beta}},$

$\zeta_5 = \ell_1 - (\lambda/C_*)^{(p-1-\beta)/p} > 0$ (see Lemma 3.3 and (2.51))

$\ell_2 = C^{\frac{1+\beta-p}{p}}\Big[b(1-\beta)(\delta_*\Gamma)^{-1}\Big((1-\delta_*\Gamma)-(1-\delta_*\Gamma)^{1-p}\Big(C/C_*\Big)^{p-1-\beta}\Big)\Big]^{\frac{p-1-\beta}{p(1-\beta)}},$

$\zeta_6 = \delta_*\Gamma\ell_2, \qquad \Gamma = 1-(C/C_*)^{\frac{p-1-\beta}{p}}, \qquad C_3 = C(1-\delta_*\Gamma)^{\frac{p}{1+\beta-p}},$ where $\delta_* \in (0,1)$ satisfies

$g(\delta_*) = \max_{[0;1]} g(\delta), \qquad g(\delta) = \delta^{\frac{1+\beta(1-p)}{p(1-\beta)}}\Big[(1-\delta\Gamma) - \Big(C/C_*\Big)^{p-1-\beta}(1-\delta\Gamma)^{1-p}\Big)\Big],$

(5) when $\beta > p-1$

$D = \Big[\frac{2(p-1)p^{p-1}}{(2-p)^{p-1}}\Big]^{1/(2-p)}$

$\xi_1 = (A_0-\epsilon)^{(p-2)/p}(1-\epsilon)^{1/p}D^{(2-p)/p}$ if $b > 0,\ 1 \le \beta < (4-p)/p,$

$\xi_1 = (A_0-\epsilon)^{(p-2)/p}D^{(2-p)/p}$ if either $b > 0,\ \beta \ge (4-p)/p$ or $b < 0,\ \beta \ge 1,$

$C_5 = (A_0 - \epsilon)\xi_1^{p/(2-p)}$

$A_0 = f(0) > 0$ (see (2.2) and lemma 3.1)

$\xi_2 = (A_0 + \epsilon)^{(p-2)/p}\left[\frac{2(p-1)p^{p-1}(p+\alpha(2-p))\mu_b}{\alpha(2-p)^p}\right]^{1/p}$

$C_6 = \left[\frac{2(p-1)p^{p-1}(p+\alpha(2-p))\mu_b}{\alpha(2-p)^p}\right]^{1/(2-p)}$

$\mu_b = 1$ if $b > 0$, $\mu_b = 1 + \epsilon$ if $b < 0$,

$\zeta_8 = \left[b(1-\beta)C_*^{\beta-1}(1-\epsilon)^{\beta-1}\left((1-\epsilon)^{p-1-\beta} - 1\right)\right]^{(p-1-\beta)/p(1-\beta)}$

II $b = 0$

$\xi_3 = (A_0/D)^{(p-2)/p}$, $\xi_4 = \xi_3\left(1 + p/\alpha(2-p)\right)^{1/p}$

$C_7 = D\left[1 + p/\alpha(2-p)\right]^{1/(2-p)}$