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Relaxation of Variational Principles for Z-problems in Effective Media Theory

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by
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ABSTRACT

Title:
Relaxation of Variational Principles for Z-problems in Effective Media Theory

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In this thesis, we consider a class of Z-problems and their associated effective operators on Hilbert spaces which arise in effective media theory, especially within the theory of composites. We provide a unified approach to obtaining solutions of the Z-problem, formulas for the effective operator in terms of generalized Schur complements, and their associated variational principles (e.g., the Dirichlet minimization principle), while allowing for relaxation of the standard hypotheses on positivity and invertibility for the classes of operators usually considered in such problems. The Hilbert space framework developed here is inspired by the methods of orthogonal projections and Hodge decompositions. However, we focus on finite-dimensional Hilbert spaces. Our theoretical development utilizes the theory of block operator matrices, the Moore-Penrose pseudoinverse (as a replacement for the inverse), the generalized Schur complement, and the generalized principal pivot transform. With this unified framework, we are able to recover the classical minimization principle for the Schur complement and derive its extension, as well as provide a new maximization principle for the generalized principal pivot transform. We also present several applications. First, we give an operator-theoretic reformulation of the discrete Dirichlet-to-Neumann (DtN) map for an electrical network on a finite linear graph and relate the DtN map to the effective operator of an associated Z-problem. Second, we show how the classical electrical conductivity of an electrical network (on a finite linear graph) is essentially the effective operator of an associated Z-problem. Next, we consider periodic linear graphs and develop a discrete analog to the periodic conductivity equation and effective conductivity in the continuum. Finally, we conclude with a discussion of future work and open problems based on this thesis.

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Acknowledgements

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Chapter 1

Introduction

In this chapter, we introduce the reader to the Z-problem and associated effective operator, a brief history of this problem and provide an overview of the chapters contained herein.

1.1 What is the Z-problem?

Although we will present the rigorous definition of the Z-problem and effective operator in a later chapter (see, Def. 13), we believe that it is to the reader's benefit to introduce them now with less precision.

A Z-problem is defined in terms of a five-tuple $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ in which $\mathcal{H}$ is a Hilbert space decomposable into an orthogonal direct sum

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{1.1}$$

of the subspaces $\mathcal{U}$, $\mathcal{E}$, $\mathcal{J}$ and σ is a bounded linear operator on $\mathcal{H}$. With this, the Z-problem is defined by the equation

$$I_0 + I = \sigma(V_0 + V), \tag{1.2}$$

where $V_0 \in \mathcal{U}$ is given and we want to solve for $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$. From this, the effective operator σ_* is defined in terms of the Z-problem as a bounded linear operator on $\mathcal{U}$ mapping V_0 to I_0, i.e.,

$$I_0 = \sigma_* V_0. \tag{1.3}$$

In physics, especially in the case that σ is the conductivity (or elasticity) in the continuum, (1.2) is considered locally. That is, $I = I(x)$, $V = V(x)$ are vector-valued functions (usually $d \times 1$ column vectors) and $\sigma = \sigma(x)$ is a scalar, matrix, or even tensor-valued function dependent on a spatial variable $x \in \mathbb{R}^d$ (usually for $d =$

2, 3). In general, the operators under consideration need not be local (e.g., convolution operators). For more information on this, see Chapter 3, Section 3.1 and also Remark 26 therein.

In our setting, when we are dealing with electrical networks, I and V are currents and voltages, where σ is the conductivity along the wires (edges). For the continuum, we replace I and V with vectors J and E, the current density and the electric field, respectively, and then σ is the local conductivity (i.e., conductivity in each space point). Although not treated here, in the case of elasticity, I and V would be replaced by τ and ϵ, the stress field and the strain field, respectively, and then σ would be replaced by C, the elasticity tensor. For more on these analogies and other Z-problems, we highly recommend [1–3].

In this thesis, we adopt the perspective that there are many problems in the theory of composites and effective media theory that have this structure and are often associated with Hodge decomposition. We also know that there are problems in other areas of science that exhibit such a structure.

Regardless of the area of interest, one would like to establish if a problem belongs to this framework. For instance, sometimes the effective operator is known, but the Z-problem and the Hodge structure are unknown. In such cases, one would like to be able to identify the Hodge decomposition and the Z-problem (if one exists) that give rise to the effective operator. On the other hand, there are sometimes problems that do not appear to have the necessary structure of a Z-problem or in which the effective operator is not apparent. In these cases, we would like techniques and tools necessary to discover if such problems can be incorporated into this framework. We believe that the content of this thesis is a step in the right direction in this regard.

In the rest of this chapter, we give an overview of the history of the Z-problem and effective operator from the theory of composites. After that, we describe the remaining content of this thesis.

1.2 A Brief History of the Z-problem

Here, we give a brief history of the Z-problem, which we divide into three eras: the era of orthogonal subspaces, the era of exact relations and the era of the Z-problem.

The first of these eras, the era of orthogonal subspaces, began with the 1981 paper by Papanicolaou and Veradhan [4]. In this paper, we find the development of an abstract framework for stationary random media under certain assumptions on a random conductivity function σ (including strictly stationary and ergodic), which includes periodic conductivity. Here, the effective conductivity σ_* is derived as a certain average in a probability space, which is reduced to the periodic average for periodic media.

This development was furthered in the 1982 paper by Kohler and Papanicolaou [5]. According to Milton [6], this was one of two papers that influenced him at first. In this paper, we are first introduced to an orthogonal decomposition $\mathcal{H} \overset{\perp}{\ominus} \mathcal{U} = \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$ of the

Hilbert space $\mathcal{H} = L^2$ and the associated orthogonal projection $\Gamma_1|_{\mathcal{H} \overset{\perp}{\ominus} \mathcal{U}} = \nabla(\Delta^{-1})\nabla\cdot$ onto $\mathcal{E}$. While brief, there is also discussion and usage of the Dirichlet and Thomson variational principles.

Although development around the Z-problem continued following 1982 (e.g., Golden and Papanicolaou 1983 [7], Milton and Golden 1985 [8] utilizing continued fraction theory), it is not until the 1986 paper by Dell'Antonio, Figari and Orlandi [9] that we see a connection to Hodge decompositions and Weyl's method of orthogonal decompositions [10, 11]. Here, Dell'Antonio, Figari and Orlandi study the response of bodies under general boundary conditions, for which they formulate the equations for conductivity σ and elasticity C in terms of appropriate projections. In particular, for periodic boundary conditions, the projections reduce to $\Gamma_0 + \Gamma_1$ onto the space of vector fields $\mathcal{U} \overset{\perp}{\oplus} \mathcal{E}$ which are (symmetrized) gradients of periodic (vector) scalar potentials in the case of (elasticity) conductivity. As Milton noted [6], this is the first instance of a dual subspace decomposition and the second of the articles that initially influenced him.

Milton introduced, in his 1987 paper [12], the usage of three mutually orthogonal subspaces $\mathcal{U}, \mathcal{E}, \mathcal{J}$ of a Hilbert space $\mathcal{H}$, that is, $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$. Development of this framework continues by Milton in his 1987 [13] and 1990 papers [14] and in a joint paper with Kohn in 1988 [15]. Of particular interest in the 1988 paper is the explicit consideration of the form of the Z-problem, which allowed for more general bounded linear operators L satisfying certain conditions (e.g. self-adjoint, positive definite and invertible) on the Hilbert space $\mathcal{H}$ (as opposed to the previous references where the focus was primarily on n-phase composites). Also, in this paper, it is clear that they are formulating effective conductivity σ_* and effective elasticity C_* from a Z-problem (which they call the "field equation").

Next is the era of exact relations, where the formulation of the Z-problem has proven to be particularly effective in obtaining exact relations of composite materials, i.e., independent microstructure relations satisfied by effective tensors. In a tour de force, Grabovsky [3, 16–18], along with Milton [19] and Sage [20, 21], developed the general theory and computational tools that allow one to explicitly describe exact relations in many physical problems. The theory achieved such a broad scope by applying the unified functional analysis framework developed in [5, 9, 12, 13, 15] for random and periodic composites.

Then comes the era of the Z-problem. This era began with the publication in 2002 of a book [1], now considered the definitive work on the theory of composites, by the preeminent authority on the subject, Graeme Milton. In Sections 12.7 and 12.8 of this text, the Z-problem and the dual Z-problem (though neither is identified as such) are established. In Chapter 13, Milton discusses weakening the hypotheses on the Dirichlet minimization principle so that the operator L needs only be self-adjoint and positive semidefinite when restricted to the $\mathcal{E}$ space, i.e., $\Gamma_1 L \Gamma_1|_{\mathcal{E}} \geq 0$. However, it seems that they did not treat this case. In particular, this serves as strong motivation for this thesis and is further motivated by Milton's remark in the last paragraph of Sect. 13.1 of [1], in which he says, "Of course, for these statements to have meaning, we need to show that the effective tensor L_* exists when L is positive semidefinite on $\mathcal{E}$ and when L^{-1} is positive semidefinite on J." The main result of this thesis, namely, Theorem 77, addresses this with even weaker hypotheses allowed, although at present we only consider finite-dimensional Hilbert spaces. In future work (see Chapter 5) we plan to extend this to infinite-dimensional Hilbert spaces.

Next, this is followed by Milton's second book with collaborators (e.g., my advisor, Aaron Welters, contributed two chapters) in 2016 [2]. Here, he discusses an idea and proposes a research program to extend the theory of composites to other areas of science. In this text, Milton sets about promoting the idea that one can bring the methods and tools of the abstract theory of composites to bear on Z-problems and their effective operators hiding within other areas of science. It is this program to which this thesis makes some contributions. For more on the book and its potential influence, we recommend Grabovsky's review [22] and strongly encourage obtaining a copy of the book.

Finally, we arrive at the modern era of the Z-problem. To provide a current look at the road ahead, Milton published a 2021 paper [23], which provides an extensive list of open problems in the theory of composites. To try and make progress on solutions of some of these problems (mostly relating to realizability), my advisor (Aaron Welters) and Anthony Stefan (then his master's student and now his doctoral student) published two papers [24, 25]. In these papers, they extend the Bessmertnyĭ realization theorem for rational functions of several complex variables and prove that every real multivariate polynomial has a symmetric determinantal representation. This furthers the work done in Stefan's 2021 MS thesis [26], where he fully develops the algebra associated with Schur complements. This brings us to the present, with this thesis.

In light of the historical development of the Z-problem, it should be clear to the reader that there are three quintessential problems (explicitly detailed in Chapter 2 Subsection 2.2.2) related to this study: What is the largest class of operators σ for which the Z-problem is solvable, and when is the solution unique? If the Z-problem is solvable, does the effective operator σ_* exist and is it unique? For such σ, what variational principles exist and do they lead to upper and lower bounds on the effective operator σ_*?

1.3 An Overview of the Thesis

We present here and in the following diagram (see Figure 1.1) an overview of this thesis.

In Chapter 2, we present the mathematical background required throughout this thesis. In particular, we introduce the definitions of external and internal direct sums and their orthogonal counterparts. This allows us to familiarize the reader with orthogonal projections, block operator matrices, Schur complements and principal pivot transforms before introducing the rigorous definition of the Z-problem, the effective operator and the dual Z-problem. We also introduce our definition of effective media theory, the quintessential problems related to the solution of the Z-problem, its effective operator and some classical results. Following this, we present the abstract Hodge decomposition for finite-dimensional Hilbert spaces from the perspective of operator theory. In addition, we include an introduction to the Moore-Penrose pseudoinverse and some of its properties.

In Chapter 3, we study several applications in effective media theory, for which the results of Chapter 4 are applicable. We begin by introducing the reader to the periodic conductivity equation in the continuum in Section 3.1 as the quintessential Z-problem, under the standard Hodge decomposition, where the effective operator in this setting is the well-known effective conductivity. This gives a baseline for analogy in the proceeding sections in which we establish results for the DtN map (a reformulation of Milton's work in the context of operator theory), effective conductivity/resistance on a finite linear digraph and conductivity on periodic linear digraphs. In particular, in Subsection 3.2.2 we prove that the DtN map is related to the effective operator of an associated Z-problem, and in Subsection 3.2.3 we prove a similar result for the classical effective conductivity. In Section 3.3, we develop the discrete analog of the periodic conductivity problem from Section 3.1.

In Chapter 4, we begin a program to address the relaxation of requirements on the operator σ. Our main result Theorem 77 shows that for $\sigma = \sigma^*$, $\sigma_{11} \geq 0$ and $\ker \sigma_{11} \subseteq \ker \sigma_{01}$ the Z-problem has non-unique solutions characterized by a generalized principal pivot transform and a unique effective operator represented by a generalized Schur complement. We also present a unified approach to obtaining solutions of constrained linear equations given in Lemma 66 and their associated variational principles given in Theorem 67, which are capable of producing both classical results and new results, such as the generalized principal pivot transform maximization principle given in Theorem 76.

In Chapter 5, we outline the continuation of this research. It is the goal of this program to also develop the Thomson variational principle when the underlying operator has lost invertibility and to extend this approach to sectorial operators in order to craft the analogy to the Gibiansky-Cherkaev-Milton method. We would also like to develop these results when the underlying Hilbert space is no longer finite-dimensional or isomorphic to a finite-dimensional space. Throughout this process, we hope to continue the work of Milton by discovering further Z-problems in other areas of science. There is also work to be done on the development of our lattice model for elasticity and viscoelasticity, in addition to the consideration of other graph topologies. Outside of the main program of interest, we also suggest methods by which the unified approach could be generalized.

In the Appendix, we include some well-known linear algebra results that are not immediately available in a form conducive to our study. We provide their statements and their proofs for completeness.

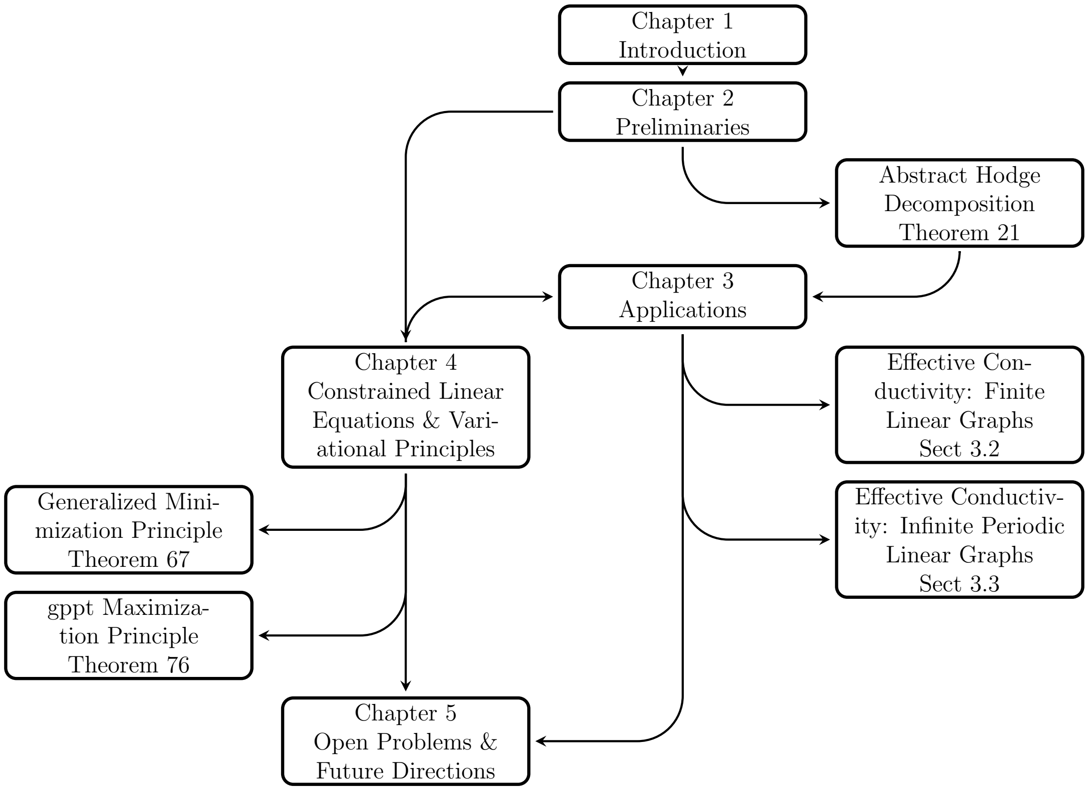


Figure 1.1: A flow diagram for this thesis.

Chapter 2

Preliminaries

In this thesis, we make the following assumptions.

(i) All Hilbert spaces are complex, i.e., over the field $\mathbb{C}$.

(ii) Any inner product $(\cdot, \cdot)_{\mathcal{H}}$ is conjugate-linear in its first argument and linear in its second argument, i.e,

$$(ax, by)_{\mathcal{H}} = \overline{a}b(x, y)_{\mathcal{H}}, \text{ for any } a, b \in \mathbb{C} \text{ and } x, y \in \mathcal{H}. \tag{2.1}$$

We make the assumption (i) for two reasons: First, it is always possible to complexify a real Hilbert space. Second, this simplifies proofs involving positive semidefinite operators, as they are always self-adjoint over complex Hilbert spaces (this is not true of real Hilbert spaces). We also make frequent use of the following notation: For $\mathcal{H}_1, \mathcal{H}_2$, Hilbert spaces, we denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear functions from $\mathcal{H}_1$ to $\mathcal{H}_2$. In the case $\mathcal{H}_1 = \mathcal{H}_2$, we denote this space by $\mathcal{L}(\mathcal{H}_1)$.

2.1 Necessary Linear Algebra

2.1.1 Direct Sums

In this subsection, we familiarize the reader with both external and internal direct sums and their orthogonal counterparts. This forms much of the background for the (abstract) Hodge decompositions introduced in Section 2.3.

The following definitions regarding direct sums are adapted from Steven Roman's book [27]. We highly recommend this text, given its thoroughness of the subject matter and ease of digestibility. Our choice to adapt these definitions over others stems from their presentation and usefulness in the current study.

We begin by introducing the definition of an external direct sum.

Definition 1 (external direct sum) *Let $\{\mathcal{V}_i\}_{i=0}^n$ be a finite collection of vector spaces over a field $\mathbb{F}$. The external direct sum of $\{\mathcal{V}_i\}_{i=0}^n$, denoted by*

$$\mathcal{V} = \boxplus_{i=0}^{n} \mathcal{V}_i = \mathcal{V}_0 \boxplus \cdots \boxplus \mathcal{V}_n, \tag{2.2}$$

is the vector space $\mathcal{V}$ whose elements are ordered n-tuples

$$\mathcal{V} = \{(v_1, \ldots, v_n) : v_i \in \mathcal{V}_i,\ i = 0, \ldots, n\}, \tag{2.3}$$

with component-wise addition and scalar multiplication.

In contrast, we now present the definition of an internal direct sum.

Definition 2 (internal direct sum) *A vector space $\mathcal{V}$ over a field $\mathbb{F}$ is the internal direct sum of an indexed collection $\{\mathcal{W}_i\}_{i\in I}$ of subspaces of $\mathcal{V}$, denoted by*

$$\mathcal{V} = \bigoplus_{i\in I} \mathcal{W}_i = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots, \tag{2.4}$$

when

$$\mathcal{V} = \sum_{i\in I} \mathcal{W}_i, \quad \text{and} \quad \mathcal{W}_i \cap \left(\sum_{j\neq i} \mathcal{W}_j \right) = \{0\}, \text{ for each } i \in I. \tag{2.5}$$

We say that $\mathcal{W}_i$ is a direct summand of $\mathcal{V}$ and when $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2$ we call $\mathcal{W}_2$ a complement of $\mathcal{W}_1$ in $\mathcal{V}$.

We remark that (2.5) implies that each $v \in \mathcal{V}$ has a unique representation given by

$$v = \sum_{i=0}^{n} v_i,\ v_i \in \mathcal{V}_i. \tag{2.6}$$

The following definition is used primarily throughout this thesis.

Definition 3 *A vector space $\mathcal{V}$ over a field $\mathbb{F}$ is the orthogonal direct sum of a finite collection of subspaces $\{\mathcal{W}_i\}_{i=0}^n$, denoted by*

$$\mathcal{V} = \overset{\perp}{\bigoplus_{i=0}^{n}} \mathcal{W}_i = \mathcal{W}_0 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \mathcal{W}_n, \tag{2.7}$$

when

$$\mathcal{V} = \bigoplus_{i=0}^{n} \mathcal{W}_i, \quad \text{and} \quad \mathcal{W}_i \perp \mathcal{W}_j,\ i \neq j. \tag{2.8}$$

When context allows, we will abuse notation and denote both orthogonal internal and external direct sums by $\overset{\perp}{\oplus}$. It is well known that internal and external direct sums are isomorphic and hence there is no loss of rigor (see [27] or [28]). Also, when context allows, we will forego definition of the direct summands in an orthogonal direct sum.

2.1.2 Orthogonal Projections and Block Operators

In this subsection, we introduce the primary object of study, block operator matrices.

First, we familiarize the reader with the notion of orthogonal projections. The following proposition is well known, as such we omit the proof and refer the reader to [27] and [28].

Proposition 4 *If $\mathcal{V} = \overset{\perp}{\oplus}{}_{i=0}^{n} \mathcal{V}_i$ then $\Gamma_{v_i} : \mathcal{V} \to \mathcal{V}_i$, defined by*

$$\Gamma_{\mathcal{V}_i} v = v_i, \; v \in \mathcal{V}, \tag{2.9}$$

is well defined, unique and linear. Furthermore, $\Gamma_{\mathcal{V}_i}$ satisfies the following properties:

(i) $\Gamma_{\mathcal{V}_i}^2 = \Gamma_{\mathcal{V}_i}$ *(idempotence),*

(ii) $\displaystyle\sum_{i=0}^{n} \Gamma_{\mathcal{V}_i} = I_{\mathcal{V}}$ *(resolution of identity),*

(iii) $\operatorname{ran} \Gamma_{\mathcal{V}_i} = \mathcal{V}_i$,

(iv) $\ker \Gamma_{\mathcal{V}_i} = \displaystyle\sum_{j \neq i} \mathcal{V}_j$.

Definition 5 (orthogonal projection) *The operator (2.9) is called an orthogonal projection onto the space $\mathcal{V}_i$.*

In this thesis, Γ will always refer to an orthogonal projection and the notation introduced above will be used to convey the necessary spaces.

We now familiarize the reader with block operator matrices and give their representation in terms of orthogonal projections.

Definition 6 *Let $\mathcal{H} = \overset{n}{\underset{i=0}{\overset{\perp}{\oplus}}} \mathcal{H}_i$ be a Hilbert space and $X_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ for $i, j = 0, \ldots, n$. The matrix X, defined by*

$$X = [X_{ij}]_{i,j=0,\ldots,n}, \tag{2.10}$$

is called a $n \times n$ block operator matrix on $\mathcal{H}$.

Proposition 7 *$\mathcal{H} = \overset{n}{\underset{i=0}{\overset{\perp}{\oplus}}} \mathcal{H}_i$ be a Hilbert space and $X \in \mathcal{L}(\mathcal{H})$. Then X has the unique block operator matrix representation, given by*

$$X = [\Gamma_{\mathcal{H}_i} X \Gamma_{\mathcal{H}_j}|_{\mathcal{H}_j}]_{i,j=0,\ldots,n}. \tag{2.11}$$

For more general results on block operator matrices, we refer the reader to [29]. In this thesis, we focus primarily on 2×2 and 3×3 block operator matrices. When $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1 \overset{\perp}{\oplus} \mathcal{H}_2$ and $X = [X_{ij}]_{i,j=0,1,2} \in \mathcal{L}(\mathcal{H})$, we can partition the space to obtain a 2×2 block operator, i.e., $\mathcal{H} = (\mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1) \overset{\perp}{\oplus} \mathcal{H}_2$ and

$$X = \left[\begin{array}{cc:c} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ \hdashline X_{20} & X_{21} & X_{22} \end{array}\right]. \tag{2.12}$$

We conclude this section with the following definition of positivity for operators and operator inequalities.

Definition 8 *Let $\mathcal{V}$ be an inner product space. An operator $X \in \mathcal{L}(\mathcal{V})$ is called positive semidefinite $X \geq 0$ when*

$$(Xv, v)_{\mathcal{V}} \geq 0, \ \forall v \in \mathcal{V}. \tag{2.13}$$

In addition, for $Y \in \mathcal{L}(\mathcal{V})$ we say that $X \geq Y$ when $X - Y \geq 0$.

2.1.3 The Schur Complement and Principal Pivot Transform

In this subsection, we define the tools used to obtain the classical results given in Subsection 2.2.3. Namely, we define the Schur complement of a block operator matrix, relate it to a block factorization and define the principal pivot transform. Generalization of these objects will be utilized to prove the results in Chapter 4 and are presented there.

The Schur complement is defined as follows.

Definition 9 (Schur complement) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space and $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix) with X_{11} invertible. The Schur complement of X with respect to X_{11} is*

$$X/X_{11} = X_{00} - X_{01} X_{11}^{-1} X_{10}. \tag{2.14}$$

Similarly, when X_{00} is invertible, the Schur complement of X with respect to X_{00} is

$$X/X_{00} = X_{11} - X_{10} X_{00}^{-1} X_{01}. \tag{2.15}$$

We now give the following proposition which relates the Schur complement to a block factorization. For the proof and additional properties of Schur complements and their associated algebras, we recommend [30] and [26].

Proposition 10 (Schur complement block factorization) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space with* $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$*. If* X_{11} *is invertible, then*

$$X = \begin{bmatrix} I_{\mathcal{H}_0} & X_{01}X_{11}^{-1} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} X/X_{11} & 0 \\ 0 & X_{11} \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ X_{11}^{-1}X_{10} & I_{\mathcal{H}_1} \end{bmatrix}, \tag{2.16}$$

and

$$\begin{bmatrix} I_{\mathcal{H}_0} & X_{01}X_{11}^{-1} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\mathcal{H}_0} & -X_{01}X_{11}^{-1} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix}, \tag{2.17}$$

$$\begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ X_{11}^{-1}X_{10} & I_{\mathcal{H}_1} \end{bmatrix}^{-1} = \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ -X_{11}^{-1}X_{10} & I_{\mathcal{H}_1} \end{bmatrix}. \tag{2.18}$$

Now that the reader has some familiarity with the Schur complement, we introduce the principal pivot transform as follows.

Definition 11 (principal pivot transform) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space and* $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ *(i.e., a* 2×2 *block operator matrix) where* X_{11} *is invertible. The principal pivot transform (ppt) of* X *with respect to* X_{11} *is the* 2×2 *block operator matrix* $\mathrm{ppt}_1(X) \in \mathcal{L}(\mathcal{H})$ *defined by*

$$\mathrm{ppt}_1(X) = \begin{bmatrix} X/X_{11} & X_{01}X_{11}^{-1} \\ -X_{11}^{-1}X_{10} & X_{11}^{-1} \end{bmatrix}. \tag{2.19}$$

Similarly, when X_{00} *is invertible, the principal pivot transform of* X *with respect to* X_{00} *is the* 2×2 *block operator matrix* $\mathrm{ppt}_0(X) \in \mathcal{L}(\mathcal{H})$ *defined by*

$$\mathrm{ppt}_0(X) = \begin{bmatrix} X_{00}^{-1} & -X_{00}^{-1}X_{01} \\ X_{10}X_{00}^{-1} & X/X_{00} \end{bmatrix}. \tag{2.20}$$

For more information on the properties of the ppt we recommend [31].

2.2 Effective Media Theory and the Z-problem

In this section, we more rigorously define the Z-problem, effective operator and the dual Z-problem. We also give our definition of effective media theory with contrasting views. In Subsection 2.2.2 we provide a more detailed description of the quintessential problems associated with Z-problems and effective operators. This is followed by a presentation of classical results in Subsection 2.2.3, which include the maximization principle for the principal pivot transform (believed to be new).

We now provide our all-encompassing definition of an effective media theory.

Definition 12 (effective media theory) *An effective media theory is an analytical or theoretical model which describes the aggregate behavior of a multicomponent system in terms of a homogeneous system (effective medium).*

Such a broad definition is not incongruous with others used throughout the community.

For instance, Kadic *et al.* say in their 2019 paper *3D-Metamaterials* in Nature Review [32]: "material properties are commonly described by effective macroscopic parameters that refer to fictitious continua...the complexity of a large system composed of many different components can be reduced. A sound mathematical basis for mapping periodic structures onto effective media or continua is the aim of homogenization theory."

Similarly, Choy in his 1999 book *Effective Media Theory* [33] says: "So what is effective medium theory?...effective medium theory functions by being able to define averages, which one hopes will be representative of the system and be connected with experimental measurements."

However, our definition has the benefit of allowing us to discuss problems that may not classically be considered effective media, but that share the same structure.

We now provide a rigorous definition (based on [1–3]) of the Z problem and the effective operator, as follows.

Definition 13 (Z-problem and effective operator) *The Z-problem*

$$(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma), \tag{2.21}$$

is the following problem associated with a Hilbert space $\mathcal{H}$, an orthogonal triple decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{2.22}$$

and a linear operator $\sigma \in \mathcal{L}(\mathcal{H})$: given $V_0 \in \mathcal{U}$, find triples $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ satisfying

$$I_0 + I = \sigma \left(V_0 + V \right), \tag{2.23}$$

such a triple (I_0, V, I) is called a solution of the Z-problem at V_0. If there exists an operator $\sigma_* \in \mathcal{L}(\mathcal{U})$ such that

$$I_0 = \sigma_* V_0, \tag{2.24}$$

whenever $V_0 \in \mathcal{U}$ and (I_0, V, I) is a solution of the Z-problem at V_0, then σ_* is called an effective operator of the Z-problem.

As will be seen in Subsection 2.2.3, we can obtain an upper bound on the effective operator through the associated minimization principles. However, as we shall see in the following section, more is needed to obtain a lower bound on the effective operator.

2.2.1 The Dual Z-problem

In this subsection, we give the definition of the dual Z-problem and briefly mention difficulties that arise when one loses invertibility.

Without delay, we provide the following definition for the dual Z-problem.

Definition 14 (dual Z-problem) *The dual Z-problem*

$$(\mathcal{H}, \mathcal{U}, \mathcal{J}, \mathcal{E}, \sigma^{-1}), \tag{2.25}$$

is the following problem associated with a Hilbert space $\mathcal{H}$, an orthogonal triple decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{J} \overset{\perp}{\oplus} \mathcal{E}, \tag{2.26}$$

and a linear operator $\sigma^{-1} \in \mathcal{L}(\mathcal{H})$: Given $I_0 \in \mathcal{U}$, find triples $(V_0, I, V) \in \mathcal{U} \times \mathcal{J} \times \mathcal{E}$ satisfying

$$V_0 + V = \sigma^{-1}(I_0 + I), \tag{2.27}$$

such a triple (I_0, V, I) is called a solution of the dual Z-problem at V_0. If there exists an operator $(\sigma^{-1})_* \in \mathcal{L}(\mathcal{U})$ such that

$$V_0 = (\sigma^{-1})_* I_0, \tag{2.28}$$

whenever $I_0 \in \mathcal{U}$ and (V_0, I, V) is a solution of the dual Z-problem at I_0, then $(\sigma^{-1})_*$ is called an effective operator of the dual Z-problem.

The main utilization of the dual Z-problem, is in obtaining a lower bound on the effective operator σ_* from its associated variational principles. We would like to be able to do the same when σ is noninvertible. However, it is not immediately clear what one means when discussing duality in this context. For further information in this regard, see Chapter 5.

2.2.2 The Quintessential Problems

In the abstract theory of composites, the most basic problems that need to be addressed are:

(i) Solvability of the Z-problem: Under what conditions does the Z-problem (2.21) have a solution; a unique solution? If possible, find a formula for all solutions in terms of σ and parameterized by those $V_0 \in \mathcal{U}$ for which solutions exist.

(ii) Existence, uniqueness and representation formulas for the effective operator: Under what conditions does the effective operator σ_* exist; is unique? If it exists, find representation formulas for it in terms of σ.

(iii) Variational principles & bounds: If $\sigma^* = \sigma$ and $\sigma \geq 0$, are there variational principles: (1) for the solutions of the Z-problem? (2) that define the effective operator σ_*? (3) that can be used to derive upper and lower bounds on an effective operator σ_* in terms of σ?

Throughout this thesis, when $\mathcal{H} = \overset{n}{\underset{i=0}{\overset{\perp}{\oplus}}} \mathcal{H}_i$ is a Hilbert space and $X_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$ for $i, j = 0, \ldots, n$, we consider the following sets of hypotheses:

(i) $X^* = X \geq 0$, X is invertible (strongest),

(ii) $X^* = X$, $X_{11} \geq 0$, X_{11} is invertible (strong),

(iii) $X^* = X \geq 0$, X is not invertible (weak),

(iv) $X^* = X$, $X_{11} \geq 0$, $\ker X_{11} \subseteq \ker X_{01}$ (weakest).

We will refer to the results obtained under Hypotheses (i) and (ii) as classical results.

2.2.3 Classical Results

The first of the results, while new, falls under the classical hypotheses and is thus included here. In particular, the following proposition uniquely solves the Z-problem and gives a Schur complement formula for the associated effective operator.

Proposition 15 (Z-problem solution and effective operator) *Consider the Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$. Decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ with $\mathcal{H}_0 = \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}, \mathcal{H}_1 = \mathcal{E}$, and let $\sigma = [\sigma_{ij}]_{i,j=0,1}$ be the 2×2 block operator representation of σ with respect to that decomposition of $\mathcal{H}$. If σ_{11} is invertible, then the Z-problem has a unique solution $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ for each $V_0 \in \mathcal{U}$, the effective operator σ_* exists, is unique and*

$$I_0 + V + I = \mathrm{ppt}_1(\sigma)(V_0), \tag{2.29}$$

$$\sigma_* = \Gamma_0 \sigma / \sigma_{11} \Gamma_0|_{\mathcal{U}}. \tag{2.30}$$

Proof. Assume the hypotheses. Suppose $V_0 \in \mathcal{U}$ and $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ such that

$$\sigma(V_0 + V) = (I_0 + I) + 0. \tag{2.31}$$

Then, $V_0 + V,\ (I_0 + I) + 0 \in \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ which implies

$$\mathrm{ppt}_1(\sigma)(V_0 + 0) = I_0 + I + V. \tag{2.32}$$

The remainder of the result follows. ∎

Alternatively, one can decompose the Hilbert space $\mathcal{H}$ by $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$ and $\sigma = [\sigma_{ij}]_{i,j=0,1,2}$ now takes the form of a 3×3 block operator matrix. In this setting, one obtains

$$\begin{bmatrix} \sigma_{00} & \sigma_{10} & \sigma_{20} \\ \sigma_{10} & \sigma_{11} & \sigma_{21} \\ \sigma_{20} & \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} V_0 \\ V \\ 0 \end{bmatrix} = \begin{bmatrix} I_0 \\ 0 \\ I \end{bmatrix}, \tag{2.33}$$

which after multiplying gives

$$\begin{bmatrix} \sigma_{00}V_0 + \sigma_{10}V \\ \sigma_{10}V_0 + \sigma_{11}V \\ \sigma_{20}V_0 + \sigma_{12}V \end{bmatrix} = \begin{bmatrix} I_0 \\ 0 \\ I \end{bmatrix}. \tag{2.34}$$

Comparing entries, one recovers more familiar formulas for the solution of the Z-problem and the effective operator as a Schur complement,

$$I_0 = \sigma_* V_0,\ V = -\sigma_{11}^{-1}\sigma_{10}V_0,\ I = \sigma_{20}V_0 + \sigma_{21}V, \tag{2.35}$$

$$\sigma_* = \begin{bmatrix} \sigma_{00} & \sigma_{10} \\ \sigma_{10} & \sigma_{11} \end{bmatrix} / \sigma_{11} = \sigma_{00} - \sigma_{01}\sigma_{11}^{-1}\sigma_{10}. \tag{2.36}$$

One can easily show that this is equivalent to (2.29) and (2.30), although we leave this calculation as an exercise for the reader.

The next two theorems are well known. For more on their history, proofs and applications, we recommend [1, 3].

Theorem 16 (Dirichlet minimization principle) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *and*

$$\sigma = [\sigma_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H}). \tag{2.37}$$

If $\sigma = \sigma^*$, $\sigma_{11} \geq 0$ *and* σ_{11} *is invertible then* σ_* *is uniquely defined by the minimization principle:*

$$(V_0, \sigma_* V_0) = \min_{V \in \mathcal{E}} (V_0 + V, \sigma(V_0 + V)),\ \forall V_0 \in \mathcal{U}, \tag{2.38}$$

and, for each $V_0 \in \mathcal{U}$, *the minimum is uniquely given by*

$$V = -\sigma_{11}^{-1}\sigma_{10}V_0. \tag{2.39}$$

In particular, we have the upper bound

$$\sigma_* \leq \sigma_{00}. \tag{2.40}$$

Theorem 17 (Thomson minimization principle) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *and*

$$\sigma = [\sigma_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H}). \tag{2.41}$$

If $\sigma^* = \sigma \geq 0$ *and* σ *is invertible then* $(\sigma_*)^{-1}$ *is uniquely defined by the minimization* *principle:*

$$(I_0, (\sigma_*)^{-1} I_0) = \min_{I \in \mathcal{J}} (I_0 + I, \sigma^{-1}(I_0 + I)), \; \forall I_0 \in \mathcal{U}, \tag{2.42}$$

and, for each $I_0 \in \mathcal{U}$, the minimum is uniquely given by

$$I = -\sigma_{22}^{-1} \sigma_{20} I_0. \tag{2.43}$$

In particular, we have the upper and lower bounds

$$0 \leq [(\sigma^{-1})_{00}]^{-1}] \leq \sigma_* \leq \sigma_{00}. \tag{2.44}$$

The following theorem, while believed to be new, follows from the extension given in Theorem 76 when the operator in question is invertible.

Theorem 18 (ppt maximization principle) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ and*

$$\sigma = [\sigma_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H}). \tag{2.45}$$

If $\sigma^* = \sigma$, $\sigma_{11} \geq 0$ and σ_{11} is invertible then $\operatorname{ppt}_1(\sigma)$ is uniquely defined by the maximization principle:

$$\frac{1}{2}(u + v, \operatorname{diag}\{I_{\mathcal{H}_0}, -I_{\mathcal{H}_1}\} \operatorname{ppt}_1(\sigma)(u + v)) = \max_{t \in \mathcal{H}_1} \left[\operatorname{Re}(v, t) - \frac{1}{2}(u + t, \sigma(u + t)) \right], \tag{2.46}$$

for each $(u, v) \in \mathcal{H}_0 \times \mathcal{H}_1$. Furthermore, for each $(u, v) \in \mathcal{H}_0 \times \mathcal{H}_1$, the maximum is unique and given by

$$t = -\sigma_{11}^{-1} \sigma_{10} u + \sigma_{11}^{-1} v \in \mathcal{H}_1. \tag{2.47}$$

A similar statement holds for $\operatorname{ppt}_0(\sigma)$ whenever $\sigma_{00} \geq 0$ and σ_{00} is invertible.

2.3 Abstract Hodge Decomposition

In this section, we adapt the results of Lim's paper [34] to operators on finite-dimensional Hilbert spaces. Although we prove many properties, the two most essential are Lemma 20 (*i*) and Theorem 21. The latter of these will be utilized in Chapter 3 when identifying the orthogonal triple decomposition needed for Z-problems and their effective operators.

We first prove the following lemma, which provides some necessary properties for the development of the proceeding lemma and theorem.

Lemma 19 *Let $\mathcal{A}, \mathcal{B}$ be finite-dimensional inner product spaces. Then* *(i)* $\ker U^*U = \ker U$,

(ii) $\operatorname{ran} U^*U = \operatorname{ran} U^*$,

(iii) $\ker U^* = (\operatorname{ran} U)^\perp$,

(iv) $\operatorname{ran} U^* = (\ker U)^\perp$,

(v) $\mathcal{A} = \ker U \overset{\perp}{\oplus} \operatorname{ran} U^*$.

Proof. Assume the hypotheses.

(i) Clearly, $\ker U \subseteq \ker U^*U$. It remains to show $\ker U^*U \subseteq \ker U$. Let $u \in \ker U^*U$. Then

$$0 = (u, U^*Uu)_{\mathcal{A}} = (Uu, Uu)_{\mathcal{B}}. \tag{2.48}$$

Hence, $Uu = 0$ which implies $u \in \ker U$. It follows that $\ker U^*U \subseteq \ker U$. Therefore, $\ker U^*U = \ker U$. This proves (i).

(ii) Obviously, $\operatorname{ran} U^*U \subseteq \operatorname{ran} U^*$. It remains to show $\operatorname{ran} U^* \subseteq \operatorname{ran} U^*U$. By the rank-nullity theorem and (i) it follows

$$\operatorname{ran} U^*U = |\mathcal{A}| - \operatorname{nullity} U^*U = |\mathcal{A}| - \operatorname{nullity} U = \operatorname{rank} U = \operatorname{rank} U^*. \tag{2.49}$$

Therefore, $\operatorname{ran} U^*U = \operatorname{ran} U$. This proves (ii).

(iii) Let $u \in \ker U^*$. Then,

$$(u, Uu')_{\mathcal{B}} = (U^*u, u')_{\mathcal{A}} = 0, \ \forall u' \in \mathcal{A}. \tag{2.50}$$

Hence, $u \in (\operatorname{ran} U)^\perp$ which implies $\ker U^* \subseteq (\operatorname{ran} U)^\perp$. It remains to show $(\operatorname{ran} U)^\perp \subseteq \ker U^*$. Let $u \in (\operatorname{ran} U)^\perp$. Then,

$$0 = (u, Uu')_{\mathcal{B}} = (U^*u, u')_{\mathcal{A}}, \ \forall u' \in \mathcal{A}. \tag{2.51}$$

Hence, $U^*u = 0$ which implies $u \in \ker U^*$. It follows that $(\operatorname{ran} U)^\perp \subseteq \ker U^*$. Therefore, $\ker U^* = (\operatorname{ran} U)^\perp$. This proves (*iii*).

(iv) By the properties of orthogonal complements, adjoints and (*iii*), it follows that

$$(\operatorname{ran} U^*)^\perp = \ker (U^*)^* = \ker U. \tag{2.52}$$

Therefore, $\operatorname{ran} U^* = (\ker U)^\perp$. This proves (*iv*).

(v) By the properties of orthogonal complements and (*iv*) it follows

$$\mathcal{A} = \ker U \overset{\perp}{\oplus} (\ker U)^\perp = \ker U \overset{\perp}{\oplus} \operatorname{ran} U^*. \tag{2.53}$$

This proves (*v*).

This completes the proof. ■

With the previous lemma under our belt, we proceed to detail the usual subspaces when we have two bounded linear operators whose composition is identically zero.

Lemma 20 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite-dimensional inner product spaces. If $U \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ and $T \in \mathcal{L}(\mathcal{B}, \mathcal{C})$ satisfy*

$$TU = 0, \ (i.e., \ U^*T^* = 0), \tag{2.54}$$

then

(i) $\ker(T^*T + UU^*) = \ker T \cap \ker U^*$,

(ii) $\ker T = \operatorname{ran} U \overset{\perp}{\oplus} \ker(T^*T + UU^*)$,

(iii) $\ker U^* = \operatorname{ran} T^* \overset{\perp}{\oplus} \ker(T^*T + UU^*)$.

Proof. Suppose $U \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ and $T \in \mathcal{L}(\mathcal{B}, \mathcal{C})$ satisfy (2.54).

(i) Clearly, $\ker T \cap \ker U^* \subseteq \ker(T^*T + UU^*)$. It remains to show $\ker(T^*T + UU^*) \subseteq \ker T \cap \ker U^*$. Let $l \in \ker(T^*T + UU^*)$. Then,

$$0 = (T^*T + UU^*)l = (T^*T)l + (UU^*)l, \tag{2.55}$$

which implies $(T^*T)l = -(UU^*)l$. It follows by (2.54),

$$(TT^*T)l = -(TUU^*)l = 0, \tag{2.56}$$

and

$$-(U^*UU^*)l = (U^*T^*T)l = 0. \tag{2.57}$$

Hence, by Lemma 19 (i) and (2.56) it follows $l \in \ker T$. Similarly, by Lemma 19 (iii) and (2.57) it follows $l \in \ker U^*$. Thus, $l \in \ker T \cap \ker U^*$ which implies $\ker T^*T + UU^* \subseteq \ker T \cap \ker U^*$. Therefore, $\ker(T^*T + UU^*) = \ker T \cap \ker U^*$. This proves (i).

(ii) Since $\ker U^* \cap \ker U = \{0\}$ it follows by (i) and (2.54)

$$\ker T = \mathcal{B} \cap \ker T = (\ker U^* \overset{\perp}{\oplus} \operatorname{ran} U) \cap \ker T \tag{2.58}$$

$$= (\ker U^* \cap \ker T) \overset{\perp}{\oplus} (\operatorname{ran} U \cap \ker T) = \ker(T^*T + UU^*) \overset{\perp}{\oplus} \operatorname{ran} U. \tag{2.59}$$

This proves (ii).

(iii) Since $\ker T \cap \operatorname{ran} T^* = \{0\}$ it follows by (i) and (2.54)

$$\ker U^* = \mathcal{B} \cap \ker U^* = (\ker T \overset{\perp}{\oplus} \operatorname{ran} T^*) \cap \ker U^* \tag{2.60}$$

$$= (\ker T \cap \ker U^*) \overset{\perp}{\oplus} (\operatorname{ran} T^* \cap \ker U^*) = \ker(T^*T + UU^*) \overset{\perp}{\oplus} \operatorname{ran} T^*. \tag{2.61}$$

This proves (iii).

This completes the proof. ■

We now have all the necessary tools to prove one of the most vital preliminaries, the abstract Hodge decomposition theorem. This theorem will be utilized throughout Chapter 3 and Chapter 4.

Theorem 21 (abstract Hodge decomposition) *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be finite-dimensional inner product spaces. If $U \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ and $T \in \mathcal{L}(\mathcal{B}, \mathcal{C})$ satisfy*

$$TU = 0 \ \ (i.e., \ U^*T^* = 0), \tag{2.25}$$

then

$$\mathcal{B} = \operatorname{ran} T^* \overset{\perp}{\oplus} \ker(T^*T + UU^*) \overset{\perp}{\oplus} \operatorname{ran} U. \tag{2.62}$$

Furthermore,

$$\operatorname{ran}(T^*T + UU^*) = \operatorname{ran} T^* \overset{\perp}{\oplus} \operatorname{ran} U, \tag{2.63}$$

$$\ker(T^*T + UU^*) = \ker T \cap \ker U^*. \tag{2.64}$$

Proof. Suppose $U \in \mathcal{L}(\mathcal{A}, \mathcal{B})$ and $T \in \mathcal{L}(\mathcal{B}, \mathcal{C})$ satisfy (2.54). It follows by Lemma 20 (*ii*)

$$\mathcal{B} = \ker T \overset{\perp}{\oplus} \operatorname{ran} T^* = \operatorname{ran} U \overset{\perp}{\oplus} \ker(T^*T + UU^*) \overset{\perp}{\oplus} \operatorname{ran} T^*. \tag{2.65}$$

Furthermore, by the properties of orthogonal complements and Lemma 20 (*i*) we have (2.64) and

$$\operatorname{ran}(T^*T + UU^*) = [\ker(T^*T + UU^*)]^{\perp} = (\ker T \cap \ker U^*)^{\perp} = \operatorname{ran} T^* \overset{\perp}{\oplus} \operatorname{ran} U. \tag{2.66}$$

This completes the proof. ■

Definition 22 *We call the decomposition (2.62) an (abstract) Hodge decomposition.*

Throughout this thesis, we will drop the term abstract from the above definition.

2.4 The Moore-Penrose Pseudoinverse

In this section, we introduce the reader to the Moore-Penrose pseudoinverse X^+ and prove a lemma detailing some of its properties. Although this operation will be heavily utilized in Chapter 4 it also has uses in Chapter 3.

We now give the definition of the Moore-Penrose pseudoinverse and prove some properties used in Chapter 3 and Chapter 4. We recommend [35] for more details regarding the Moore-Penrose pseudoinverse.

Definition 23 (Moore-Penrose pseudoinverse) *Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) < \infty$ and $X \in \mathcal{L}(\mathcal{H})$. Then the Moore-Penrose pseudoinverse (MPP) $X^+ \in \mathcal{L}(\mathcal{H})$ is the unique linear operator that satisfies the properties:*

(i) $X^+XX^+ = X^+$, *(ii)* $XX^+X = X$, *(iii)* $(X^+X)^* = X^+X$, *(iv)* $(XX^+)^* = XX^+$.

Properties (i)-(iv) are called the Penrose equations.

Lemma 24 *Let $\mathcal{H}$ be a Hilbert space with $\dim(\mathcal{H}) < \infty$ and $X \in \mathcal{L}(\mathcal{H})$. Then the following properties hold for $X^+ \in \mathcal{L}(\mathcal{H})$:*

(i) $(X^+)^* = (X^*)^+$,

(ii) $XX^+ = \Gamma_{\operatorname{ran} X} = \Gamma_{(\ker X^*)^\perp}$,

(iii) $X^+X = \Gamma_{\operatorname{ran} X^*} = \Gamma_{(\ker X)^\perp}$,

(iv) If $X = X^*$ then $X^+X = XX^+$.

Proof. Assume the hypotheses.

(i) We will prove (i) by showing that $(X^+)^*$ and $(X^*)^+$ satisfy the Penrose equations and therefore must be equal by the uniqueness of the MPP. Hence,

$$(X^+)^* = (X^+XX^+)^* = (X^+)^*X^*(X^+)^*, \tag{2.67}$$
$$X^* = (XX^+X)^* = X^*(X^+)^*X^*, \tag{2.68}$$
$$[(X^+)^*X^+]^* = (X^+)^*X^+, \tag{2.69}$$
$$[X^+(X^+)^*]^* = X^+(X^+)^*, \tag{2.70}$$

and

$$(X^*)^+ = (X^*)^+X^*(X^*)^+, \tag{2.71}$$
$$X^* = X^*(X^*)^+X^*, \tag{2.72}$$
$$[(X^*)^+X^*]^+ = (X^*)^+X^*, \tag{2.73}$$
$$[X^*(X^*)^+]^+ = X^*(X^*)^+. \tag{2.74}$$

This proves (i).

(*ii*) First, we prove that XX^+ is an orthogonal projection. It follows from Definition 23 (*i*) and (*iv*)

$$(XX^+)^2 = XX^+XX^+ = XX^+, \text{ and } (XX^+)^* = XX^+, \tag{2.75}$$

which implies that XX^+ is an orthogonal projection. Next, we prove that $\operatorname{ran} XX^+ = \operatorname{ran} X$ since by the properties of orthogonal complements $\operatorname{ran} X = (\ker X^*)^\perp$. Clearly, $\operatorname{ran} XX^+ \subseteq \operatorname{ran} X$ is the composition of linear operators. There is still to be proved $\operatorname{ran} X \subseteq \operatorname{ran} XX^+$. Let $y \in \operatorname{ran} X$ and $x \in \ker XX^+$. Then $y = Xz$ for some $z \in \mathcal{H}$ and

$$(Xz, x)_\mathcal{H} = (XX^+Xz, x)_\mathcal{H} = (Xz, XX^+x)_\mathcal{H} = 0. \tag{2.76}$$

Hence, $y \in (\ker XX^+)^\perp = \operatorname{ran} XX^+$ which implies $\operatorname{ran} X \subseteq \operatorname{ran} XX^+$. Thus, $\operatorname{ran} XX^+ = \operatorname{ran} X$. Since any orthogonal projection is uniquely defined by its range, this proves (*ii*).

(*iii*) First, we prove that X^+X is an orthogonal projection. It follows from Definition 23 (*ii*) and (*iii*)

$$(X^+X)^2 = X^+XX^+X = X^+X \text{ and } (X^+X)^* = X^+X, \tag{2.77}$$

which implies X^+X is an orthogonal projection. Next, we will prove $\operatorname{ran} X^+X = (\ker X)^\perp$ since by the properties of orthogonal complements $(\ker X)^\perp = \operatorname{ran} X^*$. Let $y \in \operatorname{ran} X^+X$ and $x \in \ker X$. Then $y = X^+Xz$ for some $z \in \mathcal{H}$. Hence, by Definition 23 (*iii*)

$$(X^+Xz, x)_\mathcal{H} = (z, X^+Xx)_\mathcal{H} = 0, \tag{2.78}$$

which implies $y \in (\ker X)^\perp$. Thus, $\operatorname{ran} X^+X \subseteq (\ker X)^\perp$. Let $y' \in \ker X^+X$. Then

$$X^+Xy' = 0 \Rightarrow XX^+Xy' = 0. \tag{2.79}$$

Hence, $y' \in \ker X$ and $\ker X^+X \subseteq \ker X$. By taking the orthogonal complement, it follows that

$$(\ker X)^\perp \subseteq (\ker X^+X)^\perp = \operatorname{ran}(X^+X)^* = \operatorname{ran} X^+X. \tag{2.80}$$

Thus, $\operatorname{ran} X^+X = (\ker X)^\perp$. Since any orthogonal projection is uniquely defined by its range, this proves (*iii*).

(*iv*) Suppose $X = X^*$. Then, $\operatorname{ran} X = \operatorname{ran} X^*$. Hence, $XX^+ = X^+X$ by (*ii*) and (*iii*). This proves (*iv*).

This completes the proof. ■

Chapter 3

Applications

The goal of this chapter is to present applications of the main results of this thesis (in Chapter 4) to problems from mathematical physics and electrical engineering. Some of these applications are discrete analogs of continuum problems, while others are new perspectives on well known problems from the point of view of the Z-problem and effective operators. This could lead to new results on old problems; for more on this, see Chapter 5. The contents of this chapter are broken down as follows.

In Section 3.1, we introduce the reader to the quintessential example of a Z-problem and an effective operator. This is the periodic conductivity equation for the continuum (with unit cell $\Omega \subseteq \mathbb{R}^d$, in particular, $d = 2, 3$ and $\Omega = [0, 2\pi]^d$) in the form of a Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ in which the orthogonal triple decomposition $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$ is just the standard Hodge decomposition on the Hilbert space $\mathcal{H} = [L^2_\#(\Omega)]^d$ of square-integrable periodic functions. In particular, $\mathcal{U}$ is the space of constant fields, $\mathcal{E}$ is the space of curl-free average-zero periodic fields (i.e., periodic gradients of potentials), and $\mathcal{J}$ is the space of divergence-free average-zero periodic fields. In this case, the effective operator σ_* and the effective conductivity σ_{eff} are the same, i.e., $\sigma_* = \sigma_{eff}$.

In Section 3.2, we present the analogy to the conductivity equation when the continuum is replaced by a finite linear digraph $G = (P_G, E_G)$ (i.e., finite node set P_G and finite directed edge set E_G) with conductivity σ. In subsection 3.2.1, we introduce the reader to the operator framework for electrical conductivity which is needed in the remaining sections. More specifically, we introduce the function spaces considered throughout Section 3.2, the graphs gradient operator D (in analogy to ∇), and the graphs divergence $D^\bullet$ (in analogy to $\nabla\cdot$) for a finite linear digraph $G = (P_G, E_G)$. Here, P_G is the set of nodes of the graph and E_G is the set of directed edges of the graph. In addition, we introduce the concept of an electrical network (P_G, E_G, σ) where σ is the conductivity satisfying $\sigma^* = \sigma \geq 0$ and the Kirchhoff operator K_σ is defined by $K_\sigma = -D^\bullet \sigma D$ (i.e., the negative of the weighted discrete Laplacian operator).

In Subsection 3.2.2, we consider the Dirichlet-to-Neumann (DtN) map Λ_σ which is the linear operator taking the boundary $u|_{P_{\partial G}}$ voltage (where u is the voltage potential and $P_{\partial G}$ is the set of all boundary nodes of the graph) to its boundary current ϕ, i.e., $\Lambda_\sigma u|_{P_{\partial G}} = \phi$. Following the work of Milton in [2, Ch. 2, Sec. 2.13], we show that there is an associated Z-problem $(\mathcal{H},\mathcal{U},\mathcal{E},\mathcal{J},\sigma)$ with a Hodge decomposition $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$ over the Hilbert space $\mathcal{H} = \mathcal{F}(E_G, \mathbb{C})$ of functions on E_G. In particular, $\mathcal{U}$ is the space of gradients of potentials harmonic in the interior P_{G° of the graph, $\mathcal{E}$ is the space of gradients of potentials zero on the boundary $P_{\partial G}$ of the graph, and $\mathcal{J} = \ker D^\bullet$ is the kernel of the divergence. We also prove that $\mathcal{U} \overset{\perp}{\oplus} \mathcal{E} = \operatorname{ran} D$ is the range of the gradient. Then we prove, for a connected graph G, that the DtN map Λ_σ and the effective operator σ_* of the Z-problem $(\mathcal{H},\mathcal{U},\mathcal{E},\mathcal{J},\sigma)$ are related by the identity $\Lambda_\sigma = \Pi^*(\sigma)_*\Pi$, where Π is the "lift" operator of the graph. More precisely, Π is a linear map that takes any function f on the boundary $P_{\partial G}$ of the graph to a potential u, which is harmonic in the interior, to its boundary "trace" $u|_{P_{\partial G}} = f$ and then maps that to its "gradient" Du, i.e., $\Pi(u|_{P_{\partial G}}) = Du$.

In Subsection 3.2.3, we consider the similar setting of a finite linear digraph $G = (P_G, E_G)$ with conductivity σ, but instead of the DtN map, we now focus on the effective conductivity σ_{eff} (whose inverse $1/\sigma_{eff}$ is the effective resistance r_{eff}). Here, the boundary $P_{\partial G} = \{p, q\}$ contains only two nodes ($p \neq q$). The effective conductivity is defined by the unique scalar σ_{eff} that satisfies $j = \sigma_{eff}[u(p) - u(q)]$ for all $j \in \mathbb{C}$ and every solution u to the equation $K_\sigma u = j(\delta_p - \delta_q)$, where $K_\sigma = -D^\bullet \sigma D$ is the Kirchhoff operator, and δ_p, δ_q are delta functions on the nodes (i.e., equal to zero except at node p, q, respectively, where it is equal to 1). We show that there is an associated Z-problem, different from that in Subsection 3.2.2, namely, $(\mathcal{H},\mathcal{U},\mathcal{E},\mathcal{J},K_\sigma)$ with a Hodge decomposition $\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$ over the Hilbert space $\mathcal{H} = \mathcal{F}(P_G, \mathbb{C})$ of functions on P_G. Here, $\mathcal{U}$ and $\mathcal{J}$ are the spaces of functions that are zero on all nodes except possibly at p and q, respectively, while $\mathcal{E}$ is the space of functions that are zero at both p and q. Moreover, we prove that the effective operator $(K_\sigma)_*$ of this Z-problem and the effective conductivity are essentially the same, that is, $(K_\sigma)_*$ is left multiplication by the scalar σ_{eff}, that is, $(K_\sigma)_* = \sigma_{eff} I_{\mathcal{U}}$ (where $I_{\mathcal{U}}$ denotes the identity operator on $\mathcal{U}$).

In Section 3.3, we consider the conductivity equation on a periodic linear digraph (G, L), where the graph $G = (\mathbb{Z}^d, E)$ has the set of nodes $\mathbb{Z}^d$, the set of directed edges E, and the periodic lattice L (that is, the nodes and edges are translationally invariant with respect to L, see Figure 3.2). For simplicity, we only consider nodes connected by edges to their nearest neighbors (i.e., between any two nodes with unit distance there is a directed edge, see Figure 3.3). Our goal of this section is to develop the discrete analogy of the continuum problem for the periodic conductivity equation and effective conductivity from Section 3.1. In this setting, we again obtain an analogous Z-problem and Hodge decomposition with the graph's gradient D and divergence $D^\bullet$ operators replaced with their restrictions $D_\#$ and $(D^\bullet)_\#$ to periodic node and edge functions. However, we realize that the analogy is imperfect because the associated Z-problem and its relationship to the periodic Ohm's law are more rich mathematically (and hence physically we presume) due to the loss of simple connectedness. Despite these complications (see Example 63), we prove that the effective operator σ_* for the Z-problem is the same as the effective conductivity σ_{eff} of a discrete periodic Ohm's law, i.e., $\sigma_* = \sigma_{eff}$.

We feel that this chapter has more results to offer with regard to future studies in effective media theory. For example, the results on the DtN map in Subsection 3.2.2 may lead to bounds and limitations on cloaking and inverse problems. The results in Subsection 3.2.3 on effective conductivity is a classical problem, however, our perspective is believed to be new. As such, the tools and theory developed for the theory of composites on effective operators may yield new results. Lastly, the results in Section 3.3 on the discrete analog to the periodic conductivity equation can be developed for other discrete problems such as discrete elasticity and viscoelasticity. One could also consider other classes of periodic graphs (e.g., non-Cartesian graph topologies or metric and quantum graphs). For more on all this, see Chapter 5.

3.1 Conductivity for the Continuum

Here we set the stage for Z-problems and effective operators in the following sections, by treating the classical and well known quintessential example of the periodic conductivity equation in the continuum, the associated Hodge decomposition, Ohm's law, and the effective conductivity derived from periodic averaging. As this is meant to be an overview, we omit (only in this section) a deeper discussion of all the spaces (e.g., Sobolev spaces) and domains of definition of all of the operators involved.

Consider the periodic conductivity equation

$$\nabla \cdot \sigma \nabla u = 0 \text{ on } \Omega = [0, 2\pi]^d, \tag{3.1}$$

where ∇ is the gradient, $\nabla \cdot$ is the divergence, σ is the conductivity, and d is the dimension (in particular, we focus on $d = 2$ or $d = 3$). The Hilbert space $\mathcal{H}$ is

$$\mathcal{H} = \left[L^2_{\#}(\Omega)\right]^d, \tag{3.2}$$

that is, the space of periodic square integrable functions on the domain Ω, with the standard inner product

$$(E, F)_{\mathcal{H}} = \frac{1}{|\Omega|}\int_\Omega \overline{E(x)}^T F(x) dx, \tag{3.3}$$

and (cell) average

$$\langle F \rangle = \frac{1}{|\Omega|}\int_\Omega F(x)\, dx, \tag{3.4}$$

for all $E, F \in \mathcal{H}$. More specifically, conductivity σ is a bounded linear operator, i.e.,

$\sigma \in \mathcal{L}(\mathcal{H})$, which satisfies $\sigma^* = \sigma \geq 0$ and σ^{-1} exists.

It is well known [1] that we have the standard Hodge decomposition

$$\mathcal{H} = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{3.5}$$

where

$$\mathcal{U} = \{U \in \mathcal{H} : U \equiv C,\ C \in \mathbb{C}^d\}, \tag{3.6}$$
$$\mathcal{E} = \{E \in \mathcal{H} : \nabla \times E = 0,\ \langle E \rangle = 0\}, \tag{3.7}$$
$$\mathcal{J} = \{J \in \mathcal{H} : \nabla \cdot J = 0,\ \langle J \rangle = 0\}, \tag{3.8}$$

such that $\nabla\times$ denotes the curl and we use the notation $U \equiv C$ to mean U is a constant function equal to C everywhere on Ω.

Remark 25 *An alternative definition of $\mathcal{E}$ is the space of all periodic functions that are the gradient of a potential, ∇u, with average zero, $\langle \nabla u \rangle = 0$. In fact, this definition is the one that we use as an analog in the discrete setting for periodic graphs in Section 3.3.*

The constitutive relation is the (periodic) Ohm's law, that is, in terms of a periodic conductivity σ it is

$$J = \sigma E,\ E \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{E},\ J \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}. \tag{3.9}$$

The effective conductivity σ_{eff} is then defined in terms of this constitutive relation and the average $\langle \cdot \rangle$ by the relationship

$$\sigma_{eff} \langle E \rangle = \langle J \rangle. \tag{3.10}$$

The connection to the Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ and the associated effective operator σ_* is as follows. First, the Z-problem in this setting is: given $E_0 \in \mathcal{U}$, find $(J_0, E, J) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ such that

$$J_0 + J = \sigma \left(E_0 + E\right), \tag{3.11}$$

and the effective operator σ_* is defined by the relation

$$\sigma_* E_0 = J_0. \tag{3.12}$$

Therefore, since $E_0 = \langle E_0 + E \rangle$ and $J_0 = \langle J_0 + J \rangle$, we have the fundamental identity

$$\sigma_{eff} = \sigma_*. \tag{3.13}$$

Remark 26 *Typically, $\sigma = \sigma(x)$ is a local tensor, i.e., a $d \times d$ matrix-valued periodic function of the spatial variable x, which is bounded above and below by some positive scalars, i.e., there exist scalars $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 I_3 \leq \sigma(x) \leq \gamma_2 I_3$ for almost every $x \in \Omega$. In this context, $\sigma \in \mathcal{L}(\mathcal{H})$ is the operator of left multiplication by the function $\sigma(x)$ as such, the effective operator σ_* is then equal to left multiplication by the effective conductivity σ_{eff}, i.e., this is the rigorous interpretation of* (3.13) *for such local tensors $\sigma(x)$. This should be considered as an analogy of the relationship between matrices and operators in linear algebra for finite-dimensional vector spaces.*

3.2 Conductivity for Finite Linear Graphs

There are two objectives for this section. The first objective is to bridge the gap between the periodic conductivity equation in the continuum in Section 3.1 and the analogy for the discrete setting in Section 3.3. To do this, we begin with Subsection 3.2.1 which gives the necessary background on electrical networks for finite linear graphs. Then in Subsection 3.2.2 we familiarize the reader with the work of Graeme Milton, but from our perspective using operator theory (as opposed to matrix theory). Specifically, we derive the Dirichlet-to-Neumann map from the effective operator of an associated Z-problem. This allows one to gain some familiarity with finite linear graph theory and elementary electric circuit theory that is helpful (but not necessary) when reading Subsection 3.2.3 and Section 3.3.

The second objective is to advance the program established in the 2016 book by Milton [2] to extend the theory of composites to other areas of science. We believe we have a new contribution to this program with Subsection 3.2.3, where we show that a classical problem in effective media theory in the context of electric circuits, that of effective conductivity/resistance, comes from a Z-problem and is the effective operator of that Z-problem. This was unexpected, and we feel that this will merit further study given its importance to electric circuit theory and connections to other areas of physics. For those who would like more background on finite linear graphs and circuit theory we recommend [26, 36–38].

3.2.1 Operator Framework for Electrical Conductivity

Let $G = (P_G, E_G)$ denote a finite linear digraph with finite nonempty node set $P_G = \{p_1, \ldots, p_{|P_G|}\}$ and finite directed edge set E_G. We assume E_G is nonempty, so that $E_G = \{e_1, \ldots, e_{|E_G|}\}$. For each edge $e \in E_G$, the outgoing and incoming incident nodes are denoted by e_- and e_+, respectively [more precisely, these are functions $(\cdot)_\pm : E_G \to P_G$] as shown in Figure 3.1.

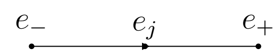


Figure 3.1: Example of a directed edge.

The following function space and its basic properties as a vector space are standard in linear algebra (see, for example, [28]).

Definition 27 *Let $\mathcal{T}$ be an arbitrary nonempty set. The set of all functions from $\mathcal{T}$ to $\mathbb{C}$ is denoted by $\mathcal{F}(\mathcal{T},\mathbb{C})$, i.e.,*

$$\mathcal{F}(\mathcal{T},\mathbb{C}) = \{f \mid f : \mathcal{T} \to \mathbb{C}\}. \tag{3.14}$$

The functions $0_{\mathcal{T}}$ and $1_{\mathcal{T}}$ are given by

$$0_{\mathcal{T}}(t) = 0, \ \forall t \in \mathcal{T}, \tag{3.15}$$
$$1_{\mathcal{T}}(t) = 1, \ \forall t \in \mathcal{T}, \tag{3.16}$$

and the identity operator on $\mathcal{F}(\mathcal{T},\mathbb{C})$ is denoted by $I_{\mathcal{F}(\mathcal{T},\mathbb{C})}$.

Lemma 28 ($\mathcal{F}(\mathcal{T},\mathbb{C})$ is isomorphic to $\mathbb{C}^{|\mathcal{T}|}$) *If $\mathcal{T}$ is an arbitrary non-empty set, then $\mathcal{F}(\mathcal{T},\mathbb{C})$ is a vector space. Furthermore, if $\mathcal{T} = \{t_1, \ldots, t_{|\mathcal{T}|}\}$ is finite, then $\mathcal{F}(\mathcal{T},\mathbb{C})$ is a Hilbert space under the inner product*

$$(f,g)_{\mathcal{T}} = \sum_{t\in\mathcal{T}} \overline{f(t)}g(t), \ \forall f,g \in \mathcal{F}(\mathcal{T},\mathbb{C}). \tag{3.17}$$

Moreover, the set

$$\alpha_{\mathcal{T}} = \{\delta_{t_i} : i = 1, \ldots, |\mathcal{T}|\}, \quad \delta_{t_i}(t_j) = \delta_{ij}, \ \ 1 \le i,j \le |\mathcal{T}|, \tag{3.18}$$

is an orthonormal basis for $\mathcal{F}(\mathcal{T},\mathbb{C})$, where δ_{ij} is the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3.19}$$

Notation 29 *Using the above notation of the ordered basis (3.18) and adapting the notation from [28], we will denote the matrix representation for any linear operator $L \in \mathcal{L}(\mathcal{F}(\mathcal{T}_1,\mathbb{C}),\mathcal{F}(\mathcal{T}_2,\mathbb{C}))$ with respect to the bases $\alpha_{\mathcal{T}_1}$, $\alpha_{\mathcal{T}_2}$ by $\left[L\right]_{\alpha_{\mathcal{T}_1}}^{\alpha_{\mathcal{T}_2}}$.*

Now consider the finite-dimensional Hilbert spaces $\mathcal{F}(P_G,\mathbb{C})$ and $\mathcal{F}(E_G,\mathbb{C})$ with inner products $(\cdot,\cdot)_{\mathcal{F}(P_G,\mathbb{C})}$, $(\cdot,\cdot)_{\mathcal{F}(E_G,\mathbb{C})}$, respectively [defined in (3.17)] and orthonormal bases α_{P_G}, α_{E_G}, respectively [defined in (3.18)]. The following definition establishes the analogs of the gradient and divergence in the continuum (see Section 3.1).

Definition 30 (analogy of ∇ and $\nabla\cdot$) *We define the function $D : \mathcal{F}(P_G, \mathbb{C}) \to \mathcal{F}(E_G, \mathbb{C})$ by*

$$(Df)(e) = f(e_+) - f(e_-),\ \forall (f, e) \in \mathcal{F}(P_G, \mathbb{C}) \times E, \tag{3.20}$$

and the function $D^\bullet : \mathcal{F}(E_G, \mathbb{C}) \to \mathcal{F}(P_G, \mathbb{C})$ by

$$(D^\bullet f)(p) = \sum_{\substack{e \in E_G, \\ e_- = p}} f(e) - \sum_{\substack{e \in E_G, \\ e_+ = p}} f(e),\ \forall (f, p) \in \mathcal{F}(E_G, \mathbb{C}) \times P_G. \tag{3.21}$$

We may refer to D and $D^\bullet$ as the graph's gradient and divergence operators, respectively, given their analogy to the gradient ∇ and divergence $\nabla\cdot$ in the continuum (see, Section 3.1), respectively. An alternative view of the operators D and $D^\bullet$ as the boundary and co-boundary operators is offered in [34] and [37].

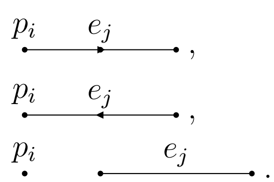
It follows that the matrix representation $\left[\cdot\right]_{\alpha_{P_G}}^{\alpha_{E_G}}$ of D is the edge-node incidence matrix A_a^T, i.e.,

$$\left[D\right]_{\alpha_{P_G}}^{\alpha_{E_G}} = A_a^T = \left[a_{ji}\right], \tag{3.22}$$

where $A_a = [a_{ij}]$ is the well known node-edge incidence matrix (see, [36] and [26]) defined by

$$a_{ij} = \begin{cases} 1 & \text{if the } j\text{th edge is incident on and directed away from the } i\text{th node,} \\ -1 & \text{if the } j\text{th edge is incident on and directed toward the } i\text{th node,} \\ 0 & \text{if the } j\text{th edge is not incident on the } i\text{th node,} \end{cases} \tag{3.23}$$

$$= \begin{cases} 1 & \\ -1 & \\ 0 & \end{cases} \tag{3.24}$$



It is an easy exercise to verify that the matrix representation $\left[\cdot\right]_{\alpha_{E_G}}^{\alpha_{P_G}}$ of $D^\bullet$ is the matrix $-A_a$, i.e.,

$$\left[D^\bullet\right]_{\alpha_{E_G}}^{\alpha_{P_G}} = -A_a. \tag{3.25}$$

In this context, the proof of the following lemma is almost immediate.

Lemma 31 (analogy of $\nabla\cdot = -\nabla^*$) *The linear operators D and $D^\bullet$, defined by 3.20 and 3.21, are well defined and are Hilbert space adjoints of each other, i.e.,*

$$D^* = -D^\bullet. \tag{3.26}$$

Proof. It is clear that the functions D and $D^{\bullet}$ are well defined and linear (i.e., left as an exercise for the reader). To prove that they are adjoints to each other, we use the identities (3.22) and (3.25) and the fact that α_{P_G} and α_{E_G} are orthonormal bases. In particular, from this and standard results from linear algebra [28], it follows that

$$\left[D^*\right]_{\alpha_{P_G}}^{\alpha_{E_G}} = (\left[D\right]_{\alpha_{P_G}}^{\alpha_{E_G}})^* = (\left[D\right]_{\alpha_{P_G}}^{\alpha_{E_G}})^T = (A_a^T)^T = A_a = -\left[D^{\bullet}\right]_{\alpha_{E_G}}^{\alpha_{P_G}} = \left[-D^{\bullet}\right]_{\alpha_{E_G}}^{\alpha_{P_G}}, \tag{3.27}$$

which implies $D^* = -D^{\bullet}$. ■

Remark 32 (Kirchhoff's Laws) *Although mathematically unnecessary, it may be helpful for the reader to see* $\ker D^{\bullet}$ *as the set of edge currents for the network and* $\operatorname{ran} D$ *as the set of edge voltages for the network. This interpretation is clearer if one notes that Kirchhoff's current law (KCL) is equivalent to* $D^{\bullet}I = 0$ *if and only if* I *is an edge current, whereas Kirchhoff's voltage law (KVL) is equivalent to* $V = Du$ *for some potential* u *if and only if* V *is an edge voltage.*

The following lemma will be useful in the remainder of Section 3.2. Although this result is well known [37], we include its proof for completeness.

Lemma 33 *Let* $G = (P_G, E_G)$ *be a finite linear digraph and* $G_1 = (P_{G_1}, E_{G_1}), \ldots, G_k = (P_{G_k}, E_{G_k})$ *be its connected components. Denote the characteristic function on the node set* P_{G_i} *of the graph* G_i *by* $\chi_{P_{G_i}}$*, i.e.,*

$$\chi_{P_{G_i}}(x) = \begin{cases} 1, & \text{if } x \in P_{G_i}, \\ 0, & \text{if } x \in P_G \setminus P_{G_i}, \end{cases} \tag{3.28}$$

for $i = 1, \ldots, k$*. Then* $\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$ *is an orthonormal basis for* $\ker D$*. In particular,*

$$\ker D = \operatorname{span}\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}, \ \dim\ker D = k, \ 1_{P_G} = \sum_{i=1}^{k} \chi_{P_{G_i}} \in \ker D \tag{3.29}$$

and G *is connected (i.e.,* $k = 1$*) if and only if* $\ker D = \operatorname{span}\{1_{P_G}\}$*.*

Proof. Assume the hypotheses. First, we prove othonormality of the set $\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$. Hence,

$$(\chi_{P_{G_i}}, \chi_{P_{G_j}})_{\mathcal{F}(P_G,\mathbb{C})} = \sum_{x\in P_G} \overline{\chi_{P_{G_i}}(x)}\chi_{P_{G_j}}(x) = \delta_{ij} \tag{3.30}$$

for all $i, j = 1, \ldots, k$. Second, we prove that the set $\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$ is a basis for

$\ker D$. Let $e \in E_G$ then $e \in E_{G_j}$ for exactly one j and hence $e_\pm \in P_{G_j}$. It follows that

$$\left(D \sum_{i=1}^{k} c_i \chi_{P_{G_i}} \right)(e) = \sum_{i=1}^{k} c_i (D\chi_{P_{G_i}})(e) = \sum_{i=1}^{k} c_i [\chi_{P_{G_i}}(e_+) - \chi_{P_{G_i}}(e_-)] = 0, \tag{3.31}$$

for each $c_i \in \mathbb{C}$ and $i = 1, \ldots, k$. Hence, $\operatorname{span}\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\} \subseteq \ker D$. It remains to prove the reverse inclusion. Let $f \in \ker D$. Fix any $l = 1, \ldots, k$ and if $|P_{G_l}| = 1$ then f is constant on P_{G_l}, otherwise let $p, q \in P_{G_l}$ with $p \neq q$. Then, as G_l is connected, there exists a sequence of nodes $\{x_j\}_{j=1}^n$ such that $x_1 = p$, $x_n = q$ and x_i, x_{i+1} are the incident nodes of an edge e_i, i.e., $\{x_i, x_{i+1}\} = \{(e_i)_-, (e_i)_+\}$. Define

$$\operatorname{sgn}(e_i) := \begin{cases} 1, \text{ if } (e_i)_- = x_i, \\ -1, \text{ if } (e_i)_+ = x_i \neq x_{i+1}. \end{cases} \tag{3.32}$$

Then

$$0 = \sum_{i=1}^{n-1} \operatorname{sgn}(e_i)(Df)(e_i) = \sum_{i=1}^{n-1} f(x_{i+1}) - f(x_i) = f(x_n) - f(x_1) = f(p) - f(q), \tag{3.33}$$

which implies $f(p) = f(q)$ and thus f is constant on G_l. This proves that $f \in \operatorname{span}\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$ and hence $\ker D \subseteq \operatorname{span}\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$. This proves $\ker D = \operatorname{span}\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$ and since $\{\chi_{P_{G_1}}, \ldots, \chi_{P_{G_k}}\}$ is an orthonormal set of vectors, it is linearly independent which proves it is a basis for $\ker D$. The remainder of the proof follows immediately from this. ■

Definition 34 (conductivity on electrical networks) *An electrical network is a triple (P_G, E_G, σ), with graph $G = (P_G, E_G)$ and operator $\sigma \in \mathcal{L}(\mathcal{F}(E_G, \mathbb{C}))$ satisfying $\sigma^* = \sigma \geq 0$. We call σ the network conductivity and we define the Kirchhoff operator $K_\sigma \in \mathcal{L}(\mathcal{F}(P_G, \mathbb{C}))$ by*

$$K_\sigma = -D^\bullet \sigma D. \tag{3.34}$$

In particular, for $\sigma = I$ [the identity operator on $\mathcal{F}(E_G, \mathbb{C})$], the "normalized" Kirchhoff operator is

$$K_I = -D^\bullet D. \tag{3.35}$$

Remark 35 *The normalized Kirchhoff operator K_I is negative of the graph's Laplacian operator $D^\bullet D$ (the latter is the analog of Δ, i.e., the Laplacian in the continuum). Similarly, the Kirchhoff operator K_σ is negative of the graph's weighted Laplacian operator $D^\bullet \sigma D$ (the latter is the analog of $\nabla \cdot \sigma \nabla$ in the continuum). In the matrix setting, the well known (refs [38] and [26]) Kirchhoff matrix is the matrix representation of the*

Kirchhoff operator K_σ with respect to the orthonormal bases $\alpha_{P_G}, \alpha_{E_G}$, i.e.,

$$\left[K_\sigma\right]_{\alpha_{E_G}}^{\alpha_{E_G}} = -\left[D^\bullet\right]_{\alpha_{E_G}}^{\alpha_{P_G}} \left[\sigma\right]_{\alpha_{E_G}}^{\alpha_{E_G}} \left[D\right]_{\alpha_{P_G}}^{\alpha_{E_G}} = A_a \left[\sigma\right]_{\alpha_{E_G}}^{\alpha_{E_G}} A_a^T, \tag{3.36}$$

where $\left[\sigma\right]_{\alpha_{E_G}}^{\alpha_{E_G}}$ is the matrix representation of the conductivity σ, i.e., the conductivity matrix. Typically, in a resistor-only network, the conductivity matrix $\left[\sigma\right]_{\alpha_{E_G}}^{\alpha_{E_G}}$ is a diagonal matrix. This corresponds to the conductivity σ being given by

$$\sigma = \sum_{i=1}^{|E_G|} \sigma_{ii}\delta_{e_i}, \tag{3.37}$$

where σ_{ii} is the conductivity (i.e., the inverse of resistance) on the ith edge of the graph $G = (P_G, E_G)$.

3.2.2 The Dirichlet-to-Neumann Map: The Z-problem and Effective Operator

In this subsection, we reformulate the work of Milton in [2, Ch. 2, Sec. 2.13] on finite linear graphs and the characterization of their discrete Dirichlet-to-Neumann (DtN) maps as effective operators. In our operator approach, we will prove that the DtN is derived from an effective operator σ_* associated to a Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ with semi-definite operator σ representing the conductivity on the graph. For further background and motivation for the DtN map in this setting, we recommend the works of Morrow and Curtis [38] and A. Stefan [26, Sec. 3.1 and 3.3].

To describe the DtN map, the node set P_G is partitioned into two nonempty subsets $P_{\partial G}$ and P_{G°, i.e.,

$$P_G = P_{\partial G} \cup P_{G^\circ}, \quad P_{\partial G} \cap P_{G^\circ} = \emptyset, \quad P_{\partial G} \neq \emptyset, \quad P_{G^\circ} \neq \emptyset. \tag{3.38}$$

In this setting, we define the set of boundary nodes and interior nodes for the graph $G = (P_G, E_G)$ to be $P_{\partial G}$ and P_{G°, respectively. We now define the discrete analog of the Dirichlet problem for the conductivity equation in the continuum.

Definition 36 (discrete Dirichlet problem) *The discrete Dirichlet problem is defined as follows: given a function $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$, find a function $u \in \mathcal{F}(P_G, \mathbb{C})$ satisfying*

$$(-D^\bullet \sigma D u)|_{P_{G^\circ}} = 0, \tag{3.39}$$
$$u|_{P_{\partial G}} = f. \tag{3.40}$$

For each such solution $u \in \mathcal{F}(P_G, \mathbb{C})$, the boundary (source) current ϕ is defined by

$$\phi = (-D^\bullet \sigma D u)|_{P_{\partial G}}. \tag{3.41}$$

Definition 37 (DtN map) *The Dirichlet-to-Neumann map is the operator* $\Lambda_\sigma \in \mathcal{L}(\mathcal{F}(P_{\partial G}, \mathbb{C}))$ *that maps the potential on the boundary* $u|_{P_{\partial G}}$ *for each solution* $u \in \mathcal{F}(P_G, \mathbb{C})$ *of the discrete Dirichlet problem (3.39) and (3.40) to the boundary current* ϕ *(3.41), i.e.,*

$$\Lambda_\sigma u|_{P_{\partial G}} = \phi. \tag{3.42}$$

Theorem 38 (the DtN map is a generalized Schur complement) *Let*

$$(P_G, E_G, \sigma), \tag{3.43}$$

be an electrical network (see Def. 34) with boundary and interior nodes, $P_{\partial G}$ *and* P_{G°*, respectively. Then the DtN map* Λ_σ *as defined in Definition 37 exists and is unique. Furthermore, it is given by the generalized Schur complement*

$$\Lambda_\sigma = (K_\sigma)_{00} - (K_\sigma)_{01}(K_\sigma)_{11}^{+}(K_\sigma)_{10}, \tag{3.44}$$

where $[(K_\sigma)_{ij}]_{i,j=0,1}$ *is the unique* 2×2 *block operator with respect to the (external) decomposition* $\mathcal{F}(P_{\partial G}, \mathbb{C}) \overset{\perp}{\oplus} \mathcal{F}(P_{G^\circ}, \mathbb{C})$ *defined by*

$$\begin{bmatrix} (K_\sigma)_{00} & (K_\sigma)_{01} \\ (K_\sigma)_{10} & (K_\sigma)_{11} \end{bmatrix} \begin{bmatrix} u|_{P_{\partial G}} \\ u|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} (K_\sigma)_{00} u|_{P_{\partial G}} + (K_\sigma)_{01} u|_{P_{G^\circ}} \\ (K_\sigma)_{10} u|_{P_{\partial G}} + (K_\sigma)_{11} u|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} (K_\sigma u)|_{P_{\partial G}} \\ (K_\sigma u)|_{P_{G^\circ}} \end{bmatrix}, \tag{3.45}$$

for all $u \in \mathcal{F}(P_G, \mathbb{C})$.

Proof. Assume the hypotheses. In terms of the identity (3.45), the discrete Dirichlet problem in Definition 3.39 is equivalent to the constrained linear system of equations

$$\begin{bmatrix} (K_\sigma)_{00} & (K_\sigma)_{01} \\ (K_\sigma)_{10} & (K_\sigma)_{11} \end{bmatrix} \begin{bmatrix} u|_{P_{\partial G}} \\ u|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} (K_\sigma u)|_{P_{\partial G}} \\ (K_\sigma u)|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix}, \; u|_{P_{\partial G}} = f. \tag{3.46}$$

Then since $\sigma^* = \sigma \geq 0$ and $K_\sigma = D^*\sigma D$ by Lemma 31, it follows that $K_\sigma^* = K_\sigma \geq 0$ and hence $[(K_\sigma)_{ij}]_{i,j=0,1}^* = [(K_\sigma)_{ij}]_{i,j=0,1} \geq 0$. Thus, by Lemma 71 the above system of linear equations is solvable for any $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$ and ϕ is given by the generalized Schur complement formula

$$\phi = [(K_\sigma)_{00} - (K_\sigma)_{01}(K_\sigma)_{11}^{+}(K_\sigma)_{10}]u|_{P_{\partial G}}. \tag{3.47}$$

As f was arbitrary, the proof of existence, uniqueness and equality (3.44) now follows immediately. This completes the proof. ∎

Remark 39 *In the matrix setting, the well known (see, [36] and [26]) response matrix of the electric network* (P_G, E_G, σ) *would be* $\left[\Lambda_\sigma\right]_{\alpha_{P_{\partial G}}}^{\alpha_{P_{\partial G}}}$*, the matrix representation of the Dirichlet-to-Neumann map* Λ_σ *with respect to the orthonormal basis* $\alpha_{P_{\partial G}}$ *for* $\mathcal{F}(P_{\partial G}, \mathbb{C})$.

We now begin the process of relating the DtN map to the rest of this thesis on the Z-problems associated with finite electric circuits and their effective operators. Here we introduce the discrete analog of the Hodge decomposition for the Dirichlet problem [11].

Theorem 40 (Hodge decomposition for the discrete Dirichlet problem) *The sets $\mathcal{U}$, $\mathcal{E}$, $\mathcal{J}$ are mutually orthogonal subspaces in the Hilbert space $\mathcal{F}(E_G, \mathbb{C})$, where*

$$\mathcal{U} = \{Du : u \in \mathcal{F}(P_G, \mathbb{C}), (D^\bullet Du)|_{P_{G^\circ}} = 0\}, \tag{3.48}$$
$$\mathcal{E} = \{Du : u \in \mathcal{F}(P_G, \mathbb{C}) \text{ and } u|_{P_{\partial G}} = 0\}, \tag{3.49}$$
$$\mathcal{J} = \ker(-D^\bullet). \tag{3.50}$$

Furthermore,

$$\mathcal{F}(E_G, \mathbb{C}) = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{3.51}$$

$$\operatorname{ran}(D) = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E}. \tag{3.52}$$

Proof. Define
$$\mathcal{K} = \{u \in \mathcal{F}(P_G, \mathbb{C}) : u|_{P_{\partial G}} = 0\}, \tag{3.53}$$
and denote by $\Gamma_{\mathcal{K}}$ the orthogonal projection onto $\mathcal{K}$. The proof now follows immediately from Lemma 31 together with the abstract Hodge decomposition theorem (i.e., Theorem 21), where

$$\mathcal{B} = \mathcal{F}(E_G, \mathbb{C}),\ \mathcal{A} = \mathcal{C} = \mathcal{F}(P_G, \mathbb{C}),\ T^* = T = D^+D = \Gamma_{\ker D^\bullet}, \tag{3.54}$$
$$U = D|_{\mathcal{K}},\ U^* = -\Gamma_{\mathcal{K}} D^\bullet,\ \operatorname{ran} T^* = \ker D^\bullet = \mathcal{J},\ \operatorname{ran} U = \operatorname{ran} D|_{\mathcal{K}} = \mathcal{E}, \tag{3.55}$$
$$\ker(T^*T + UU^*) = \operatorname{ran} D \cap \ker \Gamma_{\mathcal{K}} D^\bullet = \ker \Gamma_{\mathcal{K}} D^\bullet D = \mathcal{U}. \tag{3.56}$$

This completes the proof. ■

Having established the necessary Hodge decomposition, we now give the definition of the associated Z-problem and effective operator from which we will obtain our main result in this section on the relationship between the DtN map and this effective operator.

Definition 41 (Dirichlet Z-problem and effective operator) *The (Dirichlet) Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ associated with the Hilbert space $\mathcal{H} = \mathcal{F}(E_G, \mathbb{C})$ and the orthogonal triple decomposition of $\mathcal{H}$ in* (3.51) *is defined as follows: given $V_0 \in \mathcal{U}$, find $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ satisfying*

$$I_0 + I = \sigma(V_0 + V), \tag{3.57}$$

such a triple (I_0, V, I) is called a solution to the Z-problem at V_0. If there exists an *operator $\sigma_* \in \mathcal{L}(\mathcal{U})$ such that*

$$I_0 = \sigma_* V_0, \tag{3.58}$$

whenever $V_0 \in \mathcal{U}$ and (I_0, V, I) is a solution of the Z-problem at V_0, then σ_* is called an effective operator of this Z-problem.

Theorem 42 *Let (P_G, E_G, σ) be an electrical network with boundary and interior nodes, $P_{\partial G}$ and P_{G°, respectively. Then the effective operator σ_* of the Dirichlet Z-problem (in Definition 41) exists and is unique.*

Proof. By hypotheses $\sigma^* = \sigma \geq 0$. The result now follows immediately from Theorem 40 and Theorem 77. ∎

Remark 43 *Although all we need in this section is Theorem 42 much more can be said using Theorem 77. Specifically, we can obtain the solution of the Dirichlet Z-problem by a generalized principal pivot transform, the representation of the effective operator σ_* by a generalized Schur complement and bounds on σ_*. It is also possible to weaken the hypotheses on the network conductivity σ so that $\sigma^* = \sigma$, $\sigma_{11} \geq 0$ and $\ker \sigma_{11} \subseteq \ker \sigma_{01}$, where the 3×3 block operator $\sigma = [\sigma_{ij}]_{i,j=0,1,2}$ is with respect to the Hodge decomposition in Theorem 40. However, we do not at this time have a physical model in mind where these weak hypotheses would be meaningful.*

We are now ready to state the main result of this section that relates the DtN map to the effective operator of the Dirichlet Z-problem. However, we require the following definition and lemma.

Definition 44 (The lift operator) *The lift operator $\Pi \in \mathcal{L}(\mathcal{F}(P_{\partial G}, \mathbb{C}), \mathcal{U})$ is the operator defined by*

$$\Pi(f) = Du, \quad \textit{for every } f \in \mathcal{F}(P_{\partial G}, \mathbb{C}), \tag{3.59}$$

where u is the solution to the discrete Dirichlet problem (3.39) with conductivity $\sigma = I$ (i.e., the identity operator) and $u|_{P_{\partial G}} = f$.

Lemma 45 *If G is a connected graph, then the lift operator is well defined.*

Proof. Let $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$. We begin by proving the claim that the constrained linear system of equations (3.46) with $\sigma = I$ has a unique solution. To prove this, we only need to show that $(K_I)_{11}$ as defined by (3.45) is invertible, where K_I is the normalized Kirchhoff operator defined in (3.35). As $\dim \mathcal{F}(P_{\partial G}, \mathbb{C}) < \infty$ it suffices to prove $\ker(K_I)_{11} = \{0\}$. First, by Lemma 31 $K_I = D^* D \geq 0$ and it follows by Lemma

68 that $\ker(K_I)_{11} \subseteq \ker(K_I)_{01}$. Let $g \in \ker(K_I)_{11}$. Hence, $g \in \ker(K_I)_{01}$ which implies $(K_I)_{11}g = 0$ and $(K_I)_{01}g = 0$. Define $u \in \mathcal{F}(P_G, \mathbb{C})$ by

$$u(x) = \begin{cases} 0, \ x \in \ P_{\partial G}, \\ g(x), \ x \in P_{G^\circ}. \end{cases} \tag{3.60}$$

Then,

$$\begin{bmatrix} (K_I u)|_{P_{\partial G}} \\ (K_I u)|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} (K_I)_{00} & (K_I)_{01} \\ (K_I)_{10} & (K_I)_{11} \end{bmatrix} \begin{bmatrix} u|_{P_{\partial G}} \\ u|_{P_{G^\circ}} \end{bmatrix} = \begin{bmatrix} (K_I)_{00} & (K_I)_{01} \\ (K_I)_{10} & (K_I)_{11} \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} \tag{3.61}$$

$$= \begin{bmatrix} (K_I)_{01}g \\ (K_I)_{11}g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.62}$$

which implies $K_I u = 0$. Hence,

$$0 = (K_I u, u)_{\mathcal{F}(P_G,\mathbb{C})} = (D^* Du, u)_{\mathcal{F}(P_G,\mathbb{C})} = (Du, Du)_{\mathcal{F}(E_G,\mathbb{C})} = ||Du||^2_{\mathcal{F}(E_G,\mathbb{C})}, \tag{3.63}$$

implying $Du = 0$, that is, $u \in \ker D$. As G is a connected graph, then by Lemma 33 we have $\ker D = \operatorname{span}\{1_{P_G}\}$. As $P_{\partial G} \neq \emptyset$ then there exists $x \in P_{\partial G}$ so that $u(x) = 0$ and thus since u is also constant, it follows $u(x) = 0$ for all $x \in P_G$. Therefore, $g = 0$ and this proves $\ker(K_I)_{11} = \{0\}$, which proves the claim.

We now will prove that the function $\Pi : \mathcal{F}(P_{\partial G}, \mathbb{C}) \to \mathcal{U}$ defined by (3.59) is well-defined. Let $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$. Then from the previous claim there exists a unique solution u to the discrete Dirichlet problem (3.39) with conductivity $\sigma = I$ (i.e., the identity operator) and $u|_{P_{\partial G}} = f$. This implies, that u is in the domain of D and since it satisfies the discrete Dirichlet problem with conductivity $\sigma = I$, then $(D^\bullet Du)|_{P_{G^\circ}} = 0$ so that $Du \in \mathcal{U}$. It now follows immediately by uniqueness of the solution to that discrete Dirichlet problem that Π is well defined.

It remains to prove that $\Pi \in \mathcal{L}(P_{\partial G}, \mathbb{C})$. Since the Hilbert spaces $\mathcal{F}(P_{\partial G}, \mathbb{C})$ and $\mathcal{U}$ are finite-dimensional it suffices to prove Π is linear. Let $f_1, f_2 \in \mathcal{F}(P_{\partial G}, \mathbb{C})$ and $c \in \mathbb{C}$. Then there exist unique solution u_i to the discrete Dirichlet problem with conductivity $\sigma = I$ and $u_i|_{P_{\partial G}} = f_i$, for $i = 1, 2$. Then it follows that $u = cu_1 + u_2$ is a solution of the discrete Dirichlet problem with conductivity $\sigma = I$ and $u|_{P_{\partial G}} = cf_1 + f_2$. By the well definedness of the function Π and linearity of D we have $\Pi(cf_1 + f_2) = D(cu_1 + u_2) = cD(u_1) + D(u_2) = c\Pi(f_1) + \Pi(f_2)$. This proves that Π is linear which completes the proof of this lemma. ∎

We now prove the main result of this section on the relationship between the DtN map for the discrete Dirichlet problem and the effective operator of the Dirichlet Z-problem utilizing the lift operator.

Theorem 46 (the DtN map as an effective operator) *Let (P_G, E_G, σ) be an electrical network (in Def. 34) with connected graph $G = (P_G, E_G)$, boundary nodes $P_{\partial G}$, and interior nodes P_{G° satisfying* (3.38). *Then the DtN map Λ_σ (in Def. 37), the ef-*

fective operator σ_* (in Def. 41) and the lift operator Π (in Def. 44) satisfy the identity

$$\Lambda_\sigma = \Pi^*\sigma_*\Pi. \tag{3.64}$$

Proof. Assume the hypotheses. Let $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$. Then by definition $\Pi(f) = Du_f \in \mathcal{U}$, where u_f is the solution to the discrete Dirichlet problem (i.e., the problem in Definition 41) with conductivity $\sigma = I$ satisfying $u_f|_{P_{\partial G}} = f$. By Theorem 77 there is a solution to the Z-problem $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ at $V_0 = Du_f$, i.e.,

$$I_0 + I = \sigma(V_0 + V). \tag{3.65}$$

In particular, since $V \in \mathcal{E}$ there exists a $v \in \mathcal{F}(P_G, \mathbb{C})$ such that $v|_{P_{\partial G}} = 0$ and $V = Dv$. We claim that $u = u_f + v$ is a solution to the discrete Dirichlet problem for conductivity σ satisfying $u|_{P_{\partial G}} = f$. To prove this notice that,

$$u|_{P_{\partial G}} = (u_f + v)|_{P_{\partial G}} = f + 0 = f, \tag{3.66}$$

and since $I \in \mathcal{J} = \ker D^\bullet$ and $I_0 \in \mathcal{U}$ then,

$$0 = (D^\bullet I_0)|_{P_{G^\circ}} = (D^\bullet(I_0 + I))|_{P_{G^\circ}} = (D^\bullet\sigma(V_0 + V))|_{P_{G^\circ}} \tag{3.67}$$

$$= (D^\bullet\sigma D(u_f + v))|_{P_{G^\circ}} = (D^\bullet\sigma D(u_f + v))|_{P_{G^\circ}}. \tag{3.68}$$

This proves the claim. Because of this, $\Lambda_\sigma f = \Lambda_\sigma(u|_{P_{\partial G}}) = (-D^\bullet\sigma Du)|_{P_{\partial G}}$. Thus,

$$(f, \Pi^*\sigma_*\Pi f)_{\mathcal{F}(P_{\partial G},\mathbb{C})} = (\Pi f, \sigma_*\Pi f)_{\mathcal{F}(E_G,\mathbb{C})} = (Du_f, \sigma_* Du_f)_{\mathcal{F}(E_G,\mathbb{C})} \tag{3.69}$$

$$= (V_0 + V, \sigma(V_0 + V))_{\mathcal{F}(E_G,\mathbb{C})} = (D(u_f + v), \sigma D(u_f + v))_{\mathcal{F}(E_G,\mathbb{C})} \tag{3.70}$$

$$= (u_f + v, -D^\bullet\sigma D(u_f + v))_{\mathcal{F}(E_G,\mathbb{C})} = (u_f + v, (-D^\bullet\sigma D)(u_f + v))_{\mathcal{F}(P_G,\mathbb{C})} \tag{3.71}$$

$$= ((u_f + v)|_{P_{\partial G}}, (-D^\bullet\sigma Du)|_{P_{\partial G}})_{\mathcal{F}(P_{\partial G},\mathbb{C})} = (f, \Lambda_\sigma f)_{\mathcal{F}(P_{\partial G},\mathbb{C})}. \tag{3.72}$$

As this equality holds for all $f \in \mathcal{F}(P_{\partial G}, \mathbb{C})$ and since $\mathcal{F}(P_{\partial G}, \mathbb{C})$ is a complex Hilbert space, it follows (by the polarization identity) that $\Lambda_\sigma = \Pi\sigma_*\Pi$. This completes the proof. ■

3.2.3 Effective Conductivity/Resistance: The Z-problem and Effective Operator

Let (P_G, E_G, σ) be an electrical network (see Def. 34) with network conductivity $\sigma \in \mathcal{L}(\mathcal{F}(E_G, \mathbb{C}))$ satisfying $\sigma^* = \sigma \geq 0$.For the graph $G = (P_G, E_G)$, we assume the node set P_G has at least two nodes p, q, that is,

$$\{p, q\} \subseteq P_G, \;\; p \neq q. \tag{3.73}$$

The boundary nodes $P_{\partial G}$ and interior nodes P_{G° of the graph G will be defined by

$$P_{\partial G} = \{p, q\}, \quad P_{G^\circ} = P_G \setminus P_{\partial G}. \tag{3.74}$$

The following definition is an extension of more classical definitions found in [37, 39–41]

Definition 47 (effective conductivity/resistance) *If there exists a scalar σ_{eff} such that*

$$j = \sigma_{eff}[u(p) - u(q)], \tag{3.75}$$

whenever $j \in \mathbb{C}$ and $u \in \mathcal{F}(P_G, \mathbb{C})$ is a solution of the equation

$$(-D^\bullet \sigma D)u = j(\delta_p - \delta_q), \tag{3.76}$$

then we call σ_{eff}, the effective conductivity, and its inverse $r_{eff} = \frac{1}{\sigma_{eff}}$, the effective resistance (with the convention $r_{eff} = \infty$, if $\sigma_{eff} = 0$). Here $\delta_p, \delta_q \in \mathcal{F}(P_G, \mathbb{C})$ are the delta functions on the nodes p and q (i.e., equal to zero except at node p, q, respectively, where it is equal to 1).

Lemma 48 (Hodge decomposition for effective conductivity) *The sets $\mathcal{U}$, $\mathcal{E}$, $\mathcal{J}$ are mutually orthogonal subspaces in the Hilbert space $\mathcal{F}(P_G, \mathbb{C})$, where*

$$\mathcal{U} = \{u \in \mathcal{F}(P_G, \mathbb{C}) : u(x) = 0, \forall\, x \in P_G \setminus \{p\}\}, \tag{3.77}$$
$$\mathcal{E} = \{u \in \mathcal{F}(P_G, \mathbb{C}) : u(p) = u(q) = 0\}, \tag{3.78}$$
$$\mathcal{J} = \{u \in \mathcal{F}(P_G, \mathbb{C}) : u(x) = 0, \forall\, x \in P_G \setminus \{q\}\}. \tag{3.79}$$

Furthermore,

$$\mathcal{F}(P_G, \mathbb{C}) = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{3.80}$$
$$\mathcal{U} = \operatorname{span}\{\delta_p\},\ \mathcal{E} = \operatorname{span}\{\delta_q\},\ \mathcal{J} = \operatorname{span}\{\delta_p\}^\perp \cap \operatorname{span}\{\delta_q\}^\perp. \tag{3.81}$$

Moreover, denoting the orthogonal projections onto $\mathcal{U}$, $\mathcal{J}$, $\mathcal{E}$ by $\Gamma_{\mathcal{U}}$, $\Gamma_{\mathcal{J}}$, $\Gamma_{\mathcal{E}}$, respectively, we have the following formulas as the left multiplication operators:

$$\Gamma_{\mathcal{U}} = \delta_p I_{\mathcal{F}(P_G,\mathbb{C})}, \quad \Gamma_{\mathcal{J}} = \delta_q I_{\mathcal{F}(P_G,\mathbb{C})}, \quad \Gamma_{\mathcal{E}} = (1 - \delta_p - \delta_q) I_{\mathcal{F}(P_G,\mathbb{C})}. \tag{3.82}$$

Proof. The proof follows immediately by the abstract Hodge decomposition (i.e., Theorem 21), where

$$\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{F}(P_G, \mathbb{C}),\ T^* = T = \delta_p I_{\mathcal{F}(P_G,\mathbb{C})},\ U^* = U = \delta_q I_{\mathcal{F}(P_G,\mathbb{C})}, \tag{3.83}$$
$$\operatorname{ran} T^* = \operatorname{span}\{\delta_p\} = \mathcal{U},\ \operatorname{ran} U = \operatorname{span}\{\delta_q\} = \mathcal{J}, \tag{3.84}$$
$$\ker(T^*T + UU^*) = \operatorname{span}\{\delta_p\}^\perp \cap \operatorname{span}\{\delta_q\}^\perp = \mathcal{E}. \tag{3.85}$$

The proof of the orthogonal projections is immediate by resolution of the identity and definition of delta functions. ■

Definition 49 *The Z-problem* $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, K_\sigma)$ *associated with the Hilbert space* $\mathcal{H} = \mathcal{F}(P_G, \mathbb{C})$ *and the orthogonal triple decomposition of* $\mathcal{H}$ *in* (3.80) *and with the Kirchhoff operator* $K_\sigma = -D^\bullet \sigma D \in \mathcal{L}(\mathcal{F}(P_G, \mathbb{C}))$ *(cf. Def. 34) is defined as follows: given* $v_0 \in \mathcal{U}$*, find* $(\rho_0, v, \rho) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ *satisfying*

$$\rho_0 + \rho = K_\sigma(v_0 + v), \tag{3.86}$$

such a triple (ρ_0, v, ρ) *is called a solution to the Z-problem at* v_0*. If there exists an operator* $(K_\sigma)_* \in \mathcal{L}(\mathcal{U})$ *such that*

$$\rho_0 = (K_\sigma)_* v_0, \tag{3.87}$$

whenever $v_0 \in \mathcal{U}$ *and* (ρ_0, v, ρ) *is a solution of the Z-problem at* v_0*, then* $(K_\sigma)_*$ *is called an effective operator of this Z-problem.*

Theorem 50 (eff. conductivity as an eff. operator) *Suppose that* (P_G, E_G, σ) *is an electrical network (see, Def. 34) with conductivity* σ *satisfying* $\sigma^* = \sigma \geq 0$*,* $\sigma \neq 0$*, and with boundary and interior nodes,* $P_{\partial G} = \{p, q\}$ $(p \neq q)$ *and* $P_{G^\circ} = P_G \setminus P_{\partial G}$*, respectively. Then the effective conductivity* σ_{eff} *(see, Def. 47) exists, is unique and is related to the effective operator* $(K_\sigma)_*$ *(see, Def. 49) by*

$$\sigma_{eff} I_{\mathcal{U}} = (K_\sigma)_*, \tag{3.88}$$

where $I_{\mathcal{U}}$ *is the identity operator on* $\mathcal{U}$*, i.e.,* $(K_\sigma)_*$ *acts as left multiplication by* σ_{eff} *on* $\mathcal{U}$*.*

Proof. Assume the hypotheses. First, the effective operator $(K_\sigma)_*$ of the Z-problem (in Def. 49) exists and is unique by Theorem 77, since $K_\sigma^* = K_\sigma \geq 0$, which follows by the hypotheses $\sigma^* = \sigma \geq 0$ and Lemma 31.

We will consider two cases: $(K_\sigma)_* \neq 0$ and $(K_\sigma)_* = 0$. Let $c \in \mathbb{C} \setminus \{0\}$. Then take $v_0 = c\delta_p \in \mathcal{U}$ for which there exists a solution $(\rho_0, v, \rho) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ to the Z-problem (in Def. 49) satisfying

$$c_1\delta_p - c_2\delta_q = \rho_0 + \rho = K_\sigma(c\delta_p + v), \tag{3.89}$$

where $\rho_0 = c_1\delta_p$ and $\rho = -c_2\delta_q$ for some $c_1, c_2 \in \mathbb{C}$ and hence

$$(K_\sigma)_*(c\delta_p) = c_1\delta_p. \tag{3.90}$$

Thus, since $c \neq 0$ we can conclude that

$$(K_\sigma)_* = \frac{c_1}{c} I_{\mathcal{U}}. \tag{3.91}$$

It also follows,

$$c_1\delta_p - c_2\delta_q \in \operatorname{ran} K_\sigma = (\ker K_\sigma)^\perp, \tag{3.92}$$

and, in particular, since $1_{P_G} \in \ker K_\sigma$, where 1_{P_G} is the identity function defined on P_G (see, Def. 27 with $T = P_G$), hence

$$0 = (1_{P_G}, c_1\delta_p - c_2\delta_q)_{\mathcal{F}(P_G,\mathbb{C})} = c_1(1_{P_G}, \delta_p)_{\mathcal{F}(P_G,\mathbb{C})} - c_2(1_{P_G}, \delta_q)_{\mathcal{F}(P_G,\mathbb{C})} = c_1 - c_2 \tag{3.93}$$

which implies

$$c_1 = c_2. \tag{3.94}$$

Hence,

$$c_1(\delta_p - \delta_q) = K_\sigma(c\delta_p + v), \tag{3.95}$$

and $c\delta_p + v$ is a solution of (3.76) with constant c_1.

We will now consider the first case, i.e., suppose $(K_\sigma)_* \neq 0$. Then, by the above $0 \neq (K_\sigma)_* = \frac{c_1}{c} I_{\mathcal{U}}$ which implies $c_1 \neq 0$. Hence, for any $j_1 \in \mathbb{C}$, we have

$$j_1(\delta_p - \delta_q) = K_\sigma\left(\frac{j_1 c}{c_1}\delta_p + \frac{j_1}{c_1}v\right), \tag{3.96}$$

which implies

$$u_1 = \frac{j_1 c}{c_1}\delta_p + \frac{j_1}{c_1}v \tag{3.97}$$

is a solution to the linear equation (3.76) and

$$(K_\sigma)_* \frac{j_1 c}{c_1}\delta_p = j_1\delta_p. \tag{3.98}$$

Thus, as $c \neq 0$,

$$\frac{c_1}{c}[u_1(p) - u_1(q)] = \frac{c_1}{c}\left[\frac{j_1 c}{c_1} + \frac{j_1}{c_1}v - \frac{j_1}{c_1}v\right] = j_1. \tag{3.99}$$

In particular, we have proven that there always exists a solution to the linear equation (3.76) for any $j \in \mathbb{C}$.

Now let $j \in \mathbb{C}$ and $u \in \mathcal{F}(P_G, \mathbb{C})$ be any solution to (3.76) with constant j. We claim that

$$\frac{c_1}{c}[u(p) - u(q)] = j. \tag{3.100}$$

Then,

$$K_\sigma(v_0' + v') = K_\sigma(u - u(q)1_{P_G}) = K_\sigma u = j(\delta_p - \delta_q) = \rho_0' + \rho', \tag{3.101}$$

where

$$v_0' = [u(p) - u(q)]\delta_p \in \mathcal{U},\ \ \rho_0' = j\delta_p \in \mathcal{U}, \tag{3.102}$$
$$v' = u - u(q)1_{P_G} - [u(p) - u(q)]\delta_p \in \mathcal{E},\ \ \rho' = -j\delta_q \in \mathcal{J}. \tag{3.103}$$

This is the Z-problem $(\mathcal{F}(P_G, \mathbb{C}), \mathcal{U}, \mathcal{E}, \mathcal{J}, K_\sigma)$ (from Def. 49) with $v_0 = v_0'$ and $(\rho_0 = \rho_0', v = v', \rho = \rho') \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ is a solution at v_0'. As (3.91), it follows that

$$\frac{c_1}{c}[u(p) - u(q)]\delta_p = \frac{c_1}{c}v_0' = (K_\sigma)_* v_0' = \rho_0' = j\delta_p, \tag{3.104}$$

which proves the claim. This also proves that σ_{eff} exists, is unique,

$$\sigma_{eff} = \frac{c_1}{c} \tag{3.105}$$

and hence $\sigma_{eff} I_{\mathcal{U}} = (K_\sigma)_*$ in the case $(K_\sigma)_* \neq 0$.

Now consider the second case, i.e., $(K_\sigma)_* = 0$. Then, (3.96)-(3.103) still holds and we conclude that $(K_\sigma)_* v_0' = \rho_0' = j\delta_p$ from which we conclude that we must have $j = 0$. Hence, $0[u(p) - u(q)] = j$. This proves that σ_{eff} exists, is unique,

$$\sigma_{eff} = 0 \tag{3.106}$$

and hence $\sigma_{eff} I_{\mathcal{U}} = (K_\sigma)_* = 0$.

Therefore, since the statement is true for all cases, this completes the proof. ■

3.3 Conductivity for Periodic Linear Graphs

In this section, we study the discrete analog of the periodic conductivity equation (cf. Sect. 3.1) for linear graphs. Although we are still in a discrete setting, we must treat new complications that arise when one moves from finite linear graphs to infinite linear graphs. In order to gain a clear insight into this transition, we only consider Cartesian graphs.

Now there are some similarities and some changes that occur when transitioning from Sections 3.1 and 3.2. First, we continue to use the function spaces, $\mathcal{F}(\mathcal{T}, \mathbb{C})$ but now they are infinite-dimensional and we lose the inner product $(\cdot,\cdot)_{\mathcal{F}(T,\mathbb{C})}$. Also, the gradient D and divergence $D^\bullet$ are still well defined linear operators, but we lose their adjoint relationship. Instead, we replace these with their periodic counterparts $\mathcal{F}_\#(T, \mathbb{C})$, $D_\#$, and $(D^\bullet)_\#$. For the function space $\mathcal{F}_\#(T, \mathbb{C})$, it is a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{F}_\#(T,\mathbb{C})}$ and we recover the adjoint relationship between $D_\#$ and $(D^\bullet)_\#$, namely, $D_\#^* = -(D^\bullet)_\#$. Then we able to get a Hodge decomposition and an associated Z-problem in this setting, in analogy with the continuum problem under these periodic spaces and operators. In addition, we obtain an analogous relationship between the effective operator σ_* and the effective conductivity σ_{eff}.

3.3.1 The Lattice Z-problem and Effective Operator

Fix a d-tuple of positive integers $(\tau_1, \ldots, \tau_d) \in \mathbb{N}^d$ and consider the lattice

$$L = \left\{ \sum_{i=1}^{d} n_i a_i : n_1, \ldots, n_d \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^d, \tag{3.107}$$

spanned by the d primitive lattice vectors

$$\{a_i : a_i = \tau_i \mathbf{e}_i,\ i = 1, \ldots, d\}, \tag{3.108}$$

where $\mathbf{e}_i$ denotes the ith standard basis vector, i.e., the d-tuple in $\mathbb{Z}^d$ whose ith coordinate is 1 and all others are 0. Consider the linear directed graph $G = (\mathbb{Z}^d, E)$ with node set $\mathbb{Z}^d$ and directed edge set

$$E = \{\overrightarrow{v\,(v + \mathbf{e}_i)} : v \in \mathbb{Z}^d, i = 1, \ldots, d\}, \tag{3.109}$$

that is, each edge $e = \overrightarrow{v\,(v + \mathbf{e}_i)}$ is a directed line segment with initial node $e_- = v$ and terminal node $e_+ = v + \mathbf{e_i}$. In particular, if $e_1, e_2 \in E$ then $e_1 = e_2$ if and only if $(e_1)_\pm = (e_2)_\pm$.

Notice that, for each edge $e \in E$ and lattice vector $R \in L$, there is a unique edge $e + R \in E$ defined by translation (see Figure 3.2) as

$$e + R = \overrightarrow{(e_- + R)(e_+ + R)} \in E. \tag{3.110}$$

In particular, G becomes a *periodic digraph* under the lattice L.

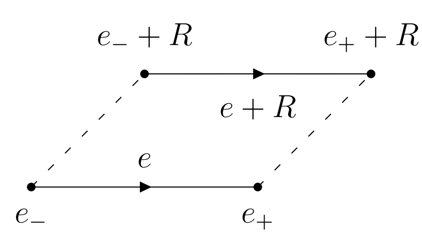


Figure 3.2: For a given edge $e \in E$ in the periodic digraph G and a lattice vector $R \in L$, the translation $e + R$ is another edge in E.

Definition 51 (primitive unit cell) *A primitive unit cell (fundamental domain) with respect to the lattice L is a subset $\mathcal{S} \subset \mathcal{T} \in \{\mathbb{Z}^d, E\}$ satisfying the properties* *(i)* $\mathcal{S} \neq \emptyset$, *(ii)* $\mathcal{T} = \mathcal{S} + L$, *(iii)* $(s, R) \in \mathcal{S} \times (L \setminus \{0\}) \Rightarrow s + R \notin \mathcal{S}$.

In particular, a primitive unit cell P for the nodes $\mathbb{Z}^d$ is given by

$$P = \mathbb{Z}^d \cap \prod_{k=1}^{d} [0, \tau_k) = \{p_1, \ldots, p_{|P|}\}, . \tag{3.111}$$

Similarly, a primitive unit cell E_P^- for the edges E is given by

$$E_P^- = \{e \in E : e_- \in P\} = \{e_1, \ldots, e_{|E_P^-|}\}. \tag{3.112}$$

An illustrated example of this construction can be found in Figure 3.3.

(a) $d = 1$, $\tau = 2$.

(b) $d = 2$, $\tau = (2, 2)$.

Figure 3.3: The periodic digraph G is generated by translation of a unit cell with node unit cell P (red) and edge unit cell E_P^- (blue).

Observe, the cardinalities of both of these primitive unit cells are nonzero and finite, i.e.,

$$0 < |P| < \infty \text{ and } 0 < |E_P^-| < \infty, \tag{3.113}$$

and the in-degree and out-degree of each node in the graph is finite and equal, i.e.,

$$0 < \deg_{out}(p) = \sum_{\substack{e \in E, \\ e_- = p}} 1 = \sum_{\substack{e \in E, \\ e_+ = p}} 1 =: \deg_{in}(p) < \infty, \quad \text{for each } p \in \mathbb{Z}^d. \tag{3.114}$$

Definition 52 (L-periodic function space) *Let $\mathcal{T}$ be an arbitrary nonempty set. We define the subspace of $\mathcal{F}(\mathcal{T}, \mathbb{C})$, containing all periodic functions with respect to a lattice L, by*

$$\mathcal{F}_{\#}(\mathcal{T}, \mathbb{C}) = \{f \in \mathcal{F}(\mathcal{T}, \mathbb{C}) : f(t + R) = f(t), \ \forall (t, R) \in \mathcal{T} \times L\}. \tag{3.115}$$

A key difference between the space $\mathcal{F}(\mathcal{T}, \mathbb{C})$ and $\mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$ is that the former is finite-dimensional only if $\mathcal{T}$ is finite-dimensional, whereas the latter is always finite-dimensional, a consequence of (3.113) and the next lemma. In progressing, we abuse notation by referring to $\mathcal{S} \in \{P, E_P^-\}$ and $\mathcal{T} \in \{\mathbb{Z}^d, E\}$ with fixed order, i.e., if $\mathcal{S} = P$ then $\mathcal{T} = \mathbb{Z}^d$ and if $\mathcal{S} = E_P^-$ then $\mathcal{T} = E$ (and vice versa).

Lemma 53 *Let* $\mathcal{S} \in \{P, E_P^-\}$ *and* $\mathcal{T} \in \{\mathbb{Z}^d, E\}$ *with lattice* L. *For each* $f \in \mathcal{F}(\mathcal{S}, \mathbb{C})$, *define its periodic extension* $f^\# \in \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$ *by*

$$f^\#(s+R) = f(s), \ \ \forall (s,R) \in \mathcal{S} \times L. \tag{3.116}$$

Then the periodic extension map, i.e., the function

$$(\cdot)^\# : \mathcal{F}(\mathcal{S}, \mathbb{C}) \to \mathcal{F}_\#(\mathcal{T}, \mathbb{C}), \tag{3.117}$$

is well-defined and, moreover, it is an invertible linear transformation with inverse the restriction map

$$(\cdot)|_\mathcal{S} : \mathcal{F}_\#(\mathcal{T}, \mathbb{C}) \to \mathcal{F}(\mathcal{S}, \mathbb{C}), \tag{3.118}$$

defined, for each $f \in \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$, *by*

$$f|_\mathcal{S}(s) = f(s), \forall s \in \mathcal{S}. \tag{3.119}$$

In particular, the following vector spaces are isomorphic:

$$\mathcal{F}(\mathcal{S}, \mathbb{C}) \cong \mathcal{F}_\#(\mathcal{T}, \mathbb{C}) \cong \mathbb{C}^{|\mathcal{S}|}. \tag{3.120}$$

Proof. We begin by proving that the periodic extension map is well-defined. By Definition 51 (*ii*) we have $\mathcal{T} = \mathcal{S} + L$, that is for each $t \in \mathcal{T}$ there exist $(s, R) \in \mathcal{S} \times L$ such that $t = s + R$. Second, by Definition 51 (*iii*) if $s \in \mathcal{S}$ and $0 \neq R \in L$ then $s + L \notin \mathcal{S}$. Now suppose, $t, t' \in \mathcal{T}$ then $t = s + R$ for some $(s, R) \in \mathcal{S} \times L$ and $t' = s' + R'$ for some $(s', R') \in \mathcal{S} \times L$. Hence, if $t = t'$ then $s + R = s' + R'$ implying $s = s' + (R' - R)$ which by Definition 51 (*iii*) further implies $R' - R = 0$ so that $s = s'$. Together with $\mathcal{T} = \mathcal{S} + L$, this proves the well-definedness of the map $(\cdot)^\# : \mathcal{F}_(S, \mathbb{C}) \to \mathcal{F}(T, \mathbb{C})$.

Next we will prove that if $f \in \mathcal{F}(\mathcal{S}, \mathbb{C})$ then $f^\# \in \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$. Let $f \in \mathcal{F}(\mathcal{S}, \mathbb{C})$ and $(t, R) \in \mathcal{T} \times L$, then there is an $(s, R') \in \mathcal{S} \times L$ such that $t = s + R'$ and so $f^\#(t+R) = f^\#(s+R'+R) = f^\#(s) = f^\#(s+R') = f^\#(t)$ which proves $f^\# \in \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$. This proves the well-definedness of the map $(\cdot)^\# : \mathcal{F}(\mathcal{S}, \mathbb{C}) \to \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$.

We will now prove that the map $(\cdot)^\# : \mathcal{F}(\mathcal{S}, \mathbb{C}) \to \mathcal{F}_\#(\mathcal{T}, \mathbb{C})$ is linear. First, both $\mathcal{F}(\mathcal{S}, \mathbb{C})$ and $\mathcal{F}_\#(\mathcal{T}, \mathbb{C})$ are vectors spaces over the field $\mathbb{C}$ with addition and scalar multiplication for functions as usual. Next, let $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{C})$ and $(c, s, R) \in \mathbb{C} \times \mathcal{S} \times L$. Then,

$$(cf + g)^\#(s + R) = (cf + g)(s) = cf(s) + g(s) = cf^\#(s + R) + g^\#(s + R) \tag{3.121}$$

$$= (cf^\# + g^\#)(s + R). \tag{3.122}$$

This proves by Definition 51 (*ii*) that $(cf + g)^\# = cf^\# + g^\#$. Hence, the periodic extension map is linear.

Next, we will prove that the restriction map is well-defined. Let $f \in \mathcal{F}_\#(\mathcal{S}, \mathbb{C})$ and

$s \in \mathcal{S}$. Then, as $f : \mathcal{T} \to \mathbb{C}$ and $\mathcal{S} \subset \mathcal{T}$ it follows $g(s) \in \mathbb{C}$ implying $f|_{\mathcal{S}} : \mathcal{S} \to \mathbb{C}$ is a well-defined function in $\mathcal{F}(\mathcal{S}, \mathbb{C})$. Suppose $f = g$ for some $g \in \mathcal{F}(\mathcal{S}, \mathbb{C})$. then $f(s) = g(s)$ for all $s \in \mathcal{T}$ and in particular $f(s) = g(s)$ for all $s \in \mathcal{S}$ which implies $f|_{\mathcal{S}} = g|_{\mathcal{S}}$, This proves $(\cdot)|_{\mathcal{S}} : \mathcal{F}_{\#}(\mathcal{T}, \mathbb{C}) \to \mathcal{F}(\mathcal{S}, \mathbb{C})$ is well-defined.

Lastly we will prove that the periodic extension map and the restriction map are inverses of each other. Let $f \in \mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$. Then for each $t \in \mathcal{T}$ there exists $(s, R) \in \mathcal{S} \times L$ with $t = s + R$ and so

$$\{[(\cdot)^{\#} \circ (\cdot)|_{\mathcal{S}}](f)\}(t) = (f|_{\mathcal{S}})^{\#}(t) = (f|_{\mathcal{S}})^{\#}(s + R) \tag{3.123}$$
$$= f|_{\mathcal{S}}(s) = f(s) = f(s + R) = f(t). \tag{3.124}$$

This implies that $[(\cdot)^{\#} \circ (\cdot)|_{\mathcal{S}}](f) = f$. As $f \in \mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$ was arbitrary it follows $(\cdot)^{\#} \circ (\cdot)|_{\mathcal{S}} = I_{\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})}$. Now let $f \in \mathcal{F}(\mathcal{S}, \mathbb{C})$ and $s \in \mathcal{S}$. Then

$$\{[(\cdot)|_{\mathcal{S}} \circ (\cdot)^{\#}](f)\}(s) = f^{\#}|_{\mathcal{S}}(s) = f^{\#}(s) = f^{\#}(s + 0) = f(s), \tag{3.125}$$

which implies $[(\cdot)|_{\mathcal{S}} \circ (\cdot)^{\#}](f) = f$. Since $f \in \mathcal{F}(\mathcal{S}, \mathbb{C})$ was arbitrary it follows $(\cdot)|_{\mathcal{S}} \circ (\cdot)^{\#} = I_{\mathcal{F}(\mathcal{S},\mathbb{C})}$. Therefore, we have prove that the periodic extension map and restriction map are inverses of each other. The remainder of the proof follows immediately. ■

It follows by Proposition 28 and Lemma 53 that, for $\mathcal{S} \in \{P, E_P^-\}$, a basis for $\mathcal{F}(\mathcal{S}, \mathbb{C})$ is given by $\alpha_{\mathcal{S}}$ defined by 3.18. Consequently, we can apply $(\cdot)^{\#}$ to the elements of $\alpha_{\mathcal{S}}$ to obtain a basis $\alpha_{\mathcal{S}}^{\#}$ for $\mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$, i.e.,

$$\alpha_{\mathcal{S}}^{\#} = \left\{(\delta_{s_i})^{\#} : \delta_{s_i} \in \alpha_{\mathcal{S}}\right\}, \quad (\delta_{s_i})^{\#}(s_j + R) = \delta_{ij}, \tag{3.126}$$

for all $R \in L$ and $i, j = 1, \ldots, |\mathcal{S}|$. For brevity we will often denote $(\delta_{s_i})^{\#} \in \alpha_{\mathcal{S}}^{\#}$ by $\delta_{s_i}^{\#}$. The following lemma is a corollary of Proposition 28 and Lemma 53.

Lemma 54 *Let $\mathcal{S} \in \{P, E_P^-\}$ and $\mathcal{T} \in \{\mathbb{Z}^d, E\}$ with lattice L. Then the vector space $\mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$ in (3.115) is finite-dimensional and*

$$0 < \dim \mathcal{F}_{\#}(\mathcal{T}, \mathbb{C}) = |\mathcal{S}| < \infty. \tag{3.127}$$

Furthermore, every $f \in \mathcal{F}_{\#}(\mathcal{T}, \mathbb{C})$ is determined by its evaluation on $\mathcal{S}$, that is,

$$f = \sum_{i=1}^{|\mathcal{S}|} f_i \delta_{s_i}^{\#}, \tag{3.128}$$

where

$$f_i = f(s_i), \quad i = 1, \ldots, |\mathcal{S}|. \tag{3.129}$$

Moreover, $\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})$ is a finite-dimensional Hilbert space with inner product

$$(f,g)_{\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})}=\sum_{i=1}^{|\mathcal{S}|}\overline{f_i}g_i,\ \textit{for all } f,g\in\mathcal{F}_{\#}(\mathcal{T},\mathbb{C}), \tag{3.130}$$

and orthornormal basis $\alpha_{\mathcal{S}}^{\#}$ as defined in (3.126).

Proof. By Lemma 53, the periodic extension map is an isomorphism from $\mathcal{F}(\mathcal{S},\mathbb{C})$ to $\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})$ and since the $\dim\mathcal{F}(\mathcal{S},\mathbb{C})=|\mathcal{S}|$ this implies $\dim\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})=|\mathcal{S}|$. Moreover, the periodic extension map takes the basis of $\mathcal{F}(\mathcal{S},\mathbb{C})$ to a basis for $\mathcal{F}_{\#}(\mathcal{T},\mathbb{C})$. In particular, $\alpha_{\mathcal{S}}\mapsto\alpha_{\mathcal{S}}^{\#}$ under the periodic extension map. The rest of the corollary follows immediately from this. ■

Similarly to Definition 30, we define an analogue to the gradient and divergence operators with periodic boundary conditions (per. b.c.).

Definition 55 (analogy of ∇ and $\nabla\cdot$ with per. b.c.) *The function*

$$D_{\#}:\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})\to\mathcal{F}_{\#}(E,\mathbb{C}), \tag{3.131}$$

the restriction of D to periodic functions, is defined by

$$(D_{\#}f)(e)=(Df)(e)\ \textit{for all } (f,e)\in\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})\times E, \tag{3.132}$$

and the function

$$(D^{\bullet})_{\#}:\mathcal{F}_{\#}(E,\mathbb{C})\to\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C}), \tag{3.133}$$

the restriction of $D^{\bullet}$ to periodic functions, is defined by

$$[(D^{\bullet})_{\#}g](p)=(D^{\bullet}g)(p),\ \textit{for all } (g,p)\in\mathcal{F}_{\#}(E,\mathbb{C})\times\mathbb{Z}^d. \tag{3.134}$$

In contrast to our setting in Section 3.2, $E_G=E$ and $P_G=\mathbb{Z}^d$ for the lattice are not finite. However, the following lemma shows that D, $D_{\#}$, $D^{\bullet}$ and $(D^{\bullet})_{\#}$ are still well defined in this setting.

Lemma 56 *The functions D, $D_{\#}$, $D^{\bullet}$ and $(D^{\bullet})_{\#}$ are well-defined linear operators.*

Proof. Clearly, $\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})$ and $\mathcal{F}_{\#}(E,\mathbb{C})$ are subspaces of the vector spaces $\mathcal{F}(\mathbb{Z}^d,\mathbb{C})$ and $\mathcal{F}(E,\mathbb{C})$, respectively. It is also clear that D is a well-defined linear function and, since the elements of E have finite in-degree and out-degree [see (3.114)], $D^{\bullet}$ is also a well-defined linear function. Thus, to prove the lemma we need only show that $\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})$ and $\mathcal{F}_{\#}(E,\mathbb{C})$ are invariant subspaces of D and $D^{\bullet}$, respectively. For any $f\in\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})$ and $R\in L$, we have

$$\begin{aligned}&(Df)(e+R)=f((e+R)_+)-f((e+R)_-)=f(e_++R)-f(e_-+R)\\&=f(e_+)-f(e_-)=(Df)(e),\ \forall e\in E.\end{aligned}$$

Next, let $g \in \mathcal{F}_{\#}(E,\mathbb{C})$. Then, for every $p \in \mathbb{Z}^d$ and every $R \in L$, we have $-R \in L$ hence $g(e-R) = g(e)$ for every $e \in E$ and

$$\{e \in E : e_\pm = p + R\} = \{e - R \in E : (e-R)_\pm = p\} = \{e' \in E : (e')_\pm = p\},$$

which implies that

$$\begin{aligned}(D^\bullet g)(p+R) &= \sum_{\substack{e\in E,\\ e_-=p+R}} g(e) - \sum_{\substack{e\in E,\\ e_+=p+R}} g(e)\\ &= \sum_{\substack{e-R\in E,\\ (e-R)_-=p}} g(e-R) - \sum_{\substack{e-R\in E,\\ (e-R)_+=p}} g(e-R)\\ &= \sum_{\substack{e'\in E,\\ (e')_-=p}} g(e') - \sum_{\substack{e'\in E,\\ (e')_+=p}} g(e') = (D^\bullet g)(p).\end{aligned}$$

This completes the proof. ■

The following technical lemma is needed in establishing the adjoint relationship of D and $D^\bullet$.

Lemma 57 *Fix an* $e_i \in E_P^-$ *and* $p_j \in P$. *Define*

$$\mathbf{S} = \sum_{\substack{e\in E,\\ e_-=p_j}} \begin{cases} 1, & e \in e_i + L,\\ 0, & e \notin e_i + L, \end{cases} \quad \textit{and} \quad \mathbf{T} = \sum_{\substack{e\in E,\\ e_+=p_j}} \begin{cases} 1, & e \in e_i + L,\\ 0, & e \notin e_i + L. \end{cases} \tag{3.135}$$

Then

$$\mathbf{S} = \begin{cases} 1, & (e_i)_- = p_j,\\ 0, & (e_i)_- \neq p_j, \end{cases} \quad \textit{and} \quad \mathbf{T} = \begin{cases} 1, & (e_i)_+ \in p_j + L,\\ 0, & (e_i)_+ \notin p_j + L. \end{cases} \tag{3.136}$$

Proof. Suppose $\mathbf{S} \neq 0$. Then there exists an element in $A = \{e \in E : e_- = p_j, e \in e_i + L\}$. Let $e \in A$. Then $e = e_i + R$ for some $R \in L$ and $p_j = e_- = (e_i+R)_- = (e_i)_- + R$. Hence, as $p_j \in P$, $(e_i)_- \in P$ and $R \in L$ we have by Definition 51 (*iii*), $R = 0$. Thus, $(e_i)_- = p_j$. It follows

$$0 \neq \mathbf{S} \Rightarrow A \neq \emptyset \Rightarrow A = \{e_i\} \Rightarrow \mathbf{S} = 1.$$

Now suppose $\mathbf{S} = 0$. Then $A = \emptyset$ which implies $(e_i)_- \neq p_j$, otherwise $e_i \in A$. Hence

$$\mathbf{S} = \begin{cases} 1, & A \neq \emptyset,\\ 0, & A = \emptyset, \end{cases} = \begin{cases} 1, & (e_i)_- = p_j,\\ 0, & (e_i)_- \neq p_j, \end{cases}$$

since $A \neq \emptyset$ if and only if $(e_i)_- = p_j$ and $A = \emptyset$ if and only if $(e_i)_- \neq p_j$. This proves the first equality. It remains to prove the second equality. Suppose $B \neq \emptyset$. Then there exists an element in $B = \{e \in E : e_+ = p_j, e = e_i + L\}$. In particular, there exists $R \in L$ such that $e = e_i + R$ and $p_j = e_+ - (e_i + R)_+ = (e_i)_+ + R$. Now let $e' \in B$. Then, similarly there exists an $R' \in L$ such that $e' = e_i + R'$ and $p_j = (e')_+ = (e_i)_+ + R'$. However, this implies $p_j - R = (e_i)_+ = p_j - R'$. Hence, as $p_j \in P$ a primitive unit cell, $R = R' = 0$ so that $e' = e_i + R = e$. Thus $B = \{e\}$ and $\mathbf{T} = 1$. We will now prove that $B \neq \emptyset$ if and only if $(e_i)_+ \in p_j + L$. This will complete the proof. If $B \neq \emptyset$ then by the preceding argument $B = \{e\}$ where $e \in E$ and $p_j = e_+ = (e_i)_+ + R$ for some $R \in L$ implying $(e_i)_+ = p_j - R = p_j + (-R) \in p - j + L$ since $R \in L \Rightarrow (-R) \in L$ as L is a lattice. Conversely, suppose $(e_i)_+ \in p_j + L$. Then $(e_i)_+ = p_j + R$ for some $R \in L$ and we have

$$p_j = (e_i)_+ - R = (e_i - R)_+ = e_+ \text{ where } e = e_i - R \in E \cap (e_i + L).$$

In particular, $e \in B$. This completes the proof. ∎

Theorem 58 (analogy of $\nabla^* = -\nabla\cdot$ with per. b.c.) *The linear operators $D_\#$ and $-(D^\bullet)_\#$ are Hilbert space adjoints of each other,*

$$(D_\#)^* = -(D^\bullet)_\#, \quad [(D^\bullet)_\#]^* = -D_\#, \tag{3.137}$$

that is,

$$(D_\# f, g)_{\mathcal{F}_\#(E,\mathbb{C})} = (f, -(D^\bullet)_\# g)_{\mathcal{F}_\#(\mathbb{Z}^d,\mathbb{C})}, \tag{3.138}$$

$$\forall (f,g) \in \mathcal{F}_\#(\mathbb{Z}^d,\mathbb{C}) \times \mathcal{F}_\#(E,\mathbb{C})\,. \tag{3.139}$$

Proof. By known results in linear algebra (see Appendix) it is sufficient to examine the action of $D_\#$ and $(D^\bullet)_\#$ on the bases $\alpha_P^\#$ and $\alpha_{E_P^-}^\#$, respectively. Hence, for each i,j it follows that

$$(D_\#\delta_{p_j}^\#, \delta_{e_i}^\#)_{\mathcal{F}_\#(E,\mathbb{C})} = \sum_{k=1}^{|E_P^-|} \overline{(D_\#\delta_{p_j}^\#)_k}(\delta_{e_i}^\#)_k = (D_\#\delta_{p_j}^\#)_i = -\delta_{p_j}^\#[(e_i)_-] + \delta_{p_j}^\#[(e_i)_+] \tag{3.140}$$

$$= -\delta_{p_j}[(e_i)_-] + \delta_{p_j}^\#[(e_i)_+] = -\begin{cases} 1, & (e_i)_- = p_j \\ 0, & (e_i)_- \neq p_j \end{cases} + \begin{cases} 1, & (e_i)_+ \in p_j + L \\ 0, & (e_i)_+ \notin p_j + L \end{cases} \tag{3.141}$$

$$\overset{\text{Lem. 57}}{=} -\sum_{\substack{e\in E,\\ e_-=p_j}} \begin{cases} 1, & e \in e_i + L \\ 0, & e \notin e_i + L \end{cases} + \sum_{\substack{e\in E,\\ e_+=p_j}} \begin{cases} 1, & e \in e_i + L \\ 0, & e \notin e_i + L \end{cases} \tag{3.142}$$

$$= -\sum_{\substack{e\in E,\\ e_-=p_j}} \delta_{e_i}^\#(e) + \sum_{\substack{e\in E,\\ e_+=p_j}} \delta_{e_i}^\#(e) = [-(D^\bullet)_\#\delta_{e_i}^\#]_j \tag{3.143}$$

$$= \sum_{k=0}^{|P|-1} \overline{(\delta_{p_j}^\#)_k}[-(D^\bullet)_\#\delta_{e_i}^\#]_k = (\delta_{p_j}^\#, -(D^\bullet)_\#\delta_{e_i}^\#)_{\mathcal{F}_\#(\mathbb{Z}^d,\mathbb{C})}, \tag{3.144}$$

which implies

$$(\delta^{\#}_{p_j}, -(D^{\bullet})_{\#}\delta^{\#}_{e_i})_{\mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C})} = (D_{\#}\delta^{\#}_{p_j}, \delta^{\#}_{e_i})_{\mathcal{F}_{\#}(E,\mathbb{C})}. \tag{3.145}$$

Therefore, $(D_{\#})^* = -(D^{\bullet})_{\#}$. ∎

Definition 59 (cell average of periodic edge functions) *The (cell) average of a periodic edge function $f \in \mathcal{F}_{\#}(E,\mathbb{C})$, denoted by $\langle f\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})}$, is the constant function*

$$\langle f\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})} = \left(\frac{1}{|E_P^-|}\sum_{i=1}^{|E_P^-|} f_i\right) 1_E \in \mathcal{F}_{\#}(E,\mathbb{C}), \tag{3.146}$$

and the (cell) average operator $\Gamma_0 \in \mathcal{L}(\mathcal{F}_{\#}(E,\mathbb{C}))$ is given by

$$\Gamma_0(f) = \langle f\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})},\ \forall f \in \mathcal{F}_{\#}(E,\mathbb{C}). \tag{3.147}$$

Theorem 60 (a Hodge decomposition on the lattice) *The sets $\mathcal{U}$, $\mathcal{E}$, $\mathcal{J}$ are mutually orthogonal subspaces in $\mathcal{F}_{\#}(E,\mathbb{C})$, where*

$$\mathcal{U} = \{U \in \mathcal{F}_{\#}(E,\mathbb{C}) : U = \langle U\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})}\}, \tag{3.148}$$

$$\mathcal{E} = \{D_{\#}u : u \in \mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C}), \langle D_{\#}u\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})} = 0\}, \tag{3.149}$$

$$\mathcal{J} = \{J \in \mathcal{F}_{\#}(E,\mathbb{C}) : (D^{\bullet})_{\#}J = 0, \langle J\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})} = 0\}. \tag{3.150}$$

Furthermore,

$$\mathcal{F}_{\#}(E,\mathbb{C}) = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}, \tag{3.151}$$

$$\mathcal{U} \overset{\perp}{\oplus} \mathcal{J} = \ker(D^{\bullet})_{\#},\ \mathcal{U} = \operatorname{ran}\Gamma_0,\ \mathcal{E} = \operatorname{ran} D_{\#}. \tag{3.152}$$

Proof. The proof follows immediately by Theorem 21 and Theorem 58, where

$$\mathcal{B} = \mathcal{C} = \mathcal{F}_{\#}(E,\mathbb{C}),\ \mathcal{A} = \mathcal{F}_{\#}(\mathbb{Z}^d,\mathbb{C}),\ T = -(D^{\bullet})_{\#},\ T^* = D_{\#},$$
$$U = U^* = \Gamma_0,\ \mathcal{U} = \operatorname{ran}\Gamma_0,\ \mathcal{E} = \operatorname{ran} D_{\#},$$
$$\mathcal{J} = \ker(D^{\bullet})_{\#} \cap \ker\Gamma_0.$$

∎

Definition 61 (lattice Z-problem and effective operator) *The lattice Z-problem $(\mathcal{H},\mathcal{U},\mathcal{E},\mathcal{J},\sigma)$, is the following problem associated with the inner product space $\mathcal{H} = \mathcal{F}_{\#}(E,\mathbb{C})$, and the orthogonal triple decomposition of $\mathcal{H}$ as*

$$\mathcal{H} = \mathcal{U}\overset{\perp}{\oplus}\mathcal{E}\overset{\perp}{\oplus}\mathcal{J}, \tag{3.153}$$

where $\mathcal{U}, \mathcal{E}, \mathcal{J}$ are given by (3.148), (3.149) and (3.150), respectively, and an operator $\sigma \in \mathcal{L}(\mathcal{H})$: Given $V_0 \in \mathcal{U}$, find triples $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ satisfying

$$I_0 + I = \sigma\left(V_0 + V\right), \tag{3.154}$$

and such triple (I_0, V, I) is called a solution of the Z-problem at V_0. If there exists an operator $\sigma_* \in \mathcal{L}(\mathcal{U})$ such that

$$I_0 = \sigma_* V_0, \tag{3.155}$$

whenever (I_0, V, I) is a solution of the Z-problem at V_0 then σ_* is called an effective operator for this Z-problem.

3.3.2 The Discrete Periodic Ohm's Law

Here we define the discrete analog to Ohm's law in the continuum problem (3.9) in Section 3.1.

Definition 62 (periodic Ohm's law and effective conductivity) *Let*

$$\sigma \in \mathcal{L}(\mathcal{F}_{\#}(E,\mathbb{C})),\ \sigma^* = \sigma \geq 0. \tag{3.156}$$

The periodic Ohm's law is

$$I = \sigma V, \tag{3.157}$$

$$I \in \ker D^{\bullet} \cap \mathcal{F}_{\#}(E,\mathbb{C}),\ V \in \operatorname{ran} D \cap \mathcal{F}_{\#}(E,\mathbb{C})\,. \tag{3.158}$$

In addition, the effective conductivity σ_{eff} is the linear operator $\sigma_{\mathit{eff}} \in \mathcal{L}(\mathcal{U})$ which satisfies

$$\langle I\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})} = \sigma_{\mathit{eff}}\langle V\rangle_{\mathcal{F}_{\#}(E,\mathbb{C})}, \tag{3.159}$$

whenever I and V satisfy the periodic Ohm's law.

Example 63 *Although every solution to the lattice Z-problem is a solution to the periodic Ohm's law (as we will prove in Theorem 65), the converse is not true in general as we show in this example. In particular, we give an example of a periodic digraph G, lattice L, a linear positive semi-definite operator $\sigma \in \mathcal{L}(\mathcal{F}_{\#}(E,\mathbb{C}))$, $V \in \operatorname{ran} D \cap \mathcal{F}_{\#}(E,\mathbb{C})$, $I \in \ker D^{\bullet} \cap \mathcal{F}_{\#}(E,\mathbb{C})$ such that $I = \sigma V$ with $I \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}$, but $V \notin \mathcal{U} \overset{\perp}{\oplus} \mathcal{E}$.*

Consider the periodic digraph G, on the lattice L, with directed edge set (3.109) for $d = 2$ and periodicity $\tau = (2, 2)$. Let $\sigma = I_{\mathcal{F}_{\#}(E,\mathbb{C})}$ be the identity operator on $\mathcal{F}_{\#}(E,\mathbb{C})$. Consider the function $u \in \mathcal{F}(\mathbb{Z}^d,\mathbb{C})$ defined by $u(i,j) = i - j$ for each $(i,j) \in \mathbb{Z}^2$. It is mapped by D to a periodic function

$$V = Du \in \operatorname{ran} D \cap \mathcal{F}_{\#}(E,\mathbb{C}), \tag{3.160}$$

since

$$V(e) = \begin{cases} 1, & \text{if } e||\mathbf{e}_1, \\ -1, & \text{if } e||\mathbf{e}_2. \end{cases} \Rightarrow V(e+R) = V(e) \text{ for all } (e,R) \in E \times L. \tag{3.161}$$

It suffices to show $I = V \in \mathcal{J}$, i.e., $\langle I \rangle_{\mathcal{F}_\#(E,\mathbb{C})} = 0$ and $D^\bullet I = 0$. To show this notice, for $i = 1, \ldots, 4$ we have $I(e_i) = 1$ and for $i = 4, \ldots, 8$ we have $I(e_i) = -1$ so that

$$\langle I \rangle_{\mathcal{F}_\#(E,\mathbb{C})} = \frac{1}{8}(I(e_1) + \cdots + I(e_8)) = 0. \tag{3.162}$$

Also,

$$-D^\bullet I = -\sum_{p' \sim p} u(p) - u(p') = u(i,j) - u(i-1,j) + u(i,j) - u(i,j-1) \tag{3.163}$$

$$+u(i+1,j) - u(i,j) + u(i,j+1) - u(i,j) = 0, \tag{3.164}$$

where $p' \sim p$ means p is connected to $p' \in P$ by an edge in E. In particular, this example illustrates that (3.166) in the following proposition need not be an equality, in general.

In light of the example above, we require the following lemma to prove (as we do in Theorem 65 below) that the effective operator of the lattice Z-problem (Def. 61) is equal to the effective conductivity for the periodic conductivity equation defined in Def. 62.

Lemma 64 *Let $\mathcal{U}, \mathcal{E}, \mathcal{J}$ denote the subspaces in (3.148), (3.149), and (3.150). Then*

$$\ker D^\bullet \cap \mathcal{F}_\#(E,\mathbb{C}) = \ker(D^\bullet)_\# = \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}, \tag{3.165}$$

$$\operatorname{ran} D \cap \mathcal{F}_\#(E,\mathbb{C}) \supseteq \mathcal{U} \overset{\perp}{\oplus} \mathcal{E}. \tag{3.166}$$

Proof. First, $D^\bullet$ and $(D^\bullet)_\#$ are equal in value on $\mathcal{F}_\#(E,\mathbb{C})$ which implies $\ker D^\bullet \cap \mathcal{F}_\#(E,\mathbb{C}) = \ker(D^\bullet)_\#$ and hence (3.165) then follows by Theorem 60. Second, since D and $D_\#$ are equal in value on $\mathcal{F}_\#(\mathbb{Z}^d,\mathbb{C})$ it follows immediately by the definition of $\mathcal{E}$ that $\mathcal{E} \subseteq \operatorname{ran} D \cap \mathcal{F}_\#(\mathbb{Z}^d,\mathbb{C})$. As the $\operatorname{ran} D \cap \mathcal{F}_\#(E,\mathbb{C})$ is a subspace of $\mathcal{F}_\#(E,\mathbb{C})$ and $\operatorname{span}\{1_E\} = \mathcal{U}$, it suffices to show that $1_E \in \operatorname{ran} D \cap \mathcal{F}_\#(E,\mathbb{C})$. Define the function $u_{\text{pos}} : \mathcal{F}(\mathbb{Z}^d,\mathbb{C}) \to \mathcal{F}(\mathbb{Z}^d,\mathbb{C})$ by

$$u_{\text{pos}}[(x_1,\ldots,x_d)] = \sum_{j=1}^{d} x_j, \text{ for all } (x_1,\ldots,x_d) \in \mathbb{Z}^d. \tag{3.167}$$

Then, for each $e \in E$ there exists a standard basis vector $\mathbf{e}_i$ such that $e_+ = e_- + \mathbf{e}_i$ so that

$$(Du)(e) = u_{\text{pos}}(e_+) - u_{\text{pos}}(e_-) = \sum_{i=1}^{d}(e_+)_j - (e_-)_j = \sum_{j=1}^{d}(e_- + \mathbf{e}_i)_j - (e_-)_j \tag{3.168}$$

$$= \sum_{j=1}^{d}(e_- + \mathbf{e}_i - e_-)_j = \sum_{j=1}^{d}(\mathbf{e}_i)_j = 1 = 1_E(e). \tag{3.169}$$

Therefore, $Du = 1_E \in \operatorname{ran} D \cap \mathcal{F}_\#(E, \mathbb{C})$. This completes the proof of (3.166). ■

Now we prove the main result of this section which is the discrete analog to the continuum identity (3.13) in Section 3.1.

Theorem 65 (effective conductivity is the effective operator) *If*

$$\sigma \in \mathcal{L}(\mathcal{F}_\#(E, \mathbb{C})),\ \sigma^* = \sigma \geq 0, \tag{3.170}$$

then effective conductivity σ_{eff} for the periodic Ohm's law (in Def. 62) equals the effective operator σ_* of the lattice Z-problem (in Def. 61), i.e.,

$$\sigma_{\mathit{eff}} = \sigma_*. \tag{3.171}$$

Proof. Assume the hypotheses. Suppose $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ satisfies the lattice Z-problem for some $V_0 \in \mathcal{U}$, so that

$$I_0 + I = \sigma(V_0 + V). \tag{3.172}$$

Then by Theorem 77 the effective operator σ_* exists and is unique, so that

$$I_0 = \sigma_* V_0. \tag{3.173}$$

On the other hand, as $V_0 + V \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{E}$ and $I_0 + I \in \mathcal{U} \overset{\perp}{\oplus} \mathcal{J}$, satisfy the periodic Ohm's law with $\langle V_0 + V\rangle_{\mathcal{F}_\#(E,\mathbb{C})} = V_0$ and $\langle I_0 + I\rangle_{\mathcal{F}_\#(E,\mathbb{C})} = I_0$ it follows that if σ_{eff} exists, then $I_0 = \sigma_{\text{eff}} V_0$, i.e., $\sigma_* V_0 = \sigma_{\text{eff}} V_0$. By Theorem 77 for every $V_0 \in \mathcal{U}$ there exists a solution to the lattice Z-problem. This proves that if σ_{eff} exists, then $\sigma_{\text{eff}} = \sigma_*$. Suppose $(I, V) \in (\ker D^\bullet \cap \mathcal{F}_\#(E, \mathbb{C})) \times (\operatorname{ran} D \cap \mathcal{F}_\#(E, \mathbb{C}))$ satisfies the periodic Ohm's law, so that

$$I = \sigma V. \tag{3.174}$$

Then, by Lemma 64 we have $I = I_0 + I'$ for some $(I_0, I') \in \mathcal{U} \times \mathcal{J}$. Since $V \in \operatorname{ran} D \cap \mathcal{F}_\#(E, \mathbb{C}) \subseteq \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$, we have $V = V_0 + V' + I''$ for some $(V_0, V', I'') \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$. It then follows that

$$I_0 + I' = \sigma(V_0 + V' + I''), \tag{3.175}$$

and so by Theorem 77 gives

$$I_0 = \sigma/\sigma_{22}(V_0 + V'), \tag{3.176}$$

where σ/σ_{22} is the Schur complement of $\sigma = [\sigma_{ij}]_{i,j=0,1,2}$ with respect to the space $\mathcal{J}$ in the orthogonal decomposition $\mathcal{F}_{\#}(E, \mathbb{C}) = \mathcal{U} \overset{\perp}{\oplus} \mathcal{E} \overset{\perp}{\oplus} \mathcal{J}$. On the other hand,

$$\sigma(V_0 + V') = I_0' + V'' + I''', \tag{3.177}$$

for some $(I_0', V'', I''') \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ and so again by Theorem 77 we have

$$\sigma/\sigma_{22}(V_0 + V') = I_0' + V'', \tag{3.178}$$

which implies $I_0 = I_0' + V''$. Since $I_0, I_0' \in \mathcal{U}$ and $V'' \in \mathcal{E}$ follows that $V'' = 0$ and $I_0 = I_0'$. Thus, we have the Z-problem

$$I_0 + I''' = \sigma(V_0 + V'), \tag{3.179}$$

and hence

$$I_0 = \sigma_* V_0. \tag{3.180}$$

This prove the effective conductivity σ_{eff} exists. Therefore $\sigma_{\text{eff}} = \sigma_*$. ■

Chapter 4

Variational Principles and Constrained Linear Equations

4.1 Introduction

In Section 4.2 we develop a unified framework for obtaining solutions to constrained linear equations and their associated variational principles. While mathematically interesting, our main motivation in this development is to obtain the effective operator for Z-problems whose operators are less well behaved (i.e., fall under the weaker hypotheses in Subsection 2.2.2). Further research in this regard is explored in Chapter 5. In addition, we prove an important property of positive semidefinite block operators used throughout Chapter 3.

In Section 4.3, we extend the solution of constrained linear equations of the type

$$X(u+v) = w, \tag{4.1}$$

for $X = X^*,\ X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{10}$. In addition, we obtain an analog to the Dirichlet minimization principle for a generalized Schur complement X/X_{11}. Although these results are known, the approach used to obtain them is new.

In Section 4.4, we further extend the solution of linear equations of the type

$$X(u+v) = (w+z), \tag{4.2}$$

for $X = X^*,\ X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{10}$. In addition, we obtain a new maximization principle for a generalized principal pivot transform $\operatorname{gppt}_1(X)$ under the same hypotheses. The solution of such equations is then used in Subsection 4.5 to obtain solutions to the Z-problems for such operators.

In section 4.5, we present the main result of this thesis. That is, we give the characterization of solutions of the Z-problem and its effective operator under the hypotheses $\sigma = \sigma^*,\ \sigma_{11} \geq 0$ and $\ker \sigma_{11} \subseteq \ker \sigma_{10}$. In the case $\sigma^* = \sigma \geq 0$, we also obtain a lower bound on the effective operator σ_*. This result is utilized throughout Chapter 3 when the existence of solutions and the existence/uniqueness of the effective operator are required.

4.2 A Unified Framework

In this section, we present a unified framework for obtaining solutions to constrained linear equations and their associated variational principles. We accomplish this by supposing that a linear operator X on a Hilbert space $\mathcal{H}$ has a certain block factorization. We also prove that for X positive semidefinite (i.e., $X \geq 0$) we have a certain kernel inclusion needed throughout Chapter 3, Section 4.3 and Section 4.4 of this chapter when considering the weakest hypotheses.

We begin by solving constrained linear equations under a general block factorization as follows.

Lemma 66 (solving constrained linear equations) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space with* $\dim \mathcal{H} < \infty$*. If* $X \in \mathcal{L}(\mathcal{H})$ *has the* 2×2 *block factorization,*

$$X = \begin{bmatrix} I_{\mathcal{H}_0} & Y^* \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ Y & I_{\mathcal{H}_1} \end{bmatrix}, \tag{4.3}$$

$$W \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_0),\ Y \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1),\ Z \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_1), \tag{4.4}$$

$$W^* = W \text{ and } Z \geq 0 \tag{4.5}$$

then, for each $u \in \mathcal{H}_0$*, the set of all solutions of the constrained linear equation*

$$X(u+v) = w, (w,v) \in \mathcal{H}_0 \times \mathcal{H}_1,$$

is given by

$$\{(w,v) \in \mathcal{H}_0 \times \mathcal{H}_1 : v = v_0 + v_1,\ v_1 \in \ker Z,\ v_0 = -Z^+ZYu,\ w = Wu\}.$$

Proof. Assume the hypotheses. Then

$$X(u+v) = w,\ \ (w,v) \in \mathcal{H}_0 \times \mathcal{H}_1 \Leftrightarrow \begin{bmatrix} W & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ Y & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}_0} & -Y^* \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} w \\ 0 \end{bmatrix} \tag{4.6}$$

$$\Leftrightarrow \begin{bmatrix} Wu \\ Z(Yu+v) \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix} \Leftrightarrow w = Wu,\ \ -ZYu = Zv \tag{4.7}$$

$$\underset{\text{Lemma 24 }(iii)}{\Leftrightarrow} v = v_0 + v_1,\ \ v_1 \in \ker Z,\ \ v_0 = -Z^+ZYu,\ \ w = Wu. \tag{4.8}$$

This completes the proof. ■

We now show that, for the same factorization, one obtains an associated minimization principle.

Theorem 67 (generalized minimization principle) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a finite-dimensional Hilbert space. If* $X \in \mathcal{L}(\mathcal{H})$ *has the* 2×2 *block factorization (4.3)-(4.5), then* W *is uniquely defined by the minimization principle*

$$(u, Wu) = \min_{v \in \mathcal{H}_1} (u + v, X(u + v)), \text{ for each } u \in \mathcal{H}_0. \tag{4.9}$$

Furthermore, for each $u \in \mathcal{H}_0$ *the set of minimizers is given by*

$$\arg\min_{v \in \mathcal{H}_1} (u + v, X(u + v)) = \{v \in \mathcal{H}_1 : v = v_0 + v_1,\ v_1 \in \ker Z,\ v_0 = -(Z^+ZY)u\}. \tag{4.10}$$

Proof. Assume the hypotheses. Then, for any $(u, v) \in \mathcal{H}_0 \times \mathcal{H}_1$, it follows that

$$(u + v, X(u + v)) = \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} I_{\mathcal{H}_0} & Y^* \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ Y & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right) \tag{4.11}$$

$$= \left(\begin{bmatrix} I_{\mathcal{H}_0} & Y^* \\ 0 & I_{\mathcal{H}_1} \end{bmatrix}^* \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} W & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ Y & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right) \tag{4.12}$$

$$= \left(\begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ Y & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} W & 0 \\ ZY & Z \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right) \tag{4.13}$$

$$= \left(\begin{bmatrix} u \\ Yu + v \end{bmatrix}, \begin{bmatrix} W(u) \\ Z(Yu + v) \end{bmatrix} \right) \tag{4.14}$$

$$= (u, Wu) + (Yu + v, Z(Yu + v)) \geq (u, Wu), \tag{4.15}$$

with equality if and only if $(Yu + v, Z(Yu + v)) = 0$. By hypothesis $Z \geq 0$ and we have

$$Z(Yu + v) = 0 \iff Z(Yu) + Zv = 0 \iff Zv = -Z(Yu) \tag{4.16}$$

$$\iff Z^+Zv = -Z^+ZYu \iff v = v_0 + v_1,\ v_1 \in \ker Z, v_0 = -Z^+ZYu. \tag{4.17}$$

Uniqueness follows from the fact that over a complex Hilbert space [which all Hilbert spaces in this thesis are by assumption (i)], a bounded linear operator is uniquely defined by its quadratic form (this is not true for real Hilbert spaces). ■

The preceding results are used in conjunction with the following lemma to extend solutions to the Z-problem and its effective operator to the weakest hypotheses.

Lemma 68 *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a finite-dimensional Hilbert space and suppose* $X = \left[X_{ij}\right]_{i,j=1,2} \in \mathcal{L}(\mathcal{H})$ *(i.e., a* 2×2 *block operator matrix). If* $X^* = X \geq 0$ *then*

$$\ker X_{11} \subseteq \ker X_{01} \ (\textit{i.e.,}\ \operatorname{ran} X_{10} \subseteq \operatorname{ran} X_{11}). \tag{4.18}$$

Proof. Assume the hypotheses. Let $x_1 \in \ker X_{11}$. Then,

$$\left(\begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \right) = \left(\begin{bmatrix} X_{01}x_1 \\ X_{11}x_1 \end{bmatrix}, \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \right) = (X_{01}x_1, 0) + (X_{11}x_1, x_1) = 0. \tag{4.19}$$

Since $X^* = X \geq 0$ and $(Xx, x) = 0$ it follows that $Xx = 0$. Hence,

$$0 = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} X_{01}x_1 \\ X_{11}x_1 \end{bmatrix} = \begin{bmatrix} X_{01}x_1 \\ 0 \end{bmatrix}, \tag{4.20}$$

which implies $X_{01}x_1 = 0$. Thus, $x_1 \in \ker X_{01}$, which proves $\ker X_{11} \subseteq \ker X_{01}$. By the properties of orthogonal complements, it follows

$$\operatorname{ran} X_{10} = \operatorname{ran} X_{01}^* = (\ker X_{01})^\perp \subseteq (\ker X_{11})^\perp = \operatorname{ran} X_{11}^* = \operatorname{ran} X_{11}. \tag{4.21}$$

This completes the proof. ■

4.3 Results on the gsc

In this section, we define the generalization of the Schur complement (see Def. 9) when invertibility is lost by means of the Moore-Penrose pseudoinverse. We then prove that the generalized Schur complement (gsc) has a block factorization consistent with Lemma 66. This allows us to obtain solutions to a constrained linear equation under weaker hypotheses and a generalization of the Dirichlet minimization principle. Although these results are not new, the approach presented to obtain them is.

We begin this section with the following definition of the generalized Schur complement.

Definition 69 (generalized Schur complement) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space with $\dim \mathcal{H} < \infty$ and $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix). The generalized Schur complement of X with respect to X_{11} is*

$$X/X_{11} = X_{00} - X_{01}X_{11}^+X_{10}. \tag{4.22}$$

Similarly, the generalized Schur complement of X with respect to X_{00} is

$$X/X_{00} = X_{11} - X_{10}X_{00}^+X_{01}. \tag{4.23}$$

We would like a block factorization with properties similar to the case when X is invertible (see Prop. 10). The following proposition provides exactly that.

Proposition 70 (gsc block factorization) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space with $\dim \mathcal{H} < \infty$ and $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix). If $X^* = X$*

then

$$X = [X_{ij}]_{i,j=0,1} = \begin{bmatrix} I_{\mathcal{H}_0} & (X_{11}^{+}X_{10})^{*} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} X/X_{11} & 0 \\ 0 & X_{11} \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ X_{11}^{+}X_{10} & I_{\mathcal{H}_1} \end{bmatrix}, \tag{4.24}$$

if and only if

$$\ker X_{11} \subseteq \ker X_{01}, \ (i.e., \ \operatorname{ran} X_{10} \subseteq \operatorname{ran} X_{11}). \tag{4.25}$$

In particular, if $X \geq 0$ then the above statements hold.

Proof. Suppose $X = X^{*}$. Then

$$\begin{bmatrix} I_{\mathcal{H}_0} & (X_{11}^{+}X_{10})^{*} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} X/X_{11} & 0 \\ 0 & X_{11} \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ X_{11}^{+}X_{10} & I_{\mathcal{H}_1} \end{bmatrix} \tag{4.26}$$

$$= \begin{bmatrix} I_{\mathcal{H}_0} & (X_{11}^{+}X_{10})^{*} \\ 0 & I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} X/X_{11} & 0 \\ X_{11}X_{11}^{+}X_{10} & X_{11} \end{bmatrix} \tag{4.27}$$

$$= \begin{bmatrix} X/X_{11} + (X_{11}X_{10})^{*}X_{11}X_{11}^{+}X_{10} & (X_{11}^{+}X_{10})^{*}X_{11} \\ X_{11}X_{11}^{+}X_{10} & X_{11} \end{bmatrix} \tag{4.28}$$

$$= \begin{bmatrix} X_{00} - X_{01}X_{11}^{+}X_{10} + X_{01}X_{11}^{+}X_{11}X_{11}^{+}X_{10} & X_{01}X_{11}^{+}X_{11} \\ X_{11}X_{11}^{+}X_{10} & X_{11} \end{bmatrix} \tag{4.29}$$

$$= \begin{bmatrix} X_{00} & X_{01}X_{11}^{+}X_{11} \\ X_{11}X_{11}^{+}X_{10} & X_{11} \end{bmatrix}, \tag{4.30}$$

which is equal to X if and only if $X_{01}X_{11}^{+}X_{11} = X_{01}$ and $X_{11}X_{11}^{+}X_{10} = X_{10}$. By Lemma 24 $(ii) - (iv)$ we have

$$\Gamma_{\operatorname{ran} X_{11}} = X_{11}X_{11}^{+} = X_{11}^{+}X_{11} = \Gamma_{\operatorname{ran} X_{11}^{*}}. \tag{4.31}$$

Hence, we have equality if and only if $\ker X_{01} \subseteq \ker X_{11}$ or, equivalently, $\operatorname{ran} X_{11} \subseteq \operatorname{ran} X_{01}$. By Lemma 68, if $X \geq 0$ then $\ker X_{11} \subseteq X_{01}$ and the statement holds. This completes the proof. ■

For further properties of the generalized Schur complement, we refer the reader to [30]. We now move on to applying the general solution of constrained linear equations obtained in Lemma 66 to the above decomposition.

Lemma 71 (solving constrained linear equations by gsc) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space with $\dim(\mathcal{H}) < \infty$ and $X \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix). If $X = X^{*}$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$, then for each $u \in \mathcal{H}_0$, the constrained linear equation*

$$X(u + v) = w, \ \ (w, v) \in \mathcal{H}_0 \times \mathcal{H}_1, \tag{4.32}$$

is solvable, and the set of all solutions is

$$\{(w,v) \in \mathcal{H}_0 \times \mathcal{H}_1 : \; v = v_0 + v_1, \; v_1 \in \operatorname{Ker} X_{11}, \; v_0 = -X_{11}^{+} X_{10} u, \; w = X/X_{11} u\}. \tag{4.33}$$

Proof. Suppose $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$. Then, by Proposition 70 X has the block factorization (4.24) and the proof follows immediately by Lemma 66, where

$$W = X/X_{11}, \; Z = Z_{11}, \; Y = X_{11}^{+} X_{10}. \tag{4.34}$$

■

Now that we know that such systems are solvable, we obtain the associated minimization principle for the generalized Schur complement directly by Theorem 67.

Theorem 72 (gsc minimization principle) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space with* $\dim(\mathcal{H}) < \infty$ *and* $X \in \mathcal{L}(\mathcal{H})$ *(i.e., a* 2×2 *block operator matrix)). If* $X = X^*$, $X_{11} \geq 0$ *and* $\ker X_{11} \subseteq \ker X_{01}$ *then* X/X_{11} *is uniquely defined by the minimization principle*

$$(u, (X/X_{11})u) = \min_{v \in \mathcal{H}_1} (u + v, X(u+v)), \; \textit{for each } u \in \mathcal{H}_0. \tag{4.35}$$

Furthermore, for each $u \in \mathcal{H}_0$ *the set of minimizers is given by*

$$\operatorname*{arg\,min}_{v \in \mathcal{H}_1} (u + v, X(u+v)) = \left\{ v : v = v_0 + v_1, \; v_1 \in \operatorname{Ker} X_{11}, \; v_0 = -(X_{11})^{+} X_{10} x \right\}. \tag{4.36}$$

In particular, if $X \geq 0$ *then the above statements hold and*

$$0 \leq X/X_{11} \leq X_{00}. \tag{4.37}$$

Proof. Suppose $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$. Then, by Proposition 70 X has the block factorization (4.24) and the proof follows immediately by Theorem 67 and Lemma 68, where

$$W = X/X_{11}, \; Z = Z_{11}, \; Y = X_{11}^{+} X_{10}, \tag{4.38}$$

and the lower bound follows for $X \geq 0$. ■

In the proofs of both preceding results, one only needs to establish that a suitable block factorization exists and make the necessary identifications. With some nuance, this will continue to be the case in the following section.

4.4 Results on the gppt

In this section, we define the generalization of the principal pivot transform (defined in 11) when invertibility is lost, using the Moore-Penrose pseudoinverse. We then prove that the generalized principal pivot transform (gppt) can be seen as a generalized Schur complement and therefore has the block factorization of Proposition 70. This factorization of the gppt is then used in conjunction with the unified framework of Section 4.2 to produce solutions to further constrained linear equations and a new maximization principle for the gppt.

We begin this section by introducing the reader to a generalization of the principal pivot transform (see 11), as follows.

Definition 73 (generalized principal pivot transform) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space with* $\dim \mathcal{H} < \infty$ *and* $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ *(i.e., a* 2×2 *block operator matrix). The generalized principal pivot transform of* X *with respect to* X_{11} *is the* 2×2 *block operator matrix* $\mathrm{gppt}_1(X) \in \mathcal{L}(\mathcal{H})$ *defined by*

$$\mathrm{gppt}_1(X) = \begin{bmatrix} X/X_{11} & X_{01}X_{11}^+ \\ -X_{11}^+X_{10} & X_{11}^+ \end{bmatrix}. \tag{4.39}$$

Similarly, the principal pivot transform of X *with respect to* X_{00} *is the* 2×2 *block operator matrix* $\mathrm{gppt}_0(X) \in \mathcal{L}(\mathcal{H})$ *defined by*

$$\mathrm{gppt}_0(X) = \begin{bmatrix} X_{00}^+ & -X_{00}^+X_{01} \\ X_{10}X_{00}^+ & X/X_{00} \end{bmatrix}. \tag{4.40}$$

For more information on the properties of the gppt, we recommend [31, 42]. Throughout the remainder of this thesis, we will focus on developments utilizing the $\mathrm{gppt}_1(X)$, as it is more connected to the structure of the Z-problem. A similar program can be applied to $\mathrm{gppt}_0(X)$ to obtain analogous results. With the following lemma, we relate the gppt to the gsc. This will then allow us to obtain the necessary block factorization by Proposition 70 to apply Lemma 66 and Theorem 67.

Lemma 74 (representing the gppt as a gsc) *Let* $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ *be a Hilbert space with* $\dim \mathcal{H} < \infty$ *and* $X = [X_{ij}]_{i,j=0,1} \in \mathcal{L}(\mathcal{H})$ *(i.e., a* 2×2 *block operator matrix). Then*

$$B/B_{11} = \mathrm{gppt}_1(X), \tag{4.41}$$

where $B \in \mathcal{L}(\mathcal{H} \overset{\perp}{\oplus} \mathcal{H}_1)$ *is the* 2×2 *block operator matrix*

$$B = \left[\begin{array}{c:c} B_{00} & B_{01} \\ \hdashline B_{10} & B_{11} \end{array}\right] = \left[\begin{array}{cc:c} X_{00} & 0 & X_{01} \\ 0 & 0_{\mathcal{H}_1} & X_{11}^+X_{11} \\ \hdashline X_{10} & -X_{11}X_{11}^+ & X_{11} \end{array}\right]. \tag{4.42}$$

Furthermore, if we define $J \in \mathcal{L}(\mathcal{H} \overset{\perp}{\oplus} \mathcal{H}_1)$ *and* $J_{00} \in \mathcal{L}(\mathcal{H})$ *as the* 2×2 *block operator matrices*

$$J = \begin{bmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{bmatrix} = \begin{bmatrix} J_{00} & 0 \\ 0 & I_{\mathcal{H}_1} \end{bmatrix}, \tag{4.43}$$

$$J_{00} = \begin{bmatrix} (J_{00})_{00} & (J_{00})_{01} \\ (J_{00})_{10} & (J_{00})_{11} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & -I_{\mathcal{H}_1} \end{bmatrix}, \tag{4.44}$$

then the 2×2 *block operator matrix* $JB \in \mathcal{L}(\mathcal{H} \overset{\perp}{\oplus} \mathcal{H}_1)$ *has the properties:*

$$JB = \left[\begin{array}{c:c} (JB)_{00} & (JB)_{01} \\ \hdashline (JB)_{10} & (JB)_{11} \end{array}\right] = \left[\begin{array}{cc:c} X_{00} & 0 & X_{01} \\ 0 & 0_{\mathcal{H}_1} & -X_{11}^{+}X_{11} \\ \hdashline X_{10} & -X_{11}X_{11}^{+} & X_{11} \end{array}\right] \tag{4.45}$$

and

$$(JB)/(JB)_{11} = J_{00}\,\mathrm{gppt}_1(X). \tag{4.46}$$

In addition, if $X = X^*$, $X_{11} \geq 0$ *and* $\ker X_{11} \subseteq \operatorname{ran} X_{01}$, *then*

$$(JB)^* = (JB), \tag{4.47}$$

$$[(JB)/(JB)_{11}]^* = (JB)/(JB)_{11}, \tag{4.48}$$

$$\ker(JB)_{11} \subseteq \operatorname{ran}(JB)_{01}. \tag{4.49}$$

In particular, if $X \geq 0$ *then the above statements hold.*

Proof. First, we prove that $B/B_{11} = \mathrm{gppt}_1(X)$. Hence,

$$B/B_{11} = \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01} \\ X_{11}^{+}X_{11} \end{bmatrix} X_{11}^{+} \begin{bmatrix} X_{10} & -X_{11}X_{11}^{+} \end{bmatrix} \tag{4.50}$$

$$= \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01}X_{11}^{+}X_{10} & -X_{01}X_{11}^{+}X_{11}X_{11}^{+} \\ X_{11}^{+}X_{11}X_{11}^{+}X_{10} & -X_{11}^{+}X_{11}X_{11}^{+}X_{11}X_{11}^{+} \end{bmatrix} \tag{4.51}$$

$$= \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01}X_{11}^{+}X_{10} & -X_{01}X_{11}^{+} \\ X_{11}^{+}X_{10} & -X_{11}^{+} \end{bmatrix} \tag{4.52}$$

$$= \begin{bmatrix} X_{00} - X_{01}X_{11}^{+}X_{10} & X_{01}X_{11}^{+} \\ -X_{11}^{+}X_{10} & X_{11}^{+} \end{bmatrix} \tag{4.53}$$

$$= \begin{bmatrix} X/X_{11} & X_{01}X_{11}^{+} \\ -X_{11}^{+}X_{10} & X_{11}^{+} \end{bmatrix} = \mathrm{gppt}_1(X). \tag{4.54}$$

Similarly, we prove $(JB)/(JB)_{11} = J_{00}\,\mathrm{gppt}_1(X)$. Hence,

$$(JB)/(JB)_{11} = \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01} \\ -X_{11}^{+}X_{11} \end{bmatrix} X_{11}^{+} \begin{bmatrix} X_{10} & -X_{11}X_{11}^{+} \end{bmatrix} \tag{4.55}$$

$$= \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01}X_{11}^{+}X_{10} & -X_{01}X_{11}^{+}X_{11}X_{11}^{+} \\ -X_{11}^{+}X_{11}X_{11}^{+}X_{10} & X_{11}^{+}X_{11}X_{11}^{+}X_{11}X_{11}^{+} \end{bmatrix} \tag{4.56}$$

$$= \begin{bmatrix} X_{00} & 0 \\ 0 & 0_{\mathcal{H}_1} \end{bmatrix} - \begin{bmatrix} X_{01}X_{11}^{+}X_{10} & -X_{01}X_{11}^{+} \\ -X_{11}^{+}X_{10} & X_{11}^{+} \end{bmatrix} \tag{4.57}$$

$$= \begin{bmatrix} X_{00} - X_{01}X_{11}^{+}X_{10} & X_{01}X_{11}^{+} \\ X_{11}^{+}X_{10} & -X_{11}^{+} \end{bmatrix} = \begin{bmatrix} X/X_{11} & X_{01}X_{11}^{+} \\ X_{11}^{+}X_{10} & -X_{11}^{+} \end{bmatrix} \tag{4.58}$$

$$= \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & -I_{\mathcal{H}_1} \end{bmatrix} \begin{bmatrix} X/X_{11} & X_{01}X_{11}^{+} \\ -X_{11}^{+}X_{10} & X_{11}^{+} \end{bmatrix} = J_{00}\,\mathrm{gppt}_1(X). \tag{4.59}$$

Upon inspection, one may also verify that (4.47)-(4.49) hold in the case that $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \operatorname{ran} X_{01}$. The case $X \geq 0$ follows immediately by Lemma 68. ■

Lemma 75 (solving constrained linear equations by gppt) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space with $\dim(\mathcal{H}) < \infty$ and $X \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix). If $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$ then for each $u \in \mathcal{H}_0$ the constrained linear equation*

$$X(u+v) = w+z, \quad (w,v,z) \in \mathcal{H}_0 \times \mathcal{H}_1 \times \mathcal{H}_1 \tag{4.60}$$

is solvable, and the set of solutions is

$$\{(w,v,z) \in \mathcal{H}_0 \times \mathcal{H}_1 \times \mathcal{H}_1 : v = r + v_1,\ v_1 \in \ker X_{11},\ z \in \operatorname{ran} X_{11},\ w + r = \mathrm{gppt}_1(X)(u+z)\}. \tag{4.61}$$

Proof. Suppose $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$. By Lemma 74 we have $(JB)/(JB)_{11} = J_{00}\,\mathrm{gppt}_1(X)$ satisfying Proposition 70. Hence, we need only to transform to the equivalent system under JB. It follows that

$$X(u+v) = w+z \iff JB(u+z+v) = w - X_{11}^{+}X_{11}v,\ z \in \operatorname{ran} X_{11}. \tag{4.62}$$

The remainder of the proof follows immediately by Lemma 66. ■ The following variational principle for the generalized principal pivot transform, we believe, is new even for the principal pivot transform.

Theorem 76 (gppt maximization principle) *Let $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$ be a Hilbert space with $\dim \mathcal{H} < \infty$ and $X \in \mathcal{L}(\mathcal{H})$ (i.e., a 2×2 block operator matrix). If $X = X^*$, $X_{11} \geq 0$ and $\ker X_{11} \subseteq \ker X_{01}$ then $\mathrm{gppt}_1(X)$ is uniquely defined by the maximization* *principle*

$$\frac{1}{2}\left(u+v,-J_{00}\,\mathrm{gppt}_1(X)(u+v)\right)=\max_{t\in\mathcal{H}_1}\left\{\mathrm{Re}\left(v,(X_{11}X_{11}^{+})t\right)-\frac{1}{2}\left(u+t,X(u+t)\right)\right\}, \tag{4.63}$$

for each $(u,v)\in\mathcal{H}_0\times\mathcal{H}_1$*. Furthermore, for each* $(u,v)\in\mathcal{H}_0\times\mathcal{H}_1$ *the set of maximizers is given by*

$$\begin{aligned}&\arg\max_{t\in\mathcal{H}_1}\left\{\mathrm{Re}\left(v,(X_{11}X_{11}^{+})t\right)-\frac{1}{2}\left(u+t,X(u+t)\right)\right\}\\&=\{t\in\mathcal{H}_1: t=t_0+t_1,\ t_1\in\ker X_{11},\ t_0=-X_{11}^{+}X_{10}u+X_{11}^{+}v\}.\end{aligned} \tag{4.64}$$

Proof. Suppose $X=X^*$, $X_{11}\geq 0$ and $\ker X_{11}\subseteq\ker X_{01}$. By Lemma 74 we have $(JB)/(JB)_{11}=J_{00}\,\mathrm{gppt}_1(X)$ satisfying Proposition 70. The remainder of the proof follows by Theorem 67. ■

4.5 Z-problem and Effective Operator: Main Results

We would like to take a moment to recall the Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ (defined in 2.21). Classically, the operator $\sigma \in \mathcal{L}(\mathcal{H})$ had to be positive semidefinite, self-adjoint and invertible for the effective operator σ_* to be uniquely defined and for the Z-problem to have a unique solution. We have weakened those hypotheses through the unified approach presented in Section 4.2 and the results in Sections 4.3 and 4.4. We use the following result coupled with Theorem 21 to characterize and solve the Z-problems presented in Chapter 3. Hence, this is the quintessential result of this thesis which allows for the solution of Z-problems and construction of the effective operator for relaxed σ. We hypothesize that the same techniques would work in the case of the dual Z-problem. However, the notion of duality under the loss of invertibility is more mathematically rich. For a further discussion on this, we refer the reader to Chapter 5.

Theorem 77 (solution of the Z-problem and effective operator) *Consider the Z-problem $(\mathcal{H}, \mathcal{U}, \mathcal{E}, \mathcal{J}, \sigma)$ with $\dim \mathcal{H} < \infty$. If*

$$\sigma = \sigma^*, \quad \sigma_{11} \geq 0, \quad \ker \sigma_{11} \subseteq \ker \sigma_{01}, \tag{4.65}$$

then the set of all solutions $(I_0, V, I) \in \mathcal{U} \times \mathcal{E} \times \mathcal{J}$ is given by

$$\{(I_0, V, I) : V = r + V_1,\ V_1 \in \ker \sigma_{11},\ I_0 + r + I = \operatorname{gppt}_1(\sigma)(V_0)\}, \tag{4.66}$$

where the gppt is with respect to the decomposition $\mathcal{H} = (\mathcal{U} \overset{\perp}{\oplus} \mathcal{J}) \overset{\perp}{\oplus} \mathcal{E}$, and the effective operator $\sigma_* \in \mathcal{L}(\mathcal{U})$ is given uniquely by the gsc

$$\sigma_* = \sigma_{00} - \sigma_{01}\sigma_{11}^+\sigma_{10} = \begin{bmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{bmatrix} / \sigma_{11}. \tag{4.67}$$

In particular, if $\sigma \geq 0$ then the above statements hold and

$$0 \leq \sigma_* \leq \sigma_{00}. \tag{4.68}$$

Proof. Suppose $\sigma = \sigma^*$, $\sigma_{11} \geq 0$ and $\ker \sigma_{11} \subseteq \ker \sigma_{01}$. Then, the proof follows immediately by Theorem 72 and Lemma 75, where

$$X(u + v) = w + z,\ \mathcal{H}_0 = \mathcal{U} \overset{\perp}{\oplus} \mathcal{J},\ \mathcal{H}_1 = \mathcal{E}, \tag{4.69}$$

$$X = \sigma,\ u = \begin{bmatrix} V_0 \\ 0 \end{bmatrix},\ v = V,\ w = \begin{bmatrix} I_0 \\ I \end{bmatrix},\ z = 0. \tag{4.70}$$

This completes the proof. ■

Chapter 5

Future Directions and Open Problems

We now conclude this thesis with a discussion of some future directions and open problems.

First, we are actively working on the analogy to the Thomson minimization principle in the case that σ is no longer invertible. My collaborators (Anthony Stefan, Robert Viator and Aaron Welters) and I, intend to include these results in a manuscript in preparation.

Second, we intend to relax our hypotheses to accommodate other classes of operators (e.g., sectorial [43]). This would be accomplished by extending the Gibiansky-Cherkaev-Milton method [44].

Our final step, will be to extend these results to infinite dimensional Hilbert spaces.

Throughout this process, we will continue to make progress on identifying Z-problems in the theory of composites, effective media theory, as well as other areas of science.

Outside of our main research program, there is the potential to further generalize the framework developed in Chapter 4. This interest is motivated by a 1998 paper [45] by Wei, in which he shows many pseduoinverses (e.g., the Drazin, Bott-Duffin and group inverses) can be obtained from a more general $A^{(2)}_{T,S}$ inverse. Development of our framework for this inverse may further extend the class of operators for which Z-problems have effective operators. It may also afford one studying such problems the choice of inverse to be employed in their solution.

We might also like to examine other physical problems within a discrete setting, where finite linear graph theory plays a role, such as discrete elasticity, viscoelasticity and even electromagnetism.

In a similar context, it could be interesting to develop the Hodge decompositions, the Z-problems and the effective operators in the lattice setting for more intricate graph topologies, e.g., non-Cartesian, metric and quantum graphs. One could also incorporate spin properties in order to discuss magnetic materials.

In all instances above, there is work to be done with regards to numerics and computation.

Beyond the Z-problem, it may be interesting to consider relaxations associated with the Y-problem and the Y-operator. For more on this see [1] and [2].

Appendix

In this section, we include a number of results useful in the derivation of adjoint relations. While these statements are well known, we had some difficulty finding them in a form conducive to the work in this thesis.

We begin by defining the adjoint between two inner product spaces, as opposed to the usual definition on a single inner product space.

Definition 78 (adjoint between inner product spaces) *Let V and W be inner product spaces over $\mathbb{C}$ with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, respectively. Let $T : V \to W$ be linear. Then any function $U : W \to V$ satisfying*

$$(T(v), w)_W = (v, U(w))_V \text{ for all } v \in V, w \in W,$$

*is called the **adjoint** of T which we denote by T^*.*

With the definition of an adjoint clearly in mind, we prove the following lemma regarding their existence and uniqueness.

Lemma 79 *Let $T : V \to W$ be a linear transformation between two inner product spaces V and W. If the adjoint of T exists then it is unique and linear.*

Proof. Let V, W be inner product spaces over a field F with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, respectively, and let $T : V \to W$ be linear. Suppose $U : W \to V$ and $S : W \to V$ are both adjoints of T. Then,

$$(v, U(w))_V = (T(v), w)_W = (v, S(w))_V, \text{ for each } (v, w) \in V \times W.$$

Hence,

$$0 = (v, U(w))_V - (v, S(w))_V = (v, U(w) - S(w))_V$$

and letting $v = U(w) - S(w)$ implies $U(w) = S(w)$ for all $w \in W$, hence $U = S$. This prove the uniqueness of the adjoint of T. We will now prove that $T^* \in \mathcal{L}(W, V)$ provided T^* exists. Note that we are considering the inner product to be conjugate linear in the second argument. Suppose the adjoint T^* of T exists. Then for all w_1,

$w_2 \in W$, $v \in V$ and for some $\lambda \in \mathbb{C}$,

$$\begin{aligned}
(v, T^*[\lambda w_1 + w_2])_V &= (Tv, \lambda w_1 + w_2)_W \\
&= (Tv, \lambda w_1)_W + (Tv, w_2)_W \\
&= \lambda (Tv, w_1)_W + (Tv, w_2)_W \\
&= \lambda (v, T^* w_1)_V + (v, T^* w_2)_V \\
&= (v, \lambda T^* w_1 + T^* w_2)_V
\end{aligned}$$

this implies $T^*[\lambda w_1 + w_2] = \lambda T^* w_1 + T^* w_2$. This proves that T^* is linear. ■

The following result is useful in the proof of Theorem 58.

Corollary 80 *Let V, W be finite dimensional inner product spaces over a field F with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_W$ and orthonormal bases $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, $\beta = \{\beta_1, \ldots, \beta_m\}$, respectively. Let $T : V \to W$ be linear. Then T^*, the adjoint of T, exists. In addition, if there exists a linear operator $U : W \to V$ satisfying*

$$(T(\alpha_i), \beta_j)_W = (\alpha_i, U(\beta_j))_V, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m \tag{5.1}$$

then $T^* = U$.

Proof. First, note that since α and β are orthonormal bases for V and W, respectively, we have that for each $v \in V$ and $w \in W$ they have unique representations

$$v = \sum_{i=1}^{n} a_i \alpha_i, \quad a_i \in F, \tag{5.2}$$

$$w = \sum_{j=1}^{m} b_j \beta_j, \quad b_j \in F. \tag{5.3}$$

Second, note that any linear function $T \in \mathcal{L}(V, W)$ is uniquely determined by its matrix representation $[T]_\alpha^\beta$, that is,

$$T(\alpha_j) = \sum_{i=1}^{m} c_{ij} \beta_i, \quad c_{ij} \in F \tag{5.4}$$

for $j = 1, \ldots, n$ and

$$[T]_\alpha^\beta = [c_{ij}].$$

Let $v \in V$ and $w \in W$. Then

$$\begin{aligned}(T(v), w)_W &= \left(T\left(\sum_{i=1}^{n} a_i\alpha_i\right), \sum_{j=1}^{m} b_j\beta_j\right)_W = \left(\sum_{i=1}^{n} a_iT(\alpha_i), \sum_{j=1}^{m} b_j\beta_j\right)_W \\ &= \left(\sum_{i=1}^{n} a_i\sum_{k=1}^{m} c_{ik}\beta_k, \sum_{j=1}^{m} b_j\beta_j\right)_W = \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{j=1}^{m} \overline{c_{ik}a_i}b_j\,(\beta_k, \beta_j)_W \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{c_{ij}a_i}b_j.\end{aligned}$$

Let $L \in \mathcal{L}(W, V)$ with the matrix representation $[L]_\beta^\alpha = [\overline{c_{ji}}]$ thus

$$L(\beta_j) = \sum_{i=1}^{n} \overline{c_{ij}}\alpha_i$$

for all $j = 1, \dots, m$. Then,

$$\begin{aligned}(v, L(w))_V &= \left(\sum_{i=1}^{n} a_i\alpha_i, L\left(\sum_{j=1}^{m} b_j\beta_j\right)\right)_V = \left(\sum_{i=1}^{n} a_i\alpha_i, \sum_{j=1}^{m} b_jL(\beta_j)\right)_V \\ &= \left(\sum_{i=1}^{n} a_i\alpha_i, \sum_{j=1}^{m} b_j\sum_{k=1}^{n}\overline{c_{kj}}\alpha_k\right)_V = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{n} \overline{c_{kj}a_i}b_j\,(\alpha_i, \alpha_k)_V \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} \overline{c_{ij}a_i}b_j\end{aligned}$$

Hence, the adjoint of T exists and is given by L. Suppose there exists $U \in \mathcal{L}(W, V)$ satisfying (5.1) and let $v \in V$ and $w \in W$. Then, using the representations (5.2) and (5.3), we have

$$\begin{aligned}(T(v), w)_W &= \left(T\left(\sum_{i=1}^{n} a_i\alpha_i\right), \sum_{j=1}^{m} b_j\beta_j\right)_W = \left(\sum_{i=1}^{n} a_iT(\alpha_i), \sum_{j=1}^{m} b_j\beta_j\right)_W \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{a_i}b_j\,(T(\alpha_i), \beta_j)_W = \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{a_i}b_j\,(\alpha_i, U(\beta_j))_V \\ &= \left(\sum_{i=1}^{n} a_i\alpha_i, \sum_{j=1}^{m} b_jU(\beta_j)\right)_V = (v, U(w))_V\,.\end{aligned}$$

Hence, $U = T^*$. ■

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