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ABSTRACT

Title:

Some Free Boundary Problems for The Nonlinear Degenerate Multidimensional Parabolic Equations Modeling Reaction-Diffusion Processes

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This dissertation presents a full classification of the short-time behavior of the interfaces or free boundaries for the nonlinear second order degenerate multidimensional parabolic partial differential equation (PDE)

$$u_t - \Delta u^m + bu^\beta = 0,\ x \in \mathbb{R}^N, 0 < t < T \tag{1}$$

with $m > 0, \beta > 0, b \in \mathbb{R}$, arising in various applications in fluid mechanics, filtration of oil or gas in a porous media, plasma physics, reaction-diffusion equations in chemical kinetics, population dynamics in mathematical biology etc. as a mathematical model of nonlinear diffusion phenomena in the presence of the absorption or release of energy. Cauchy problem with compactly supported and nonnegative initial function u_0 such that

$$supp\ u_0 = \{|x| < R\},\ u_0 \sim C(R - |x|)^\alpha, \quad as\ |x| \to R - 0,$$

with $C, \alpha > 0$ is analyzed. There is a finite speed of propagation property, and interface or free boundary emerge from the boundary of the support of the initial function either in slow diffusion regime ($m > 1$), or in fast diffusion regime ($0 < m < 1$) accompanied

with strong absorption ($b > 0, 0 < \beta < m$). Interface surface $t = \eta(x)$ may shrink, expand or remain stationary depending on the relative strength of the diffusion and reaction or absorption terms near the boundary of support, expressed in terms of the parameters $m,\beta,\alpha, sign\ b$ and C. In all cases we prove explicit formula for the interface asymptotics, and local solution near the interface. In the fast diffusion regime ($0 < m < 1$) with weak absorption ($b > 0,\beta \geq m$) or reaction ($b < 0,\beta \geq 1$), there is an infinite speed of propagation, and interfaces are absent. In all these cases we prove explicit asymptotic formula for the solution at infinity. The methods of proof are based on rescaling and blow-up techniques to establish the asymptotics of solution along some interface type manifolds, followed by application of the comparison theorems in non-cylindrical domains with non-smooth and characteristic boundary manifolds. The latter is developed in *U.G. Abdulla, Trans. Amer. Math. Soc. 357, 1, 2005, 247-265,* while the former is based on the generalization of the methods developed in *U.G. Abdulla & J. King, SIAM J. Math. Anal., 32, 2, 2000, 541-560* & *U.G. Abdulla, Nonlinear Analysis, 50, 2, 2002, 541-560.*

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Dedication

I dedicate my dissertation to my father, Ali Abuweden, who first taught me the value of education. I also commit this work to my beloved and supportive husband, Bahloul Dwaish. Words cannot express how grateful I am to him for his prayers and assistance, which never missing during all steps of my study. I devote my research to my beloved daughters Judy, Genan, Taima, Thana, and my son Ahmed, for allowing me time to accomplish my goals.

Chapter 1

Introduction

1.1 Physical Motivation: Flow in a Porous Media

Consider the flow of gas in a porous medium $\Omega \in \mathbb{R}^N, N = 3$. Let $T > 0$ and $D = \Omega \times (0, T]$. The flow is mathematically described by the following laws:

- Conservation of mass law:

$$f u_t + \operatorname{div}(u\vec{v}) = 0 \tag{1.1}$$

  where $u : D \to \mathbb{R}^+$ is a density of the gas, $\vec{v} : D \to \mathbb{R}^N$ is a velocity vector field, $f \in (0, 1]$, is a number characterizing the porosity of the medium.

- Darcy's constitutive law of gradient flow:

$$\vec{v} = -\frac{\kappa}{\mu}\nabla p, \tag{1.2}$$

  where $p : D \to \mathbb{R}$ is the pressure, $\mu > 0$ is a viscosity and $\kappa > 0$ is permeability of the gas. The negative sign in (1.2) indicates that the direction of the gas flow is opposite to the direction of the pressure gradient ∇p, i.e. the gas flows from the regions with high pressure towards the regions with low pressure.

- A phenomenological density vs. pressure power law:

$$u = u_0 p^\gamma, \tag{1.3}$$

where $0 < \gamma \leq 1$.

Combining (1.1)-(1.3), after rescaling a time variable one can derive the nonlinear diffusion equation for the density u:

$$u_t = div(u^{\frac{1}{\gamma}} \nabla u) \tag{1.4}$$

Equation (1.4) is a second order nonlinear parabolic partial differential equation (PDE) with implicit degeneration. It is a classical nonlinear parabolic equation in the regions with positive density u. On the contrary, in the regions with $u = 0$, diffusion coefficient $u^{\frac{1}{\gamma}}$ vanishes, and equation degenerates due to loss of ellipticity of the diffusion operator. Notably, degeneration is implicit, i.e. its whereabouts is a priori unknown and depends on the unknown solution u. Combination of the nonlinearity with implicit degeneration causes significantly different features of solutions of the PDE (1.4) which are relevant for physical applications. Prelude of the theory of second order nonlinear degenerate parabolic equations began with the papers [40, 96] (see also [41]), where instantaneous point source solution of the PDE (1.4) is derived. The feature of a finite speed of propagation and the existence of compactly supported nonclassical solutions and interfaces became a motivating force of the general theory.

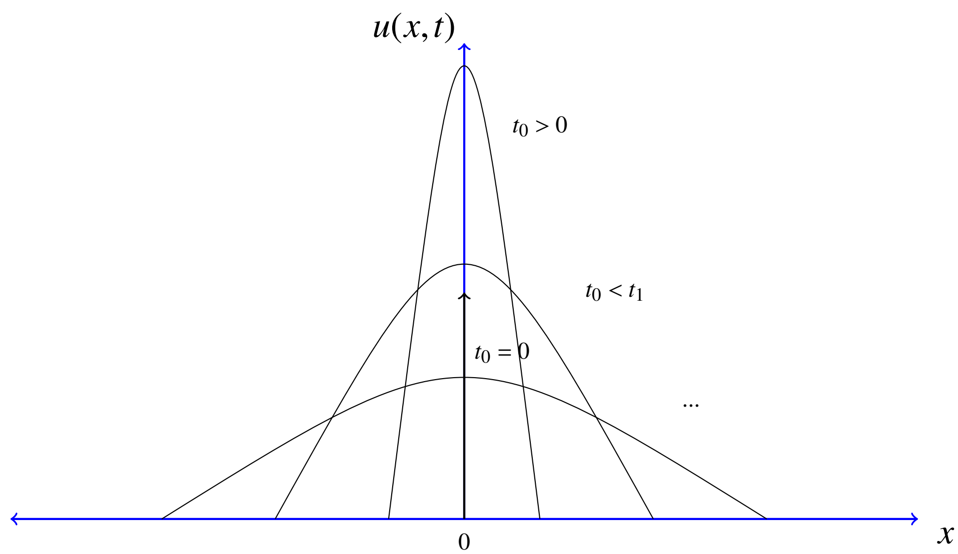

Figure 1.1: Instantaneous Point Source Solution

Following is the instantaneous point-source problem for the nonlinear diffusion equation (1.4) in $\mathbb{R}^N \times \{t > 0\}$ under the conditions

$$\begin{cases} u(x,0) = \delta(x) & x \in \mathbb{R}^N \\ \int_{\mathbb{R}^N} u(x,t)\,dx = 1 & t \geq 0 \end{cases}$$

where $\delta(\cdot)$ is Dirac's point mass with support at the origin ([96, 40]):

$$u(x,t) = t^{-\frac{N\gamma}{2\gamma+N}} \left[\frac{1}{2(2\gamma+N)} \left(C^2 - |x|^2 t^{-\frac{2\gamma}{2\gamma+N}} \right)_+ \right]^{\gamma}$$

where $(\chi)_+ = \{\chi, \quad if\ \chi > 0;\ 0, \quad if\ \chi \leq 0\}$. Figure 1.1 demonstrates the profile of the density function $u(\cdot,t)$ in different moments of time. Two key characteristic properties of the instantaneous point source solution turned out to be crucial both for application of the nonlinear diffusion type degenerate parabolic PDEs, and for the development of the extremely rich mathematical theory. First is the *finite speed of propagation* property expressed by the fact that the support of the solution is compact

$$spt\ u := \overline{\{(x,t) : u(x,t) > 0\}} = \{|x| \leq Ct^{\frac{\gamma}{2\gamma+N}}\}.$$

Boundary of the support, called an interface or free boundary, marks the boundary of the region penetrated by gas. Thus, nonlinear degenerate diffusion equation demonstrates finite speed of propagation property like classical d'Alembert wave equation, which is in contrast to infinite speed of propagation property of the classical linear diffusion equation. This is a clear indication that nonlinear diffusion equation presents more accurate mathematical model for real physical applications than its classical linear counterpart. The second vital property is that despite being a more accurate model, the instantaneous point source solution is not a solution of the PDE in the classical sense. Its first and second order derivatives are discontinuous along interfaces or free boundaries. However, it is essential to observe that despite the discontinuity of the gradient of the density, the flux vector field $u^\gamma \nabla u$ is continuous along the interface, which is in accordance with conservation laws of physics. Therefore, instantaneous point source solution is physically relevant non-classical solution of the nonlinear diffusion equation. This became a motivation for the development of general and qualitative theory of weak solutions of the nonlinear degenerate parabolic equations within last half a century accompanied with many outstanding applications in various fields of science and engineering.

1.2 Historical Review

Solution of the Cauchy problem

$$Lu = u_t - \Delta u^m + bu^\beta = 0, \ x \in \mathbb{R}^N, 0 < t < T, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N, \tag{1.6}$$

where $m > 0$, $b \in \mathbb{R}$, $\beta > 0$ is understood in the following weak sense:

Definition 1.2.1. The function $u(x,t)$ is said to be a solution (respectively, super- or subsolution) of the Cauchy Problem (1.5),(1.6), if

- u is nonnegative and continuous in $\mathbb{R}^N \times [0,T)$, locally Hölder continuous in $\mathbb{R}^N \times (0,T)$, satisfying (1.6) (respectively, satisfying (1.6) with = replaced by $\geq$ or $\leq$),
- for any t_0, t_1 such that $0 < t_0 < t_1 < T$ and for any bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$ the following integral identity holds:

$$\int\limits_{\Omega\times\{t=t_1\}} uf dx = \int\limits_{\Omega\times\{t=t_0\}} uf dx + \int\limits_{\Omega\times\{t_0<t<t_1\}} (uf_t + u^m \Delta f - bu^\beta f) dxdt$$
$$- \int\limits_{\partial\Omega\times(t_1,t_2)} u^m \frac{\partial f}{\partial \nu} dxdt, \tag{1.7}$$

  (respectively, (1.7) holds with = replaced by $\geq$ or $\leq$), where $f \in C^{2,1}_{x,t}(\overline{\Omega})$ is an arbitrary function (respectively, nonnegative function) that equals to zero on $\partial\Omega \times [t_0,t_1]$ and ν is the outward-directed normal vector to $\partial\Omega$.

Instantaneous point source solution ([40, 96]) demonstrated in the previous section is a weak solution of the Cauchy problem in the sense of the Definition 1.2.1.

Mathematical theory of the second order nonlinear degenerate parabolic PDEs begins with the work [87]. Currently there is a well established theory of well-posedness of main boundary value problems, and local regularity properties of weak solutions [38, 37, 47, 56, 57, 33, 84, 36, 79, 44, 50, 49, 94, 75, 93, 88, 34]. Without any ambition to present full survey of outstanding contributions by many mathematicians, we refer to [58, 95] which outline the modern well established theory and contain extensive list of references. General theory of boundary value problems in non-cylindrical domains with non-smooth boundary manifolds under minimal regularity assumptions on the boundaries is developed in [5, 10, 12]. In particular general theory in non-cylindrical non-smooth domains was motivated by the problem about the evolution of interfaces. To present complete classification of the development of interfaces it is essential to apply general theory of boundary-value problems in non-cylindrical domains with boundary surfaces which has the same kind of behaviour as the interface. In many cases this may be nonsmooth and characteristic.

We now make precise the meaning of the solution to Dirichlet problem (DP) in general domains. Let Ω be an open subset of $\mathbb{R}^{N+1}, N \geq 2$. Let the boundary $\partial\Omega$ of Ω consist of the closure of a domain $B\Omega$ lying on $t = 0$, a domain $D\Omega$ lying on $t = T \in (0, \infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. Assume that $\Omega(\tau) := \Omega \cap \{t = \tau\} \neq \emptyset$ for $t \in (0, T)$. The set $\mathcal{P}\Omega = \overline{B\Omega} \cup S\Omega$ is called a parabolic boundary of Ω. The class of domains with described structure is denoted by $\mathcal{D}_{0,T}$. Let $\Omega \in \mathcal{D}_{0,T}$ be given and let ψ be an arbitrary continuous non-negative function defined on $\mathcal{P}\Omega$. DP consists of finding a solution to equation(1.5) in $\Omega \cup D\Omega$ satisfying the initial-boundary condition

$$u = \psi \ on \ \mathcal{P}\Omega \tag{1.8}$$

Definition 1.2.2 (Weak Solution of The DP)**.** ([10, 12]) We say that a function $u(x,t)$ is a solution (resp., super- or subsolution) of DP (1.5),(1.8) if

- u is nonnegative, bounded and continuous in $\overline{\Omega}$, and locally Hölder continuous in $\Omega \cup D\Omega$ satisfying (1.8) (respectively satisfying (1.8) with = replaced by $\geq$ or $\leq$)

- for any t_0, t_1 such that $0 < t_0 < t_1 < T$, and for any domain $\overline{\Omega}_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subset \Omega \cup D\Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds:

$$\int_{D\Omega_1} uf\,dx = \int_{B\Omega_1} uf\,dx + \int_{S\Omega_1} (uf_t + u^m \Delta f)\,dxdt - \int_{S\Omega_1} u^m \frac{\partial f}{\partial \nu}\,dxdt \qquad (1.9)$$

  (respectively (1.9) holds with = replaced by $\geq$ or $\leq$, where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function (respectively non-negative function) that equals zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

In [5, 10, 12] existence, boundary regularity, uniqueness and comparison theorems for the DP are proved under minimal pointwise assumption on the local modulus of lower semicontinuity of the boundary manifold $S\Omega$ (see Assumption $\mathcal{A}$ and Assumption $\mathcal{M}$ in [5, 10, 12]). In particular, the following comparison theorem will be of essential use in this paper:

Theorem 1.2.3. *([10, 12]). Let u be a solution of DP and let g be a supersolution (respectively subsolution) of DP. Assume that the assumption Assumption $\mathcal{A}$ and Assumption $\mathcal{M}$ of [10] are satisfied. Then $u \leq$ (respectively $\geq$) g in Ω.*

The initial development of interfaces and local structure of solutions near the interfaces is very well understood in the one dimensional case. Full classification of evolution of interfaces and local behavior of solutions near the interfaces for the problem (1.5)-(1.6) with space dimension $N = 1$ was presented in [27] for slow diffusion case ($m > 1$), and in [6] for the fast diffusion case ($m = 1$). The results and methods of [27, 6] are extended to solve interface problem for p-Laplacian type reaction-diffusion equations in [25, 26]:

$$u_t - (|u_x|^{p-1} u_x)_x + cu^\beta = 0, \tag{1.10}$$

where $p > 0, \beta > 0$, and for the reaction-diffusion equations with double degenerate diffusion in [23]. The method of the proof developed in [27, 6] is based on rescaling and application of the one-dimensional theory of reaction-diffusion equations in general non-cylindrical domains with non-smooth boundary curves developed in [3, 2]. Sharp asymptotic estimates for the interfaces and local solutions of the Dirichlet problem for the equation (1.5) in bounded cylindrical domains domains was proved in [1]. Estimation for the interfaces via energy methods is pursued in [34].

1.3 Open Problems

Consider the Cauchy problem for the reaction-diffusion equation (1.5)-(1.6). Equation (1.5) is a nonlinear degenerate parabolic equation arising in various applications in fluid mechanics, plasma physics, population dynamics etc. as a mathematical model of nonlinear diffusion phenomena in the presence of absorption of energy [35, 40, 75, 42]. Assume that $u_0 \in C(\mathbb{R}^N; \mathbb{R}^+)$ is radially symmetric with

$$supp\ u_0 = \overline{B_R}$$

where $B_R := \{x \in \mathbb{R}^N,\ |x| < R\}$, and

$$u_0(x) \sim C(R - |x|)^\alpha \ as\ |x| \to R - 0 \tag{1.11}$$

for some $C > 0, \alpha > 0$. Typical example is

$$u_0(x) = C(R - |x|)_+^{\alpha}, \ x \in \mathbb{R}^N \tag{1.12}$$

where $\kappa_+ = \max\{\kappa; 0\}$. Solution of the Cauchy Problem (1.5),(1.6) is understood in a weak sense (Definition 1.2.1, Section 2.2). Furthermore, we will assume that $\beta \geq 1$ if $b < 0$, which is essential to guarantee uniqueness of the solution. If $m > 1$, i.e. in the slow diffusion regime the weak solution possesses a finite speed of propagation property, meaning that it is compactly supported for any $t > 0$ [40]. Boundary manifolds of the support of solution are called "free boundaries" or "interfaces". The main goal of the dissertation is to analyze the structure of interfaces emerging from sphere $\partial B_R \times \{t = 0\}$, and local solutions near the interfaces. If $0 < m < 1$, i.e. in the fast diffusion regime the weak solution may or may not be compactly supported depending on the relative strength of the diffusion and absorption terms near the boundary of support of the initial function. The goal of the dissertation is to classify short-time behavior of interfaces and local solutions near the interfaces in all cases with finite speed of propagation, and to identify the short-time asymptotics of the solution at infinity in all cases with infinite speed of propagation.

For all $x \in B_R$ near the boundary define the interface surface as

$$t = \eta_-(x) := \sup\{\tau : u(x,t) > 0, 0 < t < \tau\}.$$

If $\eta_-(x)$ is defined and finite for all x such that $0 << |x| < R$ and

$$\eta_-(x) = o(1) \quad \text{for } |x| \to R - 0, \tag{1.13}$$

then we say that the interface initially shrinks at ∂B_R.

For all $x \in B_R^c = \{|x| > R\}$ near the boundary define interface surface

$$t = \eta_+(x) := \inf\{\tau \geq 0 : u(x,t) > 0, \tau < t < \tau + \epsilon \quad \text{for some } \epsilon > 0\}.$$

If $\eta_+(x)$ is defined, positive and finite for all x such that $R < |x| << +\infty$ and satisfies (1.13), then we say that the interface initially expands at ∂B_R.

If

$$supp\ u(x,t) \equiv supp\ u_0(x)$$

for all $0 \leq t \leq \delta$, for some $\delta > 0$, then we say that interface remain stationary, or solution has a waiting time near the support of the initial function.

If for some $\delta > 0$

$$u(x,t) > 0,\ x \in \mathbb{R}^N, 0 < t \leq \delta$$

we say that there is an infinite speed of propagation and interfaces, or free boundaries are absent.

The existence and the direction of the movement of the interface, and its asymptotics is an outcome of the competition between the diffusion and reaction forces and depends on the parameters m,β,b,C, and α. The goal of this dissertation is to present full classification of the existence and short time behavior of the interfaces $\eta_\pm$, and local solution near $\eta_\pm$ in terms of the parameters m,β,b,C,α. Full classification of the interface development for the reaction-diffusion equation (1.5),(1.6),(1.11) in the slow diffusion regime ($m > 1$) is presented in Figure 2.1. In the region (1) diffusion dominates over the reaction, and interface expands; region (2) is the borderline case when diffusion and reaction are in balance, and the direction of the movement of the interface depends on the constant C. There is a critical value C_* such that interface expands if $C > C_*$, and shrinks if $C < C_*$; in region (3) absorption dominates over the reaction, and interface shrinks; finally, in region(4) diffusion weakly dominates over the absorption, and interface remains stationary. Full classification of the interface existence and development for the reaction-diffusion equation (1.5),(1.6),(1.11) in the fast diffusion regime ($0 < m < 1$) is presented in Figure 3.1. The finite interfaces exist in regions (1)-(3). The behavior of interfaces is similar to the behavior in the slow diffusion regime: in region (1) diffusion dominates and the interface expands; in region (2) diffusion and absorption are in balance, and the constant C, with critical value being C_*, plays a crucial role for the behavior of interfaces: if $C > C_*$, then interface expands, while if $C < C_*$, then interface shrinks; There is an infinite speed of propagation, and interfaces are absent in regions (4) and (5): in region (4) solution has an exponential decay at infinity, while in region (5) it has power type decay at infinity.

The outline of the dissertation is organized as follows: In Chapter 2 we consider slow diffusion case ($m > 1$). It contains three sections.

- In Section 2.1 we formulate the main results. Theorems 2.1.1-2.1.6 of Section 2.1 present full classification of the short-time behavior and asymptotics of the interfaces with respect to relative strength of diffusion versus reaction/absorption expressed in respective four regions of the parameter space (α,β). Some essential technical details of the main results are outlined in Subsection 2.1.1.

- In Section 2.2 we prove important asymptotic properties of local solutions along the special interface-type manifolds by using rescaling and application of the general theory of nonlinear degenerate parabolic equations.

- Lastly, in Section 2.3 we prove the main results by using asymptotic estimations of Section 2.2 and by constructing local super- and subsolutions based on the special comparison theorems in general non-cylindrical domains with irregular and characteristic boundary manifolds.

In Chapter 3 we consider the fast diffusion case ($0 < m < 1$). It includes three sections.

- In Section 3.1 we formulate the main results in Theorems 3.1.1-3.1.5. In Subsection 3.2 we summarize some critical technical details of the main results.
- In Section 3.3 we prove the asymptotic analysis of solutions through rescaling analysis.
- In Section 3.4 we prove the main results by using asymptotic estimations established in Section 3.3 .

Finally, Chapter 4 presents main conclusions and open problems for a future research.

Chapter 2

Interfaces and Local Solutions for the Nonlinear Degenerate Multidimensional Reaction-Diffusion Equations: Slow Diffusion Versus Reaction/Absorption

The results of this Chapter are published in [21].

2.1 Formulation of Main Results on the Classification of Interfaces

Throughout this section we assume that u is a unique weak solution of the CP (1.5)-(1.6). There are four different subcases, as shown in Figure 1. The main results are outlined below in Theorems 2.1.1-2.1.6 corresponding directly to the cases (1), (2), (3) and (4).

Theorem 2.1.1. *If* $\alpha < \frac{2}{m-min\{1,\beta\}}$, *then the interface initially expands and*

$$\eta_+(x) \sim \Big(\frac{R-|x|}{\xi_*}\Big)^{2+\alpha(1-m)} \text{ as } |x| \to R+, \tag{2.1}$$

where $\xi_* = \xi_*(C,\alpha,m) < 0$. *For arbitrary* $\xi_* < \rho < 0$, *there exists a positive number* $f(\rho)$ *depending on* C, m, α *such that*

$$u\Big|_{|x|=R-\rho t^{\frac{1}{2+\alpha(1-m)}}} \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \text{ as } t \downarrow 0. \tag{2.2}$$

Theorem 2.1.2. *Suppose that* $b > 0, 0 < \beta < 1, \quad \alpha = \frac{2}{m-\beta}$ *and* $m+\beta = 2$,

$$C_* = \Big\{\frac{b(1-\beta)^2}{(2-\beta)}\Big\}^{\frac{1}{2(1-\beta)}}, \qquad \zeta_* = b(1-\beta)C^{\beta-1}\Big(1-(C/C_*)^{2(1-\beta)}\Big)$$

If $C > C_*$ *then interface initially expands and*

$$\eta_+(x) \sim \frac{|x|-R}{-\zeta_*},\ |x| \to R+.$$

If $C < C_*$ the interface initially shrinks and

$$\eta_-(x) \sim \frac{|x| - R}{-\zeta_*}, \ |x| \to R - .$$

For arbitrary $\rho > \zeta_*$ we have

$$u(x,t) \sim C\big((\rho - \zeta_*)t\big)^{\frac{1}{1-\beta}} \qquad \text{as } |x| = R - \rho t, t \downarrow 0. \tag{2.3}$$

Theorem 2.1.3. *Let $b > 0$, $0 < \beta < 1$, $\alpha = \frac{2}{m-\beta}$ and*

$$C_* = \Big\{\frac{b(m-\beta)^2}{2m(m+\beta)}\Big\}^{\frac{1}{m-\beta}}$$

If $C > C_*$ then interface initially expands and

$$\eta_+(x) \sim \Big(\frac{|x| - R}{\zeta_*}\Big)^{\frac{2(1-\beta)}{m-\beta}} \ \text{as} \ \ |x| \to R+, \tag{2.4}$$

while if $C < C_*$ then interface initially shrinks and

$$\eta_-(x) \sim \Big(\frac{R - |x|}{\zeta_*}\Big)^{\frac{2(1-\beta)}{m-\beta}} \ \text{as} \ \ |x| \to R-, \tag{2.5}$$

where $\zeta_* = \zeta_*(C,m,\beta,b) \gtrless 0$ according to as $C \lessgtr C_*$. For arbitrary $\rho > \zeta_*$ there exists $h(\rho) > 0$ such that

$$u(x,t)\Big|_{|x|=R-\rho t^{\frac{m-\beta}{2(1-\beta)}}} \sim h(\rho) t^{\frac{1}{1-\beta}} \qquad \text{as } \ t \downarrow 0. \tag{2.6}$$

Corollary 2.1.4. *If conditions of Theorem 2.1.3 are satisfied and $m + \beta = 2$, then claims*

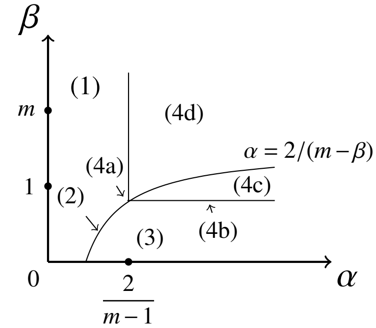

Figure 2.1: Classification of different cases in the (α,β) plane for interface development in problem (1.5), (1.6), (1.11).

(2.4),(2.5),(2.6) *are valid with*

$$\zeta_+ = \zeta_- = \zeta_* = b(1-\beta)C^{\beta-1}\Big(1-(C/C_*)^{2(1-\beta)}\Big),\ h(\rho) = C(\rho-\zeta_*)_+. \tag{2.7}$$

Theorem 2.1.5. *Let $b > 0, 0 < \beta < 1, \alpha > \frac{2}{m-\beta}$. Then interface initially shrinks and*

$$\eta_-(x) \sim \Big(\frac{R-|x|}{l_*}\Big)^{\alpha(1-\beta)} \text{ as } |x| \to R-, \tag{2.8}$$

where $l_* = C^{-\frac{1}{\alpha}}(b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$. For $\forall l > l_*$ we have

$$u\Big|_{|x|=R-lt^{\frac{1}{\alpha(1-\beta}}} \sim \{C^{1-\beta}l^{\alpha(1-\beta)} - b(1-\beta)\}^{\frac{1}{1-\beta}}t^{\frac{1}{1-\beta}} \qquad \texttt{as } t\downarrow 0. \tag{2.9}$$

Theorem 2.1.6. *If $\beta \geq 1$, $\alpha \geq \frac{2}{m-1,}$ then the interface initially remains stationary.*

2.1.1 Technical Details of the Main Results

In this section we outline some essential details of the main results described in Theorems 2.1.1-2.1.6 of section 2.1.

Technical details of Theorem 2.1.1: Precise values of the constant ξ_* and the function f are associated with the one-dimensional Cauchy Problem [27]

$$w_t = (w^m)_{yy}, \qquad y \in \mathrm{R},\ 0 < t < +\infty \tag{2.10}$$

$$w(y,0) = C(y)_+^\alpha, \qquad y \in \mathrm{R} \tag{2.11}$$

There exists a unique solution of the problem (2.10),(2.11), which is of self-similar form

$$w(y,t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f(\xi),\ \text{where}\ \xi = \frac{y}{t^{\frac{1}{\alpha(1-m)+2}}}, \tag{2.12}$$

and the shape function f solves nonlinear ODE problem

$$\begin{cases} \frac{d^2 f^m}{d\xi^2} + \frac{1}{2+\alpha(1-m)}\xi\frac{df}{d\xi} - \frac{\alpha}{2+\alpha(1-m)} f = 0, \quad \xi \in \mathbb{R} \\ f \sim C\xi^\alpha \qquad as\ \xi \to +\infty,\ f(\xi) = o(|\xi|^\alpha)\ \ as \quad \xi \downarrow -\infty, \end{cases} \tag{2.13}$$

with finite interface $\xi_* = \xi_*(C,\alpha,m) < 0$ such that

$$f(\xi) > 0,\ \xi_* < \xi < +\infty;\ f(\xi) \equiv 0,\ \xi < \xi_* \tag{2.14}$$

Through rescaling one can find dependence of f and ξ_* on C [27]:

$$f(\rho) = C^{\frac{2}{2-\alpha(m-1)}} f_1(C^{\frac{m-1}{2-\alpha(m-1)}}\rho) \tag{2.15}$$

$$f_1(\rho) = w_1(\rho,1),\ \xi_*' = inf\{x : f_1(\rho) > 0\} < 0 \tag{2.16}$$

$$\xi_* = C^{\frac{m-1}{2-\alpha(m-1)}}\xi_*' \tag{2.17}$$

where w_1 and f_1 are solutions of (2.10),(2.11), and (2.13), respectively, with the constant $C = 1$; ξ'_* is a negative number depending on m and α only.

Technical Details of Theorem 2.1.3: Precise values of the constant ζ_* and the function h are associated with the one-dimensional Cauchy Problem

$$w_t = w^m_{yy} - bw^\beta, \qquad y \in \mathrm{R},\ 0 < t < +\infty \tag{2.18}$$

$$w_0(y) = C(y)_+^{\frac{2}{m-\beta}}, \qquad y \in \mathrm{R} \tag{2.19}$$

There exists a unique solution of the problem (3.30),(3.31), which is of self-similar form

$$w(y,t) = t^{\frac{1}{1-\beta}} h(\zeta), \text{ where } \zeta = \frac{y}{t^{\frac{m-\beta}{2(1-\beta)}}}, \tag{2.20}$$

and the shape function h solves nonlinear ODE problem

$$\begin{cases} \frac{1}{1-\beta}h - \frac{m-\beta}{2(1-\beta)}\zeta h' - (h^m)'' + bh^\beta = 0, \qquad \zeta \in \mathbb{R}. \\ h(\zeta) \sim C\zeta^{\frac{2}{m-\beta}} \quad \text{ as } \zeta \uparrow +\infty,\ h(\zeta) = o(|\zeta|^{\frac{2}{m-\beta}}) \text{ as } \zeta \downarrow -\infty. \end{cases} \tag{2.21}$$

There exists a finite interface ζ_* such that $\zeta_* = \zeta_*(C,m,\beta,b)$ such that $\zeta_* \gtreqless 0$ according to as $C \lesseqgtr C_*$, and

$$h(\zeta) > 0,\ \zeta_* < \zeta < +\infty;\ h(\zeta) \equiv 0,\ \zeta \le \zeta_* \tag{2.22}$$

Technical details of Theorem 2.1.6: There are four subcases.

(5a) If $\beta = 1,\ \alpha = \frac{2}{m-1}$ then $\forall \epsilon > 0\ \exists R_\epsilon \in (0,R)$ and $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(R-|x|)_+^{\frac{2}{m-1}} e^{-bt} \le u \le$$
$$(C+\epsilon)(R-|x|)_+^{\frac{2}{m-1}} e^{-bt}\Big(1-(C/\bar{C})^{m-1} b^{-1}(1-e^{-b(m-1)t})\Big)^{\frac{1}{1-m}} \tag{2.23}$$

for $R_\epsilon \le |x| < +\infty,\ 0 \le t \le \delta_\epsilon$, and

$$T = \begin{cases} +\infty, & if \quad b \ge (C/\bar{C})^{m-1} \\ \frac{\ln\left(1-b(C/\bar{C})^{m-1}\right)}{b(1-m)}, & if\ -\infty < b < (C/\bar{C})^{m-1}, \end{cases} \tag{2.24}$$

$$\bar{C} = \Big[\frac{(m-1)^2}{2m(m+1)}\Big]^{\frac{1}{m-1}}. \tag{2.25}$$

(5b) If $\beta = 1,\ \alpha > \frac{2}{m-1}$, then $\forall \epsilon > 0,\ \exists R_\epsilon > 0,\ and\ \delta_\epsilon > 0$ such that

$$(C-\epsilon)(R-|x|)_+^\alpha e^(-bt) \le u \le (C+\epsilon)(R-|x|)_+^\alpha e^{-bt} \times$$
$$\left(1-\epsilon(b(m-1))^{-1}(1-e^{-b(m-1)t})\right)^{\frac{1}{1-m}},\ |x| \ge R_\epsilon,\ 0 \le t \le \delta_\epsilon. \tag{2.26}$$

(5c) If $1 < \beta < m,\ \alpha \ge \frac{2}{m-\beta}$, then $\forall \epsilon > 0,\ \exists R_\epsilon > 0,\ and\ \delta_\epsilon > 0$ such that

$$g_{-\epsilon} \le u(x,t) \le g_\epsilon, \quad |x| \ge R_\epsilon, \quad 0 \le t \le \delta_\epsilon, \tag{2.27}$$

$$g_\epsilon(x,t) = \begin{cases} [(C+\epsilon)^{1-\beta}(R-|x|)^{\alpha(1-\beta)} + b(\beta-1)(1-d_\epsilon)t]^{\frac{1}{1-\beta}}, & R_\epsilon \le |x| \le R \\ 0, & |x| \ge R, \end{cases} \tag{2.28}$$

$$d_\epsilon = \begin{cases} \epsilon\ sign\ b, & if\ \alpha > \frac{2}{m-\beta} \\ \left((C+\epsilon/\bar{C})^{m-\beta} + \epsilon\right) sign\ b & if\ \alpha = \frac{2}{m-\beta}. \end{cases} \tag{2.29}$$

(5d) Let either $1 < \beta < m, \frac{2}{m-1} \le \alpha < \frac{2}{m-\beta}$ or $\beta \ge m.\ \alpha \ge \frac{2}{m-1}$. If $\alpha = \frac{2}{m-1}$, then for arbitrary small $\epsilon > 0\ \exists R_\epsilon > 0,\ and\ \delta_\epsilon > 0$ such that for $|x| \ge R_\epsilon, \quad 0 \le t \le \delta_\epsilon$ we have

$$(C-\epsilon)(R-|x|)_+^\alpha (1-\gamma_{-\epsilon}t)^{\frac{1}{1-m}} \le u(x,t) \le (C+\epsilon)(R-|x|)^{\frac{2}{m-1}}(1-\gamma_\epsilon t)^{\frac{1}{1-m}}, \tag{2.30}$$

where

$$\gamma_\epsilon = \Big[\frac{2m(m+1)(C+\epsilon)^{m-1}}{m-1}\Big] + \epsilon \tag{2.31}$$

If $\alpha > \frac{2}{m-1}$ then for arbitrary small $\epsilon > 0$ $\exists R_\epsilon > 0,$ *and* $\delta_\epsilon > 0$ such that

$$(C-\epsilon)(R-|x|)_+^\alpha \le u(x,t) \le (C+\epsilon)(R-|x|)_+^\alpha (1-\epsilon t)^{\frac{1}{1-m}},\ |x| \ge R_\epsilon,\ 0 \le t \le \delta_\epsilon. \tag{2.32}$$

2.2 Preliminary Results

In the following three lemmas we establish asymptotic properties of the solution to the Cauchy problem (1.5)-(1.11) based on the scaling laws corresponding to the PDE (1.5).

Lemma 2.2.1. *Let u solves CP* (1.5),(1.6),(1.11)*, and one of the following conditions is satisfied.*
(i) $b = 0,\ 0 < \alpha < \frac{2}{m-1}$
(ii) $b > 0,\ 0 < \beta < 1,\ 0 < \alpha < \frac{2}{m-\beta}$
(iii) $b \neq 0,\ \beta \ge 1, 0 < \alpha < \frac{2}{m-1}$
Then u(x,t) satisfies (2.2).

Proof. (i) First consider the global case (1.12). Change the variable $y = x + \bar{x}$ with $\bar{x} = (R, 0, ..., 0)$. Function $v(y,t) = u(y-\bar{x}, t)$ solves the problem

$$v_t(y,t) - \Delta v^m(y,t) = 0,\ y \in \mathbb{R}^N,\ t > 0 \tag{2.33}$$

$$v(y,0) = C(R-|y-\bar{x}|)_+^\alpha,\ y \in \mathbb{R}^N \tag{2.34}$$

Since nonlinear diffusion equaton is invariant under the scaling

$$y \to k^{-\frac{1}{\alpha}} y,\ \ t \to k^{\frac{\alpha(m-1)-2}{\alpha}} t$$

rescaled function

$$v_k(y,t) = kv(k^{-\frac{1}{\alpha}}y, k^{\frac{\alpha(m-1)-2}{\alpha}}t)$$

solves the problem

$$\begin{cases} w_t(y,t) - \Delta w^m(y,t) = 0 & y \in \mathbb{R}^N \ t > 0 \\ w(y,0) = C(k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|)_+^\alpha & y \in \mathbb{R}^N. \end{cases}$$

Since

$$\lim_{k\to\infty} C\{k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|\}_+^\alpha = C(y_1)_+^\alpha, \tag{2.35}$$

the limit function solves the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)_+^\alpha$. Due to uniqueness of the solution to the CP ([43]), the latter coincides with the solution of the 1D CP (2.10),(2.11), which is of self-similar form (2.12) with the shape function $f(\xi)$ solving nonlinear ODE problem (2.13) and having finite interface $\xi_* < 0$. Therefore, we have

$$\lim_{k\to\infty} kv(k^{-\frac{1}{\alpha}}y, k^{\frac{\alpha(m-1)-2}{\alpha}}t) = t^{\frac{\alpha}{2+\alpha(1-m)}} f\Big(y_1 t^{-\frac{1}{2+\alpha(1-m)}}\Big).$$

By choosing $y_2 = \ldots.. = y_n = 0$, $k^{\frac{\alpha(m-1)-2}{\alpha}}t = \tau$, $y_1 = \rho t^{\frac{1}{2+\alpha(1-m)}}$, $\rho > \xi_*$, we have

$$u(x,t)|_{\Gamma_\rho} \sim f(\rho)t^{\frac{\alpha}{2+\alpha(1-m)}} \ as \ t \downarrow 0, \tag{2.36}$$

where,

$$\Gamma_\rho = \{(x,t) : x_1 = -R + \rho t^{\frac{1}{2+\alpha(1-m)}}, \ x_2 = \ldots\ldots\ldots = x_n = 0, \ t \geq 0\}$$

Since the initial condition is radially symmetric, the solution of the CP is radially symmetric for any fixed $t > 0$, and therefore, from (2.36), (2.2) follows. Equivalently, for all

ρ with $\xi_* < \rho < 0$ we have

$$u\Big|_{t=\eta(x)=\left(\frac{R-|x|}{\rho}\right)^{2+\alpha(1-m)}} \sim f(\rho)t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{as } |x| \to R+. \tag{2.37}$$

If $u_0(x)$ satisfies (1.11), then for arbitrary $\epsilon > 0$ $\exists 0 < R_\epsilon < R$ such that

$$\Big(C - \epsilon/2\Big)(R - |x|)_+^\alpha \le u_0(x) \le \Big(C + \epsilon/2\Big)(R - |x|)_+^\alpha, \; |x| \ge R_\epsilon \tag{2.38}$$

Let $u_{\pm\epsilon}$ be a solution of the CP (1.5),(1.6) with initial function $(C \pm \epsilon)(R - |x|)_+^\alpha$. Since the solution of CP (1.5),(1.6) is continuous, there exists a $\delta_\epsilon > 0$ such that

$$u_{-\epsilon}(x,t) \le u(x,t) \le u_{+\epsilon}(x,t), \quad |x| = R_\epsilon, 0 \le t \le \delta_\epsilon \tag{2.39}$$

From (2.38),(2.39) and a comparison Theorem 1.2.3 applied in $\{|x| > R_\epsilon, 0 < t \le \delta_\epsilon\}$ we have

$$u_{-\epsilon} \le u \le u_\epsilon \quad \text{for} \quad |x| > R_\epsilon, 0 < t \le \delta_\epsilon \tag{2.40}$$

As we have already proved for all ρ such that $\xi_*(C \pm \epsilon) < \rho < 0$, $u_{\pm\epsilon}$ satisfies (2.2) with f replaced with solution $f_{C\pm\epsilon}$ of the problem (2.13) with C replaced with $C \pm \epsilon$. Due to continuous dependence of f and ξ_* on C, from (2.40), (2.2) for u easily follows.

(ii) & (iii) Assume that the condition of the case (ii) or (iii) with $b > 0$ is fulfilled. As before, from (1.11), (2.38)-(2.40) follows. Changing the variable $y = x + \bar{x}$ with $\bar{x} = (R, 0, ..., 0)$, the function $v_{\pm\epsilon}(y,t) = u_{\pm\epsilon}(y - \bar{x}, t)$ solves the problem

$$v_t - \Delta v^m + bv^\beta = 0, \; y \in \mathbb{R}^N, \; t > 0 \tag{2.41}$$

$$v(y,0) = (C \pm \epsilon)(R - |y - \bar{x}|)_+^\alpha, \; y \in \mathbb{R}^N \tag{2.42}$$

Rescaled function

$$v_{\pm\epsilon}^{k}(y,t) = kv_{\pm\epsilon}(k^{-\frac{1}{\alpha}}y, k^{\frac{\alpha(m-1)-2}{\alpha}}t) \tag{2.43}$$

solves the problem

$$\begin{cases} w_t - \Delta w^m + k^{\frac{\alpha(m-\beta)-2}{\alpha}} b w^\beta = 0, \ y \in \mathbb{R}^N, \ t > 0. \\ w(y,0) = (C \pm \epsilon)(k^{\frac{1}{\alpha}} R - |y - k^{\frac{1}{\alpha}} \bar{x}|)_+^\alpha, \ y \in \mathbb{R}^N \end{cases}$$

From the comparison Theorem 1.2.3 and (2.35) it follows that the sequence $\{v_{\pm\epsilon}^k\}$ is uniformly bounded by the solution of the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)_+^\alpha$. Since $\alpha(m-\beta) - 2 < 0$, it easily follows that the sequence $\{v_{\pm\epsilon}^k\}$ converges to the solution of the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)_+^\alpha$. The rest of the proof coincides with the one given in case (i) above.

Finally, consider the case (iii) with $b < 0$. Let $u_{\pm\epsilon}$ be a solution of the problem

$$u_t - \Delta u^m + bu^\beta = 0, \ |x| < 2R, \ 0 < t < \delta \tag{2.44}$$

$$u(x,0) = (C \pm \epsilon)(R - |x|)_+^\alpha, \ |x| \le 2R \tag{2.45}$$

$$u|_{|x|=2R} = 0, \ 0 \le t \le \delta. \tag{2.46}$$

Due to finite speed of propagation property, the solution of the CP (1.5),(1.6) will vanish as $|x| = 2R, 0 \le t \le \delta$ for some $\delta > 0$. Therefore, by comparison theorem we have (2.39) for $R_\epsilon \le |x| \le 2R, 0 \le t \le \delta_\epsilon$. Now, the function $v_{\pm\epsilon}(y,t) = u_{\mp\epsilon}(y - \bar{x}, t)$ solves the problem

$$v_t - \Delta v^m + bv^\beta = 0, \ |y - \bar{x}| < 2R, \ 0 < t < \delta \tag{2.47}$$

$$v(y,0) = (C \pm \epsilon)(R - |y - \bar{x}|)_+^\alpha, \ |y - \bar{x}| < 2R \tag{2.48}$$

$$v|_{|y-\bar{x}|=2R} = 0, \ 0 \le t \le \delta \tag{2.49}$$

and the function $w = v^k_{\pm\epsilon}$ rescaled as in (2.43), solves the problem

$$L_k w \equiv w_t - \Delta w^m + k^{\frac{\alpha(m-\beta)-2}{\alpha}} b w^\beta = 0,\ |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}},\ 0 < t < k^{\frac{2-\alpha(m-1)}{\alpha}}\delta, \tag{2.50}$$

$$w(y,0) = (C \pm \epsilon)(k^{\frac{1}{\alpha}} R - |y - k^{\frac{1}{\alpha}}\bar{x}|)^\alpha_+,\ |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}}, \tag{2.51}$$

$$w\big|_{|y-k^{\frac{1}{\alpha}}\bar{x}|=2Rk^{\frac{1}{\alpha}}} = 0,\ 0 \le t \le k^{\frac{2-\alpha(m-1)}{\alpha}}\delta. \tag{2.52}$$

To prove the convergence of the sequence $\{v^k_{\pm\epsilon}\}$ we first prove the uniform boundedness. Consider a function

$$g(x,t) = (C+1)(1+|y|^2)^{\frac{\alpha}{2}}(1-\mu t)^{\frac{1}{1-m}},\ y \in \mathbb{R}^N,\ 0 \le t \le t_0 := \mu^{-1}/2,$$

with

$$\mu = \bar{H} + 1, \bar{H} = \max_{y \in \mathbb{R}^N} H(y),$$
$$H(y) = \alpha m(m-1)(C+1)^{m-1} \max(1; N+2-\alpha m)\Big(1+|y|^2\Big)^{\frac{\alpha(m-1)-2}{2}}.$$

We have

$$L_k g = (m-1)^{-1}(C+1)(1+|y|^2)^{\frac{\alpha}{2}}(1-\mu t)^{\frac{m}{1-m}} S,\ S \ge 1 + R$$
$$R = b(m-1)(C+1)^{\beta-1} k^{\frac{\alpha(m-\beta)-2}{\alpha}}(1+|y|^2)^{\frac{\alpha(\beta-1)}{2}}(1-\mu t)^{\frac{\beta-m}{1-m}},$$

and

$$R = O\Big(k^{\frac{\alpha(m-1)-2}{\alpha}}\Big), \text{ uniformly for } |y - k^{\frac{1}{\alpha}}\bar{x}| \le 2Rk^{\frac{1}{\alpha}},\ 0 \le t \le t_0, \text{ as } k \to +\infty$$

We also have

$$g(y,0) \geq v^k_{\pm\epsilon}(y,0),\ |y-k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}};\ g\big|_{|y-k^{\frac{1}{\alpha}}\bar{x}|=2Rk^{\frac{1}{\alpha}}} \geq 0. \tag{2.53}$$

Therefore, for all sufficiently large k g is a supersolution of the problem (2.50)-(2.52). From the Theorem 1.2.3 it follows that

$$0 \leq v^k_{\pm\epsilon}(y,t) \leq g(y,t),\ |y-k^{\frac{1}{\alpha}}\bar{x}| \leq 2Rk^{\frac{1}{\alpha}},\ 0 \leq t \leq t_0. \tag{2.54}$$

Hence, the sequence $\{v^k_{\pm\epsilon}\}$ is uniformly bounded in a strip $\{0 \leq t \leq t_0\}$. Standard regularity result for the nonlinear degenerate parabolic equations [58] imply that the sequence is uniformly Hölder continuous on compact subsets of $\{0 < t \leq t_0\}$. Arzela-Ascoli theorem and standard diagonalization argument imply that there is a pointwise convergent subsequence in $\{0 < t \leq t_0\}$, with uniform convergence on compact subsets. Since, $\alpha(m-\beta) < 2$ it easily follows that the limit function is a solution of the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)^{\alpha}_{+}$. Due to uniqueness of the latter, the whole sequence converges to its unique limit point, and the rest of the proof is completed as in previous cases. □

Lemma 2.2.2. *Let* $b > 0,\ 0 < \beta < 1, \alpha = \frac{2}{m-\beta}$. *Then solution u of the CP* (1.5),(1.6),(1.11) *satisfies* (2.6).

Proof. As before, from (1.11) we deduce (2.38)-(2.40) in the context of this lemma. Changing the variable $y = x + \bar{x}$ with $\bar{x} = (R, 0, ..., 0)$, the function $v_{\pm\epsilon}(y,t) = u_{\pm\epsilon}(y - \bar{x}, t)$ solves the problem (2.41),(2.42) with $\alpha = 2/(m-\beta)$. Rescaled function

$$v^k_{\pm\epsilon}(y,t) = kv_{\pm\epsilon}(k^{\frac{\beta-m}{2}}y, k^{\beta-1}t) \tag{2.55}$$

solves the problem

$$\begin{cases} w_t - \Delta w^m + bw^\beta = 0, \ y \in \mathbb{R}^N, \ t > 0. \\ w(y,0) = (C \pm \epsilon)(k^{\frac{m-\beta}{2}} R - |y - k^{\frac{m-\beta}{2}} \bar{x}|)_+^{\frac{2}{m-\beta}}, \ y \in \mathbb{R}^N \end{cases}$$

Since (2.35) is valid with $\alpha = 2/(m-\beta)$, the limit of the sequence $\{v^k_{\pm\epsilon}\}$ solves the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)_+^{\frac{2}{m-\beta}}$. Due to uniqueness of the solution to the CP ([76]), the latter coincides with the solution of the 1D CP (3.30),(3.31), which is of self-similar form (2.20) with the shape function $h(\zeta)$ solving nonlinear ODE problem (2.21) and having finite interface ζ_* [27]. Therefore, we have

$$\lim_{k\to\infty} kv(k^{\frac{\beta-m}{2}} y, k^{\beta-1} t) = t^{\frac{1}{1-\beta}} h\Big(y_1 t^{-\frac{m-\beta}{2(1-\beta)}}\Big).$$

The remainder of the proof of (2.6) proceeds similar to the proof of (2.2) in Lemma 2.2.1 (i) and (ii). In particular, if $C > C_*$ we have $\zeta_* < 0$ and for $\forall \rho \in (\zeta_*, 0)$

$$u(x,t) \sim h(\rho) t^{\frac{1}{1-\beta}}, \ t = \eta(x) = \Big(\frac{R - |x|}{\rho}\Big)^{\frac{2(1-\beta)}{m-\beta}} \ as \ |x| \to R+, \tag{2.56}$$

while If $C < C_*$ we have $\zeta_* > 0$ and for $\forall \rho > \zeta_*$

$$u(x,t) \sim h(\rho) t^{\frac{1}{1-\beta}}, \ t = \eta(x) = \Big(\frac{R - |x|}{\rho}\Big)^{\frac{2(1-\beta)}{m-\beta}} \ as \ |x| \to R-. \tag{2.57}$$

□

Lemma 2.2.3. *Let $b > 0$, $0 < \beta < 1, \alpha > \frac{2}{m-\beta}$. Then solution u of the CP* (1.5)*,*(1.6)*,*(1.11) *satisfies* (2.9).

Proof. As in the proof of Lemma 2.2.1, case (iii) we set (2.44)-(2.46), deduce (2.39) for

$R_\epsilon \le |x| \le 2R, 0 \le t \le \delta_\epsilon$, and derive the transformed problem (2.47)-(2.49) in the context of this lemma. Rescaled solution according to invariant scale for reaction equation

$$v^k_{\pm\epsilon}(y,t) = kv_{\pm\epsilon}(k^{-\frac{1}{\alpha}}y, k^{\beta-1}t) \tag{2.58}$$

solves the problem

$$\tilde{L}_k w \equiv w_t - k^{\frac{2-\alpha(m-\beta)}{\alpha}}\Delta w^m + bw^\beta = 0,\ |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}},\ 0 < t < k^{1-\beta}\delta, \tag{2.59}$$

$$w(y,0) = (C \pm \epsilon)(k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|)^\alpha_+,\ |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}}, \tag{2.60}$$

$$w\big|_{|y-k^{\frac{1}{\alpha}}\bar{x}|=2Rk^{\frac{1}{\alpha}}} = 0,\ 0 \le t \le k^{1-\beta}\delta. \tag{2.61}$$

To prove the uniform boundedness of $\{v^k_{\pm\epsilon}\}$ consider a function

$$g(x,t) = (C+1)(1+|y|^2)^{\frac{\alpha}{2}}e^t,\ y \in \mathbb{R}^N,\ 0 \le t \le T,$$

for some fixed $T > 0$. We have

$$\begin{gathered}\tilde{L}_k g \ge g(1-\Gamma),\\ \Gamma = (C+1)^m \alpha m e^{(m-1)t}(1+|y|^2)^{\frac{\alpha(m-1)-4}{2}}(N + (\alpha m + N - 2)|y|^2)k^{\frac{2-\alpha(m-\beta)}{\alpha}}\\ \Gamma = O(k^\gamma),\ \text{uniformly for } |y - k^{\frac{1}{\alpha}}\bar{x}| \le 2Rk^{\frac{1}{\alpha}},\ 0 \le t \le T,\ \text{as } k \to +\infty,\end{gathered} \tag{2.62}$$

where,

$$\gamma = \begin{cases} \frac{2-\alpha(m-\beta)}{\alpha}, & if\ \alpha < \frac{2}{m-1} \\ \beta - 1, & if\quad \alpha \ge \frac{2}{m-1} \end{cases}$$

The estimation (2.53) is clearly satisfied. Therefore, for sufficiently large k, g is a supersolution of (2.59)-(2.61) and (2.54) is true in this context in $0 \le t \le T$. The proof of the convergence of the sequence $\{v^k_{\pm\epsilon}\}$, and desired estimation (2.9) is completed as in the proof case (iii) of Lemma 2.2.1. □

2.3 Proofs of Main Results

2.3.1 Expanding interface

Proof of Theorem 2.1.1. The estimation (2.2), and its equivalent (2.37) are proved in Lemma 2.2.1. They imply that η_+ is defined and finite, and

$$\limsup_{|x|\to R+} \frac{\eta_+(x)}{(|x|-R)^{2+\alpha(1-m)}} \leq (-\xi_*)^{\alpha(m-1)-2}. \tag{2.63}$$

As before, we deduce (2.38)-(2.40) from (1.11), and consider the problem (2.41)-(2.42) for $v_\epsilon(y,t) = u_\epsilon(y-\bar{x},t)$. Let $w_\epsilon(y,t) = w_\epsilon(y_1,t)$ be a solution of the Cauchy problem (2.10),(2.11) with C replaced by $C+\epsilon$. Assume that $b \geq 0$. Since

$$(y_1)^\alpha_+ \geq (R-|y-\bar{x}|)^\alpha_+,\ y\in\mathbb{R}^N, \tag{2.64}$$

from the comparison theorem it follows that

$$0 \leq v_\epsilon(y,t) \leq w_\epsilon(y_1,t),\ y\in\mathbb{R}^N, t>0. \tag{2.65}$$

From (2.12)-(2.17)it follows that

$$v_\epsilon(y,t) = 0, \qquad for\ y_1 \leq (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}}\xi'_* t^{\frac{1}{2+\alpha(1-m)}},\ t\geq 0 \tag{2.66}$$

that is to say,

$$u_\epsilon(x,t) = 0, \qquad for\ x_1 \le -R + (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_* t^{\frac{1}{2+\alpha(1-m)}},\ t \ge 0 \tag{2.67}$$

From (2.38)-(2.40) it follows that for arbitrary $\epsilon > 0$ $\exists 0 < R_\epsilon < R$ and δ_ϵ such that

$$u(x,t) = 0, \qquad for\ x_1 \le -R + (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_* t^{\frac{1}{2+\alpha(1-m)}},\ 0 \le t \le \delta_\epsilon. \tag{2.68}$$

Due to radial symmetricity of u we have

$$u(x,t) = 0, \qquad for\ |x| \ge R - (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_* t^{\frac{1}{2+\alpha(1-m)}},\ 0 \le t \le \delta_\epsilon. \tag{2.69}$$

This implies that for some $\gamma_\epsilon > 0$ we have

$$\eta_+(x) \ge \Big(\frac{R-|x|}{(C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_*} \Big)^{2+\alpha(1-m)} \text{ for } R < |x| \le R + \gamma_\epsilon \tag{2.70}$$

Therefore, we have

$$\liminf_{|x| \to R+} \frac{\eta_+(x)}{(|x|-R)^{2+\alpha(1-m)}} \ge (-(C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi'_*)^{\alpha(m-1)-2}.$$

By taking $\epsilon \downarrow 0$ we have

$$\liminf_{|x| \to R+} \frac{\eta_+(x)}{(|x|-R)^{2+\alpha(1-m)}} \ge (-\xi_*)^{\alpha(m-1)-2}. \tag{2.71}$$

From (2.63) and (2.71) desired estimation (2.1) follows if $b \geq 0$. If $b < 0$, $\beta \geq 1$, we consider a function

$$\bar{u}_\epsilon = exp(-bt)u_\epsilon\Big(x, \Big(b(1-m)\Big)^{-1}\Big[exp\Big(b(1-m)t\Big) - 1\Big]\Big)$$

where u_ϵ is a solution of the CP (1.5), (1.12) with $b = 0$ and $C+\epsilon$. Accordingly, $\bar{u}_\epsilon$ solves the CP (1.5), (1.12) with $\beta = 1$ and $C+\epsilon$. By continuity of solution we can choose $\delta_\epsilon > 0$, $0 < R_\epsilon < R$ such that

$$\bar{u}_\epsilon < 1 \qquad for\ |x| > R_\epsilon, 0 < t < \delta_\epsilon.$$

Therefore, $\bar{u}_\epsilon$ is a supersolution of the PDE (1.5) with $b < 0$, $\beta \geq 1$. Similar arguments used in the derivation of (2.69) imply that

$$u(x,t) = 0, \qquad for\ |x| \geq R - (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}}\xi'_*\Big(\frac{e^{b(1-m)t)} - 1}{b(1-m)}\Big)^{\frac{1}{2+\alpha(1-m)}},\ 0 \leq t \leq \delta_\epsilon. \quad (2.72)$$

Therefore, for some $\gamma_\epsilon > 0$ we have

$$\eta_+(x) \geq \frac{1}{b(1-m)}\log\Big[1 + b(1-m)\Big(\frac{R-|x|}{(C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}}\xi'_*}\Big)^{2+\alpha(1-m)}\Big],\ R < |x| \leq R + \gamma_\epsilon \quad (2.73)$$

Passing to *liminf* as $|x| \to R+$, and then passing to limit as $\epsilon \to 0$, (2.71) follows. As before, from (2.63) and (2.71) desired estimation (2.1) again follows. □

2.3.2 Expanding or Shrinking Interface at the Borderline Case

Proof of Theorem 2.1.3. Asymptotic estimation (2.6), and its equivalents (2.56), (2.57) are proved in Lemma 2.2.2. If $C > C_*$, from (2.56) it follows that η_+ is defined and finite,

and

$$\limsup_{|x|\to R+} \frac{\eta_+(x)}{\left(|x|-R\right)^{\frac{2(1-\beta)}{m-\beta}}} \le (-\zeta_*)^{\frac{2(1-\beta)}{\beta-m}}. \tag{2.74}$$

Similarly, if $C < C_*$, from (2.57) it easily follows that

$$\liminf_{|x|\to R-} \frac{\eta_-(x)}{\left(R-|x|\right)^{\frac{2(1-\beta)}{m-\beta}}} \ge (\zeta_*)^{\frac{2(1-\beta)}{\beta-m}}. \tag{2.75}$$

First, consider the global case of initial function (1.12). Changing the variable $y = x+\bar{x}$ with $\bar{x} = (R,0,....,0)$, the function $v(y,t) = u(y-\bar{x},t)$ solves the problem (2.41)-(2.42) with $\alpha = 2/(m-\beta)$ and $\epsilon = 0$. As before, from (2.64) with $\alpha = 2/(m-\beta)$ and comparison theorem, (2.65) with $\epsilon = 0$ follows. In our context, $w(y_1,t)$ is a unique solution of the CP (3.30),(3.31), which is of self-similar form (2.20) with the shape function h solving nonlinear ODE problem (2.21), and having a finite interface ζ_*. If $C > C_*$ from [27] it follows that

$$w(y_1,t) \le C_1 t^{\frac{1}{1-\beta}}(-\zeta_1+\zeta)_+^{\frac{2}{m-\beta}} \ for \ -\infty < y_1 \le 0,\ 0 \le t < +\infty. \tag{2.76}$$

(see Appendix for explicit values of the constants C_1, ζ_1). From (2.65) with $\epsilon = 0$ we have

$$u(x,t) \le C_1\left(x_1+R-\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}}\right)_+^{\frac{2}{m-\beta}},\ -\infty < x_1 \le -R,\ 0 \le t < +\infty. \tag{2.77}$$

Due the radial symmetricity of u from (2.77) we deduce that

$$u(x,t) \le C_1\left(R-|x|-\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}}\right)_+^{\frac{2}{m-\beta}},\ |x| \ge R,\ 0 \le t < +\infty, \tag{2.78}$$

which imply

$$\eta_+(x) \geq \Big(\frac{|x|-R}{-\zeta_1}\Big)^{\frac{2(1-\beta)}{m-\beta}},\ |x| \geq R$$

and therefore,

$$\liminf_{|x|\to R+} \frac{\eta_+(x)}{(|x|-R)^{\frac{2(1-\beta)}{m-\beta}}} \geq (-\zeta_1)^{\frac{2(1-\beta)}{\beta-m}} \tag{2.79}$$

From (2.74) and (2.79), (2.4) follows. Assume that $0 < C < C_*$. From [27] it follows that if $m+\beta > 2$, then

$$0 \leq w(y_1,t) \leq \Big[C^{1-\beta}(y_1)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1-(C/C_*)^{m-\beta})t\Big]_+^{\frac{1}{1-\beta}},\ y_1 \in \mathbb{R},\ 0 \leq t < +\infty$$

From (2.65) with $\epsilon = 0$ we have

$$0 \leq u(x,t) \leq \Big[C^{1-\beta}(x_1+R)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1-(C/C_*)^{m-\beta})t\Big]_+^{\frac{1}{1-\beta}},\ x \in \mathbb{R}^N,\ 0 \leq t < +\infty.$$

Due the radial symmetricity of u it follows that

$$0 \leq u(x,t) \leq \Big[C^{1-\beta}(R-|x|)_+^{\frac{2(1-\beta)}{m-\beta}} - b(1-\beta)(1-(C/C_*)^{m-\beta})t\Big]_+^{\frac{1}{1-\beta}},\ x \in \mathbb{R}^N,\ t \geq 0. \tag{2.80}$$

If $1 \leq m < 2-\beta$, then from [27] it follows that

$$0 \leq w(y_1,t) \leq C_2\Big(y_1 - \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}},\ y_1 \leq l_1 t^{\frac{m-\beta}{2(1-\beta)}}, 0 \leq t < +\infty$$

(the values of the constants $C_2 > 0, 0 < \zeta_2 < l_1$ are given in Appendix). From (2.65) with $\epsilon = 0$ we have

$$0 \leq u(x,t) \leq C_2\Big(x_1 + R - \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}},\ x_1 \leq -R + l_1 t^{\frac{m-\beta}{2(1-\beta)}}, 0 \leq t < +\infty$$

and due to radial symmetricity of the solution u it follows that

$$0 \le u(x,t) \le C_2\Big(R-|x|-\zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}},\ |x| \ge R - l_1 t^{\frac{m-\beta}{2(1-\beta)}}, 0 \le t < +\infty. \tag{2.81}$$

From (2.80) and (2.81) it follows that for some $\mu > 0$

$$\eta_-(x) \le \Big(\frac{R-|x|}{\bar{\zeta}}\Big)^{\frac{2(1-\beta)}{m-\beta}},\ R-\mu \le |x| < R,$$

which imply

$$\limsup_{|x|\to R-} \frac{\eta_-(x)}{\Big(R-|x|\Big)^{\frac{2(1-\beta)}{m-\beta}}} \le (\zeta_2)^{\frac{2(1-\beta)}{\beta-m}}. \tag{2.82}$$

From (2.75) and (2.82), (2.5) follows. In the local case when initial condition satisfies (1.11), we first deduce (2.38)-(2.40) in the context of this theorem, and then apply the presented proof to $u_\pm$ and subsequently pass to limit as $\epsilon \to 0$. Note that in the special case $m+\beta = 2$ as in Corollary 2.1.4, $w_\epsilon(y_1,t)$ is a unique solution of the CP (3.30),(3.31) with C replaced by $C+\epsilon$ given as follows

$$w_\epsilon(y_1,t) = (C+\epsilon)(y_1 - \zeta_*^\epsilon t)^{\frac{1}{1-\beta}}$$

where ζ_*^ϵ is defined by (2.7) with C replaced by $C+\epsilon$. □

2.3.3 Shrinking Interface

Proof of Theorem 2.1.5. Asymptotic estimation (2.9) is proved in Lemma 2.2.3. It implies that for any $l > l_*$ there exists $\gamma_l > 0$ such that

$$\eta_-(x) \ge \Big(\frac{R-|x|}{l}\Big)^{\alpha(1-\beta)}, \quad R-\gamma_l \le |x| < R.$$

Passing to $\liminf$ as $|x| \to R-$, followed by limit as $l \to l_*$, we have

$$\liminf_{|x|\to R-} \frac{\eta_-(x)}{(R-|x|)^{\alpha(1-\beta)}} \geq (l_*)^{\alpha(\beta-1)} \tag{2.83}$$

To prove the opposite inequality, first from (1.11) we deduce (2.38)-(2.40) in the context of this theorem. Changing the variable $y = x + \bar{x}$ with $\bar{x} = (R, 0, ..., 0)$, the function $v_{\pm\epsilon}(y,t) = u_{\pm\epsilon}(y - \bar{x}, t)$ solves the problem (2.41)-(2.42) with $\alpha > 2/(m-\beta)$. Let $w_\epsilon(y,t)$ be a solution of the Cauchy-Dirichlet problem for the PDE (2.41) in

$$\mathcal{D}_\delta = \{(y,t) \in \mathbb{R}^N \times (0,\delta] : y_1 < R - R_\epsilon\}$$

under the conditions:

$$w(y,0) = (C+2\epsilon)(y_1)_+^\alpha, \ -\infty < y_1 \leq R - R_\epsilon, (y_2, ..., y_N) \in \mathbb{R}^{N-1} \tag{2.84}$$

$$w\Big|_{y_1 = R-R_\epsilon} = (C+2\epsilon)(R-R_\epsilon), \ (y_2, ..., y_N) \in \mathbb{R}^{N-1}, \ 0 \leq t \leq \delta. \tag{2.85}$$

From (2.64) it follows that

$$v_\epsilon(y,0) \leq w_\epsilon(y,0), \ -\infty < y_1 \leq R - R_\epsilon, (y_2, ..., y_N) \in \mathbb{R}^{N-1}. \tag{2.86}$$

Due to finite speed of propagation property and continuity of v_ϵ in $\overline{\mathcal{D}}_\delta$ it follows that for some $\delta_\epsilon > 0$ we have

$$v_\epsilon\Big|_{y_1=R-R_\epsilon} \leq (C+\epsilon)(R-R_\epsilon) \leq w_\epsilon\Big|_{y_1=R-R_\epsilon}, \ 0 \leq t \leq \delta_\epsilon. \tag{2.87}$$

From (2.64),(2.87) and comparison Theorem 1.2.3 it follows that

$$v_\epsilon(y,t) \le w_\epsilon(y,t),\ (y,t) \in \overline{\mathcal{D}}_{\delta_\epsilon}. \tag{2.88}$$

Due to uniqueness of the solution to the Cauchy-Dirichlet problem (2.41),(2.84),(2.85) in $\mathcal{D}_{\delta_\epsilon}$, we have $w_\epsilon(y,t) = w_{1\epsilon}(y_1,t)$, where the latter is a unique solution of the one-dimensional Cauchy-Dirichlet problem

$$w_t - w^m_{y_1y_1} + bw^\beta = 0,\ -\infty < y_1 < y_{1\epsilon} := R - R_\epsilon, 0 < t < \delta_\epsilon \tag{2.89}$$

$$w(y_1,0) = (C+2\epsilon)(y_1)^\alpha_+,\ -\infty < y_1 \le y_{1\epsilon} \tag{2.90}$$

$$w(y_{1\epsilon},t) = (C+2\epsilon)y_{1\epsilon},\ 0 \le t \le \delta_\epsilon. \tag{2.91}$$

From [27] it follows that if $m+\beta \ge 2$ then

$$0 \le w_{1\epsilon}(y_1,t) \le [(C+3\epsilon)^{1-\beta}(y_1)_+^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)t)^{\frac{1}{1-\beta}}]_+,\ -\infty < y_1 \le y_{1\epsilon}, 0 \le t \le \delta_\epsilon.$$

Therefore, from (2.40),(2.88) it follows that

$$0 \le u(x,t) \le ((C+3\epsilon)^{1-\beta}(x_1+R)_+^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)t)_+^{\frac{1}{1-\beta}},\ -\infty < x_1 \le -R_\epsilon, 0 \le t \le \delta_\epsilon$$

and due to the radial symmetricity of the solution u we deduce the estimation

$$0 \le u(x,t) \le \left\{(C+3\epsilon)^{1-\beta}(R-|x|)_+^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)t\right\}_+^{\frac{1}{1-\beta}},\ |x| \ge R_\epsilon, 0 \le t \le \delta_\epsilon.$$

Therefore, we have

$$\eta_-(x) \leq \frac{(C+3\epsilon)^{1-\beta}(R-|x|)^{\alpha(1-\beta)}}{b(1-\beta)(1-\epsilon)},\ R_\epsilon \leq |x| < R.$$

Taking $\limsup$ as $|x| \to R-$, followed by the limit as $\epsilon \downarrow 0$ we derive

$$\limsup_{|x|\to R-} \frac{\eta_-(x)}{(R-|x|)^{\alpha(1-\beta)}} \leq (l_*)^{\alpha(\beta-1)} \tag{2.92}$$

From (2.83) and (2.92), (3.4) follows. If $1 \leq m+\beta < 2$ from [27] it follows that for arbitrary $l > l_*$ and for all sufficiently small $\epsilon > 0$ there exists $\delta_\epsilon(l) > 0$ such that the solution of the Cauchy-Dirichlet problem (2.89)-(2.91) satisfies the following estimation:

$$0 \leq w_\epsilon(y_1,t) \leq C_3\Big(y_1 - \zeta_3 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^\alpha,\ -\infty < y_1 \leq l t^{\frac{1}{\alpha(1-\beta)}}, 0 \leq t \leq \delta_\epsilon, \tag{2.93}$$

where

$$\zeta_3 = (l_*/l)^{\alpha(1-\beta)}(1-\epsilon)l,\ C_3 = \Big[1-(l_*/l)^{\alpha(1-\beta)}(1-\epsilon)\Big]^{-\alpha}\Big[C^{1-\beta} - l^{-\alpha(1-\beta)}b(1-\beta)(1-\epsilon)\Big]^{\frac{1}{1-\beta}}.$$

From (2.40),(2.93) it follows that

$$0 \leq u(x,t) \leq C_3\Big(x_1 + R - \zeta_3 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^\alpha,\ -\infty < x_1 \leq -R + l t^{\frac{1}{\alpha(1-\beta)}}, 0 \leq t \leq \delta_\epsilon,$$

and due to radial symmetricity of u we derive the estimation

$$0 \leq u(x,t) \leq C_3\Big(R - |x| - \zeta_3 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^\alpha,\ |x| \geq R - l t^{\frac{1}{\alpha(1-\beta)}}, 0 \leq t \leq \delta_\epsilon. \tag{2.94}$$

From (2.94) it follows that for some $\gamma_\epsilon > 0$ we have

$$\eta_-(x) \le \Big(\frac{R-|x|}{\zeta_3}\Big)^{\alpha(1-\beta)}, \qquad R-\gamma_\epsilon \le |x| \le R$$

Taking $\limsup$ as $|x| \to R-$, followed by limits as $\epsilon \downarrow 0$ and $l \downarrow l_*$ we deduce

$$\limsup_{|x|\to R-} \frac{\eta_-(x)}{(R-|x|)^{\alpha(1-\beta)}} \le (l_*)^{\alpha(\beta-1)}. \tag{2.95}$$

From (2.83) and (2.95), (3.4) again follows. □

Theorem 2.1.6 is proved through direct application of Theorem 1.2.3 to upper and lower bounds given respectively in estimations (2.23), (2.26), (2.27), (2.30), (2.32).

2.3.4 Stationary Interface and Waiting Time

Proof of Theorem 2.1.6. (5a) Let $\beta = 1,\ \alpha = \frac{2}{m-1}$, then from [27] we find that

$$0 \le v(y,t) \le (C+\epsilon)(y_1)_+^{\frac{2}{m-1}} \exp^{-bt}\Big(1-(\frac{C}{\bar{C}})^{m-1} b^{-1}(1-e^{-b(m-1)t})\Big)^{\frac{1}{1-m}},$$

for $y \in \mathbb{R}^N,\ 0 \le t < T$. By (2.65) we have

$$0 \le u(x,t) \le C(x_1+R)_+^{\frac{2}{m-1}} e^{-bt}\Big(1-(\frac{C}{\bar{C}})^{m-1} b^{-1}(1-\exp^{-b(m-1)t})\Big)^{\frac{1}{1-m}}$$

for $x \in \mathbb{R}^N,\ 0 \le t < T$. Due the radial symmetricity of u we obtain the upper bound of (2.23). To find the lower estimation consider a function

$$g_{-\epsilon}(x,t) = \exp(-bt)(C-\epsilon)(R-|x|)_+^{\frac{2}{m-1}}$$

We have

$$
\begin{aligned}
Lg &= -\frac{2m(m+1)e^{-bmt}C^m(R-|x|)_+^{\frac{2}{m-1}}}{(m-1)^2} + \frac{2m(n-1)e^{-bmt}C^m(R-|x|)_+^{\frac{m+1}{m-1}}}{(m-1)|x|} \\
&= -\frac{2m(m+1)e^{-bmt}C^m(R-|x|)_+^{\frac{2}{m-1}}}{(m-1)^2} + \frac{2m(n-1)e^{-bmt}C^m(R-|x|)_+^{\frac{m+1}{m-1}}}{(m-1)|x|} \\
&= \frac{-2m(m+1)}{(m-1)^2}e^{-bmt}C^m(R-|x|)_+^{\frac{2}{m-1}}\Big(1+\frac{(m-1)(n-1)}{m+1}(1-\frac{R}{|x|})\Big)
\end{aligned}
$$

By choosing $R_1 = \frac{1}{\left(1+\frac{m-1}{(m+1)(n-1)}\right)}R < R$ and $\tilde{R_\epsilon} = \max(R_\epsilon, R_1)$ we have

$$Lg_{-\epsilon} \leq 0, \qquad |x| \geq \tilde{R_\epsilon},\ 0 < t \leq \delta_\epsilon \tag{2.96}$$

$$g_{-\epsilon}(x,0) \leq u_0(x), \qquad |x| \geq \tilde{R_\epsilon} \tag{2.97}$$

Since u and $g_{-\epsilon}$ are continuous functions there exists $\delta_\epsilon > 0$ such that

$$g_{-\epsilon}|_{|x|=\tilde{R_\epsilon}} \leq u|_{|x|=\tilde{R_\epsilon}},\ 0 \leq t \leq \delta. \tag{2.98}$$

and from a comparison principle, the lower bound of (2.23) follows.

(5b) Let $\beta = 1,\ \alpha > \frac{2}{m-1}$. As in first case we use the upper estimation of [27]; for sufficient small $\epsilon > 0$, there exists $R_\epsilon > 0$ and δ_ϵ such that

$$0 \leq v_\epsilon(y,t) \leq w_\epsilon(y_1,t) \leq (C+\epsilon)(y_1)_+^\alpha e^{-bt}\Big(1-\epsilon(b(m-1))^{-1}(1-e^{-b(m-1)t})\Big)^{\frac{1}{1-m}},$$

for $y \in \mathbb{R}^{\mathbb{N}},\ 0 \leq t < \delta_\epsilon$.From (2.65) we have

$$0 \leq u_\epsilon(x,t) \leq (C+\epsilon)(x_1+R)_+^\alpha e^{-bt}\Big(1-\epsilon(b(m-1))^{-1}(1-e^{-b(m-1)t})\Big)^{\frac{1}{1-m}},\ x \in \mathbb{R}^{\mathbb{N}}, 0 \leq t \leq \delta_\epsilon$$

Due to the radial symmetricity we get the right-hand side of (2.26). The prove of the lower bound is similar of the prove of last case. We consider the function

$$g_{-\epsilon}(x,t) = e^{-bt}(C-\epsilon)(R-|x|)_+^{\alpha}$$

which satisfies

$$Lg \leq 0, \qquad R_2 = \frac{1}{\left(1+\frac{\alpha m-1}{n-1}\right)}R < R$$

where

$$Lg = -\alpha m e^{-mbt}(C-\epsilon)^m(R-|x|)^{\alpha m-2}\Big[\alpha m - 1 + (n-1)(1-\frac{R}{|x|})\Big]$$

By choosing $\tilde{R_\epsilon} = \max(R_\epsilon, R_2)$ we deduce

$$Lg \leq 0, \qquad |x| \geq \tilde{R_\epsilon},\ 0 \leq t \leq \delta_\epsilon$$

and the comparison theorem implies that the left-hand side of (2.26) .

(5c) Let $1 < \beta < m$. $\alpha \geq \frac{2}{m-\beta}$. Then from [27],for sufficient small $\epsilon > 0$, there exists $R_\epsilon > 0$ *and* δ_ϵ such that

$$0 \leq v_\epsilon(y,t) \leq w_\epsilon(y_1,t) = \begin{cases} [(C+\epsilon)^{1-\beta}|y_1|^{\alpha(1-\beta)} + b(\beta-1)(1-d_\epsilon)t]^{\frac{1}{1-\beta}}, & 0 < y_1 \leq y_{1\epsilon} \\ 0, & y_1 \leq 0. \end{cases}$$

From (2.65) it follows that

$$0 \leq u_\epsilon(x,t) \leq w_\epsilon(x_1+R,t) = \begin{cases} [(C+\epsilon)^{1-\beta}|x_1+R|^{\alpha(1-\beta)} + b(\beta-1)(1-d_\epsilon)t]^{\frac{1}{1-\beta}}, \\ \qquad -R < x_1 \leq -R + y_{1\epsilon} \\ 0, \qquad x_1 \leq -R. \end{cases}$$

Due to the radial symmetricity, the right-hand side of (2.27) follows. To prove the lower bound consider the function

$$g_{-\epsilon} = [(C-\epsilon)^{1-\beta}(R-|x|)_+^{\alpha(1-\beta)} + b(\beta-1)(1-d_{-\epsilon})t]^{\frac{1}{1-\beta}}$$

For $b > 0,\ \alpha > \frac{2}{m-\beta}$ we have

$$Lg_{-\epsilon} = g_{-\epsilon}^{\beta}\Big[-\epsilon b - \alpha^2 m(m+\beta-1)(C-\epsilon)^{2(1-\beta)}(R-|x|)^{2\alpha(1-\beta)-2}g_{-\epsilon}^{m+\beta-2} +$$

$$+\alpha m(1+\alpha(\beta-1))(C-\epsilon)^{(1-\beta)}(R-|x|)^{\alpha(1-\beta)-2}g_{-\epsilon}^{m-1} - \alpha m(n-1)(1-\frac{R}{|x|})(C-\epsilon)^{(1-\beta)}\times$$

$$\times(R-|x|)^{\alpha(1-\beta)-2}g_{-\epsilon}^{m-1}\Big]$$

for $|x| > R_\epsilon,\ 0 \le t \le \delta_\epsilon$. For sufficiently small δ_ϵ, and for R_ϵ sufficiently close to R, we have $Lg \le 0$, and from the comparison theorem the lower bound of (2.27) follows for $\alpha > \frac{2}{m-\beta}$. If $b > 0,\ \alpha = \frac{2}{m-\beta}$, $Lg_{-\epsilon} = S\,g_{-\epsilon}^{\beta}$ where

$$S = -b\Big(\epsilon - (\frac{C-\epsilon}{C^*})^{m-\beta}\Big) - 4(m-\beta)^{-2}m(m+\beta-1)(C-\epsilon)^{2(1-\beta)}(R-|x|)^{2(2-m-\beta)}g_{-\epsilon}^{m+\beta-2} -$$

$$-2m(m-\beta)^{-2}(2-m-\beta)(C-\epsilon)^{(1-\beta)}(R-|x|)^{\frac{2(1-m)}{m-\beta}}g_{-\epsilon}^{m-1} - 2m(m-\beta)^{-1}(n-1)\times$$

$$\times(1-\frac{R}{|x|})(C-\epsilon)^{(1-\beta)}(R-|x|)^{\frac{2(1-m)}{m-\beta}}g_{-\epsilon}^{m-1}.$$

Since

$$\lim_{|x|\to R-} S = -b\epsilon$$

we have $Lg_{-\epsilon} \le 0$, in $|x| > R_\epsilon$, $0 \le t \le \delta_\epsilon$ and the lower bound of (2.27) follows from the comparison theorem for $\alpha = \frac{2}{m-\beta}$.

(5d) The proof in this case follows the same technique as in the cases (5a)-(5c). □

Chapter 3

Interfaces and local Solutions for the Nonlinear Degenerate Multidimensional Reaction-Diffusion Equations: Fast Diffusion versus Absorption

3.1 Formulation of Main Results on the Classification of Interfaces

Throughout this section we assume that u is a unique weak solution of the CP (1.5)-(1.6). In Figure 2, we have shown five different subcases. The main results are outlined below in Theorems 3.1.1-3.1.5 corresponding directly to the cases (1),(2),(3),(4) and (5).

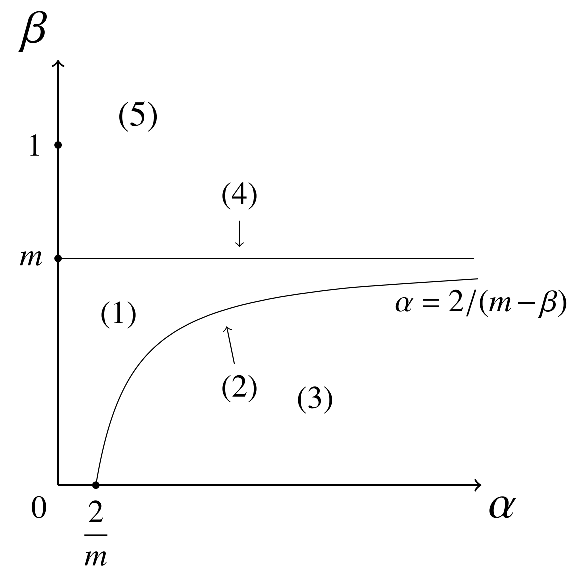

Figure 3.1: Classification of different cases in the (α,β) plane for interface development in the problem (1.5),(1.11),(1.12)

Theorem 3.1.1. *Suppose that* $0 < \beta < m$, $0 < \alpha < \frac{2}{m-\beta}$ *in this case the interface initially expands and* $\exists\, \delta > 0$ *such that*

$$\Big(\frac{R-|x|}{\xi_2}\Big)^{\frac{2(1-\beta)}{m-\beta}} \leq \eta_+(x) \leq \Big(\frac{R-|x|}{\xi_1}\Big)^{\frac{2(1-\beta)}{m-\beta}},\ 0 < t \leq \delta. \tag{3.1}$$

(see appendix for constants ξ_1,ξ_2*) Moreover, for arbitrary* $\rho \in \mathrm{R}$*, there exists a positive number* $f(\rho)$ *depending on* C, m *and* α *such that the solution* u *satisfies* (2.2) *along the surface* $|x| = R - \rho t^{\frac{1}{2+\alpha(1-m)}}$.

Theorem 3.1.2. *Suppose that* $b > 0,\ 0 < \beta < m < 1$, $\alpha = \frac{2}{m-\beta}$*.Then the interface expands or shrinks according as* $C \gtrless C_*$*. If* $C > C_*$ *the interface expands and*

$$\Big(\frac{R-|x|}{\xi_4}\Big)^{\frac{2(1-\beta)}{m-\beta}} \leq \eta_+(x) \leq \Big(\frac{R-|x|}{\xi_3}\Big)^{\frac{2(1-\beta)}{m-\beta}}. \tag{3.2}$$

while if $C < C_*$ the interface shrinks and

$$\Big(\frac{R-|x|}{\xi_5}\Big)^{\frac{2(1-\beta)}{m-\beta}} \le \eta_-(x) \le \Big(\frac{R-|x|}{\xi_6}\Big)^{\frac{2(1-\beta)}{m-\beta}} \tag{3.3}$$

(see appendix for constants $\xi_3 - \xi_6$)

Theorem 3.1.3. *Suppose $0 < \beta < m, \alpha > \frac{2}{m-\beta}$. Then interface initially shrinks and*

$$\eta_-(x) \sim \Big(\frac{R-|x|}{l_*}\Big)^{\alpha(1-\beta)}, \qquad \text{as } |x| \to R-. \tag{3.4}$$

For arbitrary $l > l_* = C^{-\frac{1}{\alpha}}(b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$, the following asymptotic formula is valid:

$$u(x,t)\Big|_{|x|=R-lt^{\frac{1}{\alpha(\beta-1)}}} \sim \{C^{1-\beta}l^{\alpha(1-\beta)} - b(1-\beta)\}^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}} \text{ as } t \downarrow 0. \tag{3.5}$$

Theorem 3.1.4. *Let $b > 0,\ 0 < \beta = m < 1,\ \alpha > 0$. Then there is an infinite speed of propagation and*

$$\log u(x,t) \sim -\frac{b^{1/2}}{m}|x|, \text{ as } |x| \to +\infty. \tag{3.6}$$

Theorem 3.1.5. *Let either $b > 0, \beta > m$ or $b < 0,\ \beta \ge 1$. Then there is an infinite speed of propagation. If either $b > 0,\ \beta \ge \frac{3-m}{2}$ or $b < 0,\ \beta \ge 1$ then $\exists \delta > 0$ such that for $\forall$ fixed $t \in (0,\delta]$*

$$u(x,t) \sim Dt^{\frac{1}{1-m}}|x|^{\frac{2}{m-1}}, \text{ as } |x| \to +\infty,\ t \in [0,\delta] \tag{3.7}$$

with $D = (2m(m+1(1-m)^{-1})^{\frac{1}{1-m}}$. If $b > 0, 1 \le \beta < \frac{3-m}{2}$, then

$$\lim_{t\to 0}\lim_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} = D \tag{3.8}$$

If $b > 0$, $m < \beta < 1$ then $\exists\delta > 0$ such that for arbitrary fixed $t \in (0, \delta]$

$$u(x,t) \sim C_*|x|^{2/(m-\beta)} \qquad as \;\; |x| \to +\infty. \tag{3.9}$$

3.2 Technical Details of the Main Results

In this section we outline some essential details of the main results described in Theorems 3.1.1-3.1.5 of section 2.1.

Technical details of Theorem 3.1.1: The solution u satisfies

$$t^{\frac{1}{1-\beta}} C_1(\xi - \xi_1)_+^{\frac{2}{m-\beta}} \le u(x,t) \le C_*(\xi - \xi_2)_+^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}}, \; 0 < t \le \delta. \tag{3.10}$$

where $\xi = (R - |x|)t^{-\frac{m-\beta}{2(1-\beta)}}$, the left-hand side of (3.10) is valid for $|x| > R$, while the right-hand side is valid for $|x| \ge R + lt^{\frac{m-\beta}{2(1-\beta)}}$. Moreover, for arbitrary $\rho \in \mathbb{R}$, there exists a positive number $f(\rho)$ depending on C, m and α which satisfies

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2+\alpha(1-m)}} \text{ as } t \downarrow 0^+, \tag{3.11}$$

along the surface $|x| = R - \rho t^{\frac{1}{2+\alpha(1-m)}}$.

Technical details of Theorem 3.1.2: In this case the behavior of the interface depends on the constant C. If $C > C_*$ we have the following estimation

$$C^{'}(\xi - \xi^{'})_+^{\frac{2}{m-\beta}} \le u(x,t) \le C^{"}\Big(R - |x| - \xi^{"} t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}}, \quad ,0 < t < +\infty. \tag{3.12}$$

where $C^{'} = C_2, C^{''} = C_*$ and $\xi^{'} = \xi_3, \xi^{''} = \xi_4$. If $C < C_*$ we have the estimation

$$C^{'}(\xi - \xi^{'})_+^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} \le u(x,t) \le C^{''}(\xi - \xi^{''})_+^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}}, \quad |x| < R, 0 < t < +\infty. \tag{3.13}$$

where the left-hand side is valid for $|x| > R + l_1 t^{\frac{m-\beta}{2(1-\beta)}}$, $0 < t \leq \delta$, and the right-hand side is valid for $|x| > R$ with $C' = C_*, C'' = C_3$ and $\xi' = \xi_5, \xi'' = \xi_6$.

Technical details of Theorem 3.1.4: For arbitrary $M > 0$ and $\epsilon > 0$, $\exists\ \delta = \delta(\epsilon, M) > 0$ such that

$$t^{1/(1-m)}\phi(x) \leq u(x,t) \leq (t+\epsilon)^{1/(1-m)}\phi(x) \qquad \text{for } R \leq |x| < +\infty,\ 0 \leq t \leq \delta_\epsilon, \tag{3.14}$$

where $\phi(x)$ is a solution of the elliptic PDE problem

$$\begin{cases} \Delta\phi^m = b\phi^m + \frac{1}{1-m}\phi, & |x| > R. \\ \phi|_{|x|=R} = M, \qquad \phi|_{|x|\to+\infty} = 0. \end{cases} \tag{3.15}$$

Technical details of Theorem 3.1.5: If either $b > 0$, $\beta \geq \frac{3-m}{2}$ or $b < 0$, $\beta \geq 1$ then $\exists\delta > 0$ such that for $\forall$ fixed $t \in (0, \delta]$

$$C_5 t^{\frac{\alpha}{(2+\alpha(1-m))}}(\zeta + \zeta_1)^{\frac{2}{m-1}} \leq u(x,t) \leq C_6 t^{\frac{\alpha}{2+\alpha(1-m)}}\Big(\zeta_2 + \frac{|x| - R}{t^{\frac{1}{2+\alpha(1-m)}}}\Big)^{\frac{2}{m-1}}, |x| > R. \tag{3.16}$$

If $b > 0, \beta > m$, the following upper estimation is valid

$$u(x,t) \leq Dt^{\frac{1}{1-m}}(|x| - R)^{\frac{2}{m-1}},\ |x| > R,\ 0 < t < \infty \tag{3.17}$$

Let $b < 0$, $\beta \geq 1$. Then for arbitrary sufficiently small $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$u(x,t) \leq D(1-\epsilon)^{\frac{1}{m-1}} t^{\frac{1}{1-m}}(|x| - R)^{\frac{2}{m-1}},\ -\infty < |x| \leq R - \mu t^{1/(2+\alpha(1-m))}\ 0 < t \leq \delta. \tag{3.18}$$

where

$$\mu = ((D^{-1}(A_0+\epsilon))^{\frac{m-1}{2}}(1-\epsilon)^{\frac{-1}{2}}$$

From (3.16) and (3.18), (3.7) follows. Let $b > 0$, $m < \beta < 1$. Then $\forall\ \epsilon > 0$ there exists a number $\delta > 0$ such that

$$C_*(1-\epsilon)t^{\frac{1}{1-\beta}}(\xi_8+\xi)^{\frac{2}{m-\beta}} \le u(x,t), \le C_*(|x|-R)^{\frac{2}{m-\beta}},\ |x| \ge R,\ 0 < t \le \delta. \tag{3.19}$$

3.3 Asymptotic Analysis of Solutions through Rescaling

In this section we formulate several Lemmas about the local asymptotics of solution to the Cauchy problem along some interface type manifolds in the positivity region. The proofs of the Lemmas is based on rescaling and application of the general theory of nonlinear degenerate parabolic PDEs.

Lemma 3.3.1. *Let u solve* (1.5) *with $b = 0$ and $u_0(x)$ satisfy* (1.12). *Then for arbitrary $\rho \in \mathbb{R}$, $\exists f(\rho) > 0$ such that*

$$u(x,t) \sim f(\rho)t^{\frac{\alpha}{2+\alpha(1-m)}} as t \downarrow 0^+, \tag{3.20}$$

along the surface $|x| = R - \rho t^{\frac{1}{2+\alpha(1-m)}}$.

The proof similar to the proof of Lemma 2.2.1 case (i) with $b = 0$.

Lemma 3.3.2. *Let u solves* (1.5)*,and u_0 satisfy* (1.11). *If one of the following conditions is satisfied*

(i) $b > 0,\ 0 < \beta < m < 1,\ 0 < \alpha < \frac{2}{m-\beta}$,

(ii) $b > 0, 0 < m < 1, \beta \ge m, ,\ \alpha > 0$

(iii) $b<0,\beta\geq 1, 0<m<1,\alpha>0$,

then u(x,t) satisfies (3.20).

Lemma 3.3.3. *Let u solves CP* (1.5),(1.11) *and* (1.12) *with* $b>0,\ 0<\beta<1, m>\beta$, *and* $\alpha=\frac{2}{m-\beta}$. *Then* $\exists\zeta_*=\zeta_*(C,m,\beta,b)$,*such that* $\forall\rho>\zeta_*\ \exists h(\rho)>0$ *such that*

$$u(x,t)\sim h(\rho)t^{\frac{1}{1-\beta}} \text{ as } t\downarrow 0_+, \text{along the surface } |x|=R-\rho t^{\frac{m-\beta}{2(1-\beta)}}. \tag{3.21}$$

If $0<C<C_*$ *then* $0<h(\rho)<C_*(\rho)^{\frac{2}{m-\beta}}$, *if* $C>C_*$ *then* $h(0)=A_1>0$, *where* $A_1=A_1(m,\beta,C,b)$.

Lemma 3.3.4. *Let u solves CP* (1.5)- (1.11) *with* $b>0,\ 0<\beta<1,\ m>\beta,\alpha=\frac{2}{m-\beta}$ *and,* $0<C<C_*$. *There exists* $l_1<0$, $\forall l\geq -l_1,\ \exists\lambda>0$ *such that*

$$u(x,t)\Big|_{|x|=R-lt^{\frac{m-\beta}{2(1-\beta)}}}\sim\lambda t^{\frac{1}{1-\beta}}, \text{ as } t\downarrow 0. \tag{3.22}$$

Lemma 3.3.5. *Let u solves* (1.5),(1.6),(1.11) *with* $b>0,\ 0<\beta<1, m>\beta$, *and* $\alpha>\frac{2}{m-\beta}, C>0$. *Then for arbitrary* $l>l_*$ (3.5) *is valid.*

3.3.1 Proof of Lemma 3.3.2: Diffusion Dominates Over the Reaction

The proof for cases (i) and (ii) coincides with the proof of case (ii) and (iii) with $b>0$ of Lemma 2.2.1. Case (iii). Let $b<0,\beta\geq 1,\alpha>0$. Change variables and define a new function $v(y,t)=v_{\mp\epsilon}(y,t)$ which solve the problem.

$$v_t-\Delta v^m+bv^\beta=0,\ |y-\bar{x}|<2R,\ 0<t<\delta \tag{3.23}$$

$$v(y,0)=(C\mp\epsilon)(R-|y-\bar{x}|)_+^\alpha,\qquad |y-\bar{x}|<2R \tag{3.24}$$

$$v(y,t)=0,\qquad |y-\bar{x}|=2R,\ 0\leq t\leq\delta. \tag{3.25}$$

The rescaled function $v_k(y,t) = kv(k^{-\frac{1}{\alpha}}y, k^{\frac{\alpha(m-1)-2}{\alpha}}t)$ satisfied the problem

$$w_t - \Delta w^m + k^{\frac{\alpha(m-\beta)-2}{\alpha}} bw^\beta = 0,\ y \in E^k = \{(y,t), |y - k^{\frac{1}{\alpha}}\bar{x}| < 2k^{\frac{1}{\alpha}}R\},\ 0 < t < \delta \tag{3.26}$$

$$w(y,0) = (C \mp \epsilon)(k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|)_+^\alpha, \qquad |y - k^{\frac{1}{\alpha}}\bar{x}| < 2k^{\frac{1}{\alpha}}R \tag{3.27}$$

$$w(y,0) = 0, \qquad |y - k^{\frac{1}{\alpha}}\bar{x}| = 2k^{\frac{1}{\alpha}}R \tag{3.28}$$

Since

$$\lim_{k\to\infty} (C \mp \epsilon)\{k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|\}_+^\alpha = C(y_1)_+^\alpha, \tag{3.29}$$

When $k \to \infty$ $\lim_{k\to\infty} v_k(y,t) = \tilde{w}(y,t) = w(y_1,t)$ which solves the problem

$$w_t - w^m_{y_1 y_1} + bw^\beta = 0,\ \ y \in \mathbb{R}^N, 0 \le t < T \tag{3.30}$$

$$w(y_1,0) = C(y_1)_+^\alpha.\ y \in \mathbb{R}^N \tag{3.31}$$

with the self-similar form (2.12). If $\beta > 1$ by [6] consider a function

$$g = \{(C+1)\big(1+|y|^2\big)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{1}{1-\beta}},\ y \in \mathbb{R}^N\}$$

we have

$$Lg = (C+1)(\beta-1)^{-1}(1+|y|^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{\beta}{1-\beta}}S$$

with

$$S\ = \nu - h(y) + bk^{\frac{\alpha(m-\beta)-2}{\alpha}}(\beta-1)(C+1)^{\beta-1}(1+|y|^2)^{\frac{\alpha(\beta-1)}{2}}$$

and

$$h(y) = \alpha m(\beta-1)(C+1)^{m-1}(1+|y|^2)^{\frac{\alpha(m-1)-4}{2}}(1-\nu t)^{\frac{m}{1-\beta}}\Big((\alpha m-1)|y|^2+1\Big)$$

$$\nu = 1 + h_*(y), h_*(\alpha,\beta,m) = \max_{y\in R^N} h(y)$$

Since $\frac{\alpha(m-\beta)-4}{2} < 0$ then $S \geq 1 + H$ where

$$H = O(k^{\frac{\alpha(m-\beta)-2}{\alpha}}(k^{\frac{2}{\alpha}})^{\frac{\alpha(\beta-1)}{2}})) = O(k^{(m-1)-\frac{2}{\alpha}})$$

Then

$$Lg \geq (C+1)(\beta-1)^{-1}(1+|y|^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{\beta}{1-\beta}}(1+H).$$

when $k \to \infty,\ Lg \geq 0, \quad in\ E^k$. Then for small $0 < \epsilon \ll 1$, the following inequality is true in wider region $\{y_1 > 0\}$

$$g(y,0) \geq (C+1)(y_1)_+^{\alpha} \geq (C \mp \epsilon)(k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|)^{\alpha}, |y - k^{\frac{1}{\alpha}}\bar{x}| < 2k^{\frac{1}{\alpha}}R, 0 < t < \delta. \quad (3.32)$$

By the comparison Theorem 4.3 and (3.29)
$0 \leq v^k_{\mp\epsilon}(y,t) \leq g(y,t)$, for $y \in E^k, 0 \leq t \leq \delta$.
Let G be a compact subset of

$$P = \{(y,t) : y \in R^N, y_1 > 0, 0 < t \leq t_0\}$$

The sequences $\{v^k\}$ is uniformly bounded and equicontinuous for some subsequence k' there exists a limit

$$v(y,t) = \lim_{k\to\infty} v^k(y,t),\ (y,t) \in P$$

The limit function is a solution of the CP (1.5),(1.6) with $b = 0$ and $u_0(x) = C(x_1)_+^{\alpha}$. Due to uniqueness of the latter, the whole sequence converges to its unique limit point. Similar arguments as in the case $b > 0$, implies (3.20).

If $\beta = 1$ by [6] consider a function

$$g(y,t) = (C+1)\exp(\nu t)(1+|y|^2)^{\frac{\alpha}{2}}$$

we have

$Lg = gS$, where $S = \nu - h(y) + bk^{\frac{\alpha(m-1)-2}{\alpha}}$

$$h(y) = \alpha m(C+1)^{m-1}(1+|y|^2)^{\frac{\alpha(m-1)-4}{2}} \exp(\nu(m-1)t)\Big[\frac{(\alpha m-1)|y|^2+1}{(1+|y|^2)}\Big]$$

and

$$\nu = 1 + \max_{y\in\mathbb{R}} h(y)$$

Since $\frac{\alpha(m-1)-2}{\alpha} < 0$, then $bk^{\frac{\alpha(m-1)-2}{\alpha}} \to 0$, as $k \to \infty$. and $S \geq 0$ which imply that $Lg \geq 0$ in E^k.Then for small $0 < \epsilon \ll 1$, we obtain (3.32). Same argument in case $\beta > 1$ (3.20) follows.

3.3.2 Proof of Lemma 3.3.3: Diffusion and Absorption in Balance

First let $u_0(x)$ satisfies (1.12), the proof of the asymptotic of the solution (3.21) is similar to proof Lemma 2.2.2.The second assertion of the lemma follows from Lemma 4.5 of [6]. If $0 < C < C_*$ then $0 < h(\rho) < C_*(\rho)^{\frac{2}{m-\beta}}$, if $C > C_*$ then $h(0) = A_1 > 0$.

Now, suppose that $u_0(x)$ satisfies (1.11), then $\forall \epsilon > 0\ \exists R_\epsilon \in (0,R)$ such that

$$\Big(C-\epsilon\Big)(R-|x|)_+^{\frac{2}{m-\beta}} \leq u_0(x) \leq \Big(C+\epsilon\Big)(R-|x|)_+^{\frac{2}{m-\beta}}, |x| \geq R_\epsilon \tag{3.33}$$

Let $u_{\mp\epsilon}(x)$ be a solution of the CP (1.5)-(1.11) with initial function $u_0(x)$ replaced by $(C\mp\epsilon)(R-|x|)_+^{\frac{2}{m-\beta}}$ since the solution $u_{\mp\epsilon}(x)$ is continuous, there exists a $\delta_\epsilon > 0$ such that

$$u_{-\epsilon}(x,t) \le u(x,t) \le u_{+\epsilon}(x,t), |x| = R_\epsilon, 0 \le t \le \delta_\epsilon \tag{3.34}$$

By applying the comparison principle in $\{|x| > R_\epsilon, 0 < t \le \delta_\epsilon\}$ we have

$$u_{-\epsilon}(x,t) \le u(x,t) \le u_\epsilon(x,t) \text{ for } |x| > R_\epsilon, 0 < t \le \delta_\epsilon \tag{3.35}$$

According to previous global case we have

$$u_{\mp\epsilon}(x,t) \sim h(\rho; C\mp\epsilon)t^{\frac{1}{1-\beta}} \text{ as } t \downarrow 0_+, \text{along } |x| = R-\rho t^{\frac{m-\beta}{2(1-\beta)}}.$$

Therefore, (3.35) is satisfied for $|x| = R-\rho t^{\frac{m-\beta}{2(1-\beta)}}, 0 < t \le \delta_\epsilon$.

$$\frac{u_{-\epsilon}(x,t)}{h(\rho)t^{\frac{1}{1-\beta}}} \le \frac{u(x,t)}{h(\rho)t^{\frac{1}{1-\beta}}} \le \frac{u_\epsilon(x,t)}{h(\rho)t^{\frac{1}{1-\beta}}},\ |x| = R-\rho t^{\frac{m-\beta}{2(1-\beta)}},\ ,0 < t \le \delta_\epsilon.$$

Passing to the limit first as $t \downarrow 0$, and then as $\epsilon \downarrow 0$ we have

$$u(x,t)\Big|_{R-\rho t^{\frac{m-\beta}{2(1-\beta)}}} \sim h(\rho)t^{\frac{1}{1-\beta}}, \quad \text{as } t \to 0+.$$

3.3.3 Proof of Lemma 3.3.4: Diffusion and Absorption in Balance

Let $w(y_1,t)$ solution of (3.30),(3.31), then $\exists l < 0\ \forall l \ge -l_1,\ \exists\lambda > 0$ such that

$$w(lt^{\frac{m-\beta}{2(1-\beta)}}, t) = \lambda t^{\frac{1}{1-\beta}}$$

$$\lim_{k\to\infty} kv(k^{\frac{\beta-m}{2}}y, k^{\beta-1}t) = w(y_1,t)$$

By choosing $y_1 = lt^{\frac{m-\beta}{2(1-\beta)}}, y_2 = ... = y_N = 0,\ k^{\beta-1}t = \tau.$

$$\lim_{\tau\to 0} u(l\tau^{\frac{m-\beta}{2(1-\beta)}} - R, 0, ..., 0, \tau) = \lambda\tau^{\frac{1}{1-\beta}}$$

By radial symmetricity we get (3.22).

3.3.4 Proof of Lemma 3.3.5: Absorption Dominates over the Diffusion

Since $u_0(x)$ satisfies (1.11) then (3.33) follows. Let $u_{\mp\epsilon}$ be a solution of the CP (1.5)(1.6) with intial function $(C\mp\epsilon)(R-|x|)_+^{\alpha}$ since the solution of CP (1.5),(1.6) is continuous, there exists a $\delta_\epsilon > 0$ such that (3.34) is satiesfies. By applying the comparison principle in $\{|x| > R_\epsilon, 0 < t \le \delta_\epsilon\}$ we get (3.35).

As in Lemma 2.2.1 case (iii) we pursue the change of the variables, and define the function $v_{\mp\epsilon}(y,t)$, which solves the problem (3.23)-(3.25). The rescaled function $v^k_{\mp\epsilon}(y,t)$ solves the problem

$$w_t - k^{\frac{2-\alpha(m-\beta)}{\alpha}}\Delta w^m + bw^\beta = 0 \ \ in \ \ E^K_\epsilon = \{(y,t):\ |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}},\ \ t > 0\} \tag{3.36}$$

$$w(y,0) = (C\mp\epsilon)(k^{\frac{1}{\alpha}}R - |y - k^{\frac{1}{\alpha}}\bar{x}|)_+^{\alpha} \quad |y - k^{\frac{1}{\alpha}}\bar{x}| < 2Rk^{\frac{1}{\alpha}} \tag{3.37}$$

$$w|_{|y-k^{\frac{1}{\alpha}}\bar{x}|=2Rk^{\frac{1}{\alpha}}} = 0,\ 0 \le t \le k^{1-\beta}\delta. \tag{3.38}$$

To prove that $\lim_{k\to\infty} v^k_{\mp\epsilon}(y,t)$ exists we have to prove the uniform boundedness. Consider a function

$$g(y,t) = (C+1)(1+|y|^2)^{\frac{\alpha}{2}}e^t, y \in \mathbb{R}^N$$

for some $T > 0$ we have $L_k g \geq g(1+S)$ *in* E^k

$$S = b(C+1)^{\beta-1}(1+|y|^2)^{\frac{\alpha(\beta-1)}{2}} e^{(\beta-1)t} - h(y)$$

$$S = O(k^{\frac{2-\alpha(m-\beta)}{\alpha}}) \text{ uniformly for } |y - k^{\frac{1}{\alpha}}\bar{x}| \leq 2Rk^{\frac{1}{\alpha}} \text{as } k \to \infty,$$

The rest of the proof coincides with the proof of Lemma 2.2.3 of slow diffusion case.

3.4 Proofs of Main Results

In this section we prove the main results characterized in Section 3.1.

3.4.1 Expanding interface

Proof of Theorem 3.1.1. The asymptotic estimation (3.20) follows from Lemma 3.3.2.Take arbitrary small $\epsilon > 0$; from (3.20) it follows that there exists a δ_ϵ such that

$$(A_0 - \epsilon)t^{\frac{\alpha}{2+\alpha(1-m)}} \leq u(x,t) \leq (A_0 + \epsilon)t^{\frac{\alpha}{2+\alpha(1-m)}}, \ |x| = R, 0 < t \leq \delta_\epsilon. \tag{3.39}$$

where $\rho = 0,\ A_0 = f(0) > 0$. Consider a function

$$g(x,t) = t^{\frac{1}{1-\beta}} f_1(\xi),\ \xi = (R - |x|)t^{-\frac{m-\beta}{2(1-\beta)}} \text{ and } f_1(\xi) = C_0(\xi - \xi_0)_+^{\frac{2}{m-\beta}}$$

Estimate Lg in $\{(x,t), R < |x| < R - \xi_0 t^{\frac{m-\beta}{2(1-\beta)}}, 0 < t \leq \delta\}$. We then have

$$Lg = bg^\beta \Big\{ 1 - (\frac{C_0}{C_*})^{m-\beta} - \frac{C_0^{1-\beta}}{b(1-\beta)} \xi_0 (\xi - \xi_0)^{\frac{2-m-\beta}{m-\beta}} + \frac{2m(n-1)}{b(m-\beta)|x|} t^{\frac{m-\beta}{2(1-\beta)}} C_0^{m-\beta}(\xi - \xi_0) \Big\}. \tag{3.40}$$

Since $\frac{2-m-\beta}{m-\beta} > 0$ we have

$$0 \le (\xi - \xi_0)^{\frac{2-m-\beta}{m-\beta}} \le (-\xi_0)^{\frac{2-m-\beta}{m-\beta}}, \quad \text{if } \xi_0 < \xi < 0,$$

and therefore

$$Lg \le bg^{\beta}\Big\{1 - (\frac{C_0}{C_*})^{m-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)}(-\xi_0)^{\frac{2(1-\beta)}{m-\beta}} + \frac{2m(n-1)C_0^{m-\beta}}{b(m-\beta)R}(-\xi_0)t^{\frac{m-\beta}{2(1-\beta)}}\Big\}$$

in $R < |x| < R - \xi_0 t^{\frac{m-\beta}{2(1-\beta)}}, 0 < t \le \delta$. For all $\epsilon > 0$ we try to choose C_0 and ξ_0 such that

$$1 - (\frac{C_0}{C_*})^{m-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)}(-\xi_0)^{\frac{2(1-\beta)}{m-\beta}} = -\epsilon \tag{3.41}$$

This can be chosen as in [6], take $C_0 = C_1$ and $\xi_0 = \xi_1$ (see Apendix 4.1). Then we have

$$Lg \le 0 \text{ for } R < |x| < R - \xi_0 t^{\frac{m-\beta}{2(1-\beta)}} \ t > 0, \tag{3.42a}$$

$$Lg = 0 \text{ for } |x| < R, \ t > 0, \tag{3.42b}$$

which implies that g is subsolution of (1.5). Since $\frac{1}{1-\beta} > \frac{\alpha}{2+\alpha(1-m)}$ it follows from (3.39) there exists a $\delta_2 > 0$ such that

$$f_1(0)t^{\frac{1}{1-\beta}} \le A_0 t^{\frac{\alpha}{2+\alpha(1-m)}}, \text{ for } 0 \le t \le \delta_2. \tag{3.43a}$$

$$u(x,0) = g(x,0) = 0, \text{ for } |x| \ge R. \tag{3.43b}$$

Fix $\epsilon = \epsilon_0$ and take $\delta = \min(\delta_1, \delta_2)$.From (3.42),(3.43) and by the comparison theorem the letf-hand side of (3.10), and the right-hand side of (3.1) follow. To prove the upper estimation, let $v(y,t)$ solves (3.23)-(3.25), and $w(y_1,t)$ satisfies (3.26)-(3.28) with $b = 0$.

We use the following rough estimation from [6]:

$$w(y_1,t) \le Dt^{\frac{1}{1-m}}(-y_1-R)^{\frac{2}{m-1}} \text{ for } -\infty < y_1 < -2R,\ 0 < t < +\infty,$$

where $D = (\frac{2m(m+1)}{1-m})^{\frac{1}{1-m}}$. From the comparison theorem it follows that

$$u(x,t) \le Dt^{\frac{1}{1-m}}(-x_1-R)^{\frac{2}{m-1}} \text{ for } -\infty < y_1 < -2R,\ 0 < t < +\infty.$$

Choose a curve $\Gamma : x_1 = -R - lt^{\frac{m-\beta}{2(1-\beta)}}, x_2 = ... = x_N = 0$

$$u(x,t)|_\Gamma \le Dt^{\frac{1}{1-m}}(lt^{\frac{m-\beta}{2(1-\beta)}})^{\frac{2}{m-1}} = Dl^{\frac{2}{m-1}}t^{\frac{1}{1-\beta}}$$

Due the radial symmetricity

$$u(x,t)|_{|x|=R+lt^{\frac{m-\beta}{2(1-\beta)}}} \le Dl^{\frac{2}{m-1}}t^{\frac{1}{1-\beta}}. \tag{3.44}$$

Now to establish more accurate estimation, consider a function

$$g(x,t) = t^{\frac{1}{1-\beta}}f_1(\xi) \text{ where, } f_1(\xi) = C_*(\xi-\xi_2)_+^{\frac{2}{m-\beta}},\ \xi = (R-|x|)t^{-\frac{m-\beta}{2(1-\beta)}}$$

We estimate Lg in $G_{l,\delta} = \{(x,t) : |x| > R + lt^{\frac{m-\beta}{2(1-\beta)}}\},\ 0 < t \le \delta\}$. From (3.40) we obtain

$$Lg = bg^\beta\Big\{-\frac{C_*^{1-\beta}}{b(1-\beta)}\xi_2(\xi-\xi_2)_+^{\frac{2-m-\beta}{m-\beta}} + \frac{2m(n-1)}{b(m-\beta)|x|}t^{\frac{m-\beta}{1-\beta}}C_*^{m-\beta}(\xi-\xi_2)_+\Big\}$$

since $\xi_2 < 0$ and $\xi - \xi_2 > 0$, it's obvious that $Lg \geq 0$ in $G_{l,\delta}$. From [6] and (3.44) we have to choose ξ_2 and l such that

$$u(x,t)|_{|x|=R+lt^{\frac{m-\beta}{2(1-\beta)}}} \leq Dl^{\frac{2}{m-1}} t^{\frac{1}{1-\beta}} \leq g(x,t)|_{|x|=R+lt^{\frac{m-\beta}{2(1-\beta)}}} = C_*(\xi - \xi_2)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} =$$

$$= C_*(-l - \xi_2)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}},\ 0 < l < -\xi_2.$$

By applying the comparison theorem in $G_{l,\delta}$, we obtain the right-hand side of (3.10) and the left-hand side of (3.1). □

3.4.2 Borderline Case for the Interface Movement

Proof of Theorem 3.1.2. Let u solves the CP (1.5)- (1.11). In both cases $C \gtrless C_*$, the proof of the right-hand side coincides with the proof of Theorem 2.1.3 of slow diffusion case when $1 \leq m < 2 - \beta$. If $C > C_*$, we pursue the change of varibles $y = x + \bar{x}$, $\bar{x} = (R, 0, ..., 0)$. The function $v(y,t) = u(y - \bar{x}, t)$ solves the problem (3.23)-(3.25) and let $w(y_1,t)$ be s solution of (3.30),(3.31). The latter has self-similar form

$$w(y_1,t) = h(\xi) t^{\frac{1}{1-\beta}}, \qquad \xi = \frac{y_1}{t^{\frac{m-\beta}{2(1-\beta)}}}$$

(see Apendix 4.1 for explicit values of the constants $C_2, \xi_3, \xi_4, \xi_5, \xi_6$). From [27] and [6] we find that

$$v(y,t) \leq w(y_1,t) \leq C_* t^{\frac{1}{1-\beta}} \Big(-\xi_4 + \frac{y_1}{t^{\frac{m-\beta}{2(1-\beta)}}} \Big)_+^{\frac{2}{m-\beta}}, -\infty < y_1 < 0,\ (y_2, ..., y_N) \in \mathrm{R}^{N-1}, 0 < t < +\infty.$$

Then

$$u(x,t) \leq C_* \Big(x_1 + R - \xi_4 t^{\frac{m-\beta}{2(1-\beta)}} \Big)_+^{\frac{2}{m-\beta}}, \text{for} x_1 \geq -R + \xi_4 t^{\frac{m-\beta}{2(1-\beta)}}, 0 < t < +\infty.$$

and

$$u(x,t) \equiv 0,\ x_1 \leq -R + \xi_4 t^{\frac{m-\beta}{2(1-\beta)}},\ 0 < t < +\infty.$$

Due the radial symmetricity of u we deduce that

$$u(x,t) \leq C_* t^{\frac{1}{1-\beta}} \Big(-\xi_4 + \frac{R-|x|}{t^{\frac{m-\beta}{2(1-\beta)}}} \Big)_+^{\frac{2}{m-\beta}}, \quad |x| > R, 0 < t < +\infty. \tag{3.45}$$

where $\xi_4 < \xi < 0$, then the right-hand side of (3.12) and the left-hand side of (3.2) follow. To prove the lower estimation let

$$g(x,t) = t^{\frac{1}{1-\beta}} f_1(\xi) \text{ where, } f_1(\xi) = C_2(\xi - \xi_3)_+^{\frac{2}{m-\beta}},\ \xi = (R - |x|)t^{-\frac{m-\beta}{2(1-\beta)}}$$

As in Theorem 3.1.1 we estimate g in $G_{\xi_3,\delta}$ where

$$G_{\xi_3,\delta} = \{(x,t) : R < |x| < R - \xi_3 t^{\frac{m-\beta}{2(1-\beta)}},\ 0 < t \leq \delta\}$$

Then form (3.40) it follows that

$$Lg \leq 0 \text{ for } R < |x| < R - \xi_3 t^{\frac{m-\beta}{2(1-\beta)}}\ t > 0, \tag{3.46a}$$

$$Lg = 0 \text{ for } |x| \geq R - \xi_3 t^{\frac{m-\beta}{2(1-\beta)}},\ t > 0, \tag{3.46b}$$

By comparing $g(x,t)$, and $u(x,t)$ in $\{|x| = R, 0 \leq t \leq T\}$, for arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that

$$g(x,t)|_{|x|=R} \leq (A_1 - \epsilon)t^{\frac{1}{1-\beta}} \leq u(x,t)|_{|x|=R},\ 0 \leq t \leq \delta_\epsilon.$$

provided that $C_2(-\xi_3)^{\frac{2}{m-\beta}} = A_1 - \epsilon$. Comparison theorem implies that $u(x,t) \geq g(x,t)$, and therefore the left-hand side of (3.12), and the right-hand side of (3.2) follow. If $C < C_*$, Then from Lemma 3.3.4 it follows that for arbitrary $\epsilon > 0$ there exists a number $\delta_\epsilon > 0$ and a constant $l_1 < 0$ such that for arbitrary $l \leq -l_1$ there exists $\lambda > 0$ such that

$$(\lambda - \epsilon)t^{\frac{1}{1-\beta}} \leq u(x,t) \leq (\lambda + \epsilon)t^{\frac{1}{1-\beta}}, \ |x| = R - lt^{\frac{m-\beta}{2(1-\beta)}}, \ 0 < t \leq \delta_\epsilon. \tag{3.47}$$

To prove the left-hand side consider a function g

$$g(x,t) = t^{\frac{1}{1-\beta}} f_1(\xi) \text{ where, } f_1(\xi) = C_*(\xi - \xi_5)_+^{\frac{2}{m-\beta}}, \ \xi = (R - |x|)t^{-\frac{m-\beta}{2(1-\beta)}}$$

We estimate Lg in

$$G_{l_1,\delta} = \{(x,t) : |x| \geq R + l_1 t^{\frac{m-\beta}{2(1-\beta)}}, \ 0 < t \leq \delta\}$$

From (3.40) we have

$$Lg \leq 0, \text{ for } R + l_1 t^{\frac{m-\beta}{2(1-\beta)}} \leq |x| \leq R - \xi_5 t^{\frac{m-\beta}{2(1-\beta)}}$$

$$g(x,t)\Big|_{|x|=R+l_1 t^{\frac{m-\beta}{2(1-\beta)}}} = C_*(-l_1 - \xi_5)^{\frac{2}{m-\beta}} t^{\frac{1}{1-\beta}} \leq (\lambda - \epsilon)t^{\frac{1}{1-\beta}} \leq u(x,t)\Big|_{|x|=R+l_1 t^{\frac{m-\beta}{2(1-\beta)}}}$$

$$\text{for } 0 < \xi_5 < -l_1.$$

For $|x| > R + l_1 t^{\frac{m-\beta}{2(1-\beta)}}, \ 0 < t \leq \delta$ the left-hand sides from (3.13) and (3.3) follow. To prove the right-hand side, we consider solution $v(y,t)$ of the Cauchy problem (3.23)-(3.25) and solution $w(y,t)$ of the problem (3.26),(3.27). Similarly as it is done in previous Chapter for the slow diffusion case, and by [27] we deduce that

$$0 \le v(y,t) \le w(y_1,t) \le C_3\Big(y_1 - \xi_6 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}}, \text{ for } y_1 \ge l_2 t^{\frac{m-\beta}{2(1-\beta)}}$$

$$u(x,t) \le C_3\Big(x_1 + R - \xi_6 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}},\ x_1 \ge -R + \xi_6 t^{\frac{m-\beta}{2(1-\beta)}} \tag{3.48}$$

Due the radial symmetricity of u from (3.48) it follows that

$$u(x,t) \le C_3 t^{\frac{1}{1-\beta}}\Big(R - |x| - \xi_6 t^{\frac{m-\beta}{2(1-\beta)}}\Big)_+^{\frac{2}{m-\beta}}, \quad R - \delta_\epsilon < |x| < R, 0 < t < +\infty. \tag{3.49}$$

$$u(x,t) \equiv 0, \qquad if \quad \Big(\frac{R-|x|}{\xi_6}\Big)^{\frac{2(1-\beta)}{m-\beta}} \le t \le \Big(\frac{R-|x|}{l_2}\Big)^{\frac{2(1-\beta)}{m-\beta}},\ R - \delta_\epsilon < |x| < R$$

Then the right-hand sides from (3.13) and (3.3) follow. □

3.4.3 Shrinking Interface

Proof of Theorem 3.1.3. Lemma 3.3.4 implies the estimation (3.5). The proof of the asymptotic formula (3.4) coincides with the proof given in Theorem 2.1.5 when $m+\beta < 2$. To prove the upper bound, let $v(y,t)$ be a solution of the Cauchy problem (3.23)-(3.25), and $w(y,t)$ be a solution of (3.30),(3.31). By using similar proof as in Theorem 2.1.3 and by [27], we have

$$0 \le v(y,t) \le w(y_1,t) \le C_4\Big(y_1 - \xi_7 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^{\alpha}, y \in \mathrm{R}^N, t > 0$$

and therefore,

$$0 \le u(x,t) \le C_6\Big(x_1 + R - \xi_7 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^{\alpha}, x \in \mathrm{R}^N, t > 0$$

and

$$u(x_1, 0, ..., 0, t) \equiv 0, \; for \; t \geq \Big(\frac{x_1 + R}{\xi_7}\Big)^{\alpha(1-\beta)}.$$

Due the radial symmetricity of u

$$u(x,t) \leq C_4\Big(R - |x| - \xi_7 t^{\frac{1}{\alpha(1-\beta)}}\Big)_+^{\alpha}, \; for \; |x| > R, t > 0.$$

$$u(x,t) \equiv 0, \; if \quad t \geq \Big(\frac{R - |x|}{\xi_7}\Big)^{\alpha(1-\beta)}$$

that is to say

$$\eta_-(x) \leq \Big(\frac{R - |x|}{\xi_7}\Big)^{\alpha(1-\beta)}, \qquad R - \delta_\epsilon < |x| < R$$

(see Appendix 4.1 for explicit value of C_4, ξ_7). Taking the limit first when $\epsilon \downarrow 0$, and then $l \to l_*$ we have

$$\limsup_{|x| \to R-} \frac{\eta_-(x)}{(R - |x|)^{\alpha(1-\beta)}} \leq \Big(\frac{1}{l_*}\Big)^{\alpha(1-\beta)}, \quad |x| \to R-. \tag{3.50}$$

From Lemma 3.3.5 for arbitrary $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\eta_-(x) \geq \Big(\frac{R - |x|}{l_*}\Big)^{\alpha(1-\beta)}, \qquad R - \delta_\epsilon \leq |x| \leq R$$

and therefore,

$$\liminf_{|x| \to R-} \frac{\eta_-(x)}{(R - |x|)^{\alpha(\beta-1)}} \geq (\frac{1}{l_*})^{\alpha(1-\beta)} \tag{3.51}$$

By combining (3.50) and (3.51) we obtain (3.4). □

3.4.4 Infinite Speed of Propagation and Asymptotics of the Local Solution at Infinity

Proof of Theorem 3.1.4. Let $b > 0, 0 < \beta = m < 1$, and $\alpha > 0$. From Lemma 3.3.2 it follows that

$$u(x,t)|_{|x|=R} \sim f(0)t^{\frac{\alpha}{2+\alpha(1-m)}},\ t \downarrow 0,\ f(0) > 0$$

Consider a function

$$u(x,t) = t^{\frac{1}{1-m}}\phi(x) \tag{3.52}$$

where ϕ is a solution of the elliptic problem (3.15). We have

$$Lu = u_t - \Delta u^m + bu^m = t^{\frac{1}{1-m}}\Big[\frac{1}{1-m}\phi - \Delta\phi^m + b\phi^m\Big] = 0.$$

Hence (3.52) is a solution of (1.5). Solution of the elliptic problem (3.15) is radially symmetric $\phi(x) = \psi(r),\ r = |x|$, where $\psi(r)$ is a solution of the ODE problem

$$\begin{cases} \mathcal{L}\psi(r) = -(\psi^m)''(r) - \frac{n-1}{r}(\psi^m)'(r) + b\psi^m(r) + \frac{1}{1-m}\psi(r) = 0,\ R < r < +\infty. \\ \psi(R) = M, \qquad \psi(+\infty) = 0. \end{cases}$$

Consider a function

$$g(r) = C\exp(-\gamma r),\quad C,\gamma > 0.$$

We have

$$\mathcal{L}\psi = C^m \exp(-\gamma mr)\Big(-(\gamma m)^2 + \frac{(n-1)\gamma m}{r} + b + \frac{C^{1-m}}{1-m}\exp(\gamma(m-1)r)\Big)$$

By choosing

$$\gamma = m^{-1}b^{\frac{1}{2}},\ C = M\exp(-m^{-1}b^{\frac{1}{2}}R),$$

we have

$$\mathcal{L}\psi = C^m \exp(-\gamma m r)\Big(\frac{(n-1)\gamma m}{r} + \frac{C^{1-m}}{1-m}\exp(\gamma(m-1)r)\Big) > 0,\ g(R) = M.$$

From the standard comparison theorem it follows that

$$\psi(r) \le M \exp(m^{-1} b^{\frac{1}{2}}(R-r)),\ R \le r < +\infty,$$

and therefore

$$0 \le \phi(x) \le M \exp(m^{-1} b^{\frac{1}{2}}(R-|x|)),\ R \le |x| < +\infty. \tag{3.53}$$

From (3.52) and (3.53) the following upper estimation follows

$$u(x,t) \le M t^{\frac{1}{1-m}} \exp(m^{-1} b^{\frac{1}{2}}(R-|x|)),\ R \le |x| < +\infty. \tag{3.54}$$

To prove the lower bound, choose arbitrary $\epsilon > 0$ and consider

$$\psi(r) = M \exp(\gamma(R-r)),$$

with $\gamma = m^{-1}(b+\epsilon)^{1/2}$. We have

$$\mathcal{L}\psi(r) = M^m e^{\gamma m(R-r)}\Big[-\epsilon + \frac{(n-1)\gamma^m}{r} + \frac{M^{1-m}e^{\gamma(1-m)R}}{1-m}e^{\gamma(m-1)r}\Big] \le$$

$$M^m e^{\gamma m(R-r)}\Big[-\epsilon + \frac{(n-1)\gamma^m}{R} + \frac{M^{1-m}}{1-m}\Big] \le 0,\ \text{if} R \gg 1 \text{ and } M \ll 1.$$

From the comparison theorem it follows

$$\psi(r) \ge M \exp\Big(m^{-1}(b+\epsilon)^{1/2}(R-r)\Big),\ R\ \le r < +\infty,$$

and therefore,

$$u(x,t) \geq Mt^{\frac{1}{1-m}} \exp\Big(m^{-1}(b+\epsilon)^{1/2}(R-|x|)\Big),\ |x| > R,\ 0 < t < +\infty. \tag{3.55}$$

From (3.54),(3.55) it follows that

$$\log Mt^{\frac{1}{1-m}} + \frac{(b+\epsilon)^{1/2}}{m}(R-|x|) \leq \log u(x,t) \leq \log Mt^{\frac{1}{1-m}} + \frac{b^{1/2}}{m}(R-|x|)$$

or

$$\frac{\log Mt^{\frac{1}{1-m}} + \frac{(b+\epsilon)^{1/2}}{m}(R-|x|)}{|x|} \leq \frac{\log u(x,t)}{|x|} \leq \frac{\log Mt^{\frac{1}{1-m}} + \frac{b^{1/2}}{m}(R-|x|)}{|x|}$$

for $R \leq |x| < +\infty,\ 0 < t < \delta.$ Passing to the limit as $|x| \to +\infty$ we deduce

$$-\frac{(b+\epsilon)^{1/2}}{m} \leq \liminf_{|x|\to+\infty} \frac{\log u(x,t)}{|x|} \leq \limsup_{|x|\to+\infty} \frac{\log u(x,t)}{|x|} \leq -\frac{b^{1/2}}{m}$$

Finally, passing to the limit as $\epsilon \downarrow 0$, desired asymptotic formula (3.6) follows. □

Proof of Theorem 3.1.5. Let either $b > 0,\ \beta > m$ or $b < 0,\ \beta \geq 1$.The asymptotic estimation (3.20) is proven in Lemma 3.3.1. Let $y = x + \bar{x}, \bar{x} = (R, 0, 0, ..., 0)$ and we consider the Cauchy problem for $v(y,t) = u(y - \bar{x}, t)$:

$$\begin{cases} v_t - \Delta v^m + bv^\beta = 0,\ y \in \mathbb{R}^N,\ t > 0 \\ v(y,0) = C(R - |y-\bar{x}|)_+^\alpha,\ y \in \mathbb{R}^N \end{cases} \tag{3.56}$$

From the asymptotic estimation (3.20) it follows that

$$v|_{|y-\bar{x}|=R} \sim f(0)t^{\frac{\alpha}{2+\alpha(1-m)}},\ t \downarrow 0.$$

Let $w(y,t)$ is a solution of the Cauchy problem

$$\begin{cases} v_t - \Delta v^m + bv^\beta = 0, \; y \in \mathbb{R}^N, \; t > 0 \\ v(y,0) = C(y_1)_+^\alpha, \; y \in \mathbb{R}^N \end{cases} \tag{3.57}$$

By comparing the initial function of $w(y_1,t)$ and the initial function of $v(y,t)$ we have

$$(R - |y - \bar{x}|)_+^{\frac{1}{1-\beta}} \leq (y_1)_+^{\frac{1}{1-\beta}}, \; y \in \mathbb{R}^{\mathbb{N}}. \tag{3.58}$$

From the Comparison theorem it follows that

$$v(y,t) \leq w(y,t), \; y \in \mathbb{R}^N, \; t > 0.$$

From the upper bound established in [6] it follows that

$$v(y,t) \leq w(y,t) \leq C_6 t^{\frac{\alpha}{2+\alpha(1-m)}} (\zeta_2 + \zeta)^{\frac{2}{m-1}}, y_1 \leq 0, \; 0 \leq t \leq \delta.$$

where

$$\zeta = \frac{-y_1}{t^{\frac{1}{2+\alpha(1-m)}}}.$$

Since u is radial symmetric, the right-hand side of (3.16) easily follows.

If $b > 0, \beta \geq 1$, then by [6] the following upper estimation is valid

$$v(y,t) \leq w(y,t) \leq Dt^{\frac{1}{1-m}} (-y_1)^{\frac{2}{m-1}}, \; -\infty < y_1 < 0, \; 0 < t < +\infty$$

which implies the upper bound

$$u(x,t) \leq Dt^{\frac{1}{1-m}} (-x_1 - R)^{\frac{2}{m-1}}, \; -\infty < x_1 < -R, \; 0 < t < +\infty$$

Due to radial symmmetricty of u, we deduce (3.17).

To prove the lower bound, let

$$g(x,t) = t^{\frac{\alpha}{(2+\alpha(1-m))}} f(\zeta), \quad \zeta = (|x|-R)t^{\frac{-1}{(2+\alpha(1-m))}},\ f(\zeta) = C_0(\zeta+\zeta_0)^{\frac{-2}{1-m}}.$$

We have

$$Lg = t^{\frac{\alpha m-2}{2+\alpha(1-m)}}\mathcal{S}f$$

where

$$\mathcal{S}f = \frac{1}{2+\alpha(1-m)}C_0(\zeta+\zeta_0)^{\frac{2}{m-1}}\Big(R(\zeta)+b(2+\alpha(1-m))t^{\frac{2-\alpha(m-\beta)}{(2+\alpha(1-m))}}C_0^{\beta-1}(\zeta+\zeta_0)^{\frac{2(1-\beta)}{1-m}}\Big) \quad (3.59)$$

$$R(\zeta) = \alpha + \frac{2}{(1-m)}\zeta(\zeta+\zeta_0)^{-1} - \frac{2m(1+m)(2+\alpha(1-m))}{(1-m)^2}C_0^{m-1}$$
$$-\frac{t^{\frac{1}{(2+\alpha(1-m))}}}{|x|}\frac{2m(n-1)(2+\alpha(1-m))}{1-m}C_0^{m-1}(\zeta+\zeta_0)$$

We choose $C_0 = C_5, \zeta_0 = \zeta_1$ (see appendix). If $b > 0$, and $1 \le \beta < \frac{3-m}{2}$ we have

$$\begin{aligned} R(\zeta) = \alpha + 2(1-m)^{-1}\zeta(\zeta+\zeta_1)^{-1} - 2m(1+m)(2+\alpha(1-m))(1-m)^{-2}C_5^{m-1} - \\ -(1-\frac{R}{|x|})2m(n-1)(2+\alpha(1-m))(1-m)^{-1}C_5^{m-1}(\zeta+\zeta_1)\zeta^{-1} \end{aligned} \quad (3.60)$$

and since the last term of $R(\zeta)$ is positive, we deduce

$$R(\zeta) \le \alpha + 2(1-m)^{-1}\zeta(\zeta+\zeta_1)^{-1} - 2m(1+m)(2+\alpha(1-m))(1-m)^{-2}C_5^{m-1} \le$$

$$\le \alpha + 2(1-m)^{-1} - 2m(1+m)(2+\alpha(1-m))(1-m)^{-2}C_5^{m-1} =$$

$$= -(2+\alpha(1-m))((1-m)(1-\epsilon)^{-1}\epsilon.$$

Then

$$\mathcal{L}f \leq 0 \text{ for } \zeta \geq 0,\ 0 \leq t \leq \delta_4 \tag{3.61}$$

where $\delta_4 = min(\delta_1, \delta_5)$ and

$$\delta_5 = \Big((b(1-m)(1-\epsilon))^{-1}\epsilon(A_0-\epsilon)^{1-\beta}\Big)^{\frac{2+\alpha(1-m)}{2+\alpha(\beta-m)}}$$

From (3.61) It follows that

$$Lg \leq 0 \text{ for } |x| > R,\ 0 < t \leq \delta_4 \tag{3.62}$$

If either $b > 0, \beta \geq \frac{3-m}{2}$ or $b < 0, \beta \geq 1$, from (3.59),(3.60) we have

$$\begin{aligned} R(\zeta) \leq \Big(\alpha - 2m(m+1)(2+\alpha(1-m))(1-m)^{-2}C_5^{m-1}\Big)(\zeta+\zeta_1)+ \\ +2(1-m)^{-1}\zeta = -2(1-m)^{-1}\zeta_1. \end{aligned} \tag{3.63}$$

which again implies (3.61), where $\delta_4 = \delta_1$ if $b < 0$, $\delta_4 = min\{\delta_1, \delta_5\}$ if $b > 0$. As before, (3.62) follows from (3.63). Also, we have

$$u(x,t) \geq (A_0-\epsilon)t^{\frac{\alpha}{(2+\alpha(1-m))}} = C_5 t^{\frac{\alpha}{(2+\alpha(1-m))}}\zeta_1^{\frac{2}{m-1}} = g(x,t), \text{for } |x| = R, 0 \leq t \leq \delta_4. \tag{3.64}$$

From (3.64), and the comparison theorem we get the lower bound of (3.16).

Assume now that either $b > 0,\ \beta \geq \frac{3-m}{2}$ or $b < 0,\ \beta \geq 1$. Note that from (3.16) we have

$$u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \leq C_6\Big(\frac{\zeta_2 t^{\frac{1}{2+\alpha(1-m)}} + |x| - R}{|x|}\Big)^{\frac{2}{m-1}}, |x| > R,\ 0 \leq t < \delta.$$

and by taking lim sup as $|x| \to +\infty$ it follows that

$$\limsup_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \le C_6 \tag{3.65}$$

In fact we have sharper upper bound whenever $b > 0, \beta \ge 1$, which is the direct consequence of (3.17):

$$\limsup_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \le D,\ 0 \le t \le +\infty. \tag{3.66}$$

If $b > 0$ and $\beta \ge \frac{3-m}{2}$, from the lower bound of (3.16) it follows that for some $\delta > 0$ which is independent of ϵ we have,

$$\liminf_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \ge D \tag{3.67}$$

and hence from (3.66), (3.67) we deduce

$$D \le \liminf_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \le \limsup_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} \le D \tag{3.68}$$

Therefore, (3.7) holds. However, if $m < \beta < \frac{3-m}{2}$, then from (3.17), (3.16), we only deduce (3.8).

Let $b < 0, \beta \ge 1$. From (3.58), and upper bound proved in [6] we have

$$v(y,t) \le w(y,t) \le D(1-\epsilon)^{\frac{1}{m-1}} t^{\frac{1}{1-m}} (-y_1)^{\frac{2}{m-1}},\ -\infty < y_1 \le -\mu t^{1/(2+\alpha(1-m))},\ 0 < t \le \delta.$$

and therefore,

$$u(x,t) \le D(1-\epsilon)^{\frac{1}{m-1}} t^{\frac{1}{1-m}} (-R-x_1)^{\frac{2}{m-1}},\ -\infty < x_1 \le -R-\mu t^{1/(2+\alpha(1-m))}\ 0 < t \le \delta.$$

Due to radial symmetricty of $u(x,t)$ we have

$$u(x,t) \le D(1-\epsilon)^{\frac{1}{m-1}} t^{\frac{1}{1-m}} \Big(|x|-R\Big)^{\frac{2}{m-1}},\ |x| \ge R + \mu t^{1/(2+\alpha(1-m))}\ 0 < t \le \delta, \tag{3.69}$$

with

$$\mu = (D^{-1}(A_0+\epsilon))^{\frac{m-1}{2}} (1-\epsilon)^{-\frac{1}{2}}.$$

From (3.16) and (3.69), (3.7) again follows.

Let $b > 0$, $m < \beta < 1$. The proof of left-hand side is similar to proof of lower bound of theorem 3.1.1. Consider the function

$$g(x,t) = t^{\frac{1}{1-\beta}} f_1(\xi),\ \xi = (|x|-R)t^{-\frac{m-\beta}{2(1-\beta)}} \text{ and } f_1(\xi) = C_*(1-\epsilon)(\xi_8+\xi)^{\frac{2}{m-\beta}}$$

We have

$$Lg = bg^{\beta}\Big\{1 + \frac{((1-\epsilon)C_*)^{1-\beta}}{b(1-\beta)}\xi_8(\xi_8+\xi)^{\frac{2-m+\beta}{m-\beta}} - \frac{2m(m+\beta)}{b(m-\beta)^2}((1-\epsilon)C_*)^{m-\beta} -$$

$$-\frac{2m(n-1)}{|x|(m-\beta)} t^{\frac{m-\beta}{2(1-\beta)}} ((1-\epsilon)C_*)^{m-\beta} (\xi_8+\xi)\Big\}$$

Since $\frac{2-m-\beta}{m-\beta} < 0$, and $m-\beta < 0$ we deduce $0 \le (\xi_8+\xi)^{\frac{2-m-\beta}{m-\beta}} \le (\xi_8)^{\frac{2-m-\beta}{m-\beta}}$, which implies

$$Lg \le 0 \text{ for } |x| > R,\ 0 < t \le \delta, \tag{3.70}$$

which implies that g is subsolution of (1.5).Then by the comparison theorem we have

$$C_*(1-\epsilon)t^{\frac{1}{1-\beta}}(\xi_8+\xi)^{\frac{2}{m-\beta}} \le u(x,t),\ |x| \ge R,\ 0 < t \le \delta. \tag{3.71}$$

To find the upper bound, we use the upper bound derived in [6]

$$v(y,t) \le w(y,t) \le C_*(-y_1)^{\frac{2}{m-\beta}},\ -\infty < -y_1 \le 0,\ 0 < t \le \delta,$$

which imply

$$u(x,t) \le C_*(-R - x_1)^{\frac{2}{m-\beta}},\ -\infty < -R - x_1 \le 0,\ 0 < t \le \delta.$$

Due to radial symmetricity, we deduce

$$u(x,t) \le C_*(|x| - R)^{\frac{2}{m-\beta}},\ |x| \ge R,\ 0 < t \le \delta. \tag{3.72}$$

From (3.71) and (3.72), (3.19) follows. Dividing by $|x|^{\frac{2}{m-\beta}}$, and passing to limit, first as $|x| \to +\infty$, and then as $\epsilon \to 0$, from (3.19), desired asymptotic formula (3.9) follows. □

Chapter 4

Conclusions

The dissertation presents full classification of the short-time behavior of the interfaces and local solution near the interfaces in the Cauchy problem for the nonlinear degenerate multidimensional parabolic equation, with nonnegative compactly supported initial function, modeling reaction-diffusion processes in both slow ($m > 1$) and fast diffusion ($0 < m < 1$) regimes:

$$Lu = u_t - \Delta u^m + bu^\beta = 0, \; x \in \mathbb{R}^N, 0 < t < T,$$

$$supp \; u(x,0) = \{|x| < R\}, \; u(x,0) \sim C(R - |x|)_+^\alpha, \text{ as } |x| \to R - 0,$$

where $m > 0, \beta > 0, b \in \mathbb{R}, C > 0, \alpha > 0.$ The classification is based on the relative strength of the diffusion and reaction/absorption forces. The following are the main results:

- If $m > 1, \alpha < \frac{2}{m - min\{1,\beta\}}$, then slow diffusion dominates over the absorption/reaction, and interface expands with asymptotics

$$\eta_+(x) \sim \Big(\frac{R-|x|}{\xi_*}\Big)^{2+\alpha(1-m)} \text{ as } |x| \to R+,$$

where

$$\eta_+(x) = \inf\{\tau \geq 0 : u(x,t) > 0, \tau < t < \tau+\epsilon \quad \text{for some } \epsilon > 0\}.$$

- If $b > 0, \alpha = \frac{2}{m-\beta}, 0 < \beta < \min(m,1)$ then diffusion and strong absorption are in balance, there is a critical value

$$C_* = \Big[b(m-\beta)^2/(2m(m+\beta))\Big]^{1/(m-\beta)}$$

such that the interface expands or shrinks corresponding to $C > C_*$ or $C < C_*$ and if $C > C_*$ we have

$$\eta_+(x) \sim \Big(\frac{R-|x|}{\zeta_*}\Big)^{\frac{2(1-\beta)}{m-\beta}}, \ |x| \to R+,$$

while if $C < C_*$ we have

$$\eta_-(x) \sim \Big(\frac{R-|x|}{\zeta_*}\Big)^{\frac{2(1-\beta)}{m-\beta}}, \ |x| \to R-,$$

where

$$\eta_-(x) = \sup\{\tau : u(x,t) > 0, 0 < t < \tau\},$$

and $\zeta_* = \zeta_*(m,\beta,b,C,\alpha) \lesseqgtr 0$ if $C \gtreqless C_*$.

- If $b > 0, \alpha > \frac{2}{m-\beta}, 0 < \beta < \min(m,1)$, then strong absorption dominates over the diffusion and interface shrinks with asymptotics

$$\eta_-(x) \sim \Big(\frac{R-|x|}{l_*}\Big)^{\alpha(1-\beta)} \text{ as } |x| \to R-,$$

where $l_* = C^{-\frac{1}{\alpha}}(b(1-\beta))^{\frac{1}{\alpha(1-\beta)}}$.

- If $m > 1, \beta \geq 1, \alpha \geq \frac{2}{m-1}$ then both slow diffusion and reaction/absorption are weak, and the interface initially remains stationary.

- If $0 < m < 1, 0 < \beta < m, 0 < \alpha < \frac{2}{m-\beta}$, then fast diffusion weakly dominates over the strong absorption and interface expands with asymptotics

$$\eta_+(x) \sim \Big(\frac{|x|-R}{\zeta_+}\Big)^{\frac{2(1-\beta)}{m-\beta}},\ |x| \to R+,$$

where $\zeta_+ = \zeta_+(m,\beta,b,C,\alpha) > 0$.

- If $b > 0, 0 < \beta = m < 1, \alpha > 0$, then domination of the fast diffusion over strong absorption is moderate, there is an infinite speed of propagation and for some $\delta > 0$ we have

$$\log u(x,t) \sim -m^{-1}b^{\frac{1}{2}}|x|, \quad \text{as } |x| \to +\infty,\ 0 < t \leq \delta$$

- If $0 < m < 1$ and either $b > 0, \beta > m$ or $b < 0$, $\beta \geq 1$, then fast diffusion strongly dominates over the absorption/reaction and there is an infinite speed of propagation. If either $b > 0$, $\beta \geq \frac{3-m}{2}$ or $b < 0$, $\beta \geq 1$, then there exists a $\delta > 0$ such that

$$u(x,t) \sim Dt^{\frac{1}{1-m}}|x|^{\frac{2}{m-1}}, \quad \text{as } |x| \to +\infty,\ t \in (0,\delta]$$

with $D = [2m(m+1)(1-m)^{-1}]^{1/1-m}$. If $b > 0,\ 1 \le \beta < \frac{3-m}{2}$ we have

$$\lim_{t\downarrow 0} \lim_{|x|\to+\infty} u(x,t)t^{\frac{1}{m-1}}|x|^{\frac{2}{1-m}} = D.$$

If $m < \beta < 1$ then for some $\delta > 0$ we have

$$u(x,t) \sim C_*|x|^{\frac{2}{m-\beta}}, \quad \text{as } |x| \to +\infty, t \in (0,\delta].$$

4.1 Future Development

The results of the dissertation will motivate the development of the implemented methods to different open problems in the field, such as

- Solving interface problem in multi-dimensional double-degenerate parabolic equation of turbulent filtration in porous media:

$$u_t = div(|\nabla u^m|^{p-2}\nabla u^m) - bu^\beta$$

- Solving interface problem for the nonlinear multi-dimensional double-degenerate parabolic equation in non-homogeneous porous media:

$$u_t = div(a(x,t)|\nabla u^m|^{p-2}\nabla u^m) - b(x,t)u^\beta.$$

- Inspired and motivated with the recent development in optimal control of parabolic free boundary problems in [18, 19, 24, 22, 28] the results of the dissertation can be applied to solve optimal control problems for the nonlinear degenerate second order parabolic equations.

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Appendix

Part A: Here we bring explicit values of the constants used in Sections 2.1 and 2.3 of Chapter 2

$$\zeta_1 = \begin{cases} -A_1^{\frac{m-1}{2}}(1+b(1-\beta)A_1^{\beta-1})^{-\frac{1}{2}}(2m(m+\beta)(1-\beta))^{\frac{1}{2}}(m-\beta)^{-1} & \text{if } m+\beta > 2 \\ -(A_1/C_*)^{\frac{m-\beta}{2}} & \text{if } 1 \le m+\beta < 2, \end{cases}$$

$$C_1 = \begin{cases} A_1(-\zeta_1)^{\frac{2}{m-\beta}} & \text{if } m+\beta > 2 \\ C_* & \text{if } 1 \le m+\beta < 2, \end{cases}$$

$$A_1 = w(0,1) = h(0).$$

$$\zeta_2 = \begin{cases} C^{\frac{\beta-m}{2}}(b(1-\beta)(1-(C/C_*)^{m-\beta}))^{\frac{m-\beta}{2(1-\beta)}} & \text{if } m+\beta > 2 \\ \delta_*\Gamma l_1 \quad \text{if } m+\beta < 2, \end{cases}$$

$$l_1 = C^{\frac{\beta-m}{2}}\Big[b(1-\beta)\Big(\delta_*\Gamma\Big)^{-1}\Big(1-\delta_*\Gamma-\Big(1-\delta_*\Gamma\Big)^{-1}(C/C_*)^{m-\beta}\Big)\Big]^{\frac{m-\beta}{2(1-\beta)}},$$

$$\Gamma = 1-(C/C_*)^{\frac{m-\beta}{2}}, \quad C_2 = C(1-\delta_*\Gamma)^{\frac{2}{\beta-m}},$$

and $\delta_* \in (0,1)$ satisfies

$$g(\delta_*) = \max_{[0,1]} g(\delta), \qquad g(\delta) = \delta^{\frac{2-\beta-m}{m-\beta}}\Big[1-\delta\Gamma-\Big(1-\delta\Gamma\Big)^{-1}(C/C_*)^{m-\beta}\Big]$$

Part B: We given here explicit values of the constants used in Section 2 in the outline of the results and later in section 4 during the proof of these results of Chapter 3.

(1) when $0<\beta<m,\ 0<\alpha<2(m-\beta)^{-1}$

$$C_*=\Big[b(m-\beta)^2/(2m(m+\beta))\Big]^{1/(m-\beta)},\ C_1=\Big(\frac{(1-\beta)(1+\epsilon)}{1-m}\Big)^{\frac{1}{m-\beta}}C_*,$$

$$\xi_1=-b^{\frac{m-1}{2(1-\beta)}}(2m)^{\frac{1}{2}}(m+\beta)^{\frac{1}{2}}(m-\beta)^{\frac{m+\beta-2}{2(1-\beta)}}\Big(\frac{1-m}{(1+\epsilon)(1-\beta)}\Big)^{\frac{1-m}{2(1-\beta)}}$$

$$\xi_2=-b^{\frac{m-1}{2(1-\beta)}}(2m)^{\frac{1}{2}}(m+\beta)^{\frac{1-m}{2(1-\beta)}}(m+1)^{\frac{m-\beta}{2(1-\beta)}}(m-\beta)^{-1}(1-\beta)(1-m)^{\frac{m+\beta-2}{2(1-\beta)}}.$$

$$l=-\frac{m-\beta}{1-\beta}\xi_2.$$

(2) $0<\beta<m,\ \alpha=2(m-\beta)^{-1}$

$$\xi_3=-(A_1-\epsilon)^{\frac{m-1}{2}}(1+b(1-\beta)(1+\epsilon))^{\frac{-1}{2}}(m-\beta)^{-1}(m(1-\beta))^{\frac{1}{2}}(2(m+\beta))^{\frac{1}{2}}$$

$$\xi_4=-(A_1-\epsilon)^{\frac{m-\beta}{2}}C_*^{-\frac{m-\beta}{2}},\ C_2=(A_1-\epsilon)\xi_3^{-2/(m-\beta)}$$

$$\xi_5=-l_1-\Big(\frac{\lambda-\epsilon}{C_*}\Big)^{\frac{m-\beta}{2}}$$

$$l_2=C^(\beta-m)/2[b(1-\beta)(\delta_*\Gamma)^{-1}(1-\delta_*\Gamma-(1-\delta_*\Gamma)^{-1}(C/C_*)^{m-\beta})]^{(m-\beta)/2(1-\beta)}$$

$$\xi_6=\delta_*\Gamma l_2,\ \Gamma=1-(C/C_*)^{(m-\beta)/2},\ C_3=C(1-\delta_*\Gamma)^{2/(\beta-m)}$$

where $\delta_*\in(0,1)$ satisfies

$$g(\delta_*)=\max_{[0;1]}g(\delta),\quad g(\delta)=\delta^{(2-m-\beta)/(m-\beta)}[1-\delta\Gamma-(1-\delta\Gamma)^{-1}(C/C_*)^{m-\beta}]$$

(2) $\beta > m$,

$$D = [2m(m+1)(1-m)^{-1}]^{1/1-m},$$

$$\zeta_1 = (A_0 - \epsilon)^{(m-1)/2}(1-\epsilon)^{1/2}D^{(1-m)/2} \quad if b > 0,\ 1 \leq \beta < (3-m)/2,$$

$$\zeta_1 = (A_0 - \epsilon)^{(m-1)/2}D^{(1-m)/2} \quad if either b > 0,\ 1 \leq \beta \geq (3-m)/2 or b < 0, \beta \geq 1,$$

$$C_5 = (A_0 - \epsilon)\zeta^{2/(1-m)}, \quad A_0 = f(0) > 0,$$

$$\zeta_2 = (A_0 + \epsilon)^{(m-1)/2}\Big[\frac{2m(m+1)(2+\alpha(1-m))\mu_b}{\alpha(1-m)^2}\Big]^{1/2},$$

$$C_6 = \Big[\frac{2m(m+1)(2+\alpha(1-m))\mu_b}{\alpha(1-m)^2}\Big]^{1/(1-m)},$$

$$\mu_b = 1\ if b > 0, \quad \mu_b = 1 + \epsilon\ if b < 0,$$

$$\xi_8 = [b(1-\beta)C_*^{\beta-1}(1-\epsilon)^{\beta-1}(1-(1-\epsilon)^{m-\beta})]^{(m-\beta)/2(1-\beta)}.$$