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Stability Analysis of Neutral Functional Differential Equations Arising in Partial Element Equivalent Circuit Models

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by

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ABSTRACT

Stability Analysis of Neutral Functional Differential Equations

Arising in Partial Element Equivalent Circuit Models

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Neutral Functional Differential Equations (NFDEs) arise in the study of the Partial Element Equivalent Circuit (PEEC) model with time delays. We present sufficient conditions for asymptotic stability and global stability in the delays of the PEEC NFDE's, using Lyapunov-Razumikhin function methods..

We develop, for the first time, a standard mixing-type nonlinearity for the PEEC NFDEs. Introducing time invariant and time varying nonlinear perturbation to the PEEC NFDEs, we develop sufficient conditions for stability of the nonlinear perturbed PEEC NFDEs and convergence of the nonlinear system to the original stable linear autonomous system. We also develop sufficient conditions for stability and convergence of the nonlinear perturbed PEEC NFDEs.

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List of Abbreviations and Notations

List of abbreviations and notations commonly used in this dissertation

C	$C = C([-\tau, 0], \mathbb{R}^n)$
$D(B,T)\phi$	$\phi(0) - \sum_{i=1}^{m} B_i \phi(-\tau_i)$. for $B = \{B_1, B_2, \ldots B_m\}$, $T = \{\tau_1, \tau_2, \ldots, \tau_m\}$
DDE	Delay Differential Equation
NFDE	Neutral Functional Differential Equation
NFDE (D, f)	$\frac{d}{dt} D(t, x_t) = f(t, x_t)$ with $D, f : \mathbb{R} \times C \to \mathbb{R}^n$
$NFDE(\sigma, \phi, f)$	$\frac{d}{dt} Dx_t = f(x_t)$, with $x_\sigma = \phi$
norms, $\lvert u \rvert$	3 types of norms are all expressed as $\lvert u \rvert$ with the definition depending on u: $\lvert u \rvert = \underset{-\tau \le \theta \le 0}{sup} \lvert u(\theta) \rvert, \lvert u(\theta) \rvert$ the Euclidean norm of vector $u(\theta)$ for $u \in C$ $\lvert u \rvert = \sqrt{\sum_{i=1}^{n} u_i^2}$, Euclidean norm. For $u \in \mathbb{R}^n$ $\lvert u \rvert = \underset{\lvert x \rvert \neq 0}{sup} \frac{\lvert ux \rvert_Y}{\lvert x \rvert_X}$, the operator norm for operator $u : X \to Y$
RFDE	Retarded Functional Differential Equation

Dedication

I thank God for being the rock on which I leaned, daily, and the light with which to see the world. As I worked to earn my PhD I received encouragement and support from my family and countless friends. I thank God for that and cannot list them all here.

I would like to thank my dissertation committee for their part in the long journey that has been my PhD degree. I have taken classes from Dr. Kozaitis, Dr. Dshalalow and Dr. Kiguradze and they have inspired me with the joy of learning. It is no small thing, it is what kept me going to complete my degree. Each of my committee members has also inspired me by the example of their individual achievements, "You can learn so much more."

My advisor, Dr. Bhaskar Gnana Tenali, I cannot thank enough. He was my instructor for my first graduate level math class and inspired me to learn beyond the pages of our Rudin textbook. I formed my first study group out of fear from the first exam in that class, a practice I kept for all my other professors which helped me learn more deeply. Thank you Dr. Tenali for your teaching, guidance, patience, "be a seeker of light", "I hope that you will listen, but not with the memory of what you already know; and this is very difficult to do..." - Jiddu Krishnamurti, what it means to be a mathematician, support and friendship. The biggest gift

I often overlooked, often took for granted and it was right in front of me: your time. You made clear that you gave your time freely but for this I owe a debt that I cannot fully repay. The only thing I can do is to continue to learn and find opportuniities to teach others.

To my wife Susan, you were the invisible fifth member of my committee. You make life worth living! You were the first one to hear each of my mathematical troubles and successes, even if you didn’t want to. No one but us two will know exactly how many math talks you listened to. You only said, ”That’s enough” a few times when it was true a thousand times. Thank you for your love, patience, support, filling out of forms, saying ”that’s due tomorrow” dozens of times, advice, time, distractions, more patience, faith and personal sacrifice. When my resolve grew thin you were the well from which I drew my strength. I am happy to share this achievement with you.

Chapter 1

Introduction

1.1 Motivation for Research

Computer technology is always looking to run at faster speeds meaning at higher signal frequencies. As the frequency of electromagnetic signals increases, a key question to be asked is, " Can the integrity of the signals be maintained when transmitting at such high frequencies through circuit board traces? " The stability of solutions to the governing electromagnetic equations is an important part of the answer to this question.

As clocking speeds increase, the time taken for the electromagnetic signals to travel between small distances can become significant relative to the period of the signal. As an example, when modeling a 10GHz signal the 10^{th} harmonic, 100GHz, may be of interest. At 100 GHz the wavelength is 3mm so electromagnetic interactions at mm distances will involve delays which are comparable to the period of this harmonic. Differential equations modeling 10 GHz signals in a circuit board will therefore need to include the time delays. Traditional electromagnetic analysis is done with the assumption that time delays are negligeable and can be ignored, resulting in ODEs while incorporation of time delays will result in Delay Differential Equations (DDEs).

When designing circuit boards to route signals at GHz and higher frequencies, industry standard layout rules exist as in [44] and [45]. These layout rules account for known issues with capacitive and inductive coupling of signals routed in nearby conductors, modeled using ODEs. These general layout rules do not consider the results of modeling electromagnetic interactions using Delay Differential Equations. Antonini and Pepe in [2], for the first time, have considered the stability of electromagnetic signals modeled using DDEs in the Partial Element Equivalent Circuit (PEEC) model. The specific DDE's studied are Neutral Functional Differential Equations (NFDEs).

The existing literature on PEEC NFDE stability analysis follows Lyapunov-Krasovskii functional methods which require a computationally intensive optimization step. In addition, the literature only covers linear autonomous NFDEs.

Here for the first time we present a stability analysis of PEEC NFDEs using Lyapunov-Razumikhin function methods. The Lyapunov-Razumikhin methods presented here do not require the computationally intensive optimization step and thus offer and advantage in numerical evaluation of stability of PEEC NFDEs.

As a linear model is often an idealization of a physical system, the effects of perturbations to the linear autonomous system are of interest. Some linear perturbations are briefly examined and a more in depth analysis is done for nonlinear perturbations. We examine mixing type nonlinear perturbations and provide sufficient conditions for stability and convergence of the solution of the nonlinear perturbed system to the solution of the linear autonomous system.

1.2 Outline of the Dissertation

In chapter 2 we begin by examining basic properties of Delay Differential Equations (DDEs). We examine existing literature relevant to the research presented in this dissertation and thus establish that the research presented here fills a gap in existing work. We list some basic properties of specific DDEs: Retarded Functional Differential Equations (RFDEs) as well as Neutral Functional Differential Equation (NFDEs). For the purposes of this outline we describe here examples of RFDEs and NFDEs, leaving detailed definitions for Chapter 2.

For a suitable interval, I, and continuous function $x(t) : I \to \mathbb{R}^n$ a linear time-invariant RFDE is of the form:

$\frac{d}{dt}x(t) = A_0x(t) + \sum_{k=1}^{m} A_kx(t - \tau_k)$.

A linear time-invariant NFDE is of the form:

$\frac{d}{dt}[x(t) - \sum_{k=1}^{m} B_kx(t - \tau_k)] = A_0x(t) + \sum_{k=1}^{m} A_kx(t - \tau_k)$.

A_k, B_k here are real, $n \times n$ matrices, $\tau > 0$, and delays $\tau_k \in [0, \tau]$.

This review of DDE literature establishes the need for the PEEC NFDE stability analysis that we present in Chapter 3. Specifically, existing research applies Lyapunov-Razumikhin stability analysis to Delay Differential Equations (DDEs) but no published research applies this method to the PEEC NFDEs as we present in Chapter 3. We show the development of PEEC NFDEs as in [2]. At the end of Chapter 2 we present an implementation of the existing PEEC NFDE stability analysis [2] which requires iteration until an optimization criteria is achieved. Such iterations require longer numerical computations compared to similar non-iterative methods and also may have undetermined computation time required to achieve optimization criteria. The research in [2] is a key example as other published PEEC NFDE stability analyses are based on [2]. These computation issues also establish the need for new, non-iterative PEEC NFDE stability analysis.

In chapter 3, we present sufficient conditions for the solution $x = 0$ of the PEEC NFDE's to be uniformly asymptotically stable and all solutions to approach zero as $t \to \infty$. We employ Lyapunov-Razumikhin stability analysis theorems, however the methods of this analysis alone do not allow determination of sufficient stability conditions, verifiable by direct calculation from PEEC NFDE parameters. We restrict PEEC NFDE time delays to be rational multiples of a common value and this allows us to cast the PEEC NFDEs as a sampled-time system. This allows us to use the theory of difference equations and we can calculate our sufficient stability conditions directly from PEEC NFDE parameters. A corollary is given which shows that for PEEC NFDEs which meet the stability conditions of Theorem 3.4.4, there exists a neighborhood G of the original delay vector T, such that if all other NFDE system parameters remain constant but a different delay vector, within neighborhood G is used, then theorem results still apply for such an NFDE. Finally, we show that if a PEEC NFDE meets our sufficient stability conditions then NFDEs are also stable with delays in some neighborhoods around each of the delay values of the original NFDE. Thus, our results also apply to these NFDEs that do not meet the restriction that time delays be rational multiples of a common value. An example NFDE is given which meets our sufficient conditions for stability and for all solutions to approach zero .

In chapter 4 we consider perturbations to the PEEC NFDEs. Some linear perturbations which arise from variations in manufactured printed circuit boards are examined. Next we consider nonlinear perturbations thus arise from nonlinear behavior of circuit board conductors. In existing literature such as [1], [2], [5] are considered only linear time invariant NFDEs for the PEEC model. As of the present date there have been no publications of the study of nonlinear NFDEs for the PEEC model. In Chapter 4 for the first time we develop nonlinear perturbations to the PEEC NFDEs.

In chapter 5 we present sufficient conditions for uniform stability of the PEEC NFDEs with nonlinear perturbations developed in chapter 4. Also, sufficient conditions for convergence of the solutions of the PEEC NFDEs with nonlinear perturbations to the solutions of the original linear time-invariant PEEC NFDEs are provided. Examples are given demonstrating that our sufficient conditions for stability and convergence can be met. This is the first time in published literature that stability of nonlinear perturbations to PEEC NFDEs have been investigated.

In chapter 6 we summarize the results developed in previous chapters. Finally, we present opportunities for further research.

Chapter 2

Literature Survey

2.1 Introduction

In this chapter we examine existing literature to establish that the research presented here is novel and fills a gap in the literature. First we list some basic properties of Retarded Functional Differential Equations (RFDEs) as well as Neutral Functional Differential Equation (NFDEs).

The basic properties of these types of DDEs highlight some differences between simpler RFDEs compared to more complex NFDEs. As expected there is more literature on RFDEs than NFDEs. A further narrowing occurs when considering published literature on PEEC NFDEs compared to NFDEs in general. This establishes the need for the PEEC NFDE stability analysis that we present in Chapter 3. Specifically we list research that applies Lyapunov-Razumikhin stability analysis to Delay Differential Equations (DDEs) but no existing literature applies this method to the PEEC NFDEs as we present in Chapter 3. At the end of Chapter 2 we present an implementation of convex optimization used in the existing PEEC NFDE stability analysis [2]. The optimization process is iterative and requires longer numerical computations compared to similar non-iterative methods and also may have undetermined computation time required to achieve optimization criteria. These computation issues also establish the need for the new, non-iterative PEEC NFDE stability analysis we present in Chapter 3.

2.2 General DDEs

2.2.1 Preliminaries on DDEs

Here we list some basic properties of DDE's. The following definitions and theorems are taken from [17] which is a classic reference on the subject of DDE's. DDEs with delay in a function but not in its derivative are called Retarded Functional Differential Equations (RFDE's), whereas DDEs with delay in the function and in its derivative are called Neutral Functional Differential Equations (NFDE's).

Definition: Let $r \in \mathbb{R}^+$, and $C = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions,with the topology of uniform convergence.

For $\phi \in C$, let $|\phi| = \sup_{-r \le \theta \le 0} |\phi(\theta)|$.

Note: Banach space C and norm $|\phi|, \phi \in C$ defined here will be used throughout this dissertation. Let $\sigma \in \mathbb{R}, A \geq 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. For all $t \in [\sigma, \sigma + A]$, define $x_t \in C$ by [17], $x_t(\theta) = x(t + \theta),\ \theta \in [-r, 0]$.

Let $D \subseteq \mathbb{R} \times C, f : D \to \mathbb{R}^n$ is a given function and " $\dot{}$ " represents the right hand derivative. The relation

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

is called a Retarded Functional Differential Equation on D, denoted by RFDE or $RFDE(f)$[17]. A function x is said to be a solution of (2.1) on $[\sigma - r, \sigma + A]$ if there are $\sigma \in \mathbb{R}, A > 0$ such that $x \in C([\sigma - r, \sigma + A], (t, x_t) \in D$ and $x(t)$ satisfies (2.1) for $t \in [\sigma, \sigma + A]$.

For a given $\sigma \in \mathbb{R}, \phi \in C$, we say that $x(\sigma, \phi, f)$ is a solution of (2.1) with history function ϕ at σ or simply a solution through (σ, ϕ) if there is an $A > 0$ such that $x(\sigma, \phi, f)$ is a solution of (2.1) on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi, f) = \phi$ [17].

For a solution of (2.1), $\sigma \in \mathbb{R}, \phi \in C$ and $f(t, \phi)$ continuous, it is easy to see [17], this is equivalent to :

$$\begin{aligned} x_\sigma &= \phi \\ x(t) &= \phi(0) + \int_\sigma^t f(s, x_s) ds \quad t \geq \sigma \end{aligned} \tag{2.2}$$

The following existence and uniqueness theorem is a classical result [17].

Theorem 2.2.1. *Let Ω be an open set in $\mathbb{R} \times C$ and $f \in C(\Omega, \mathbb{R}^n)$. If $(\sigma, \phi) \in \Omega$ then there is a solution of the $RFDE(f)$ passing through (σ, ϕ).*

If in addition, $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω, then there is a unique solution of the RFDE(f) passing through(σ, ϕ), for $(\sigma, \phi) \in \Omega$.

Definition 2.2.2. [17]: Let $\Omega \subset \mathbb{R} \times C$ be open, $f \in C(\Omega, \mathbb{R}^n)$. Let $D : \Omega \to \mathbb{R}^n$ be a continuous operator that is atomic at zero. The relation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \tag{2.3}$$

is called a *Neutral Functional Differential Equation*, NFDE (D, f).

We present a few standard definitions of stability of the zero solution of (2.3) [17].

Suppose $f(t,0) = 0, \ \forall t \in \mathbb{R}$. The solution $x = 0$ of ((2.3)) is said to be

- *stable* if for any $\sigma \in \mathbb{R}, \epsilon > 0 \ \exists \ \delta = \delta(\epsilon, \sigma)$ such that $\phi \in B(0, \delta)$ implies $x_t(\sigma, \phi) \in B(0, \epsilon)$ for $t \geq \sigma$.

- *uniformly stable* if it is stable and δ is independent of σ.

- *asymptotically stable* if it is stable and $\exists \ b_0 = b_0(\sigma) > 0$ such that $\phi \in B(0, b_0)$ implies $x(\sigma, \phi)(t) \to 0$ as $t \to \infty$

- *uniformly asymptotically stable* if it is uniformly stable and $\exists \ b_0 > 0$ such that $\forall \ \eta > 0, \ \exists \ t_0(\eta)$ such that $\phi \in B(0, b_0)$ implies $x_t(\sigma, \phi) \in B(0, \eta)$ for $t \geq \sigma + t_0(\eta), \ \forall \ \sigma \in \mathbb{R}$.

2.2.2 Lyapunov-Razumikhin Stability Analysis of RFDEs

In [28] Magpantay and Humphries use Razumikhin analysis to investigate stability of RFDE's and RFDE's with state-dependent delays of the following form:

$$\dot{u}(t) = f(t, u(t), u(t - \tau_1(t, u(t))), \ldots u(t - \tau_N(t, u(t)))), \quad t \geq 0,$$
$$u(t) = \phi(t), \qquad\qquad t \leq 0$$

In [6] Chaillet, Pogromsky and Ruffer use Razumikhin analysis to investigate stability of the RFDE:

$$\dot{x}(t) = f(t, x(t), x_t), \tag{2.4}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n$ x $C([-\theta, 0])^n \to \mathbb{R}^n$.

In addition, the following non-linear RFDE is investigated:

$$\begin{aligned}\dot{x} &= Ax + g(t, x_t) \\ g_i(t, x_t) &= S_i\left(u_i(t) + \sum_{j=1}^{n} c_{ij} x_j(t - \delta_{ij})\right)\end{aligned} \tag{2.5}$$

where $c_{ij} \in \mathbb{R}$ and $\delta_{ij} \in [0, \theta]\ \forall\ i, j \in \{1, \ldots, n\}$ for some $\theta > 0$.

In [30] Ning, He, et al consider a Razumikhin-Type theorem for Input to State Stability (ISS) (see Definition A.1.5 in the Appendix A) of the following RFDE:

$$\dot{x}(t) = f(t, x_t, u(t)) \tag{2.6}$$

where $x : \mathbb{R} \to \mathbb{R}$ and input $u : \mathbb{R}^+ \to \mathbb{R}^m$ are assumed to be measureable and locally essentially bounded. $f : \mathbb{R}^+ \times C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^m$ is assumed to be Lipschitz in (t, x), uniformly continuous in u, and to satisfy $f(t, 0, 0) = 0$.

2.2.3 A Fundemental Difference Between RFDEs and NFDEs

Traditionally RFDE theory was developed more extensively and is then applied to NFDEs.

While RFDEs and NFDEs are both DDEs there are certain fundamental differences [17]. For example (2.2) can be used to find a solution $x_{(k-1)r}$ on successive intervals $[(k-1)r, kr]$ for k a non-negative integer. Integration will result in solution $x(t)$ of the RFDE becoming smoother in each successive interval.

Now, consider the linear autonomous NFDE:

$$\dot{x}(t) - C\dot{x}(t - r) = Ax(t) + Bx(t - r) + f(t) \tag{2.7}$$

where A, B, C, and r are constants with $r > 0, C \neq 0$ and f is a continuous function on $\mathbb{R}$.

The following theorem highlights the differences in the properties of the solutions of NFDEs compared to the soltuions to RFDEs.

Theorem 2.2.3. *[17]: If $C \neq 0$, and ϕ is a continuously differentiable function on $[-r, 0]$, then $\exists$ a unique function $x : (-\infty, \infty) \to \mathbb{R}$ that coincides with ϕ on $[-r, 0]$, is continuously differentiable and satisfies ((2.7)) except maybe at points $kr, k = 0, \pm 1, \pm 2, \ldots$. This solution x can have no more derivatives than ϕ and is continuously differentiable if and only if:*

$$\dot{\phi}(0) = C\dot{\phi}(-r) = A\phi(0) + B\phi(-r) + f(0) \tag{2.8}$$

2.2.4 NFDE Stability Theorems

Previously, we examined the use of Lyapunov-Krasovskii Functionals and Lyapunov-Razumikhin Functions to analyze stability of RFDE's. In the literature these are two major methods for analyzing stability of RFDE's. This section extends these two methods to stability analysis of NFDE's.

First, a few standard results on the stability of NFDEs are stated. These results can be found in [17].

2.2.5 PEEC Numerical Methods

Some research has been done to improve the performance of the PEEC model compared to actual behavior of electromagnetic signals travelling in solid conductors. In [25] Kochetov and Wollenberg investigate improving the accuracy of the PEEC model for high frequencies. PEEC model volume and surface elements can respond linearly to arbitrarily large frequencies while the physical materials being modeled do not. In this case unwanted high frequency oscillations can occur. The authors improve the PEEC model by adding a damping resistor in parallel with PEEC model partial inductance components. In [1] Antonini, Deschrijver and Dhaene investigate instability in the rPEEC model instability due to discrete representation of the electromagnetic continuous problem. One approach is to use the PEEC method to generate the PEEC circuit network, then add filters to this circuit to reduce discretization effects. This article refers to the methods used in Kochetov and Wollenberg [25]. In [3], Bellen, Guglielmi and Ruehli study the stability of numeric methods mainly focusing on integration techniques, used to solve the PEEC model. The following NFDE:

$$\begin{aligned} y'(t) - Ly(t) &= N(y'(t-\tau)) - Ly(t-\tau)) + (M + NL)y(t-\tau), \quad t \geq 0 \\ y(t) &= g(t) \end{aligned} \tag{2.9}$$

is transformed to an algebraic recursion by setting $\Phi(t) := y'(t) - Ly(t)$

$$\begin{aligned} y'(t) &= Ly(t) + \Phi(t) \qquad t \geq 0 \\ y(t_0) &= g(t_0) \\ \Phi(t) &= N\Phi(t-\tau)) + (M + NL)y(t-\tau), \quad t \geq \xi_1 \end{aligned} \tag{2.10}$$

$$= Mg(t-\tau) + Ng'(t-\tau), \qquad t_0 \leq t < \xi_1 \tag{2.11}$$

In [5] the authors propose use of the PEEC model to perform an analysis of perturbations of input parameters on solutions to equations for electromagnetic fields in a circuit board. Specifically, the optimal value of design parameters are determined based on system performance requirements. As an example this paper generates a frequency domain form of the PEEC model, with g a system parameter of interest:

$$V_p(s, g) = Z(g)I_p(s, g) \tag{2.12}$$

$$Z(g) = L^T(g)(sC(g) + G(g))^{-1}B(g) \tag{2.13}$$

From (2.12), (2.13) a state space form is generated. Here the notation $\hat{f} = \frac{df}{dg}$ is used:

$$\begin{pmatrix} V_p(s) \\ \hat{V}_p(s) \end{pmatrix} = \begin{pmatrix} Z & 0 \\ \hat{Z} & Z \end{pmatrix} \begin{pmatrix} I_p(s) \\ \hat{I}_p(s) \end{pmatrix} \tag{2.14}$$

$$\hat{Z} = \hat{L}^T(sC + G)^{-1}B + L^T(sC + G)^{-1}\hat{B}$$

$$-L^T(sC + G)^{-1}\ (s\hat{C} + \hat{G})\ (sC + G)^{-1}B \tag{2.15}$$

Traditional analysis involves perturbing system input parameters and repeatedly solving electromagnetic equations for each value of the perturbed parameters: a very computationally expensive process.

The authors in [5] use expressions (2.14) and (2.15) along with cubic spline interpolation to solve electromagnetic equations for each value of the perturbed parameters. This approach has a much smaller computational cost of interpolation compared to the traditional approach of repeated solutions of the electromagnetic equations.

Note that another application of the work in [5] would be for analysis of small linear perturbations to a PEEC NFDE model such as may be seen during variations in manufacture of printed circuit boards. If these linear perturbations were to be investigated then the cubic spline interpolation method in [5] would allow reduced calculations for a range of such linear perturbations.

A connection between Razumikhin function type ISS stability analysis and the non-linear small gain theorem is established in [43] and is often cited in control theory and DDE literature.

In [33] Rabah, Skylar, and Barkhayev present stability of RFDE's and NFDE's such as:

$$\frac{d}{dt}[z(t) - Kz_t] = Lz_t + Bu(t) \quad t \geq 0 \tag{2.16}$$

In [13], Gil' focuses on stability analysis of NFDE's avoiding Lyapunov type stability analysis.

In [18], Hale and Lunel consider the solution map: $T(t) : C \to C$ as

$$(T(t)\varphi)(\theta) = x_t(\theta; \varphi) = x(t + \theta; \varphi) \quad \text{for } \theta \in [-h, 0], \tag{2.17}$$

study the stability properties for linear NFDE's and show that a strongly continuous semigroup is generated by the solution map. The focus is on modifying NFDE systems with feedback to create stable systems.

2.3 PEEC NFDEs

2.3.1 Stability in the Delays of PEEC NFDEs

In [19], Hale and Lunel investigate the effects of small changes in delays to the stability of a system. In [21], Ize' and Molfetta extend the analysis of NFDE's with constant delay to NFDEs with time-dependent time delays. In [47], Yue and Han study the stability in the delays of PEEC NFDE's. In [29], Niculescu covers Lyapunov-Krasovskii and Lyapunov-Razumikhin stability analysis methods. This includes standard methods of generating sufficient conditions for stability as well as more complex methods for generating necessary and sufficient conditions for stability.

2.3.2 Stability of PEEC NFDE's Using Lyapunov-Krasovskii Functionals

The pioneering work on PEEC was initiated by Ruehli in [37] in the context of VLSI inductance calculations. The interest in PEEC methods increased during last two decades due to developments in technology that required joint circuit and EM simulations. The following formulation of Delay Differential Equation (DDE) of neutral type for PEEC circuits is obtained starting from integral equation formulation of Maxwell's equation. (see [2], [35]).

The mixed potential integral equation for electric field at a point $\mathbf{r}$ that may be located either inside a conductor or inside a dielectric region is given by:

$$\mathbf{E}^i(\mathbf{r},t) = \frac{\mathbf{J}(\mathbf{r},t)}{\sigma} + \mu \int_{u'} G(\mathbf{r},\mathbf{r}')\frac{\partial J}{\partial t}(\mathbf{r}',t_d)\ du' + \nabla\frac{1}{\varepsilon}\int_{S'} G(\mathbf{r},\mathbf{r}')q(\mathbf{r}',t_d)\ dS' \quad (2.18)$$

where

$\mathbf{E}^i(\mathbf{r},t)$ - applied electric field at a field point $\mathbf{r}$ and time t,

$\mathbf{J}(\mathbf{r},t)$ - current density in the conductor

σ - electrical conductivity; ε - the dielectric permittivity ($> \varepsilon_0$, perimittivity of

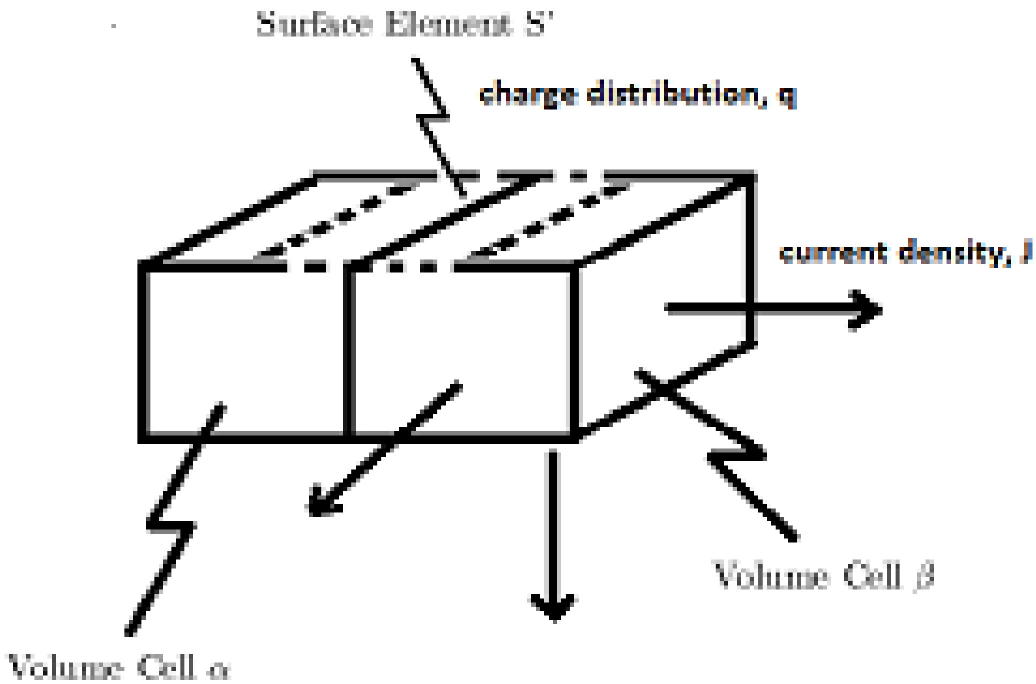

Figure 2.1: PEEC Model Volume and Surface Elements

free space)

$G(\mathbf{r}, \mathbf{r}') := \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ - Green's function

$t_d = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$ where the delay is due to the free space travel time of EM waves from $\mathbf{r}$ to $\mathbf{r}'$. Here u' is the volume where current and change densities are defined and S' is the surface of the conductor where a charge density q is assumed to be located. The unknowns in the equation are the charge density $\mathbf{J}$ and the surface charge density q.

Volumes u' and surfaces S' are discretized into elementary regions and patches. Examples of these surface and volume elements are shown in figure 2.1. The unknowns J and q are expanded into a series of pulse basis functions. These basis functions are also used to weight the discretized mixed integral equation. Each term of the weighted mixed integral equation is interpreted as voltage drop and then the spatial discretized version of (2.18) corresponds to Kirchoff's Voltage Law enforced at each volume cell.

For conductive regions, this reads as:

$$-Av_n(t) - Ri_L(t) - L_p \frac{di_l}{dt} = v_s(t) \tag{2.19}$$

where v_n denotes the potentials to infinity, i_L are the inductive currents flowing in volume cells, A is the connectivity matrix and R is a diagonal matrix containing the resistances of volume cells, L_p is the partial inductances matrix, $v_s(t)$ is the vector of voltages induced by the external electric field. That is, the continuous electromagnetic problem is converted into a discrete circuital one whose topological structures are nodes and branches and whose unknowns are potentials to infinity v and the currents i_L. Requiring that the continuity equations hold implies that in a discrete form, the Kirchoff's Current Law

$$\frac{dQ(t)}{dt} - A^T i_L(t) = i_s(t) \tag{2.20}$$

holds at each node, where $Q(t)$ is a vector containing the charge of surface cells and $i_s(t)$ is the vector of lumped current sources. After considering the magnetic field coupling between two inductive cells and electric field couplings between two surfaces patches, and using a relation between charges and potentials, one arrives at the following following set of functional differential equations to be solved, known as (L_p, P, R, τ) PEEC model or retarded PEEC model (rPEEC) given by:

$$-Av_n(t) - Ri_L(t) - L_P \frac{di_L}{dt}(t - \tau^v) = v_s(t) \tag{2.21}$$

$$C_s \frac{dv_n}{dt} + P_n i_{lumped}(t - \tau^s) - P_n A^T i_L(t - \tau^s) = P_n i_s(t - \tau^s) \tag{2.22}$$

where C_s is a diagonal matrix containing the pseudo self-capacitances, P_n is dense matrix containing the normalized coefficients of potentials, τ^v is the vector of all the involved time delays between volume cells, τ^s is the vector of vector of all time delays between surface cells.

Further, introducing a state variable vector

$$\mathbf{x}(t) = \begin{bmatrix} v(t) \\ i_L(t) \end{bmatrix} \tag{2.23}$$

and decomposing L_p and P_n into not retarded and retarded components, collecting the time delays in vectors τ^s and τ^v in a unique vector of time delays $\tau = (\tau_1, \tau_2, \cdots, \tau_m)$ where m is the number of distinct time delays, sorted in ascending order, one arrives at the following NFDE in the Hale's form [2]:

$$\frac{d}{dt}\left(\mathbf{x}(t) - \sum_{i=1}^{m} B_i \mathbf{x}(t - \tau_i)\right) = \mathbf{A}_0 \mathbf{x}(t) + \sum_{i=1}^{m} \mathbf{A_i} \mathbf{x}(t - \tau_i) + \mathbf{G}\mathbf{u}(t) \tag{2.24}$$

where the matrices $\mathbf{A}_i$ and $\mathbf{B_i}, i = 1, 2, \cdots, m$ are suitably obtained matrices involving the non retarded component of partial inductance L_p and the diagonal matrix C_s. $\mathbf{G}$ is also a suitable matrix multiplying the vector of non delayed and delayed sources $\mathbf{u}(t)$.

Chapter 3

Lyapunov-Razumikhin Stability for PEEC NFDEs

3.1 Introduction

Here we use Lyapunov-Razumikhin stability analysis methods to develop sufficient conditions for uniform asymptotic stability of the PEEC NFDEs.

In [2], Antonini and Pepe consider the Partial Element Equivalent Circuit (PEEC) model presented by A. Ruehli [36]. This PEEC model with time delays results in Neutral Functional Differential Equations (NFDEs).

Three theorems are presented providing sufficient conditions for the solution $x = 0$ of the PEEC NFDE's to be uniformly asymptotically stable and all solutions to approach zero as $t \to \infty$.

Theorem 3.4.1 provides sufficient conditions for uniform asymptotic stability and for solutions to approach zero as $t \to \infty$ but does not allow evaluation of these conditions directly from the PEEC NFDE system parameters.

Making an assumption about $T = \{\tau_1, \tau_2, \ldots, \tau_m\}$ Theorem 3.4.4 provides sufficient conditions for uniform asymptotical stability of the solution $x = 0$ and all solutions to approach zero as $t \to \infty$, and these conditions can be evaluated directly from the PEEC NFDE system parameters. A corollary is then given showing that for PEEC NFDEs that meet the stability conditions of Theorem 3.4.4, there exists a neighborhood of the delay vector T such that if all other NFDE system parameters remain constant but a different delay vector, within the neighborhood is used then uniform asymptotical stability and all solutions approaching zero as $t \to \infty$ also apply for such an NFDE. This means that Theorem 3.4.4 is of practical use as it allows evaluation of sufficient conditions directly from the PEEC NFDE system parameters, and the original restrictions on the delay vector are not as rigid.

For Theorem 3.4.7, the PEEC NFDEs are recast to provide a different formulation for the system. Sufficient conditions are provided for uniform asymptotic stability of the modified PEEC NFDE and that all solutions approach zero as $t \to \infty$.

We consider the following homogeneous linear autonomous NFDE that arises in a PEEC model:

$$\frac{d}{dt}\left(x(t) - \sum_{i=1}^{m} B_i x(t - \tau_i)\right) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i)$$

$$x_\sigma = \phi, \quad \text{with } \sigma \in \mathbb{R}. \tag{3.1}$$

Here A_i, B_i are real $n \times n$ matrices, for $i \in 1 : m,$ and the delays

$T = \{\tau_1, \tau_2, \ldots, \tau_m\}$ are such that $0 < \tau_1 < \tau_2 < \cdots < \tau_m = \tau$.

Further $x_t, \phi \in C([-\tau, 0]; \mathbb{R}^n)$.

Define the operators D, f and L as follows:

Definition 3.1.1. $D : C \to \mathbb{R}^n, f : C \to \mathbb{R}^n$ and $L : C \to \mathbb{R}^n$,

$D\phi = \phi(0) - \sum_{i=1}^{m} B_i\phi(-\tau_i)$,

$f(\phi) = A_0\phi(0) + \sum_{i=1}^{m} A_i\phi(-\tau_i)$.

In some settings, operator L is used to express the right hand side of the NFDE (3.1)

$L\phi = A_0\phi(0) + \sum_{i=1}^{m} A_i\phi(-\tau_i)$.

3.2 Conditions for the PEEC NFDE D Operator to be Atomic at 0 and $-\tau$

Here we examine conditions for the D operator to be atomic at 0 and $-\tau$. These properties are prerequisites for existence and uniqueness theorems used in this chapter and in chapter 5.

Using Definition A.1.1 of atomic for the PEEC NFDEs (3.1), we have

$$\begin{aligned} D\phi &= \int_{-\tau}^{0} d[\eta(\theta)]\phi(\theta) \\ &= \phi(0) - \sum_{k=1}^{m} B_k\phi(-\tau_k) \\ &= \sum_{k=0}^{m} B_k\phi(-\tau_k) \end{aligned}$$

with $\tau_0 = 0, B_0 = I_n,$ the $n \times n$ identity matrix.

Thus,

$D\phi = \int_{-\tau}^{0} \sum_{k=0}^{m} \delta(\theta + \tau_k)B_k\phi(\theta), \quad d[\eta(\theta)] = \sum_{k=0}^{m} \delta(\theta + \tau_k)B_k d\theta,$

where $\delta(\theta)$ is the Dirac-delta function.

Also

$\eta(\beta^+) - \eta(\beta^-) = \int_{\beta^-}^{\beta^+} d[\eta(\theta)] = B_k$ for $\beta = -\tau_k, k \in 0 : m$.

For $\beta = 0$,

$$\lim_{h \to 0^+} \Big| \int_h^s + \int_{-s}^{-h} d[\eta(\theta)]\phi(\theta) = \int_h^s + \int_{-s}^{-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) \Big|.$$

For $s < \tau,$ since $-\tau_k \notin [-s, -h] \cup [h, s] \; \forall \; k \in 0 : m,$ we have

$$\int_h^s + \int_{-s}^{-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) = 0.$$

For $\tau_1 \leq s < \tau_j,$ we have

$$\lim_{h \to 0^+} \Big| \int_h^s + \int_{-s}^{-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) \Big| = \Big| \sum_{k=1}^{j-1} B_k \phi(-\tau_k) \Big|$$

$$\leq \sum_{k=1}^{j-1} |B_k||\phi(-\tau_k)|.$$

Let $\gamma(s) = \sum_{k=1}^{j-1} |B_k||\phi(-\tau_k)|, \quad j$ the least integer such that $s < \tau_j \leq \tau$. Then $\gamma(s) = \sum_{k=1}^{m} |B_k||\phi(-\tau_k)|$ for $s \geq \tau$ meets the requirements given in Definition A.1.1.

Further, we have:

$A(\beta, D)|_{\beta=0} = \eta(0^+) - \eta(0^-) = B_0 = I_n$, which is non-singular.

For $\beta = -\tau_m = -\tau$,

$$\lim_{h \to 0^+} \Big| \int_{\beta+h}^{\beta+s} + \int_{\beta-s}^{\beta-h} d[\eta(\theta)]\phi(\theta) \Big| = \Big| \int_{\beta+h}^{\beta+s} + \int_{\beta-s}^{\beta-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) \Big|.$$

For $s < \tau - \tau_{m-1},$ since $-\tau_k \notin [\beta - s, \beta - h] \cup [\beta + h, \beta + s] \; \forall \; k \in 0 : m,$

$$\int_{\beta+h}^{\beta+s} + \int_{\beta-s}^{\beta-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) = 0.$$

For $\tau - \tau_{m-1} \leq s < \tau - \tau_j$,

$$\lim_{h \to 0^+} \Big| \int_{\beta+h}^{\beta+s} + \int_{\beta-s}^{\beta-h} \sum_{k=0}^{m} \delta(\theta + \tau_k) B_k \phi(\theta) \Big|$$
$$= \Big| \sum_{k=j+1}^{m-1} B_k \phi(-\tau_k) \Big| \leq \sum_{k=j+1}^{m-1} |B_k||\phi(-\tau_k)|.$$

Let $\gamma(s) = \sum_{k=j+1}^{m-1} |B_k||\phi(-\tau_k)|$, j the least integer such that $s < \tau_j \leq \tau$ and $\gamma(s) = \sum_{k=1}^{m} |B_k||\phi(-\tau_k)|$. For $s \geq \tau$ meets the requirements given in Definition A.1.1.

Also,

$A(\beta, D)|_{\beta=-\tau} = \eta(-\tau^+) - \eta(-\tau^-) = B_m$. The atomic at $-\tau$ condition is met iff B_m is non-singular. Thus, we have the following Lemma.

Lemma 3.2.1. *The PEEC NFDE D operator will always be atomic at* 0 *and will also be atomic at* $-\tau$ *iff* $det(B_m) \neq 0$.

3.3 Properties of D Operators

Let $B = \{B_1, B_2, \ldots B_m\}$. We may also emphasize the dependence of D on the set of delays T and the set of coefficient matrices B and write:

$$D(B, T)\phi = \phi(0) - \sum_{i=1}^{m} B_i \phi(-\tau_i). \tag{3.2}$$

The operator $D : C \to \mathbb{R}^n$, which is atomic at zero (3.2.1), clearly is linear and continuous, is said to be *stable* if the zero solution of the homogeneous difference equation:

$$Dy_t = 0, \quad t \geq 0 \tag{3.3}$$

with $y_0 = \psi \in C_D$ is uniformly asymptotically stable. The following result from [17] is of importance to our work.

Theorem 3.3.1.

The following statements are equivalent:

i) D is stable.

ii) There exist constants $\alpha, b(\alpha) > 0$ such that for all $h \in C([0,\infty]; \mathbb{R}^n)$ any solution y of the nonhomogeneous equation:

$$Dy_t = h(t), \quad t \geq 0 \tag{3.4}$$

satisfies

$$|y(\phi, h)(t)| \leq b(\alpha)[|\phi|e^{-\alpha t} + \sup_{0 \leq s \leq t} |h(s)|]. \tag{3.5}$$

For linear autonomous D operators, the notion stability in the delays, is given as follows:

Definition 3.3.2. [17]: $D(B,T)$ is said to be stable locally in the delays if there exists an open neighborhood $I(T) \subset (\mathbb{R}^+)^m$ of T such that $D(B,S)$ is stable for all $S \in I(T)$. $D(B,T)$ is stable globally in the delays if it is stable for all $T \in (\mathbb{R}^+)^m$.

3.4 Lyapunov-Razumikhin Stability for PEEC NFDEs

3.4.1 Lyapunov-Razumikhin Stability Conditions for the PEEC NFDE

Here we obtain sufficient conditions for uniform asymptotic stabililty of the PEEC NFDEs and global stability in the delays of the associated D operator. Using the coefficient matrices in (3.1) we define the following constants.

(i) $K = |D| = \sup_{\phi \in C} \frac{|D\phi|}{|\phi|} = 1 + \sum_{i=1}^{m} |B_i|$,

(ii) $\Sigma_A = \sum_{i=1}^{m} |A_i|, \quad \Sigma_B = \sum_{i=1}^{m} |B_i|$,

(iii) $P = -(A_0)^{-1}, \; P_s = \frac{1}{2}(P + P^T), \quad P_a = \frac{1}{2}(P - P^T), \quad \rho = |P_a|/(|P|)$

(iv) $\lambda_{Pmax}, \lambda_{Pmin} =$ are the eigenvalues of P_s with maximum, minimum magnitude respectively and ,

(v) $r_{\lambda_P} = \sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}} \geq 1.$, Define $\varsigma = \rho |P||A_0|$.

Theorem 3.4.1. *We assume the following conditions hold:*

(i) A_0 *is negative definite.*

(ii) $0 < \Sigma_B < 1/r_{\lambda_P}$

(iii) $\dfrac{K}{(1 + K r_{\lambda_P})} < b < \dfrac{K}{(1 + K r_{\lambda_P}) \Sigma_B r_{\lambda_P}}$

(iv) $\varsigma < [\dfrac{K}{(1 + K r_{\lambda_P}) b r_{\lambda_P}} - \Sigma_B]$

$(v)\ \ \Sigma_A < \dfrac{1}{|P_s|}[\dfrac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - (\Sigma_B + \varsigma)]$

Then the D operator is stable globally in the delays and solution $x = 0$ of system (3.1) *is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.*

Remark 3.4.2. If condition ii) of Theorem 3.4.1 is met then $\frac{K}{(1+Kr_{\lambda_P})} < \frac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}}$ and the constraint in condition iii) will not refer to an empty interval. Clearly $\frac{K}{(1+Kr_{\lambda_P})} < 1$. For given P, $\frac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}}$ can be made arbitrarily large by changing suitable matrices B_i thus making the magnitude of Σ_B small.

Remark 3.4.3. In the above conditions i) - v) of Theorem 3.4.1 each term can be evaluated directly from the system (3.1) except for b, the coefficient in the exponential bound expression given in (3.5).

Next, we develop sufficient conditions to determine uniform asymptotic stability of system (3.1) which can be evaluated directly from the system parameters. Without loss of generality, assume the delays $\tau_i, i \in 1:m$, in the physical model can be obtained as a rational number times a common factor:

$$\tau_i = q_i \Delta t, i \in 1:m, \quad q_i \text{ rational}, q_i, \Delta t > 0. \tag{3.6}$$

We convert system (3.4) to a sampled time system with uniform interval *timeStep* between indexed time values. Let $\tau_i = n_i \cdot timeStep \quad \forall\ i \in 1:m$. In particular $\tau = \tau_m = n_m \cdot timeStep$ with $n_m = intervalCount$.

Next, we generate the next state transition matrix or companion matrix, $A_{advstep}$, as follows:

$$\text{Let } Bind_i = \begin{pmatrix} 0\ldots 0 & B_i(1,1) & 0\ldots 0 & \ldots & B_i(1,n) & 0\ldots 0 \\ 0\ldots 0 & B_i(2,1) & 0\ldots 0 & \ldots & B_i(2,n) & 0\ldots 0 \\ \vdots\ldots\vdots & \vdots & \vdots\ldots\vdots & \ldots & \vdots & \vdots\ldots\vdots \\ 0\ldots 0 & B_i(n,1) & 0\ldots 0 & \ldots & B_i(n,n) & 0\ldots 0 \end{pmatrix}. \tag{3.7}$$

With $Bind_i$ a real $n \times n \cdot intervalCount$ matrix, we define matrices $\left(A_{i,j}\right)$:

$$\left(A_{i,j}\right) = \begin{pmatrix} 0 & \delta(i,j) & 0 & \ldots & 0 \\ 0 & 0 & \delta(i,j) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \delta(i,j) \\ 0 & 0 & Bind_1(i,j) & \ldots & Bind_m(i,j) \end{pmatrix}. \tag{3.8}$$

Here $\delta(i,j)$ is the Kronecker delta function and $Bind_k(i,j)$ is in the column corresponding to integer n_k, for all $k \in 1 : m$.

Now, define the companion matrix:

$$A_{advstep} = \begin{pmatrix} \left(A_{1,1}\right) & \left(A_{1,2}\right) & \cdots & \left(A_{1,n}\right) \\ \left(A_{2,1}\right) & \left(A_{2,2}\right) & \cdots & \left(A_{2,n}\right) \\ \vdots & \vdots & \ldots & \vdots \\ \left(A_{n,1}\right) & \left(A_{n,2}\right) & \cdots & \left(A_{n,n}\right) \end{pmatrix}. \tag{3.9}$$

Theorem 3.4.4. *Generate companion matrix $A_{advstep}$ as described in* (3.7), (3.8) *and* (3.9) *above. We assume the following conditions hold:*

i) A_0 *is negative definite,*

ii) $0 < \Sigma_B < 1/r_{\lambda_P}$.

Let S_A *be the set of eigenvalues of* $A_{advstep}$, *and* $\lambda_{Amax} = \max_{\lambda \in S_A} |\lambda|$,

the spectral radius of $A_{advstep}$.

iii) $\lambda_{Amax} < 1$ *and the eigenvectors of* $A_{advstep}$ *span its column space.*

Let $b = 1/(1 - \lambda_{Amax})$

For this value of b if conditions iv) - vi) are met

iv) $\dfrac{K}{(1 + Kr_{\lambda_P})} < b < \dfrac{K}{(1 + Kr_{\lambda_P})\Sigma_B r_{\lambda_P}}$

v) $\varsigma < [\dfrac{K}{(1 + Kr_{\lambda_P})br_{\lambda_P}} - \Sigma_B]$

vi) $\Sigma_A < \dfrac{1}{|P_s|}[\dfrac{K}{(1 + Kr_{\lambda_P})br_{\lambda_P}} - (\Sigma_B + \varsigma)]$.

Then, the solution $x = 0$ *of system 3.1 is uniformly asymptotically stable and all solutions approach zero as* $t \to \infty$. *Further, the* D *operator is stable globally in the delays.*

Next we consider relaxing the conditions on the delays, (3.6) using global stability in the delays of the D operator and Theorem A.5.1 (see Appendix A).

Theorem A.5.1 (see Appendix A) allows operators D and L to be time dependent which is more general than needed for linear autonomous PEEC NFDEs. Here we let $\lambda = T$, the vector of m delays and Banach space $\Lambda = \mathbb{R}^m$. From Lemma 3.2.1 we have that D is atomic at zero independent of the delays, thus uniformly with respect to λ which meets Theorem A.5.1 condition i). In Theorem A.1.4 (see Appendix A) are established existence and uniqueness conditions for stable PEEC NFDEs which meets Theorem A.5.1 conditions ii) and iii). Thus we have the following Lemma.

Lemma 3.4.5. *There exists a neighborhood, $N'(T)$ of the set of delays T, within which the solution of stable PEEC NFDEs will be continuous with respect to parameter T.*

We now state the following result:

Corollary 3.4.6. *For PEEC NFDE (D, T) with T rationally dependent, meeting the conditions of Theorem 3.4.4, there is a neighborhood G of T such that for any $T_0 \in G$, the solution $x = 0$ of NFDE $(D(B, T_0), T_0)$ is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$, with the $D(B, T_0)$ operator stable globally in the delays.*

This corollary states that although Theorem 3.4.4 assumes the PEEC NFDEs have rationally dependent delays, T, the theorem results will apply for delays in a neighborhood of T which will include rationally independent delays.

3.4.2 Lyapunov-Razumikhin Stability Conditions for a Reformulated PEEC NFDE

A reformulation applicable to PEEC NFDEs was suggested by Kolmanovskii and Myshkis [26] and also Niculescu in [29] in the following manner. Clearly:

$$x(t - \tau_{i-1}) - x(t - \tau_i) = \frac{d}{dt} \int_{-\tau_i}^{-\tau_{i-1}} x(t + \theta) d\theta.$$

For (3.1) we now have $D_1 x_t = (x(t) - \sum_{i=1}^{m} B_i x(t - \tau_i))$.

For $\tau_m = \tau, \tau_0 = 0$, rewrite sum: $\sum_{i=1}^{m} A_i x(t-\tau_i)$, considering pairs of terms:

$$A_m x(t-\tau_m) + A_{m-1} x(t-\tau_{m-1}) = -A_m[x(t-\tau_{m-1}) - x(t-\tau_m)] + \sum_{j=m-1}^{m} A_j x(t-\tau_{m-1}).$$

Thus, $\sum_{i=1}^{m} A_i x(t-\tau_i)$

$$= -\sum_{j=m}^{m} A_j[x(t-\tau_{j-1}) - x(t-\tau_j)] + \sum_{j=m-1}^{m} A_j x(t-\tau_{m-1}) + \sum_{i=1}^{m-2} A_i x(t-\tau_i).$$

For the k^{th} term:

$$A_k x(t-\tau_k) + \sum_{j=k+1}^{m} A_j x(t-\tau_{k+1}) = -\sum_{j=k+1}^{m} A_j[x(t-\tau_k) - x(t-\tau_{k+1})] + \sum_{j=k}^{m} A_j x(t-\tau_k)$$

Thus, $\sum_{i=1}^{m} A_i x(t-\tau_i)$

$$= -\sum_{i=k}^{m-1} \sum_{j=i+1}^{m} A_j[x(t-\tau_i) - x(t-\tau_{i+1})] + \sum_{j=k}^{m} A_j x(t-\tau_k) + \sum_{i=1}^{k-1} A_i x(t-\tau_i).$$

Finally, also including $A_0 x(t-\tau_0) = A_0 x(t)$, then $\sum_{i=0}^{m} A_i x(t-\tau_i)$

$$= -\sum_{i=0}^{m-1}\sum_{j=i+1}^{m} A_j[x(t-\tau_i) - x(t-\tau_{i+1})] + \sum_{j=0}^{m} A_j x(t-\tau_0)$$
$$= -\sum_{i=1}^{m}\sum_{j=i}^{m} A_j[x(t-\tau_{i-1}) - x(t-\tau_i)] + \sum_{j=0}^{m} A_j x(t-\tau_0).$$

Let $C_i = \sum_{j=i}^{m} A_j, \quad \forall\, i \in 0 : m$. Rewrite the original NFDE:

$$\frac{d}{dt}\big(x(t) - \sum_{i=1}^{m} B_i x(t-\tau_i)\big) = C_0 x(t) + \sum_{i=1}^{m} C_i[x(t-\tau_{i-1}) - x(t-\tau_i)]$$

Thus,

$$\frac{d}{dt}\Big(x(t) - \sum_{i=1}^{m} B_i x(t-\tau_i) + \sum_{i=1}^{m} C_i \int_{-\tau_i}^{-\tau_{i-1}} x(t+\theta)d\theta\Big) = C_0 x(t)$$

Define $Dx_t = x(t) - \sum_{i=1}^{m} B_i x(t-\tau_i) + \sum_{i=1}^{m} C_i \int_{-\tau_i}^{-\tau_{i-1}} x(t+\theta)d\theta,$ (3.10)

initial condition: $x_\sigma = \phi,$ with $B_i, C_i \in \mathbb{R}^{n\times n}$ for $i \in 1 : m,$

$x_t, \phi \in C([-\tau, 0]; \mathbb{R}^n).$

For the modified NFDE in (3.10) define:

$$K = |D| = \underset{\phi \in C}{sup} \frac{|D\phi|}{|\phi|} = 1 + \sum_{i=1}^{m} |B_i| + \sum_{i=1}^{m} |C_i|(\tau_i - \tau_{i-1}),$$

$$\Sigma_B = \sum_{i=1}^{m} |B_i|, \quad C_K = 1 + \sum_{i=1}^{m} |C_i|(\tau_i - \tau_{i-1}), \quad P = -(C_0)^{-1},$$

Define $P_s = \frac{1}{2}(P + P^T), \quad P_a = \frac{1}{2}(P - P^T), \quad \rho = |P_a|/(|P|).$

Denote the eigenvalues of P_s, with maximum and minimum magnitudes as

λ_{Pmax}, λ_{Pmin} respectively.

$$r_{\lambda_P} = \sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}} \geq 1, \qquad C_{max} = \max_{i \in 1:m} |C_i|.$$

Theorem 3.4.7. *If the following conditions are met:*

i) $C_0 < 0$, *negative definite.*

ii) $0 < \Sigma_B < 1/r_{\lambda_P}$

iii) $\dfrac{K}{(1+Kr_{\lambda_P})} < b < \dfrac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}}$

iv) maximum delay $\tau < \dfrac{\frac{K}{br_{\lambda_P}(1+Kr_{\lambda_P})} - \Sigma_B}{C_{max}}$

v) $\rho|P||C_0| < \dfrac{K}{br_{\lambda_P}(1+Kr_{\lambda_P})} - (\Sigma_B + C_{max}\tau)$

Then the solution $x = 0$ *of the modified system* (3.10) *is uniformly asymptotically stable and all solutions approach zero as* $t \to \infty$.

3.5 Proofs and Example

Here we prove Theorem 3.4.1 for the original system 3.1.

Let $u(s) = \lambda_{Pmin}s^2, v(s) = \lambda_{Pmax}s^2$.

For $\alpha(\eta) = \sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}}K\eta, v(K\eta) \leq u(\alpha(\eta))$

$$\text{Let } \gamma = b^2c_1^2\frac{(1+K\sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}})^2}{K^2} = b^2c_1^2\frac{(1+Kr_{\lambda_P})^2}{K^2} \tag{3.11}$$

From Theorem 3.4.1 constraint iii), $\gamma > 1, \quad \forall\ c_1 > 1$.

$\beta(\eta) = bc\eta(K\sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}} + 1)$ for some c such that $c_1 > c > 1$.

We may verify that $\beta(\eta) > b(\eta + \alpha(\eta))$.

Let $F(s) = \gamma s$ then $F(v(K\eta)) > v(\beta(\eta))$.

Define $V(s) = \frac{1}{2}s^T P_s s, \quad \forall\ s \geq 0$.

Then:

$$\frac{d}{dt}V(s) = s^T P_s \frac{ds}{dt} \tag{3.12}$$

Consider functions ϕ meeting the Razumikhin condition on ϕ in A.3.5.

Let ϕ be such that $F(V(D\phi) \geq V(\phi(\theta)),\ \forall\theta \in [-\tau, 0]$.

From the definition of $P, \lambda_{Pmax}, \lambda_{Pmin}$, we get

$$\begin{aligned}\gamma\lambda_{Pmax}|D\phi|^2 \geq \gamma D\phi^T P_s D\phi = F(V(D\phi)) \geq V(\phi(\theta))\\ \geq \lambda_{Pmin}|\phi(\theta)|^2 \quad \forall\theta \in [-\tau, 0].\end{aligned} \tag{3.13}$$

Thus $|\phi(\theta)| \leq \sqrt{\frac{\gamma\lambda_{Pmax}}{\lambda_{Pmin}}}|D\phi|, \quad \forall\theta \in [-\tau, 0]$.

Let $\delta = \sqrt{\frac{\gamma\lambda_{Pmax}}{\lambda_{Pmin}}} = \sqrt{\gamma}r_{\lambda_P}$.

From 3.1, we find a lower bound of $\frac{\phi(0)\cdot D\phi}{|D\phi|^2}$:

$$\begin{aligned}\phi(0) &= D\phi + \sum_{i=1}^{m} B_i\phi(-\tau_i)\\ \frac{\phi(0)\cdot D\phi}{|D\phi|^2} &= \frac{D\phi\cdot D\phi + \sum_{i=1}^{m} B_i\phi(-\tau_i)\cdot D\phi}{|D\phi|^2}\\ &\geq \frac{|D\phi|^2 - (|\sum_{i=1}^{m} B_i\phi(-\tau_i)\cdot D\phi|)}{|D\phi|^2}\\ &\geq 1 - \frac{\sum_{i=1}^{m}|B_i||\phi(-\tau_i)||D\phi|}{|D\phi|^2}.\end{aligned} \tag{3.14}$$

From 3.13 then

$$\frac{\phi(0)\cdot D\phi}{|D\phi|^2} \geq 1 - \frac{\delta\sum_{i=1}^{m}|B_i||D\phi|^2}{|D\phi|^2}$$
$$= 1 - \delta(\sum_{i=1}^{m}|B_i|) = 1 - \delta\Sigma_B. \tag{3.15}$$

$$\text{Thus } \frac{\phi(0)\cdot D\phi}{|D\phi|^2} > 0 \text{ if } \delta\Sigma_B < 1. \tag{3.16}$$

From 3.4.1, iii) we have $0 < b < \dfrac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}}, \dfrac{(1+Kr_{\lambda_P})b\Sigma_B r_{\lambda_P}}{K} < 1.$

Thus we can select in (3.11) $c_1 > 1$ such that $\dfrac{(1+Kr_{\lambda_P})bc_1\Sigma_B r_{\lambda_P}}{K} < 1.$
Thus $\delta\Sigma_B < 1$.

$$\text{Define } \epsilon = 1 - \delta\Sigma_B,\ \text{thus } 0 < \epsilon < 1. \tag{3.17}$$

An upper bound of $\frac{\phi(0)\cdot D\phi}{|D\phi|^2}$ can be found similarly:

$$\phi(0) = D\phi + \sum_{i=1}^{m} B_i\phi(-\tau_i),$$
$$\frac{\phi(0)\cdot D\phi}{|D\phi|^2} \leq 1 + \frac{\sum_{i=1}^{m}|B_i||\phi(-\tau_i)||D\phi|}{|D\phi|^2}, \tag{3.18}$$
$$\leq 1 + \delta\Sigma_B = 2 - \epsilon. \text{ Also, } 1 < (2-\epsilon) < 2.$$

$$\text{Then } \frac{\phi(0)\cdot D\phi}{|D\phi|^2} \geq \epsilon. \tag{3.19}$$

Thus $|D\phi|^2\epsilon \leq \phi(0)\cdot D\phi \leq |D\phi|^2(2-\epsilon)$,

$$\frac{d}{dt}V(D\phi) = D\phi^T P_s(\frac{d}{dt}D\phi),$$
$$= D\phi^T (P - P_a)A_0\phi(0) + D\phi^T P_s \sum_{i=1}^{m} A_i\phi(-\tau_i).$$

Since A_0 is negative definite, condition i) of 3.4.1,

$$\frac{d}{dt}V(D\phi) \leq -D\phi^T\phi(0) - D\phi^T P_a A_0\phi(0)$$
$$+ |D\phi^T P_s \Sigma_A \max_{i\in 1:m} \phi(-\tau_i)|.$$

From 3.13 and 3.19 we say

$$\frac{d}{dt}V(D\phi) \leq -\epsilon|D\phi|^2 + |D\phi|\rho|P||A_0|\delta|D\phi| + |P_s|\Sigma_A\delta|D\phi|^2,$$
$$\leq -|D\phi|^2[\epsilon - (\rho|P||A_0|\delta + |P_s|\Sigma_A\delta)].$$

From (3.4.1), iv) and (3.19) : we say,

$\epsilon_1 = \epsilon - (\rho|P||A_0|\delta + |P_s|\Sigma_A\delta) > 0.$

Let $w(s) = \epsilon_1 s^2 > 0 \quad \forall\ s > 0.$

Thus, $\frac{d}{dt}V(D\phi) \leq -w(|D\phi|)$ and by Theorem A.3.5 (see Appendix A) it follows that the solution $x = 0$ of system (3.1) is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.

To prove stability in the delays for the D operator in system (3.1) we use Theorem A.3.4, (iii) (see Appendix A):

$$sup\{\gamma(\sum_{k=1}^{m} B_k e^{i\theta_k}) : \theta_k \in [0, 2\pi], k \in 1 : m\}$$
$$\leq sup\{|\sum_{k=1}^{m} B_k e^{i\theta_k}| : \theta_k \in [0, 2\pi], k \in 1 : m\}$$
$$\leq sup\{\sum_{k=1}^{m} |B_k|\ |e^{i\theta_k}| : \theta_k \in [0, 2\pi], k \in 1 : m\}$$

$$\leq \sum_{k=1}^{m} |B_k| = \Sigma_B$$

from Theorem A.3.5 ii): $\Sigma_B < 1/r_{\lambda_P} \leq 1$.

Thus, any system meeting the given sufficient requirements for uniform asymptotic stability will also have $D(B,T)$ stable globally in the delays and Theorem 3.4.1 is proved.

Now we prove Theorem 3.4.4. We begin by using properties of inhomogeneous difference equations to determine the value of coefficient b in equation (3.5). In Theorem 3.4.4 the continuous time system in (3.4) is represented by the sampled time system:

$$\begin{aligned} &Y(n+1) = A_{advstep}Y(n) + H(n) \\ &\quad Y(n), H(n) \in \mathbb{R}^{n \cdot intervalCount}, A_{advstep} \in \mathbb{R}^{n \cdot intervalCount \times\ n \cdot intervalCount} \end{aligned} \tag{3.20}$$

with initial condition vector $Y(n_0)$.

From [27], the linear time-invariant difference equation (3.20) has the solution:

$$\begin{aligned} &Y(n) = A_{advstep}^{n-n_0} Y(n_0) + \sum_{j=n_0}^{n-1} A_{advstep}^{n-(j+1)} H(j) \\ &|Y(n)| \leq |A_{advstep}|^{n-n_0} |Y(n_0)| + \\ &\quad [\sum_{j=n_0}^{n-1} |A_{advstep}|^{n-(j+1)}] \max_{\jmath \in n_0:n} |H(j)|. \end{aligned} \tag{3.21}$$

From Theorem 3.4.4 condition iii), system (3.20) is stable.

Also $0 < |A_{advstep}| \leq \lambda_{Amax} < 1$.

Thus, $|A_{advstep}|^{n-n_0} = e^{-\alpha(n-n_0)}$ for some $\alpha < 0$.

Since $0 < |A_{advstep}| < 1$ then for all $n \geq n_0$,

the power series in (3.21) has the bound:

$[\sum_{j=n_0}^{n-1} |A_{advstep}|^{n-(j+1)}] < \dfrac{1}{(1 - |A_{advstep}|)} < \dfrac{1}{(1 - \lambda_{Amax})}$,

with $1 < \dfrac{1}{(1 - \lambda_{Amax})} < \infty$.

Thus for $b = \dfrac{1}{(1 - \lambda_{Amax})}$ we can write:

$$|Y(n)| \leq b\big(|Y(n_0)|e^{-\alpha(n-n_0)} + \max_{j \in n_0:n} |H(j)|\big)$$

and the value $\dfrac{1}{(1 - \lambda_{Amax})}$ meets the requirements for the coefficient b in equation (3.5).

Since the value $\frac{1}{(1-\lambda_{Amax})}$ meets the requirements for the coefficient b in equation (3.5) then if (3.4.4) conditions i) - vi) are met then application of Theorem 3.4.1 proves that the D operator is stable globally in the delays, the solution $x = 0$ of system (3.1) is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$. Thus Theorem 3.4.4 is proved.

Now we prove Corollary 3.4.6, that is of practical use. Here we are using the notation described in (3.1.1) and (3.2).

Assume NFDE $(D(B,T), f)$ in (3.1) meets conditions i)-vi) for Theorem (3.4.4) thus the operator $D(B,T)$ is stable globally in the delays, the solution $x = 0$ of system (3.1) is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$. Since operator $D(B,T)$ in (3.1) is stable globally in the delays then in $\mathbb{R}^n$ an neighborhood of T, G_1, can be chosen such that operator $D(B,T')$ is still stable for each $T' \in G_1$.

From Lemma 3.4.5 there is a neighborhood, G_2 of the set of delays T, within which the solution of the PEEC NFDEs will be continuous with respect to parameter T. It is useful to point out some properties of the constants used in Theorem 3.4.4 to evaluate sufficient conditions for stability and convergence.

(i) All constants: $K, \Sigma_A, \Sigma_B, |P_s|, b, r_{\lambda_P}$ and ς are > 0.

(ii) Varying the delays, T can only affect constants: $|P_s|, b, r_{\lambda_P}$ and ς.

For Theorem 3.4.4 conditions i) - vi), it can be seen by observation that if these conditions hold for a specific value of T then these conditions will continue will be true for NFDE $(D(B,T'), f)$ with $T' \in G_3$, a sufficiently small subset of G_2. Thus if we define $G_4 = G_1 \cap G_3$, then for each $T' \in G_4$, application of Theorem 3.4.4 will give that:

Operator $D(B,T')$ will be stable globally in the delays and the solution $x = 0$ of system (3.1) is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$. Thus the Corollary 3.4.6 to Theorem 3.4.4 is proved.

The following example, evaluated using Matlab R2018a (9.4.0.13654) Windows 64-bit version, demonstrates that the Theorem 3.4.4 conditions are consistent.

The following single delay NFDE is of the form of (3.1) and meets Theorem 3.4.4 conditions.

Example 3.5.1.

Let $n = 3, p = 3, m = 1$ and $\tau_1 = 3.3000x10^{-11}sec$. Choose system matrices as:

$$A0 = \begin{pmatrix} -1000 & 100 & 200 \\ 150 & -900 & 0 \\ 150 & 100 & -600 \end{pmatrix},$$

$$A1 = \begin{pmatrix} 15.0000 & 0 & -45.0000 \\ -7.5000 & -7.5000 & -15.0000 \\ -7.5000 & -22.5000 & 0 \end{pmatrix},$$

$$B1 = \begin{pmatrix} -0.0035 & 0.0174 & 0.0069 \\ 0.0139 & 0 & 0.0104 \\ -0.0069 & 0.0139 & 0.0035 \end{pmatrix}.$$

The eigenvalues of the symmetric component of A0, eigsA =

$$\lambda_1 = -1.1150 \times 1.0e + 03$$

$$\lambda_2 = -0.8730 \times 1.0e + 03$$

$$\lambda_3 = -0.5120 \times 1.0e + 03$$

i) Since all eigenvalues are < 0 we have A0 is negative definite.

The following constants can be calculated easily:

$K = 1.0246,\ \Sigma_B = 0.024560,\ r_{\lambda_P} = 1.4368,\ 1/r_{\lambda_P} = 0.6960.$

ii) Thus Theorem 3.4.4 condition ii) is met, $\Sigma_B < 1/r_{\lambda_P}$.

numSubdivisions = Number of discretization intervals per time step = 2.

$$A_{advstep} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0035 & 0 & 0 & 0.0174 & 0 & 0 & 0.0069 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & .0139 & 0 & 0 & 0 & 0 & 0 & 0.0104 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -.0069 & 0 & 0 & 0.0139 & 0 & 0 & 0.0035 & 0 \end{pmatrix}$$

Now we examine requirements for eigenvalues of $A_{advstep}$.

The eigenvalues of $A_{advstep}$ are:

$0.0000 + 0.0000i$, $-0.0000 + 0.1380i$,

$0.0000 + 0.0000i$, $-0.0000 - 0.1380i$,

$0.0000 + 0.0000i$, $-0.0268 + 0.0000i$,

$-0.1354 + 0.0000i$, $0.0268 + 0.0000i$,

$0.1354 + 0.0000i$.

iii.a) Thus, we see that the maximum eigenvalue magnitude, maxEigA, is $0.1380 < 1$.

Here we have $rank(A_{advstep}) = 6$ and $nullity(A_{advstep}) = 3$.

iii.b) There are 6 distinct non-zero eigenvalues, thus eigenvectors of $A_{advstep}$ span its column space.

Choose $b = 1/(1 - maxEigA) = 1/(1 - 0.1380) = 1.1601$

$\frac{K}{(1+Kr_{\lambda_P})} = 0.40987 < b = 1.1601 < \frac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}} = 11.4011.$

Thus b is in the required range.

$\varsigma = \rho|P||A_0| = 0.10107 < [\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - \Sigma_B] = 0.21681$

iv.1) Thus ς is in the required range.

$\Sigma_A = 49.0359 < \frac{1}{|P_s|}[\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - (\Sigma_B + \varsigma)] = 59.4318$

iv.2) Thus Σ_A is in the required range. Since all the conditions in Theorem 3.4.4 are met it follows that the D operator is stable globally in the delays and the solution $x = 0$ of system 3.1 is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.

We now prove Theorem 3.4.7 for the modified system (3.10).

Let $u(s) = \lambda_{Pmin}s^2, v(s) = \lambda_{Pmax}s^2,$

$\alpha(\eta) = \sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}}K\eta$ thus $v(K\eta) \le u(\alpha(\eta)),$

$\beta(\eta) = bc\eta(K\sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}} + 1)$ for some $c > 1,$

then $\beta(\eta) > b(\eta + \alpha(\eta)).$

$$\text{Let } \gamma = b^2c_1^2\frac{(1 + K\sqrt{\frac{\lambda_{Pmax}}{\lambda_{Pmin}}})^2}{K^2} = b^2c_1^2\frac{(1 + Kr_{\lambda_P})^2}{K^2} \tag{3.22}$$

Let $F(s) = \gamma s$ then $F(v(K\eta)) > v(\beta(\eta)).$

Define $V(s) = \frac{1}{2}s^TP_ss \quad \forall\ s \ge 0$ then $\frac{d}{dt}V(s) = s^TP_s\frac{ds}{dt}.$

Using the restriction on ϕ from the Razumikhin theorem A.3.5:

let ϕ be such that $F(V(D\phi) \ge V(\phi(\theta))\ \forall\theta \in [-\tau, 0].$

From the definition of $P, \lambda_{Pmax}, \lambda_{Pmin}$, we have

$\gamma\lambda_{Pmax}|D\phi|^2 \ge \gamma D\phi^TP_sD\phi = F(V(D\phi)) \ge V(\phi(\theta))$

$$\geq \lambda_{Pmin}|\phi(\theta)|^2 \quad \forall\theta \in [-\tau, 0]. \tag{3.23}$$

Thus $|\phi(\theta)| \leq \sqrt{\dfrac{\gamma\lambda_{Pmax}}{\lambda_{Pmin}}}|D\phi| \quad \forall\theta \in [-\tau, 0].$

Let $\delta = \sqrt{\frac{\gamma\lambda_{Pmax}}{\lambda_{Pmin}}} = \sqrt{\gamma} r_{\lambda_P}$. Define $C_{max} = \underset{i\in 0:m}{max}(|C_i|)$.

From 3.10, we get,

$$\phi(0) = D\phi + \sum_{i=1}^{m} B_i\phi(-\tau_i) - \sum_{i=1}^{m} C_i \int_{-\tau_i}^{-\tau_{i-1}} \phi(\theta)d\theta$$

$$\frac{\phi(0)\cdot D\phi}{|D\phi|^2} = \frac{D\phi\cdot D\phi + \sum_{i=1}^{m} B_i\phi(-\tau_i)\cdot D\phi - \sum_{i=1}^{m} C_i \int_{-\tau_i}^{-\tau_{i-1}} \phi(\theta)d\theta \cdot D\phi}{|D\phi|^2}$$

$$\geq \frac{|D\phi|^2 - (|\sum_{i=1}^{m} B_i\phi(-\tau_i)\cdot D\phi| + |\sum_{i=1}^{m} C_i \int_{-\tau_i}^{-\tau_{i-1}} \phi(\theta)d\theta \cdot D\phi|)}{|D\phi|^2}$$

$$\geq 1 - \left(\frac{\sum_{i=1}^{m} |B_i||\phi(-\tau_i)| + \sum_{i=1}^{m} |C_i| \int_{-\tau_i}^{-\tau_{i-1}} |\phi(\theta)|d\theta}{|D\phi|}\right)$$

Since $\phi \in C([-\tau, 0], \mathbb{R}^n)$, for some $\theta_i \in [-\tau_i, -\tau_{i-1}]$

$$= 1 - \frac{\sum_{i=1}^{m} |B_i||\phi(-\tau_i)| + \sum_{i=1}^{m} |C_i||(\tau_i - \tau_{i-1})|\phi(\theta_i)|}{|D\phi|}.$$

From (3.23)

$$\frac{\phi(0)\cdot D\phi}{|D\phi|^2} \geq 1 - \frac{\delta\sum_{i=1}^{m} |B_i||D\phi|^2 + \delta C_{max}\sum_{i=1}^{m}(\tau_i - \tau_{i-1})|D\phi|^2}{|D\phi|^2}$$

$$= 1 - \delta(\sum_{i=1}^{m} |B_i| + C_{max}\tau) = 1 - \delta(\Sigma_B + C_{max}\tau) \tag{3.24}$$

From 3.4.7 iv): since $\tau < \dfrac{\frac{K}{br_{\lambda_P}(1+Kr_{\lambda_P})} - \Sigma_B}{C_{max}}$, there exists $c_1 > 1$ such that $\tau < \dfrac{\frac{K}{bc_1 r_{\lambda_P}(1+Kr_{\lambda_P})} - \Sigma_B}{C_{max}}$.

Note: from iii) then $\dfrac{K}{bc_1 r_{\lambda_P}(1 + Kr_{\lambda_P})} - \Sigma_B > 0$.

Let $\Delta_1 = \dfrac{K}{br_{\lambda_P}(1 + Kr_{\lambda_P})} - \Sigma_B$.Choose c_1 in (3.22) such that $\dfrac{K}{bc_1 r_{\lambda_P}(1 + Kr_{\lambda_P})} - \Sigma_B = \Delta_1/2$.

Thus $c_1 = \dfrac{2K}{br_{\lambda_P}(1 + Kr_{\lambda_P}) + K} > \dfrac{2K}{K + K} = 1$ as required in 3.22.

Thus $\delta(\Sigma_B + C_{max}\tau) < 1$ for this c_1.

$$\text{Let } \epsilon = 1 - \delta(\Sigma_B + C_{max}\tau) > 0 \text{ then } \frac{\phi(0) \cdot D\phi}{|D\phi|^2} \geq \epsilon > 0. \tag{3.25}$$

Now, we have

$\dfrac{d}{dt}V(D\phi) = D\phi^T P(\dfrac{d}{dt}D\phi) = D\phi^T PC_0\phi(0) = -D\phi^T \cdot \phi(0)$.

From above:

From 3.25 we get $\dfrac{d}{dt}V(D\phi) \leq -\epsilon|D\phi|^2$

Let $w(s) = \epsilon s^2$, then $\dfrac{d}{dt}V(D\phi) \leq -w(|D\phi|)$,

for all functions ϕ satisfying $F(V(D\phi)) \geq V(\phi(\theta)), -\tau \leq \theta \leq 0$.

Thus we see that, all the sufficient conditions for uniform asymptotic stability of the solution $x = 0$ of the system and for all solutions to approach zero as $t \to \infty$ given in (A.3.5) (see Appendix A) are met.

3.6 Results

Theorem 3.4.1 provides sufficient conditions for uniform asymptotic stability of the solution $x = 0$ of the PEEC NFDEs, that all solutions approach zero as $t \to \infty$ and for global stability in the delays of the D operator. This theorem does not allow evaluation of these conditions directly from the PEEC NFDE system parameters. Theorem 3.4.4 assumes that the delays $\tau_i, i \in 1 : m$, in the physical model can be obtained as a rational number times a common factor (3.6). This theorem then provides sufficient conditions for uniform asymptotic stability of the solution $x = 0$ of the PEEC NFDEs, that all solutions approach zero as $t \to \infty$ and for global stability in the delays of the D operator. These conditions can be evaluated directly from the PEEC NFDE system parameters. A corallary is then given which shows that these results do not apply just to the NFDE with a single set of rationally dependent delay values, but also for NFDEs with delays in some neighborhood around these delay values. In addition this Lyapunov-Razumikhin Stability theorem 3.4.4 does not require an optimization step as was seen in chapter 2 for Lyapunov-Krasovskii methods. As such the Lyapunov-Razumikhin Stability theorem 3.4.4 could be used as a pre-check for stability of a PEEC NFDE. If the conditions for this Lyapunov-Razumikhin Stability theorem 3.4.4 are not met then lengthier optimization methods of Lyapunov-Krasovskii stability theorems could be used to determine if stability criteria may be met.

For Theorem 3.4.7 the PEEC NFDEs are recast to provide a different formulation for the system. Sufficient conditions are provided for uniform asymptotically stability of the solution $x = 0$ of these modified PEEC NFDE's and that all solutions approach zero as $t \to \infty$.

Chapter 4

Perturbed PEEC Model

We consider a perturbed linear time-invariant NFDE that may arise in the PEEC model. We first consider the perturbations which arise from variations during the manufacture of circuit boards and these result in mathematical model linear perturbations. Next, we consider nonlinear perturbations that arise from nonlinear behavior of circuit board conductors. So far, nonlinear perturbations to the PEEC NFDEs have not been studied.

4.1 Linear Perturbations to the PEEC NFDEs

Assume a circuit board is laid out using industry standard practices for 10GHz signals. A standard circuit board is composed of fiberglass insulating material with signals routed in layers of etched copper. High speed signals are routed in differential pairs of traces with the goal that each trace in the pair would experience the same external electromagnetic effects which would be eliminated when the differential signal is examined.

Vias are narrow cylinders of copper used to make connections between copper signal routing layers. Vias are necessary but result in discontinuities which are typically modeled by their capacitance and inductance effects at high frequencies [41]. The best practice is for high speed differential pairs to be routed a minimum distance from vias but this is not always possible. If differential pairs are routed through vias then it is desired that both traces of the pair are routed through the same number and types of vias. Typically routing through a via will result in an unused signal path or stub. Such stubs are to be kept at a minimal length to avoid oscillations at higher frequencies.

The perturbations to PEEC NFDEs that we examine are based on (minor) violations of industry standard layout practices and expected variations in circuit board materials. When manufacturing multiple copies of a circuit board it is expected that there will be minor variations in trace positions and lengths. The perturbations we would examine are:

1) Variations in static permittivity, ϵ_r, of the insulating fiberglass material: higher for fiberglass areas and lower for epoxy areas. This variation will affect the original EFIE used to derive the NFDEs, specifically the transmission of electrostatic fields which are due to charge distributions on surface elements. A basic model might use a uniform ϵ_r with minor step discontinuities. Such variations would result in changes to the calculated coefficients of potential but the governing NFDEs would remain linear.

2) Variations in length of one trace of a differential pair. This would result in a constant shift in time and a frequency dependent phase shift. At high frequencies this would result in different impedances of the 2 traces in a differential pair. One approach to modeling would be have a variation in u(t), the driving function to

the system. This variation would add or subtract volume elements from a given PEEC model. The goverining NFDEs would remain linear.

3) Variations in via stub lengths. This would create another impedance mismatch between traces in a differential pair. Modeling would be similar to the perturbation of variations in length of one trace of a differential pair. Variations in via stub lengths could result in oscillation in one trace of a differential pair that does not occur in the other trace. This variation would also add or subtract volume elements from a given PEEC model. The goverining NFDEs would remain linear.

4) Minor gaps in the ground plane near or under a differential pair of traces, or the undesired effects of vias near a differential pair of traces. Either of these perturbations may be modeled by adding capacitors and/or inductors to the model, [41]. For PEEC NFDEs, this may result in minor variations in system parameters. This variation would also add or subtract volume elements from a given PEEC model. The goverining NFDEs would remain linear.

In conclusion, modeling the listed variations in manufactured circuit boards result in linear perturbations to an existing PEEC NFDE model. A stability analysis of the perturbed model would therefor be analogous to the original unperturbed model and so, we do not discuss it here. As mentioned in chapter 2, the PEEC model cubic spline interpolation method developed in [5] would be well suited for analyzing these linear perturbations.

4.2 Non-Linear Perturbations to the PEEC NFDEs

4.2.1 Mixing Effects in Modeling of Nonlinear Circuits

Here we consider nonlinear behavior of capacitive and inductive materials in the PEEC NFDE Model described in previous chapters. Later, nonlinear capacitive behavior is introduced to the PEEC model as a nonlinear perturbation.

It has long been known that nonlinear circuits often result in a mixing effect, or the output of 2 inputs being proportional their product. A specific example is the Gilbert Multiplier Cell described in [15]. This is a transistor circuit which has the mixing effect as a desired output. For V_1, V_2 two input voltages to the Gilbert Multiplier Cell, the output is given as:

$$\Delta I = I_{EE}\Big(\frac{V_1}{2V_T}\Big)\Big(\frac{V_2}{2V_T}\Big) \qquad V_1, V_2 << V_T$$

With I_{EE} a constant bias current, V_T a constant voltage.

For general nonlinear circuits a mixing effect is often used to model nonlinearity. In [34] single frequency voltage signals $V_1 cos(\omega_1 t)$ and $V_2 cos(\omega_2 t)$ are applied to a nonlinear circuit with a general output current modeled as:

$$I[V(t)] = \sum_{r,s} I^c_{r,s} cos(r\omega_1 + s\omega_2)t$$
$$+ I^s_{r,s} sin(r\omega_1 + s\omega_2)t$$

Here integers r, s are used to represent multiple harmonics of the original input frequencies. Identities such as $cos(a)cos(b) = \frac{1}{2}[cos(a+b) - cos(a-b)]$ have been used to represent products terms as single frequency signals. Terms $I^c_{r,s}, I^s_{r,s}$ are used to allow the output to have cosine terms in phase with the inputs and sine terms out of phase with the inputs. In this general model it would be assumed that the product terms of the original input frequencies such as $cos(\omega_1 t)cos(\omega_2 t)$ would be the largest amplitude and of primary interest.

4.2.2 Introducing Non-Linear Elements to the PEEC NFDEs

In [2], the delayed electromagnetic interactions are described using:

$$
\begin{aligned}
Lp_{\alpha\beta} &= \frac{\mu}{4\pi}\frac{1}{a_\alpha a_\beta}\int_{u_\alpha}\int_{u_\beta}\frac{1}{R_{\alpha\beta}}du_\alpha du_\beta \\
v_{\alpha\beta} &= Lp_{\alpha\alpha}\frac{di_\alpha}{dt}(t) + Lp_{\alpha\beta}\frac{di_{L\beta}}{dt}(t-\tau_{\alpha\beta}) \\
v_L(t) &= L_p\frac{di_L}{dt}(t-\tau^v)
\end{aligned}
\tag{4.1}
$$

with v_L, L_p, i_L, τ^v vectors comprised of elements described above

α, β two volume cells in the model,

$R_{\alpha\beta}$ the center to center distance between cells α, β

Thus state variable v_L is defined with delays. State variables i_L have an analogous definition with the roles of v, i reversed. Both use the discrete idealization of assuming all current flow/ free charges are located at the centers of volume/ surface elements α, β.

For the LTI case:

$-Av_n(t) - Ri_L(t) - L_p\frac{di_L}{dt}(t-\tau^\nu) = v_s(t) \qquad (20a)$

$-Av_n(t)$: A is the connectivity matrix showing nodes connected by branches. $v_n(t)$ are nodal voltages.

$Ri_L(t)$: R is the resistance of a volume elements in the directions of $i_L(t)$ which is a vector of currents flowing through volume elements. This term is the Ohm's law voltage drop of volume elements, $V = I \cdot R$

$L_p \frac{di_L}{dt}(t-\tau^\nu)$ is the voltage drop caused by mutual inductance of 2 volume elements with $\frac{di_L}{dt}(t-\tau^\nu)$ the derivative of current flowing in volume elements inductively coupled to other elements at delays given by vector τ^ν.

$v_s(t)$ are voltage drops between pairs of volume elements due to externally applied electric fields.

$$C_s \frac{dv_n(t)}{dt} + P_n i_{lumped}(t-\tau^s) - P_n A^T i_L(t-\tau^s) = P_n i_s(t-\tau^s) \qquad (20b)$$

$C_s \frac{dv_n(t)}{dt}$: C_s is the vector of capacitance of surface elements, $\frac{dv_n(t)}{dt}$ is the vector of derivatives of voltages of nodes which are centered in surface elements. This term gives current due to change in node voltage.

$P_n i_{lumped}(t-\tau^s)$: P_n is the matrix of coefficients of potential between surface elements. $i_{lumped}(t-\tau^s)$ is the delayed vector of currents flowing in external resistive memoryless elements connecting nodes to complete circuits in the PEEC model.

$P_n i_{lumped} A^T i_L(t-\tau^s)$: P_n is the matrix of coefficients of potential between surface elements. A is the connectivity matrix showing nodes connected by branches. $i_L(t-\tau^s)$ is the delayed vector of currents from volume elements. This term is currents into a node from connected volume elements with delays due to distance to centers of volume elements.

$P_n i_s(t-\tau^s)$: P_n is the matrix of coefficients of potential between surface elements. $i_s(t-\tau^s)$ is the delayed vector of external current sources.

Here i_{lumped} represents external components connected to nodes in the model. This is related to voltages through:

$$P_n i_{lumped}(t-\tau^s) = P_n G_l v_n(t-\tau^s)$$

G_l are conductances representing external connections between nodes, completing circuit paths. In this case the external connections are assumed to be memoryless resistive components.

4.2.3 Nonlinear Response of Inductive and Capacitive Elements

For nonlinear response of inductive elements, let
$i_{L1-nonlinear} = i_{L1}(t) \sum_{j=1}^{N_v} f_{1,j} Lp_{1,j} \frac{di_{Lj}}{dt}(t - \tau^{\nu}_{1,j})$, N_v = number of volume elements.

This equation shows a detailed representation considering element to element interactions. $Lp_{1,j}$ is the partial inductance linking volume elements 1 and j. i_{Lj} is the current in volume element j. Nonlinear weighting coefficient $f_{1,j}$ will only be non-zero if current i_{Lj} induces a current in cell 1 parallel to the direction of i_{L1}. Delay $\tau^{\nu}_{1,j}$ is the delay between volume cells $1, j$. Here $Lp_{1,j} \frac{di_{Lj}}{dt}(t - \tau^{\nu}_{1,j})$ is the voltage induced at cell 1 due to the derivative of the current in cell j. It is assumed that this voltage is still described by this linear relation and that the nonlinearity is due to the current response of volume cell 1 to this voltage.
The vector of all such nonlinear current responses is given by:

$$i_{L-nonlinear} = F I_L(t) f_j Lp_r \frac{di_L}{dt}(t - \tau^{\nu}), \text{ where}$$

$$I_L(t) = \begin{pmatrix} i_1(t) & 0 & \dots & 0 \\ 0 & i_2(t) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & i_n(t) \end{pmatrix}.$$

Here, τ^{ν} is a vector of all distinct time delays between volume elements. Some pairs of elements may have identical time delays between them due to the regularity of the PEEC element spacing.

Similarly for nonlinear response of capacitive elements we have the following: $v_{n1-nonlinear} = v_{n,1}(t) \sum_{j=1}^{N_s} f_{1,j} P_{1,j} G_{l,j} v_{n,j}(t - \tau_{1,j}^s)$ $\quad N_s$ = number of surface elements.

$P_{1,j}$ is the coefficient of potential capacitively linking surface elements 1 and j. $v_{n,j}$ is the voltage in surface element j. Nonlinear weighting coefficient $f_{1,j}$ will only be non-zero if voltage $v_{n,j}$ induces a voltage in cell 1 parallel to the direction of $v_{n,1}$. Delay $\tau_{1,j}^s$ is the delay between surface cells $1, j$. Here $P_{1,j} G_{l,j} v_{n,j}(t - \tau_{1,j}^s)$ is the current induced at cell 1 due to the voltage in cell j. It is assumed that this current is still described by this linear relation and that the nonlinearity is due to the voltage response of surface cell 1 to this current.

The vector of all such nonlinear voltage responses is given by:

$$v_{n-nonlinear} = F V_C(t) P_{n,r} G_l v_n(t - \tau^s), \text{ where}$$

$$V_C(t) = \begin{pmatrix} v_1(t) & 0 & \dots & 0 \\ 0 & v_2(t) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & v_n(t) \end{pmatrix}$$

Here τ^s is a vector of all distinct time delays between surface elements. Some pairs of elements may have identical time delays between them due to the regularity of the PEEC element spacing.

4.2.4 PEEC NFDE Equations with Nonlinear Contributions

For the LTI PEEC NFDE model [2]:

$$-Av_n(t) - Ri_L(t) - Lp_{nr}\tfrac{di_L}{dt}(t) = v_s(t) + Lp_r\tfrac{di_L}{dt}(t-\tau^v) \qquad (24a)$$

$$C_s\tfrac{dv_n(t)}{dt} + P_{n,nr}G_l v_n(t) - P_{n,nr}A^T i_L(t)$$
$$= P_n i_s(t-\tau^s) - P_{n,r}G_l v_n(t-\tau^s) + P_{n,r}A^T i_L(t-\tau^s) \qquad (24b).$$

We introduce nonlinear capacitance voltage effects here as:

$$-Av_n(t) - Ri_L(t) - Lp_{nr}\tfrac{di_L}{dt}(t)$$
$$= v_s(t) + Lp_r\tfrac{di_L}{dt}(t-\tau^v) + F_C V_C(t) P_{n,r} G_l v_n(t-\tau^s) \qquad (24c)$$

We introduce nonlinear inductance current effects here as:

$$C_s\frac{dv_n(t)}{dt} + P_{n,nr}G_l v_n(t) - P_{n,nr}A^T i_L(t)$$
$$= P_n i_s(t-\tau^s) - P_{n,r}G_l v_n(t-\tau^s) + P_{n,r}A^T i_L(t-\tau^s) +$$
$$F_L I_L(t) Lp_r \frac{di_L}{dt}(t-\tau^{\nu_j}) \qquad (24d)$$

$$\begin{pmatrix} -A & -(R+Lp_{nr})\frac{d}{dt} \\ C_s\frac{d}{dt} + P_{n,nr}G_l & +P_{n,nr}A^T \end{pmatrix} \cdot \begin{pmatrix} v_n(t) \\ i_L(t) \end{pmatrix} =$$
$$\begin{pmatrix} v_s(t) + Lp_r\frac{di_L}{dt}(t-\tau^v) \\ P_n i_s(t-\tau^s) - P_{n,r}G_l v_n(t-\tau^s) + P_{n,r}A^T i_L(t-\tau^s) \end{pmatrix}$$
$$+ \begin{pmatrix} F_C V_C(t) P_{n,r} G_l v_n(t-\tau^s) \\ F_L I_L(t) Lp_r \frac{di_L}{dt}(t-\tau^{\nu_j}) \end{pmatrix}$$

In [2], these matrix variables are converted to the A_k, B_k matrices in the final

PEEC NFDEs. For linear time-invariant PEEC NFDEs:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} \tilde{A}_i x(t - \tau_i^s) + \sum_{j=1}^{h} \tilde{B}_j x(t - \tau_j^\nu)$$

$$A_0 = \begin{pmatrix} 0 & -Lp_{nr} \\ C_s & 0 \end{pmatrix}^{-1} \begin{pmatrix} A & R \\ -P_{n,nr} G_l & P_{n,nr} A^T \end{pmatrix}$$

$$\tilde{A}_i = \begin{pmatrix} 0 & -Lp_{nr} \\ C_s & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ -K_i \bigodot P_{n,nr} G_l & K_i \bigodot P_{n,nr} A^T \end{pmatrix}$$

$$\tilde{B}_j = \begin{pmatrix} 0 & -Lp_{nr} \\ C_s & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & H_j \bigodot Lp_r \\ 0 & 0 \end{pmatrix}$$

The matrix A used here is the connectivity matrix and is not the same as A_k used in PEEC NFDEs.

For linear autonomous PEEC NFDEs:

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{m} B_k x(t - \tau_k)] = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k)$$

$$\text{initial condition: } x_\sigma = \phi$$

Adding both nonlinear inductive and capacitive elements would result in:

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{m} B_k x(t - \tau_k)] + \sum_{k=1}^{m} F_{Lk} X(t) B_k \frac{d}{dt} x(t - \tau_k) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k)$$

$$+ \sum_{k=1}^{m} F_{Ck} X(t) A_k x(t - \tau_k) \qquad (4.2)$$

with history function $x_\sigma = \phi$.

Here, the inductive terms on the left hand side of the equation make the D operator nonlinear. We chose to introduce only the nonlinear capacitive elements to the PEEC NFDE model which results in a linear D operator.

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{m} B_k x(t-\tau_k)] = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k) + \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k)$$

$$\text{with history function } x_\sigma = \phi. \tag{4.3}$$

4.3 Summary

Various perturbations to the PEEC NFDEs were considered. We determined that linear pertubations would be analogous to the original unperturbed model and did not study them further. Of nonlinear perturbations considered we chose the nonlinear capacitive elements since the D operator remains linear in this case. The linear D operator allows a more straightforward analysis for a first nonlinear PEEC NFDE model.

Chapter 5

Stability and Convergence of LTI PEEC NFDEs With Nonlinear Perturbations

5.1 Introduction

In this chapter, we discuss stability and convergence of linear autonomous PEEC NFDEs with nonlinear perturbations. Nonlinear perturbations to the PEEC NFDEs have not been previously studied in existing literature. Here, for the first time we develop sufficient conditions for uniform stability and convergence of linear autonomous PEEC NFDEs with nonlinear perturbations.

5.2 LTI PEEC NFDE and Nonlinear Perturbed Systems

For the Linear Autonomous PEEC system define $NFDE(\sigma, \phi, f)$ as:

$$\frac{d}{dt}Dx_t = f(x_t), \quad \text{with initial condition} \quad x_\sigma = \phi,$$
$$Dx_t = x(t) - \sum_{k=1}^{m} B_k x(t - \tau_k) \tag{L}$$
$$f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k)$$

with $x_t, \phi \in C,$ and set $\{\tau_i\} = \{\tau_0,\ \tau_1, \dots,\ \tau_m\}$,

with delays $-\tau = -\tau_m < -\tau_{m-1} < \cdots < -\tau_1 < -\tau_0 = 0.$

A time invariant nonlinear perturbation is introduced by adding a mixing term to each delayed term at a given PEEC model element:

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{m} B_k x(t - \tau_k)] =$$
$$A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k) + \sum_{k=1}^{m} F_k X(t) A_k x(t - \tau_k),$$

Here $X(t) = \begin{pmatrix} x_1(t) & 0 & \dots & 0 \\ 0 & x_2(t) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & x_n(t) \end{pmatrix}$, $F_k, X(t) \in \mathbb{R}^{n \times n}$.

Consider the autonomous nonlinear $NFDE$ as:

$$\frac{d}{dt}Dy_t = f(x_t) + g(y_t), \quad y_\sigma = \phi$$
$$\text{with } D(x_t) = x(t) - \sum_{k=1}^{m} B_k x(t-\tau_k)$$
$$f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k) \qquad (P_a)$$
$$g(x_t) = \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k).$$

Let a time varying nonlinear perturbation be given by:

$\bar{g}(x_t) = h_1(t)\sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k)$, with $h_1(t) > 0$, $h_1(t)$ continuous and $\int_0^\infty h_1(t)dt < \infty$.

Consider the time varying nonlinear $NFDE$ as:

$$\frac{d}{dt}Dy_t = f(x_t) + g(y_t), \quad y_\sigma = \phi$$
$$\text{with } D(x_t) = x(t) - \sum_{k=1}^{m} B_k x(t-\tau_k),$$
$$f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k), \qquad (P_b)$$
$$\bar{g}(x_t) = \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k).$$

5.3 Preliminaries

We recall a few relevant lemmas and results from the literature that are used in this chapter.

We use Theorem A.1.4 (see Appendix A) to prove existence and uniqueness of the solution to (L). We have $(\sigma, \phi) \in \Omega = \mathbb{R} \times\ C$ which is an open set.

To prove Lipschitz property of f we consider $f(\phi) = A_0\phi(t) + \sum_{k=1}^{m} A_k\phi(t-\tau_k)$ with A_k constant for $k =\in 0 : m$, thus also $|A_k| < \infty$ for $k =\in 0 : m$. Let $|\phi| \leq \delta$, with $\delta > 0$ then $|f(\phi)| \leq |A_0||\phi| + |\phi| \sum_{k=1}^{m} |A_k| \leq \delta \sum_{k=0}^{m} |A_k|$
and f is Lipschitzian in ϕ independent of σ. From Theorem A.1.4 then we have the following result:

Lemma 5.3.1. *For all $(\sigma, \phi) \in \Omega = \mathbb{R} \times C$ there exists a unique solution of (L), through (σ, ϕ).*

Define set $U = \{(t, \sigma, \phi) \in \mathbb{R} \times \ \mathbb{R} \times \ C : x_t(\sigma, \phi) \text{ is defined for } NFDE(D, f)\}$.
For all fixed $(t, \sigma) \in \mathbb{R} \times \ \mathbb{R}$ define $U(t, \sigma) = \{\phi \in C : (t, \sigma, \phi) \in U\}$.
We define set W, the image of solution map $T(t, \sigma)$.

$$W = \{\phi \in C : \exists\psi \in C, \text{ such that } T(t, \sigma) = \phi \text{ for some } \sigma \in \mathbb{R}, t \geq \sigma\}. \tag{5.1}$$

From Lemma 5.3.1 then for system (L), $U = \mathbb{R} \times \mathbb{R} \times C$ and $U(t, \sigma) = C$. From Lemma 3.2.1 and Theorem A.6.1 (see Appendix A) we have this result.

Lemma 5.3.2. *If $det(B_m) \neq 0$ then D is atomic at $-\tau$ and 0 and for LTI NFDE (L), solution map $T(t, \sigma) : C \to W$ is a homeomorphism.*

For the LTI NFDE, (L), define:

$$K = |D| = \sup_{\phi \in C} \frac{|D\phi|}{|\phi|} = 1 + \sum_{i=1}^{m} |B_i|,$$

$$\Sigma_A = \sum_{i=1}^{m} |A_i|, \quad \Sigma_B = \sum_{i=1}^{m} |B_i|, \quad P = -(A_0)^{-1},$$

$$P_s = \frac{1}{2}(P + P^T), \quad P_a = \frac{1}{2}(P - P^T), \quad \rho = |P_a|/(|P|)$$

$\lambda_{Pmax}, \lambda_{Pmin} =$ eigenvalues of P_s with maximum,

minimum magnitude,

$r_{\lambda_P} = \sqrt{\dfrac{\lambda_{Pmax}}{\lambda_{Pmin}}} \geq 1, \quad$ Define $\varsigma = \rho|P||A_0|$

For system (L) by Riesz representations of operators D, f we have:

$D\phi = \phi(0) - \int_{-\tau}^{0} d\mu(t)\phi(\theta), \quad f(\phi) = \int_{-\tau}^{0} d\eta(t)\phi(\theta).$

The characteristic matrix for system (L), as in [17], is given by

$\Delta(z) = z\big(I - \int_{-\tau}^{0} e^{zt} d\mu(t)\big) - \int_{-\tau}^{0} e^{zt} d\eta(t) \quad z \in \mathbb{C}.$

From [17] we get an exponential bound for the solution of a linear NFDE.

Let $\alpha_{D,f} = \sup\{Rez : det(\Delta(z)) = 0\}$.

Then for any $a > \alpha_{D,f}$ there is a constant $c = c(a)$ such that solution x_t of (L) satisfies:

$|x(t;\phi)| \leq ce^{at}|\phi| \quad t \geq \sigma.$

Thus also $|x_t(\sigma, \phi)| \leq ce^{at}|\phi| \quad t \geq \sigma.$

For stable systems we have that $\alpha_{D,f} < 0$.

Lemma 5.3.3. *For a stable system (L) there exists an $\alpha < 0$, $a \in (\alpha, 0)$, and constant $c = c(a) > 0$ such that*

$|x_t(\sigma, \phi)| \leq ce^{at}|\phi|, \qquad t \geq \sigma.$

Remark 5.3.4. For the stable LTI PEEC NFDE, (L) if the solution map $T(t, \sigma)$ is a homeomorphism then we have this result.

Lemma 5.3.5. $|T(t, \sigma)\phi|$, *with* $|\phi| = 1$ *has a non-zero lower bound. Also there exists* $q > 0$ *such that* $\inf_{\psi \in C, \ |\psi|=1} |T(t, \sigma)\psi| > q \quad \forall t \geq \sigma.$

Proof: If the solution map $T(t,\sigma)$ is a homeomorphism then $|T(t,\sigma)\phi| = 0$ iff $\phi = 0$. Also $T(t,\sigma)$ has the closed mapping property and for $S = \{\phi \in C : |\phi| = 1\}$ then $T(t,\sigma)S$ is a closed set that does not contain zero.
Thus there exists $q > 0$ such that $\underset{\psi \in C,\ |\psi|=1}{inf} |T(t,\sigma)\psi| > q$ for each $t \geq \sigma$.

Here, we list the definition of the linear variational equation of (L) with respect to the solution $x_t(s, y_s(\sigma,\phi))$:

$$\begin{aligned} &\frac{d}{dt} D u_t = f_x(t, x_t(s, y_s(\sigma,\phi))) u_t, \quad t \geq s \geq \sigma, \\ &u_\sigma = \phi. \end{aligned} \tag{5.2}$$

5.4 Stability and Convergence Conditions for Nonlinear Perturbations of the LTI PEEC NFDEs

Here we present the analysis of PEEC NFDEs with nonlinear perturbations. We have developed results giving sufficient conditions for stability and convergence of the PEEC NFDEs with nonlinear perturbations.

Theorem 5.4.1. *If system (L) is uniformly stable and satisfies the following conditions:*
a) initial condition $x_\sigma = \phi$ satisfies $D\frac{d\phi}{dt} = f(\phi)$ (Compatability Condition)
b) $det(B_m) \neq 0$,
then nonlinear systems (P_a), (P_b) are uniformly stable.
For solutions y_t of (P_b), x_t of (L) for all $\epsilon > 0$, $c > 0$ there exists real $\Theta = \Theta(\epsilon,\ c)$ such that if $|\phi| < c$

then the solutions of (P_b) *are bounded in the future and*

$|y_t(\tau,\phi) - x_t(\tau,\phi)| < \epsilon \qquad \forall\ t \geq \tau \geq \Theta.$

Remark 5.4.2. condition a) in this theorem is necessary to allow use of (A.6.3). In the application however, this may be a restrictive condition: it may be desired to model behavior for the system having any starting state, $x_\sigma = \phi$ for all $\phi \in C$. Here we consider one way to mitigate the restriction imposed by condition a).
If initial condition $x_\sigma = \phi$ does not satisfy $D\frac{d\phi}{dt} = f(\phi)$ then a related system, $(\bar{L})$, using the first step of the solution of linear (L) will.

From original system

$$\frac{d}{dt}Dx_t = f(x_t), \quad \text{with initial condition } \ x_\sigma = \phi, \quad (L)$$

we generate

$$\frac{d}{dt}D\bar{x}_t = f(\bar{x}_t), \quad \text{with initial condition } \ \bar{x}_{\bar{\sigma}} = x_{\sigma+\tau}, \quad (\bar{L}).$$

Here $\bar{\sigma} = \sigma + \tau$.

In the application this is reasonable since typically $\tau = 10^{-11}$ and solutions are calculated for $t > \sigma + 10^{-3}$.

Remark 5.4.3. Condition b), $det(B_m) \neq 0$, is required so that the solution map of the linear autonomous PEEC NFDE will be a homeomorphism. This is important to the analysis done here. However, in the application, matrix B_m may be very sparse due to connectivity of the finite elements in the PEEC model. Thus, in the application $det(B_m) = 0$, may occur. So, we consider an arbitrariliy small linear perturbation to guarantee $det(B_m) \neq 0$ which is appropriate for the application. For any $\epsilon > 0$, let $B'_m = B_m + \epsilon I$. For system (L) if the atomic at $-\tau$ condition is not met, $det(B_m) = 0$, then $\epsilon > 0$ with $det(B'_m) \neq 0$ can be chosen arbitrarily small. For what $\epsilon > 0$ does this not work? Suppose $B'_m = B_m + \epsilon I$ for such an ϵ. If $det(B'_m) = 0$ then there exists $V \in \mathbb{R}^n, V \neq 0$ such that $B'_m V = 0, B_m V + \epsilon I = 0$. Thus $-\epsilon$ is a negative eigenvalue of the original matrix B_m.

Thus we have this result.

Lemma 5.4.4. *For $\epsilon > 0, det(B'_m) \neq 0$ iff $-\epsilon$ is not an eigenvalue of the original matrix B_m.*

It is not difficult to choose $\epsilon > 0$ such that $det(B'_m) \neq 0$. An example is given at the end of this chapter. We now generate new NFDE systems from systems $(L), (P_a)$ and (P_b) using the linear perturbation $B'_m = B_m + \epsilon I$. The NFDE systems, $(L'), (P'_a)$ and (P'_b) will be used to obtain sufficient conditions for stability and convergence of the PEEC NFDEs with nonlinear perturbations. For the Linear Autonomous PEEC system replace B_m with B'_m to obtain the $NFDE$:

$$\frac{d}{dt}Dx_t = f(x_t), \quad \text{with initial condition} \quad x_\sigma = \phi,$$

$$f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k), \qquad (L')$$

$$D(x_t) = x(t) - \sum_{k=1}^{m-1} B_k x(t - \tau_k) + B'_m x(t - \tau).$$

with $x_t, \phi \in C$. Define autonomous nonlinear $NFDE$ as:

$$\frac{d}{dt}Dy_t = f(y_t) + g(y_t), \quad y_\sigma = \phi,$$

$$\text{with } f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k), \qquad (P'_a)$$

$$g(x_t) = \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k),$$

$$D(x_t) = x(t) - \sum_{k=1}^{m-1} B_k x(t-\tau_k) + B'_m x(t-\tau).$$

Define a time varying nonlinear perturbation:

$\bar{g}(x_t) = h_1(t) \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k)$ with $h_1(t)$ such that $\int_0^\infty h_1(t)dt < \infty$.

Using this, we consider a time varying nonlinear $NFDE$ given by:

$$\frac{d}{dt} D y_t = f(y_t) + \bar{g}(y_t), \quad y_\sigma = \phi,$$

$$\text{with } f(x_t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t-\tau_k), \qquad (P'_b)$$

$$\bar{g}(x_t) = h_1(t) \sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k),$$

$$D(x_t) = x(t) - \sum_{k=1}^{m-1} B_k x(t-\tau_k) + B'_m x(t-\tau).$$

Theorem 5.4.5. *Let system (L) satisfy condition a) for Theorem 5.4.1 but $det(B_m) = 0$ so that D from (L) is not atomic at $-\tau$.*
Assume delays $\{\tau_i\}$ are rational multiples of a common factor as in (3.6). From D we create a companion matrix $A_{advstep}$ as defined in Theorem 3.4.4. Assume conditions i) - vi) from Theorem 3.4.4 hold for the original system (L).
Then there exists $\epsilon > 0, \epsilon$ arbitrarily small, with matrix $A_{advstep}$ and systems (L'), (P'_a), and (P'_b) defined as above such that $det(B'_m) \neq 0$, and conditions i) - vi) from Theorem 3.4.4 hold for system (L') : thus system (L') is uniformly

asymptotically stable and nonlinear systems (P'_a), (P'_b) *are uniformly stable,*
For solutions y_t *of* (P'_b), x_t *of* (L') *then* $\forall\ \epsilon > 0,\ c > 0$ *there exists real* $\Theta = \Theta(\epsilon,\ c)$ *such that if* $|\phi| < c$,
then the solutions of (P'_b) *are bounded for all* $t \geq \sigma$ *and*
$|y_t(\tau, \phi) - x_t(\tau, \phi)| < \epsilon \qquad \forall\ t \geq \tau \geq \Theta$

Here we prove Theorem 5.4.1. Since system (L) is stable and $det(B_m) \neq 0$, then from (5.3.2)
D is atomic at $-\tau, 0$ and for LTI NFDE (L), solution map $T(t, \sigma) : C \to C$ is a homeomorphism.
Define $\Phi(t, \sigma, \phi) = \frac{\partial}{\partial \phi} x_t(\sigma, \phi)$.
The Frechet derivative $\frac{\partial}{\partial \phi} x_t(\sigma, \phi)$ is given by operator $A : C \to C$ such that:
$\lim_{\phi_n \to 0} |(x_t(\sigma, \phi + \phi_n) - x_t(\sigma, \phi) - A\phi_n|/|\phi_n| = 0, \quad$ for $\phi_n \in C$.
If this limit is zero for operator A then $\frac{\partial}{\partial \phi} x_t(\sigma, \phi) = A$.

$|(x_t(\sigma, \phi + \phi_n) - x_t(\sigma, \phi) - A\phi_n|/|\phi_n|$

$= |T(t, \sigma)(\phi + \phi_n) - T(t, \sigma)\phi - A\phi_n|/|\phi_n|$

$= |T(t, \sigma)\phi_n - A\phi_n|/|\phi_n|$

for operator $A = T(t, \sigma)$ (5.3)

$= |T(t, \sigma)\phi_n - T(t, \sigma)\phi_n|/|\phi_n| = 0 \quad$ for all $\phi_n \in C, \phi \neq 0$.

Thus $\lim_{\phi_n \to 0} |(x_t(\sigma, \phi + \phi_n) - x_t(\sigma, \phi) - A\phi_n|/|\phi_n| = 0$,
and $\dfrac{\partial}{\partial \phi} x_t(\sigma, \phi) = T(t, \sigma)$.

From (5.3) and Theorem A.6.1 (see Appendix A) then $\Phi(t, \sigma, \phi) = \frac{\partial}{\partial \phi} x_t(\sigma, \phi)$ exists and is continuous $\forall\ t \geq \sigma, \phi \in C, \Phi^{-1}(t, \sigma, \psi)$ exists $\forall\ t \geq \sigma, \psi \in W$. Theorem

A.13 states that its results are valid as long as z_t exists as a solution of:

$$\frac{d}{dt}z_t(\sigma,\phi) = \Phi^{-1}(t,\sigma,z_t)Y_\sigma g(t, x_t(\sigma, z_t)). \tag{5.4}$$

We now show that for the NFDE (L) then such z_t will exist for all $t \geq \sigma$.

For the linear variational equation (5.2), let $f_x = \frac{\partial}{\partial x_t} f(x_t)$.

We find the Frechet derivative as below:

$$\begin{aligned}
&|f(x_t + x_t^1) - f(x_t) - Ax_t^1|/|x_t^1|,\\
&= |f(x_t^1) - Ax_t^1|/|x_t^1| \quad \text{by linearity of } f,\\
&\text{for operator } Ax_t^1 = f(x_t^1) \text{ then,}\\
&= |f(x_t^1) - f(x_t^1)|/|x_t^1| = 0 \quad \text{for } x_t^1 \in C, x_t^1 \neq 0.\\
&\text{Then } \lim_{x_t^1 \to 0} |f(x_t^1) - f(x_t^1)|/|x_t^1| = 0 \quad \text{for } x_t^1 \in C\\
&\text{and } \frac{\partial}{\partial x_t} f(x_t) = f(x_t).
\end{aligned} \tag{5.5}$$

The variational equation then becomes:

$\frac{d}{dt}Du_t = f(u_t), \quad t \geq s \geq \sigma \geq 0.$

This is equivalent to the original NFDE, (L) .

For $z_t \in C$, let $y_t \in C$ be defined as:

$$y_t(\sigma,\phi) = x_t(\sigma, z_t), \qquad x_t \in C,\ x_t \text{ a solution of } (L). \tag{5.6}$$

We select z_t such that $\frac{d}{dt}z_t(\sigma,\phi) = \Phi^{-1}(t,\sigma,z_t)Y_\sigma g(t, x_t(\sigma, z_t)), \quad z_\sigma = \phi.$

Thus also $\frac{d}{dt}z_t(\sigma,\phi) = \Phi^{-1}(t,\sigma,z_t)Y_\sigma g(t, y_t(\sigma, \phi)).$

[46] shows that such $y_t = x_t(\sigma, z_t)$ is a solution of perturbed

nonlinear systems $(P_a), (P_b)$.

For (L) we define $\Phi(t,\sigma,\phi) = \frac{\partial}{\partial t} x_t(\sigma,\phi)$.

From (5.3), for linear autonomous (L):

$\Phi(t,\sigma,\phi)\psi = \Phi(t,\sigma)\psi = T(t,\sigma)\psi, \quad \forall\ t \geq \sigma,\ \phi,\psi \in C.$

Also: $\Phi^{-1}(t,\sigma,\phi)\psi = \Phi^{-1}(t,\sigma)\psi = T^{-1}(t,\sigma)\psi, \quad \forall\ t \geq \sigma,\ \phi,\psi \in W.$

From (A.6.2), $Y_\sigma g(t, y_t(\sigma,\phi))$ is a solution of the linear variational equation (5.2).

From (5.6), for some $\psi \in C$, we have

$Y_\sigma g(t, x_t(\sigma, z_t(\sigma,\phi))) = Y_\sigma g(t, y_t(\sigma,\phi)) = T(t,\sigma)\psi = \Phi(t,\sigma)\psi.$

The theorem condition (5.4) then becomes for $\psi \in C$:

$$\frac{d}{dt} z_t(\sigma,\phi) = \Phi^{-1}(t,\sigma)\Phi(t,\sigma)\psi$$

$$= \psi. \tag{5.7}$$

Then

$$z_t(\theta) = z_t(-\tau) + \int_{-\tau}^{\theta} \psi(s)ds \qquad \theta \in [-\tau, 0]. \tag{5.8}$$

Since such ψ is continuous the solution z_t exists for all $t \geq \sigma$.

Thus from A.13 we see that the solution $y_t(\sigma,\phi)$ of (P_a) satisfies:

$$y_t(\sigma,\phi) = x_t\big(\sigma, \int_\sigma^t \Phi^{-1}(s,\sigma,z_s) Y_\sigma g(s, y_s(\sigma,\phi))ds\big).$$

Also

$$y_t(\sigma,\phi) = x_t(\sigma,\phi) + \int_\sigma^t \Phi(t,\sigma,z_s)\Phi^{-1}(s,\sigma,z_s) Y_\sigma g(s, y_s(\sigma,\phi))ds, \quad \forall\ t \geq \sigma.$$

Similarly, solution $\bar{y}_t(\sigma,\phi)$ of (P_b) satisfies:

$$\bar{y}_t(\sigma,\phi) = x_t\big(\sigma, \int_\sigma^t \Phi^{-1}(s,\sigma,z_s) Y_\sigma \bar{g}(s, y_s(\sigma,\phi)) ds\big)$$

also

$$\bar{y}_t(\sigma,\phi) = x_t(\sigma,\phi) + \int_\sigma^t \Phi(t,\sigma,z_s)\Phi^{-1}(s,\sigma,z_s) Y_\sigma \bar{g}(s, y_s(\sigma,\phi)) ds \quad \forall\ t \geq \sigma.$$

For Theorem A.6.4 (see Appendix A), we find the norm $|\Phi^{-1}(t,\sigma,\phi)|$:

$$|\Phi^{-1}(t,\sigma,\phi)| = \sup_{\psi \in W,} \Big[|\Phi^{-1}(t,\sigma,\phi)\psi| / |\psi| \Big] = \sup_{\psi \in W,} \Big[|\ T^{-1}(t,\sigma)\psi| / |\psi| \Big]$$

Since $T^{-1}(\sigma,\phi)\psi$ exists $\forall\ \psi \in W, \quad T(\sigma,\phi)C = W.$

$$\begin{aligned} |\Phi^{-1}(t,\sigma,\phi)| &= \sup_{\psi \in C,} [|T^{-1}(t,\sigma)T(t,\sigma)\psi| / |T(t,\sigma)\psi|]. \\ &= \sup_{\psi \in C,} [|\psi| / |T(t,\sigma)\psi|] = \sup_{\psi \in C,\ |\psi|=1} \frac{1}{|T(t,\sigma)\psi|} \\ &= \frac{1}{\inf_{\psi \in C,\ |\psi|=1} |T(t,\sigma)\psi|}. \end{aligned} \tag{5.9}$$

Also

$$\begin{aligned} |g(t, x_t(\sigma,\phi)| &= |\sum_{k=1}^{m} F_k X(t) A_k x(t-\tau_k)| \\ &\leq |x(t)|\ |x_t| \sum_{k=1}^{m} |F_k||A_k|\ \leq |x_t|^2 \sum_{k=1}^{m} |F_k||A_k|. \end{aligned} \tag{5.10}$$

From (5.9), (5.14) and $|Y_\sigma| = 1$,

$|\Phi^{-1}(t,\sigma,\phi) Y_\sigma g(t, x_t(\sigma,\phi)| \leq \frac{|x(t)|\ |x_t| \sum_{k=1}^{m} |F_k||A_k|}{\inf_{\psi \in C,\ |\psi|=1} |x_t(\sigma,\psi)|} \leq \frac{|x_t|^2 \sum_{k=1}^{m} |F_k||A_k|}{\inf_{\psi \in C,\ |\psi|=1} |x_t(\sigma,\psi)|}.$

From (5.3.3) then

For a stable system (L), $\exists\ \alpha < 0,$ and constant

$c = c(a) > 0$ such that

$$|\Phi(t,\sigma,\phi)| = |x_t(\sigma,\phi)| \leq ce^{at}|\phi| \qquad t \geq \sigma, \phi \in C. \tag{5.11}$$

From (5.3.5) then

$|T(t,\sigma)\phi|,$ with $|\phi| = 1$ has a non-zero lower bound. Then

$\exists\ q > 0$ such that

$$|\Phi^{-1}(t,\sigma,\phi)| = \frac{1}{\underset{\psi \in C,\ |\psi|=1}{inf} |T(t,\sigma)\psi|} < 1/q \quad \forall t \geq \sigma, \phi \in W \tag{5.12}$$

and

$|\Phi^{-1}(t,\sigma,\phi)Y_\sigma g(t, x_t(\sigma,\phi)| \leq c^2 e^{2at}|\phi|^2 \sum_{k=1}^m |F_k||A_k|/q.$

In Theorem A.6.4 (see Appendix A) define $h(t,u) = c_1 e^{2at}u^2$, with $c_1 = c^2/q \sum_{k=1}^m |F_k||A_k| > 0$.

Then for the nonlinear system (P_a), generate the nonlinear ODE:

$\dot{u} = h(t,u), \quad u \geq 0$ and $h(t,0) = 0$ with nontrivial initial condition $u(\sigma) = k > 0$.

Thus: $u^{-2}\dot{u} = c_1 e^{2at}$.

We solve: $\frac{1}{2}(u^{-1}(t) - u^{-1}(\sigma)) = \int_\sigma^t c_1 e^{2a\theta} d\theta = \frac{c_1}{-2a}(e^{2a\sigma} - e^{2at}) \geq 0, \quad$ for $c_1 > 0, a < 0, t \geq \sigma$. Express the solution at the currrent time in terms of the initial value:

$$u(t) = 1/[1/k + \frac{c_1}{-a}(e^{2a\sigma} - e^{2at})] = k/[1 + \frac{c_1 k}{-a}(e^{2a\sigma} - e^{2at})] \leq k = u(\sigma). \tag{5.13}$$

Thus for $\dot{u} = h(t,u), u(\sigma) = k$ then $u(t) \leq u(\sigma) \quad$ for all $t \geq \sigma$.

Thus for all $\epsilon > 0$, let $\delta = \epsilon$ then if $u(\sigma) < \delta$, then $u(t) < \epsilon$

and the zero solution of $u(t)$ is uniformly stable.

From Theorem A.6.4 (see Appendix A) then the trivial solution of (P_a) is uniformly stable.

For system (P_b),

$\bar{g}(x_t) = h_1(t)g(x_t) \quad$ with $h_1(t) > 0,\ h_1(t)$ continuous and $\int_\sigma^\infty h_1(t)dt < \infty \quad t \geq \sigma$.

Let $n_1 = \int_\sigma^\infty h_1(t)dt$. Thus $0 < n_1 < \infty$

$$|\bar{g}(t, x_t(\sigma, \phi)| = |h_1(t) \sum_{k=1}^{m} F_k X(t) A_k x(t - \tau_k)| \ \leq n_1|x(t)|\ |x_t| \sum_{k=1}^{m} |F_k||A_k|,$$

$$\leq \ n_1|x_t|^2 \sum_{k=1}^{m} |F_k||A_k|. \tag{5.14}$$

$|\Phi^{-1}(t, \sigma, \phi)Y_\sigma \bar{g}(t, x_t(\sigma, \phi)| \leq \frac{n_1|x_t|^2 \sum_{k=1}^{m} |F_k||A_k|}{\underset{\psi \in C,\ |\psi|=1}{inf} |x_t(\sigma,\psi)|} \ \leq n_1|x_t|^2 \sum_{k=1}^{m} |F_k||A_k|/q$

In theorem A.6.4 (see Appendix A) define $h(t, u) = c_1 e^{2at} u^2$, with

$c_1 = n_1 c^2/q \sum_{k=1}^{m} |F_k||A_k| > 0$.

Again, $\dot{u} = h(t, u)$, with condition $\quad u \geq 0$ will result in the zero solution of $u(t)$ being uniformly stable.

From Theorem A.6.4 (see Appendix A) then the trivial solution of (P_b) is uniformly stable.

From (5.11), There exists $a < 0, c > 0$ such that $|\Phi(t, \sigma, \phi)| \leq ce^{at}|\phi| \leq c|\phi|$ for all $\phi \in C$.

From (5.12), there exists $q > 0$ such that $|\Phi^{-1}(t, \sigma, \phi)| < 1/q$ for all $\phi \in W$.

Let $N(\alpha) = min(c\alpha, 1/q)$. If $\phi| < \alpha$

then $|\Phi(t, \sigma, \phi)|, |\Phi^{-1}(t, \sigma, \phi)| < N(\alpha)$ for all $\phi \in W$ and this $N(\alpha)$ meets condition ii) for Theorem A.6.54.3 for system (L).

Define $h(t, \alpha) = c_1 e^{2at} \alpha^2, a < 0$ and $c_1 = n_1 c^2 \sum_{k=1}^{m} |F_k||A_k| > 0$.

Then for all $\alpha > 0$

if $|\phi| < \alpha$ then $|\bar{g}(t, x_t(\sigma, \phi)| < h(t, \alpha) \quad t \geq \sigma$, which meets condition iii) of Theorem A.6.5.

Thus, for solutions y_t of $(P_b),\ x_t$ of (L)

for all $\epsilon > 0,\ c > 0$ there exists real $\Theta = \Theta(\epsilon,\ c)$ such that

if $|\phi| < c$ then the solutions of (b) are bounded in the future and

$|y_t(\tau, \phi) - x_t(\tau, \phi)| < \epsilon \qquad \forall\ t \geq \tau \geq \Theta.$

Here we prove Theorem 5.4.5. Since system (L') meets the conditions for Razumikhin Theorem 3.4.4 then (L') is uniformly asymptotically stable.

From Theorem 5.4.1 then nonlinear systems $(P_a'),\ (P_b')$ are uniformly stable.

Further, for solutions y_t of $(P_b'),\ x_t$ of (L') then for all $\epsilon > 0,\ c > 0$ there exists real $\Theta = \Theta(\epsilon,\ c)$ such that if $|\phi| < c$ then the solutions of (P_b') are bounded for $t \geq \sigma$ and

$|y_t(\tau, \phi) - x_t(\tau, \phi)| < \epsilon \qquad \forall\ t \geq \tau \geq \Theta.$

5.5 Examples

Example 5.5.1. Here, we give an example to demonstrate that Theorem 5.4.5 conditions are consistent. Computations in the following example are done using Matlab R2018a (9.4.0.13654) Windows 64-bit version.

The following single delay system meets the criteria for the Theorem 5.4.1 to prove asymptotic stability yet has $det(B_m) = 0$. This means the D operator is not atomic at $-\tau$ and Theorem 5.4.1 can not be applied. Instead, we will have to find a suitable perturbation ceofficient, ϵ, then wiill apply Theorem 5.4.5.

Here we give an example of linear autonomous NFDEs (L) which meets Theorem 3.4.4 conditions. With $n = 3, p = 3, m = 1$ and $\tau_1 = 3.3000x10^{-11}sec$. System matrices are:

$$A0 = \begin{pmatrix} -1000 & 100 & 200 \\ 150 & -900 & 0 \\ 150 & 100 & -600 \end{pmatrix}, \tag{5.15}$$

$$A1 = \begin{pmatrix} 15.0000 & 0 & -45.0000 \\ -7.5000 & -7.5000 & -15.0000 \\ -7.5000 & -22.5000 & 0 \end{pmatrix}, \tag{5.16}$$

$$B1 = \begin{pmatrix} -.003473 & .017361 & .006945 \\ -.003473 & .017361 & .006945 \\ -.006945 & .013889 & .003472 \end{pmatrix}, \tag{5.17}$$

$$F1 = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}. \tag{5.18}$$

The eigenvalues of the symmetric component of A0, eigsA = $-1.1150 \times 1.0e + 03, -0.8730 \times 1.0e + 03, -0.5120 \times 1.0e + 03$

i) Since all eigenvalues are < 0 we have A0 is negative definite

The following constants can be calculated easily:

$K = 1.031,\ \Sigma_B = 0.030988,\ r_{\lambda_P} = 1.4368,\ 1/r_{\lambda_P} = 0.6832$

ii) Thus the condition ii) is met, $\Sigma_B < 1/r_{\lambda_P}$

Here numSubdivisions = Number of discretization intervals per time step = 2.

$$A_{advstep} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0035 & 0 & 0 & 0.0174 & 0 & 0 & 0.0069 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -.0035 & 0 & 0 & 0.0174 & 0 & 0 & 0.0069 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -.0069 & 0 & 0 & 0.0139 & 0 & 0 & 0.0035 & 0 \end{pmatrix} \tag{5.19}$$

Here we check $A_{advstep}$ eigenvalue requirements.

The eigenvalues of $A_{advstep}$ are:

$0.0000 + 0.0000i$, $0.13176 + 0.0000i$,

$0.0000 + 0.0000i$, $-0.13176 + 0.0000i$,

$0.0000 + 0.0000i$, $0.0000 + 0.00063i$,

$-0.0000 + 0.1107e - 06i$, $0.0000 - 0.00063i$,

$-0.0000 - 0.1107e - 06i$.

iii.a) Thus, we see that the maximum eigenvalue magnitude, maxEigA, is $0.1318 < 1$.

Here we have $rank(A_{advstep}) = 6$ and $nullity(A_{advstep}) = 3$.

iii.b) There are 6 distinct non-zero eigenvalues, eigenvectors of $A_{advstep}$ span its column space.

With $b = 1/(1 - maxEigA) = 1/(1 - 0.1318) = 1.1518$, we can easily calculate:

$\frac{K}{(1+Kr_{\lambda_P})} = 0.4109 < b = 1.1518 < \frac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}} = 9.0588.$

iii) Thus b is in the required range.

Here we have $\varsigma = \rho|P||A_0| = 0.10107 < [\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - \Sigma_B] = 0.21274$

iv.1) Thus ς is in the required range.

$\Sigma_A = 49.0359 < \frac{1}{|P_s|}[\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - (\Sigma_B + \varsigma)] = 57.3428$

iv.2) Thus Σ_A is in the required range. Since all the conditions in 3.4.1 are met it follows that the D operator is stable globally in the delays and the system 3.1.1 is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.

So NFDE (L) is uniformly asymptotically stable but has $det(B_m) = 0$ and does not meet the conditions for Theorem 5.4.1. Next we will choose an $\epsilon > 0$ to generate B'_m such that Theorem 5.4.5 can be used instead.

The following 1 delay system meets the criteria for the Theorem 5.4.5 to prove asymptotic stability and $det(B_m) \neq 0$. $\epsilon = \frac{|B_m|}{n^2} = 0.0034$ was used to generate $B'_m = B_m + \epsilon I$.

Here we give an example of NFDEs $(L'), (P'_a)$ and (P'_b) such that system (L') meets Theorem 3.4.4 conditions and these 3 systems together meet Theorem 5.4.5 conditions. For NFDE (L') with $n = 3, p = 3, m = 1$ and $\tau_1 = 3.3000x10^{-11}sec.$

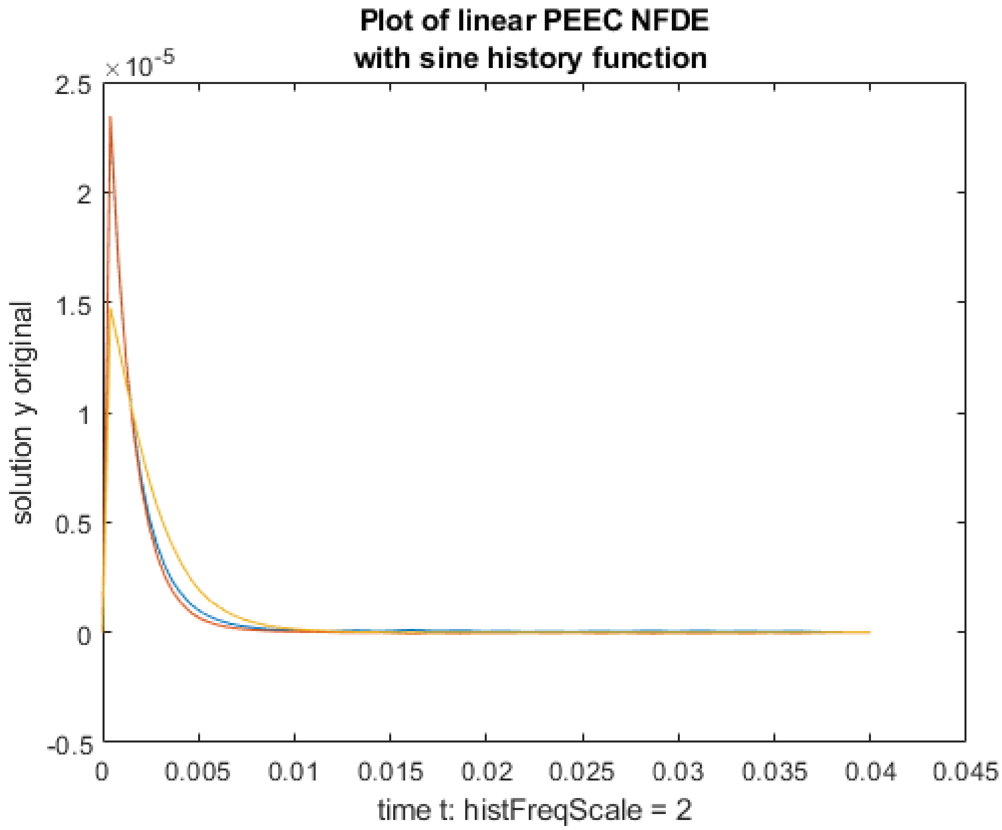

Figure 5.1: Plot of Solution to Original Linear NFDE

The system matrices are:

$$A0 = \begin{pmatrix} -1000 & 100 & 200 \\ 150 & -900 & 0 \\ 150 & 100 & -600 \end{pmatrix}, \tag{5.20}$$

$$A1 = \begin{pmatrix} 15.0000 & 0 & -45.0000 \\ -7.5000 & -7.5000 & -15.0000 \\ -7.5000 & -22.5000 & 0 \end{pmatrix}, \tag{5.21}$$

$$B1 = \begin{pmatrix} -.00002990 & .017361 & .006945 \\ -.003473 & .0208041 & .006945 \\ -.006945 & .013889 & .0069151 \end{pmatrix}, \tag{5.22}$$

$$F1 = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}. \tag{5.23}$$

So $det(Bm') = 2.4659e - 07$. The value chosen for ϵ has successfully met this condition for Theorem 5.4.5. Now we check that this perturbation of B_m has still allowed the remaining Theorem 5.4.5 conditions to be met.

The eigenvalues of the symmetric component of A0, eigsA =

$-1.1150 \times 1.0e + 03, -0.8730 \times 1.0e + 03, -0.5120 \times 1.0e + 03$

i) Since all the eigenvalues are < 0 we have that A0 is negative definite.

We easily calculate $K = 1.0332,\ \Sigma_B = 0.033178,\ r_{\lambda_P} = 1.4368,\ 1/r_{\lambda_P} = 0.6832.$

ii) Thus the condition ii) is met, $\Sigma_B < 1/r_{\lambda_P}$.

numSubdivisions = Number of discretization intervals per time step = 2

$$A_{advstep} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.0035 & 0 & 0 & 0.0174 & 0 & 0 & 0.0069 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -.0035 & 0 & 0 & 0.0208 & 0 & 0 & 0.0069 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -.0069 & 0 & 0 & 0.0139 & 0 & 0 & 0.0069 & 0 \end{pmatrix} \tag{5.24}$$

Theorem 3.4.4 additional requirements:

The eigenvalues of $A_{advstep}$ are:

0.0000000, 0.0586778,

0.0000000, 0.0586744,

0.0000000, −0.0586778,

−0.1442341, −0.0586744,

0.1442341.

iii.a) Thus, we see that the maximum eigenvalue magnitude, maxEigA, is $0.1442341 < 1$.

$rank(A_{advstep}) = 6$ and $nullity(A_{advstep}) = 3$.

iii.b) There are 6 distinct non-zero eigenvalues thus the eigenvectors of $A_{advstep}$ span its column space.

With $b = 1/(1 - maxEigA) = 1/(1 - 0.1318) = 1.1685$, we can easily calculate:

$$\frac{K}{(1+Kr_{\lambda_P})} = 0.41124 < b = 1.1685 < \frac{K}{(1+Kr_{\lambda_P})\Sigma_B r_{\lambda_P}} = 8.468$$

Thus b is in the required range.

Here we have $\varsigma = \rho|P||A_0| = 0.10107 < [\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - \Sigma_B] = 0.20725.$

iv.1) Thus ς is in the required range.

$\Sigma_A = 49.0359 < \frac{1}{|P_s|}[\frac{K}{(1+Kr_{\lambda_P})br_{\lambda_P}} - (\Sigma_B + \varsigma)] = 54.5243$

iv.2) Thus Σ_A is in the required range. Since all the conditions in 3.4.1 are met it follows that the D operator is stable globally in the delays and the solution $x = 0$ of system (L') is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.

Thus the conditions for Theorem 5.4.5 are met and:

- systems (P_a') and (P_b') are uniformly stable
- For solutions y_t of (P_b), x_t of (L) for all $\epsilon > 0$, $c > 0$ there exists real $\Theta = \Theta(\epsilon, c)$ such that if $|\phi| < c$

  then the solutions of (P_b) are bounded in the future and

  $|y_t(\tau, \phi) - x_t(\tau, \phi)| < \epsilon \qquad \forall\ t \geq \tau \geq \Theta.$

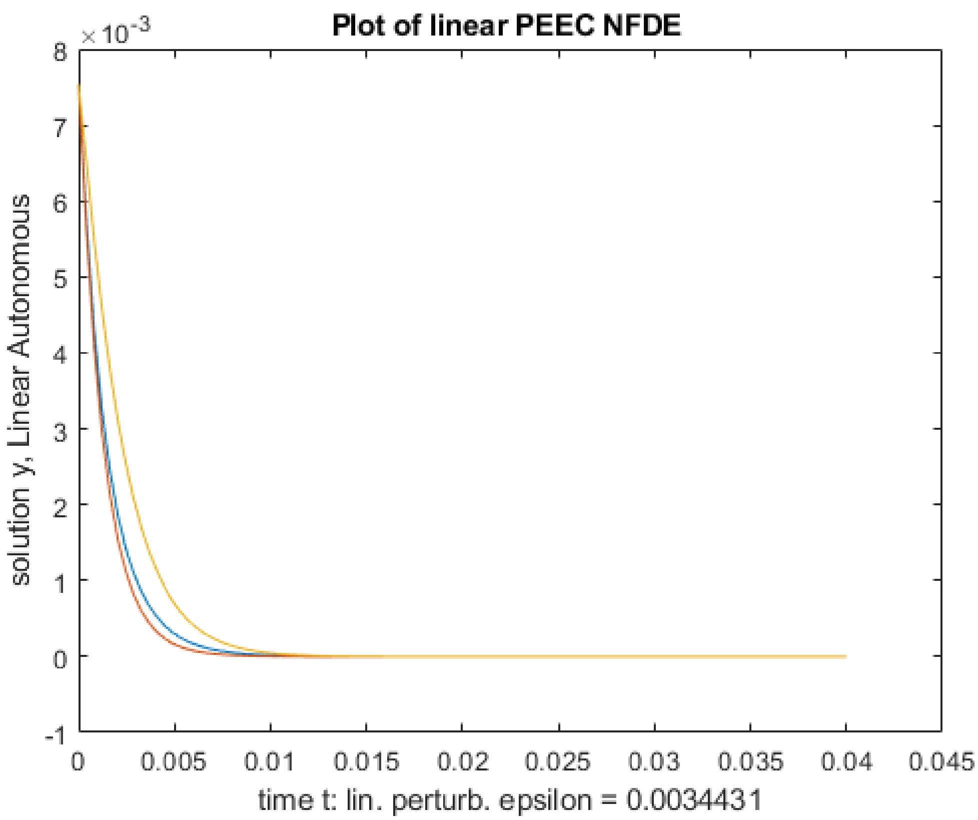

Figure 5.2: Plot of Solution to Linear NFDE

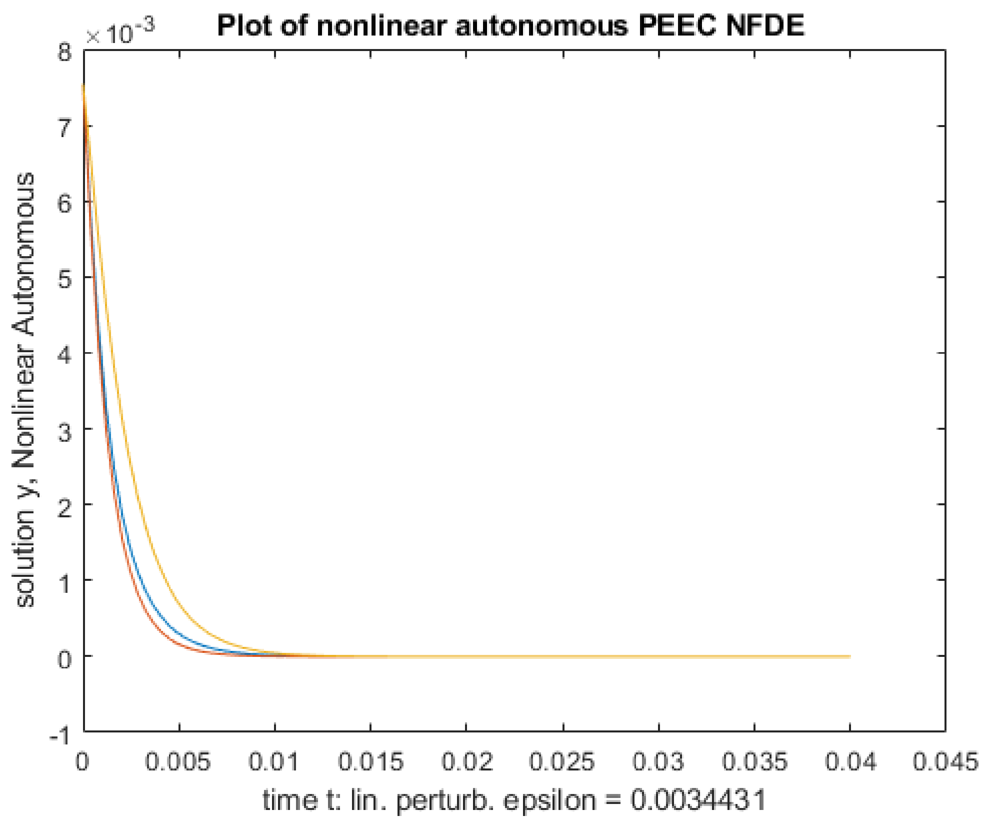

Figure 5.3: Plot of Solution to Nonlinear Autonomous NFDE

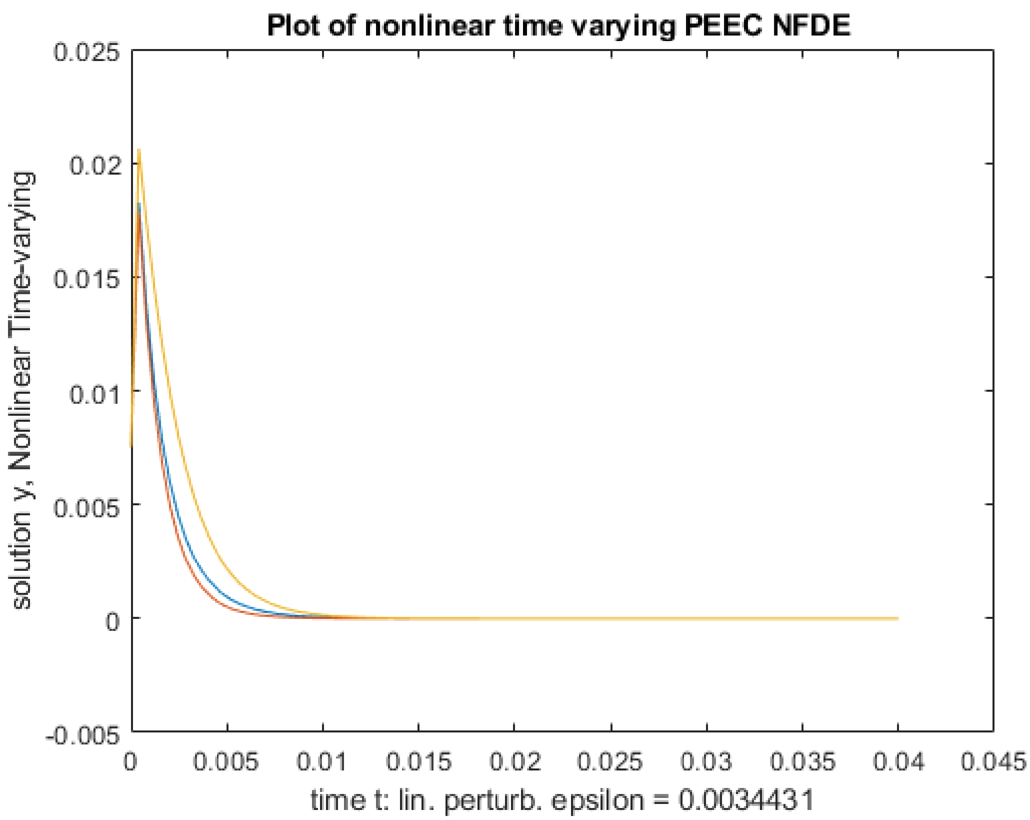

Figure 5.4: Plot of Solution to Nonlinear Time-Varying NFDE

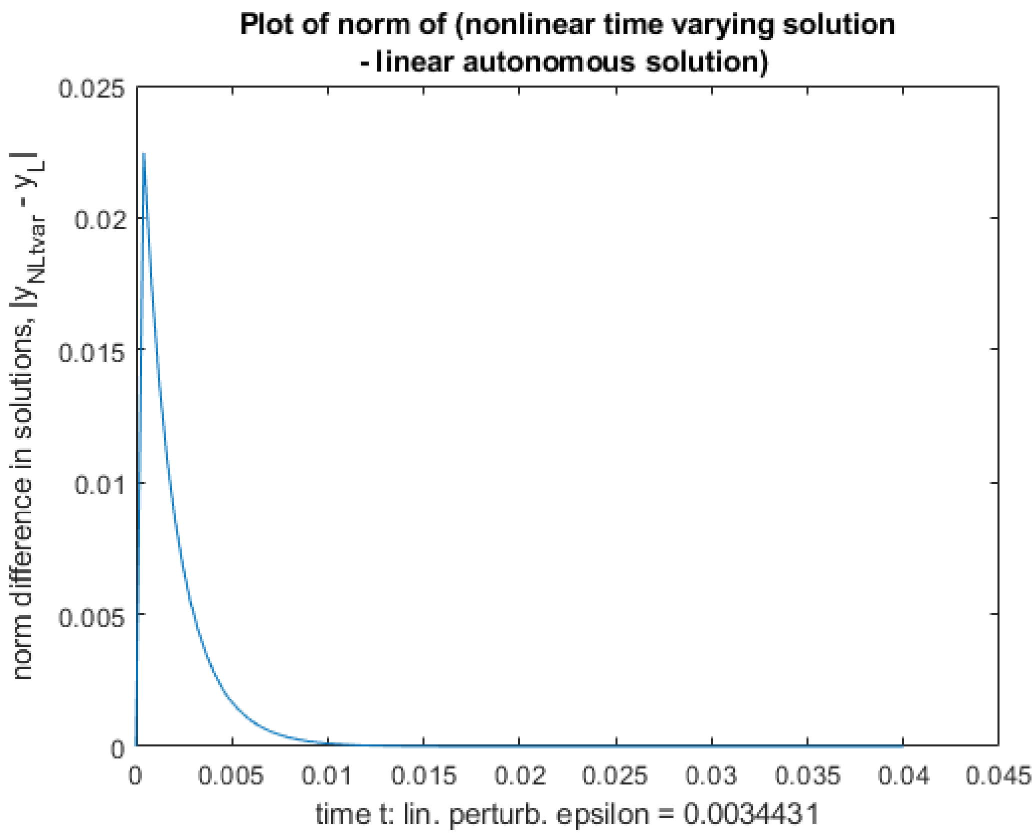

Figure 5.5: Plot of Norm of Nonlinear Time-Varying Solution minus Linear Solution NFDEs

Chapter 6

Conclusions

In the literature only the Lyapunov-Krasovskii functional methods are used for determining stability of PEEC NFDE's are studied. In out studies we use the Lyapunov-Razumikhin function methods applied to determine the stability properties of the NFDE's arising in the PEEC model . In addition the Lyapunov-Razumikhin stability Theorem 3.4.4 provides sufficient conditions for the stability and convergence allowing direct evaluation of criteria from system parameters and does not require an optimization step as is needed with Lyapunov-Krasovskii stability analysis methods. The application of results of [46], to nonlinear perturbations of an LTI NFDE resulted in two new theorems, one requiring the general condition of stability of the orginal NFDE, the other requiring the stability specific conditions given in a Razumikhin Stability theorem. The general stability condition allowed a more general application but left out specific details of how this would be implemented. The specific conditions given in a Razumikhin Stability theorem resulted in a narrower application but allowed direct calculation from the matrices of the original PEEC NFDE systems.

Application of results from [46] to the PEEC NFDEs showed that most of the conditions were met by the homeomorphism property of $T(t, \sigma, \phi)$, the solution operator of the LTI PEEC NFDE. $T(t, \sigma, \phi)$, is a homeomorphism iff $det(B_m) \neq 0$. It was seen that even if a PEEC NFDE does have $det(B_m) = 0$, the desired condtion can be met by a linear perturbation of $B'_m = B_m + \epsilon I$ with $\epsilon > 0$ arbitrarily small. In practical application of the PEEC model this would allow the presented nonlinear perturbation conditions to be examined for a PEEC NFDE system even if $det(B_m) = 0$.

The condition that the history function ϕ meet the compatability condition, $D\frac{d\phi}{dt} = f(\phi)$, is more restrictive. The approach presented here should be acceptable in practical application of the PEEC model. We create a system related to the original system with $x_\sigma = \phi$ but use the first step of this solution as the inital condtions for a related system, $x'_{\sigma+\tau} = x_{\sigma+\tau}(\sigma, \phi)$.

6.1 Areas for Further Research

Here we presented sufficient conditions for stability of the original PEEC NFDEs, and also sufficient conditions for stability and convergence of the nonlinearly perturbed PEEC NFDEs. Development of necessary and sufficient conditions for stability and convergence of PEEC NFDEs is an open area of research.

In the sufficient conditions for stability and convergence of the nonlinearly perturbed PEEC NFDEs we required that the history function ϕ meet the compatability condition: ϕ, $\frac{d}{dt}D\phi = f(\phi)$. Developing sufficient conditions which did not require that the history function ϕ meet the compatability condition would be another area for further research. Removal of this condition would allow application of the methods developed here to a wider set of system starting condtions. In chapter 4 we introduced nonlinear capacitive elements to the PEEC model. This resulted in a nonlinear perturbation to the PEEC NFDEs but kept a linear D operator in the NFDE. For further study, the noniinear inductive elements could be added to the PEEC model which would result in a nonlinear D operator in the PEEC NFDEs. In this case standard properties of the NFDEs may no longer be maintained: existence, uniqueness and continuous dependence on system parameters would have to be re-examined.

Appendix A

Background Material

Here we list some definitions and theorems relevant to this dissertation.

A.1 Basic DDE Properties

Definition A.1.1. [17]: For Banach spaces X and $Y, \mathcal{L}(X,Y)$ is the Banach space of bounded linear mappings from X to Y with the operator topology. If $L \in \mathcal{L}(X,Y)$ then the Riesz representation theorem implies there is an $n \times n$ matrix function η on $[-\tau, 0]$ of bounded variation such that

$$L\phi = \int_{-\tau}^{0} d[\eta(\theta)]\phi(\theta).$$

For such η we have extended the definition such that $\eta(\theta) = \eta(-\tau)$ for $\theta \leq -\tau$, and $\eta(\theta) = \eta(0)$ for $\theta \geq 0$.

Let Λ be an open subset of a metric space. We say $L : \Lambda \to \mathcal{L}(C, \mathbb{R}^n)$ has smoothness on the measure if for any $\beta \in \mathbb{R}$, there is a scalar function $\gamma(\lambda, s)$ continuous for $\lambda \in \Lambda, s \in \mathbb{R}, \gamma(\lambda, 0) = 0$, such that if $L(\lambda)\phi = \int_{-\tau}^{0} d[\eta(\lambda,\theta)]\phi(\theta), \lambda \in \Lambda, s > 0$, then

$$\left| \lim_{h \to 0^+} \int_{\beta+h}^{\beta+s} + \int_{\beta-s}^{\beta-h} d[\eta(\lambda,\theta)]\phi(\theta) \right| \leq \gamma(\lambda, s).$$

If, in addition, $\beta \in \mathbb{R}$ and matrix $A(\lambda; \beta, L) = \eta(\lambda, \beta^+) - \eta(\lambda, \beta^-)$ is non-singular at $\lambda = \lambda_0$, we say that $L(\lambda)$ is *atomic* at β on λ_0. If matrix $A(\lambda; \beta, L)$ is non-singular on a set $K \subset \Lambda$, we say that $L(\lambda)$ is *atomic* at β on K.

Definition A.1.2. [17] : If $D(t, \phi) = D_0(t)\phi - g(t), f(t, \phi) = L(t)\phi + h(t)$ where $D_0(t)$ and $L(t)$ are linear in ϕ, the NFDE (D, f) is called linear. It is linear homogeneous if $g \equiv 0, h \equiv 0$ and linear nonhomogeneous if either $g \not\equiv 0$ or $h \not\equiv 0$. An NFDE (D, f) is called autonomous if $D(t, \phi)$ and $f(t, \phi)$ do not depend on t. (See [17])

Theorem A.1.3. *[17]: Let Ω be an open set in $\mathbb{R} \times C$ and $(\sigma, \phi) \in \Omega$ then there is a solution of the NFDE (D, f) passing through (σ, ϕ).*

Theorem A.1.4. *[17]: If Ω is an open set in $\mathbb{R} \times C$, and $f : \Omega \to \mathbb{R}^n$ is Lipschitizian in ϕ on compact sets in Ω then, for any $(\sigma, \phi) \in \Omega$, $\exists$ a unique solution of the NFDE (D, f) through (σ, ϕ).*

Definition A.1.5. The notion of Input to State Stability (ISS) is described Sontag [40]: Let $\mathbb{R}^+ = [0, +\infty)$. Define a function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ to be class K if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is class K_∞ if it is class K and $\gamma(s) \to \infty$ as $|s| \to \infty$. Define a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ to be class KL if for each fixed t the mapping $\beta(., t)$ is of class K and for each fixed

$s,\ \beta(s,t)$ is decreasing to zero on t as $t \to \infty$. Define

$$|\phi_0| \text{ the Euclidean norm of } \phi_0 \text{ for } \phi_0 \in \mathbb{R}^n$$

$$\|u\| = ess\ sup\{|u(t)|, t \geq 0\}.$$

A system with state x is ISS if for each measurable, essentially bounded control u with initial state ϕ_0, the solution x exists for each $t \geq 0$ and x satisfies:

$|x(t)| \leq \beta(|\phi_0|, t) + \gamma(\|u\|)$

for some function β of class KL and function γ of class K.

A.2 Summary of Differential Equation Stability Analysis Methods

This section defines some standard stability analysis methods for ODEs, then defines related concepts for RFDEs.

For Differential Equations various methods of stability analysis exist. Let the state of a system be given by vector $x = (x_1, x_2, \ldots, x_n)$ with $x \in \mathbb{R}^n$. Consider the Ordinary Differential Equation (ODE):

$$\dot{x} = Ax \tag{A.1}$$

with A a real constant $n \times n$ matrix. Brauer and Nohel [4] give the classic result:

Theorem A.2.1. *[4]: If all eigenvalues of A have nonpositive real parts and all those eigenvalues with zero real parts are simple, then the solution $x = 0$ of (A.1) is stable. If and only if all eigenvalues of A have negative real parts, the zero solution of (A.1) is asymptotically stable.*

For more general systems stability properties may not be determined so easily. In [26], Kolmanovskii and Myshkis give a general framework for stability analysis of ODE's. Let the state of a system be given by vector $x \in n-$dimensional domain D and the state of the system given by the ODE:

$$\dot{x} = f(x) \tag{A.2}$$

with f a smooth (continuously differentiable) function. Let $V : D \to \mathbb{R}$ be a smooth function.

Definition A.2.2. [26]: Define the derivative of V with respect to system ((A.2)) as:

$$\dot{V}(x) = dV(x(t))/dt := gradV(x) \cdot f(x) = \sum_{\mathrm{i}=1}^{n} \frac{\partial V}{\partial x_i} f_i(x). \tag{A.3}$$

Definition A.2.3. [26]: A smooth function, $V : D \to \mathbb{R}$ is called a *guiding function* for (A.2) if $\dot{V}(x) \leq 0$ in D. A *Lyapunov function* in $\mathbb{R}^n$ is any scalar smooth function V defined in some neighborhood G of the origin such that:

$$V(0) = 0, V(x) > 0 \; \forall \; x \in G, x \neq 0.$$

Theorem A.2.4. *[26]: Let $0 \in G, f(0) = 0$ and assume a Lyapunov function $V : G \to \mathbb{R}$ exists for which $\dot{V}(x) \leq 0$ $(\forall x \in G)$. Then the solution $x \equiv 0$ of ((A.2)) is stable. If in addition $\dot{V}(x) < 0$ $(\forall x \in G, x \neq 0)$. Then the zero solution of ((A.2)) is asymptotically stable.*

For RFDE's such as ((2.1)) the concept of a Lyapunov Function is generalized to a Lyapunov Functional. In [17], Hale and Lunel generalize Lyapunov Function analysis of ODE's to RFDE's.

If $V : \mathbb{R} \times C \to \mathbb{R}$ is continuous and $x(\sigma, \phi)$ is a solution of ((2.1)) through (σ, ϕ), define

$$\dot{V} t, \phi) = \underset{h \to 0^+}{limsup} \frac{1}{h} \left[V\left(t+h, x_{t+h}(t, \phi)\right) - V\left(t, \phi\right) \right].$$

Theorem A.2.5. *[17]: Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times C \to \mathbb{R}$ such that*

$$u(|\phi(0)|) \leq V(t, \phi) \leq v(|\phi|)$$
$$\dot{V}(t, \phi) \leq -w(|\phi(0)|)$$

then the solution $x = 0$ of ((2.1)) is uniformly stable. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ of ((2.1)) is uniformly asymptotically stable. In this case V is called a Lyapunov-Krasovskii Functional.

Another common method of analyzing stability of RFDE's uses Lyapunov Functions instead of Lyapunov Functionals used above. The following theorem is an example [17]:

Theorem A.2.6. *[17] Suppose $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$ and consider the RFDE(f), (2.1). Supose $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0, v$ strictly increasing. If there is a continuous function $V : \mathbb{R} \times$*

$\mathbb{R}^n \to \mathbb{R}$ *such that*

$$u(|x|) \leq V(t,x) \leq v(|x|), \qquad t \in \mathbb{R}, x \in \mathbb{R}^n$$

and

$$\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|) \quad \text{if } V(t+\theta,\phi(\theta)) \leq V(t,\phi(0)),$$

then the solution $x = 0$ *of ((2.1)) is uniformly stable. In this case* V *is called a* Lyapunov-Razumikhin Function.

A.3 Properties of D Operators

Definition A.3.1. [17]: If $V : \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous function and $x(\sigma,\phi)$ is a solution of an NFDE (D,f) through (σ,ϕ) then define:

$$\dot{V}(\sigma,\phi) = \limsup_{h\to 0^+} \frac{1}{h}\left[V\left(t+h, x_{t+h}(\sigma,\phi)\right) - V\left(\sigma,\phi\right)\right]$$

Theorem A.3.2. *[17] Stability analysis using Lyapunov-Krasovskii functionals: Let* D *be stable, and* $f : \mathbb{R} \times C \to \mathbb{R}^n$ *be such that* f *takes bounded sets of* $\mathbb{R} \times C$ *into bounded sets of* $\mathbb{R}^n$*. Let* $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous, nonnegative, nondecreasing functions:* $u(s), v(s) > 0$ *for* $s \neq 0, u(0) = v(0) = 0$*. If there exists continuous function* $V : \mathbb{R} \times C \to \mathbb{R}^n$ *such that* $u(|D\phi|) \leq V(t,\phi) \leq v(|\phi|), \quad t \in \mathbb{R}, \phi \in C$*, and* $\dot{V}(t,\phi) \leq -w(|D\phi|)$ *then the solution* $x = 0$ *of NFDE* (D,f) *is uniformly stable. If* $u(s) \to \infty$ *as* $s \to \infty$ *then the solutions of NFDE* (D,f) *are uniformly bounded. If* $w(s) > 0$ *for* $s > 0$ *then the solution* $x = 0$ *of NFDE* (D,f) *is uniformly asymptotically stable. The same conclusion holds if the upper bound on* $\dot{V}(t,\phi)$ *is given by:* $\dot{V}(t,\phi) \leq -w(|D\phi(0)|)$

Definition A.3.3. A set of delays T is said to be *rationally dependent* if there exists a set of m rational numbers, $\{c_j\}, \sum_{j=1}^{m} |c_j| \neq 0$, such that $\sum_{j=1}^{m} c_j \tau_j = 0$. If a set of delays T is not rationally dependent then it is *rationally independent.*

It is known that, Theorem 6.1 [17], the following are equivalent:

Theorem A.3.4.

(i) *For some $\tau_1, \dots \tau_m$ rationally independent, $D(B,T)$ is stable*

(ii) *For $\gamma(M)$ the spectral radius of M*

(iii) $sup\{\gamma(\sum_{k=1}^{m} B_k e^{i\theta_k}) : \theta_k \in [0, 2\pi], k \in 1 : m\} < 1$

(iv) *$D(B,T)$ is stable locally in the delays.*

(v) *$D(B,T)$ is stable globally in the delays.*

The following Lyapunov-Razumikhin stability theorem [17] is useful to obtain suitable verifiable conditions for stability of the PEEC model.
Suppose D is a stable operator, $|D| = K$. Let $0 \leq u(s) \leq v(s), s \geq 0$, be continuous, nondecreasing functions, $u(s) \to \infty$ as $s \to \infty$. Suppose there exists a continuous function $\alpha(\eta), \eta \geq 0$, satisfying $v(K\eta) \leq u(\alpha(\eta))$. Let $\beta(\eta) > b(\eta + \alpha(\eta))$ be a continuous function where $b > 0$ is defined in (3.5) . Let $F : [0, \infty) \to \mathbb{R}^+$ be a nondecreasing function such that $F(v(K\eta)) > v(\beta(\eta))$ for $\eta > 0$. Under these conditions,

Theorem A.3.5. *[17] Stability analysis using Lyapunov-Razumikhin functions: Suppose the above notation, D is stable, $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, and takes*

$\mathbb{R}\times$ (bounded sets of C) into bounded sets of $\mathbb{R}^n$ and consider the NFDE (D, f). If there exists a continuous, positive function $w(s), s \geq 0$ and a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $u(|x|) \leq V(x) \leq v(x)$ for all $x \in \mathbb{R}^n$,
and $\dot{V}(D\phi) \leq -w(|D\phi|)$ for all functions ϕ satisfying $F(V(D\phi)) \geq V(\phi(\theta)), -r \leq \theta \leq 0$,
then the solution $x = 0$ of the NFDE (D, f) is uniformly asymptotically stable and all solutions approach zero as $t \to \infty$.

In [26] Kolmanovskii and Myshkis give a theorem comparing the stability of NFDE's to stability of related ODE's.

Theorem A.3.6. *[26]: Consider the initial value problem*

$$\dot{x}(t) = f(t, x(t)),\ t \geq 0,\ x(t_0) = x_0,\ t_0 \geq 0,\ f(t, 0) \equiv 0 \tag{A.4}$$

and an analogous problem for an NFDE:

$$\begin{aligned} \dot{x}(t, x_t) &= f(t, x(t, x_t)),\ t \geq 0,\ x_0 = \phi \in C([-h, 0], \mathbb{R}) \\ x(t, x_t) &= x(t) - g(t, x_t),\ g(t, 0) \equiv 0 \end{aligned} \tag{A.5}$$

Where, $f, g : [t_0, \infty) \times Q_H \to \mathbb{R}$ are continuous maps, $Q_H = \{u \in C|\ |u| < H\}$, $0 < H < \infty, f(t, 0) \equiv g(t, 0) \equiv 0, \quad |f(t, \psi)| \leq C, |g(t, \psi)| \leq C, t \geq t_0, \psi \in Q_H$ [26]: If the trivial solution of ODE (A.4) *is unstable then the trivial solution of NFDE A.5 is also unstable.*

Ize' and Ventura [22] study a variation of constants formula applied to NFDEs.

The following system of nonlinear NFDEs is considered:

$$
\begin{aligned}
(\frac{d}{dt})Dy_t &= f(t, y_t) \\
(\frac{d}{dt})Dx_t &= f(t, x_t) + g(t, x_t)
\end{aligned}
\tag{A.6}
$$

where

$D : C \to E$ is a continuous linear autonomous operator,

$f : \Omega \to E^n, \Omega \subset \mathbb{R} \times C$, an open set, is a non-linear continuous function with a continuous first Frechet derivative with respect to $\phi \in C$ and

$g : \Omega \to E^n$ is a nonlinear continuous function.

(A.6) is a nonlinear system of NFDEs with f and g mapping bounded closed sets into bounded sets.

In [16] Haddock et. al. use Razumikhin analysis to investigate the stability of NFDE's with infinite delays. In [14], Gorgodze, la Ramishvili and Tadumadze study the continuous dependence of a solution of an NFDE such as

$\frac{d}{dt}D(t, x_t) = f(t, x_t)$

on the right-hand side, $f(t, x_t)$, and initial data, x_0, taking into account perturbations of variable delays.

A.4 PEEC NFDE Linear Matrix Inequalities

Antonini and Pepe then generate the following Linear Matrix Inequality (LMI) which, if met, will give sufficient conditions for ISS of the NFDE[2]:

If $\exists$ in $\mathbb{R}^{n \times n}$ positive symmetric matrices $P, S, Q_{i,j}, i \in 1 : m, j \in 1, 2 s.t.$:

$Q_{i,2} > Q_{i,1}, \quad i \in 1 : m$

$Q_{i,1} > Q_{i+1,2}, \quad i \in 1 : m - 1$

$$\equiv \; Q_{1,2} > Q_{1,1} > Q_{2,2} > Q_{2,1} > Q_{3,2} > \dots$$

$$\dots > Q_{m-1,2} > Q_{m-1,1} > Q_{m,2} > Q_{m,1} > 0$$

Note: for $A, B \in \mathbb{R}^{n\times n}$ define $A > B$ if $A - B$ is positive definite

$$M \;=\; \begin{pmatrix} A_0^T P + PA_0 + Q_{1,2} & PC + Q_{1,2}B \\ C^T P + B^T Q_{1,2} & -Q + B^T Q_{1,2} B \end{pmatrix} < 0 \tag{A.7}$$

Note: this matrix is $\begin{pmatrix} n\times n & n\times mn \\ mn\times n & mn\times mn \end{pmatrix} \in \mathbb{R}^{n(m+1)\times n(m+1)}$

where

$C = [C_1 C_2 \dots C_m] \quad \in \mathbb{R}^{n\times mn}$

$C_k = A_k + A_0 B_k, \quad k \in 1:m \quad \in \mathbb{R}^{n\times n}$

$B = [B_1 B_2 \dots B_m] \quad \in \mathbb{R}^{n\times mn}$

$Q = \text{diag}(Q_{1,1} - Q_{2,2}, Q_{2,1} - Q_{3,2} Q_{m-1,1} - Q_{m,2}, Q_{m,1} \quad \in \mathbb{R}^{mn\times mn}$

So $Q = \begin{pmatrix} (Q_{1,1} - Q_{2,2}) & 0 & \dots & 0 \\ 0 & (Q_{2,1} - Q_{3,2}) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & Q_{m,1} \end{pmatrix}$

then for all values of time delays $0 < \tau_1 < \tau_2 < \dots \tau_m$, $\exists$ function β of class KL and a function γ of class K such that the solution of the resulting NFDE satisfies:

$$\int_{t-\tau_m}^{t} |x(\theta)|^2 d\theta \le \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t)}\|_\infty), \quad t \ge 0 \tag{A.8}$$

also if there exists m nonnegative reals $\eta_1, \eta_2, \dots \eta_m$ with $\sum_{k=1}^{m} \eta_k < 1 s.t.$

$$B^T SB - \eta \bigotimes S \le 0 \tag{A.9}$$

$$\eta = \begin{pmatrix} \eta_1 & 0 & \dots & 0 \\ 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \eta_m \end{pmatrix} \in \mathbb{R}^{m\times m},$$

$$\eta \bigotimes S = \begin{pmatrix} \eta_1 S & 0 & \dots & 0 \\ 0 & \eta_2 S & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \eta_m S \end{pmatrix} \in \mathbb{R}^{nm\times nm}$$

then, for all values of time delays $0 < \tau_1 < \tau_2 < \dots \tau_m$ s.t. the solution of the resulting NFDE (2.24) is ISS.

With the LMI (A.7), Antonini and Pepe use the following Lyapunov-Krasovskii functional to prove ISS of the NFDE:

$$V(\phi) = (D\phi)^T P(D\phi) + \sum_{i=0}^{m-1} \int_{-\tau_{i+1}}^{-\tau_i} \phi^T(\theta) Q_{i+1}(\theta)\phi(\theta)d\theta \qquad \phi \in C([-\tau, 0]; \mathbb{R}^n) \tag{A.10}$$

The stability analysis of Antonini and Pepe in [2] relies on the work of Pepe in [31] which deals with an RFDE and Continuous Time Difference Equation (CTDE) that generalize NFDEs:

$$\begin{aligned} \dot{\xi} &= f(\xi_t, x_t, u(t)),\ t \geq 0, a.e. \\ x(t) &= g(\xi_t, x_t),\ t \geq t_0, \\ \xi(t_0 + \tau) = \xi_0(\tau), &\quad x(t_0+\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0] \end{aligned} \tag{A.11}$$

It is known [31] that the derivative of a Lyapunov functional involves the derivative of the solution which is not practical for numeric methods. Following [8] defines an upper right hand Dini derivative that uses quantities that have already been solved for up to the current time value. This derivative is equal almost everywhere to the derivative found using the derivative of the solution. An NFDE is then derived from the above RFDE and CTDE by setting $\xi(t) = x(t) - g(t, x_t)$. Sufficient conditions to prove ISS for this NFDE are given in [31].

Pepe and Verriest [32] extended Pepe's 2007 paper, [31] , using the same RFDE and CTDE as in Pepe's 2007 paper. In [32] an upper right hand Dini derivative is defined using the L_p norm and using a Lyapunov functional approach, sufficient conditions are found for ISS of the RFDE and CTDE system. Note that [2], [31] and [32] are the main works on ISS of PEEC NFDE's and all focus on use of Lyapunov-Krasovskii functionals to prove ISS of the NFDE's.

A search of the literature shows that a Lyapunov-Razumikhin function analysis for the PEEC NFDE's has not been done. In general, most work uses Lyapunov-Krasovskii functionals to analyze stability, not Lyapunov-Razumikhin functions. When Razumikhin functions are used to analyze DDE's, analysis is typically done for RFDE's but not NFDE's.

In [29], Niculescu analyzes Razumikhin functions used to analyze RFDE's and NFDE's and gives some general methods for creating LMIs for sufficient stability conditions.

A.5 Implementation of LMIs for PEEC NFDE

In [2], it is stated that the LMI can be solved using software such as Matlab. In [39] are given methods for casting LMI's as a convex optimization problems. Using methods shown in [39] we cast the LMI (A.7) as a convex optimization problem to demonstrate the matrix sizes involved in the optimization process.

In the original LMI we must find $n \times n$ symmetric positive definite matrices $P, Q_{i,j}, i \in 1:m, j \in 1,2$ such that:

$$Q_{1,2} > Q_{1,1} > Q_{2,2} > Q_{2,1} > Q_{3,2} > \cdots > Q_{m-1,2} > Q_{m-1,1} > Q_{m,2} > Q_{m,1} > 0$$

$$\equiv \; Q_{1,2} - Q_{1,1} > 0,\; Q_{1,1} - Q_{2,2} > 0,\; Q_{2,2} - Q_{2,1} > 0,\; Q_{2,1} - Q_{3,2} > 0,$$

$$\cdots > Q_{m-1,2} - Q_{m-1,1} > 0,\; Q_{m-1,1} - Q_{m,2} > 0,\; Q_{m,2} - Q_{m,1} > 0,\; Q_{m,1} > 0$$

and $\quad M \;=\; \begin{pmatrix} A_0^T P + PA_0 + Q_{1,2} & PC + Q_{1,2}B \\ C^T P + B^T Q_{1,2} & -Q + B^T Q_{1,2} B \end{pmatrix} < 0$

Generate a basis for $n \times n$ symmetric matrices: $\{E_{i,j}\} \quad i = 1:n, j = 1:2$

$\forall i \in 1:n,\; j \in 1:2$

$$\begin{aligned} E_{i,j}(u,v) &= 1 \quad \text{for } u,v = i,j \text{ or } j,i \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{A.12}$$

Represent matrices as linear combinations of basis matrices:

$P = \sum_{u=1}^{n} \sum_{v=u}^{n} p_{u,v} E_{u,v}$

$Q_{i,j} = \sum_{u=1}^{n} \sum_{v=u}^{n} q_{i,j,u,v} E_{u,v} \qquad i \in 1:m, j \in 1:2$

Define matrix objFuncMatrix =

$$\begin{pmatrix} -P & 0 & \dots & 0 & 0 & 0 \\ 0 & -(Q_{1,2} - Q_{1,1}) & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & -(Q_{m,2} - Q_{m,1}) & 0 & 0 \\ 0 & 0 & \dots & 0 & -Q_{m,1} & 0 \\ 0 & 0 & \dots & 0 & 0 & M \end{pmatrix}$$

Define objective function, $f: \; \mathbb{R}^{(2m+1)n(n+1)/2} \times \mathbb{R}^{n(m+1)\times n(m+1)} \to \mathbb{R}$

$f(\{p_{u,v}\}, \{q_{1,1,u,v}\}, \ldots, \{q_{m,2,u,v}\}, M) = max$ Eigenvalue($objFuncMatrix$)

Minimize the objective function. If $f < 0$ then all Eigenvalues of objFuncMatrix are < 0, all component matrices are negative definite.

Thus $P > 0, M < 0$ and

$Q_{1,2} > Q_{1,1} > Q_{2,2} > Q_{2,1} > Q_{3,2} > \cdots > Q_{m-1,2} > Q_{m-1,1} > Q_{m,2} > Q_{m,1} > 0$

This illustrates the size of the matrix that will be subjected to convex optimization methods to prove sufficient conditions for stability of the PEEC NFDE using Lyapunov-Krasovskii methods. In contrast, the Lyapunov-Razumikhin methods developed in this thesis will not require a convex optimization step, and are therefor computationally less intense.

Theorem A.5.1. *(Continuous dependence) [17]: Suppose* $\Omega \subset \mathbb{R} \times C$ *is open,* Λ *is a subset of a Banach space,*

$D : \Omega X \Lambda \to \mathbb{R}^n, L : \Omega X \Lambda \to \mathbb{R}^n$ *satisfy the following hypotheses:*

i) $D(t, \phi, \lambda)$ *is atomic at zero for all* $(t, \phi) \in \Omega$ *uniformly with respect to* λ.

ii) $D(t, \phi, \lambda), L(t, \phi, \lambda)$ *are continuous in* $(t, \phi) \in \Omega \ \forall \ \lambda \in \Lambda$

and continuous in (t, ϕ, λ) *for* $(t, \phi) \in \Omega$

iii) $NFDE(D(\cdot, \lambda_0), L(\cdot, \lambda_0))$ *has a unique solution through* $(\sigma, \phi) \in \Omega$

that exists on interval $[\sigma - r, b]$ *then there exists a neighborhood*

$N(\sigma, \phi, \lambda_0)$ *of* $(\sigma, \phi, \lambda_0)$ *such that for any* $(\sigma', \phi', \lambda') \in N(\sigma, \phi, \lambda_0)$,

all solutions $x(\sigma', \phi', \lambda')$ *of NFDE*$(D(\cdot, \lambda'), L(\cdot, \lambda')$ *through* (σ', ϕ')

exist on $[\sigma' - r, b]$ *and* $x(\sigma', \phi', \lambda')$ *is continuous at* $(t, \sigma', \phi', \lambda_0)$

for $t \in [\sigma, \sigma + A]$, *and* $(\sigma', \phi', \lambda') \in N(\sigma, \phi, \lambda_0)$.

A.6 Nonlinear NFDEs

Here we list material that is used to develop our uniform stability conditions for PEEC NFDEs with nonlinear perturbations.

Theorem A.6.1. *[46]: Suppose* $\Omega \subset \mathbb{R} \times C$ *is open,* D *is atomic at* 0 *and* $-\tau$. *If the functions* $D, A(t,0),\ A(t,-\tau)$ *of Definition A.1.1 are uniformly continuous on closed bounded sets of* Ω *and* $f(x_t)$ *is continuous and takes bounded sets into bounded sets, and the initial value problem has a unique solution, then the solution map* $T(t,\sigma) : U(t,\sigma) \to T(t,\sigma)U(t,\sigma)$ *is a homeomorphism.*

Lemma A.6.2. *[22]: If* $x_t(s, y_s(\sigma,\phi))$ *is a differentiable solution with respect to* t *of* (L) *and* $y_s(\sigma,\phi)$ *is a differentiable solution with respect to* s *of* (P_a) *or* (P_b) *then* $\frac{d}{dt}x_t(s, y_s(\sigma,\phi))$ *and* $T(t,s; y_s(\sigma,\phi)Y_\sigma g(t,y_t))$ *are solutions of* 5.2 *that coincide with* $Y_\sigma g(t, y_t(\sigma,\phi))$ *at* $t = s$.

Theorem A.6.3. *[46]: Define* $Y_\sigma, Y_\sigma(\theta) \in \mathbb{R}^n,\ \theta \in [-\tau, 0]$ *as:*

$Y_\sigma(\theta) = 0 \qquad \forall\ \theta \in [-\tau, 0),$

$Y_\sigma(\theta) = I \qquad \theta = 0.$

Let Ω *be an open subset of* $\mathbb{R} \times C$

For systems:

$\frac{d}{dt}Dx_t = f(x_t), \quad (a)$

$\frac{d}{dt}Dy_t = f(y_t) + g(y_t) \quad (b)$

with $f, g : \Omega \to \mathbb{R}^n$ *continuous functions,* $f(\phi)$ *takes bounded sets into bounded sets, the initial value problem has a unique solution,* $D : \Omega \to \mathbb{R}^n$ *is a continuous linear function satisfying* $D(\phi) = \phi(0) - \int_{-\tau}^{0} d\mu(t,\theta)\phi(\theta) \quad (t,\phi) \in \Omega$ *for some* $n \times n$ *matrix function of bounded variation* $\mu(t,\theta),\ \theta \in [-\tau, 0]$ *and the* D *operator is atomic at* $-\tau$ *and* 0.

If (a) *admits unique solution* $x_t(\sigma,\phi)\forall\ t \geq \sigma$ *and if* $\Phi(t,\sigma,\phi) = \frac{\partial}{\partial\phi}x_t(\sigma,\phi)$ *exists and is continuous for all* $t \geq \sigma, \Phi^{-1}(t,\sigma,\phi)$ *exists for all* $t \geq \sigma$ *then solution* $y_t(\sigma,\phi)$

of (b) satisfies:

$$
\begin{aligned}
&y_t(\sigma,\phi) = x_t\big(\sigma, \int_\sigma^t \Phi^{-1}(s,\sigma,z_s)Y_\sigma g(s, y_s(\sigma,\phi))ds\big)\, \textit{also}\\
&y_t(\sigma,\phi) = x_t(\sigma,\phi) + \int_\sigma^t \Phi(t,\sigma,z_s)\Phi^{-1}(s,\sigma,z_s)Y_\sigma g(s,y_s(\sigma,\phi))ds\\
&\textit{as far as } z_t(\sigma,\phi) \textit{ exists for } t \geq \sigma \textit{ as a solution of:}\\
&\frac{d}{dt}z_t(\sigma,\phi) = \Phi^{-1}(t,\sigma,z_t)Y_\sigma g(t, x_t(\sigma, z_t(\sigma,\phi)))\\
&\textit{This requires that } \phi \textit{ meets the compatability condition:}\\
&\frac{d}{dt}D\phi = f(\phi)
\end{aligned}
\tag{A.13}
$$

The following theorem [46] gives sufficient conditions for stability of stable linear NFDEs with nonlinear perturbations.

Theorem A.6.4. *[46]: We assume the conditions of A.13. In addition*

i) If $|\Phi^{-1}(t,\sigma,\phi)Y_\sigma g(t, x_t(\sigma,\phi)| \leq h(t, |\phi|)$ *with*

$h \in C([\sigma,\infty) \times \mathbb{R}^+; \mathbb{R}^+)$, $h(t,0) \equiv 0$,

and the trivial solution of $\dot{u} = h(t,u)$, $u(\sigma) \geq 0$ *is uniformly stable,*

ii) and if the trivial solution of (a) *is uniformly stable,*

Then the trivial solution of (b) *is uniformly stable.*

Theorem A.6.5. *[46]: For the systems of NFDE* (a) *and* (b)

We assume

i) There exists at least one bounded solution of (a).

ii) For all $\alpha > 0$, $\exists$ *a constant* $N(\alpha) > 0$ *such that if*

$\|\Psi\| < \alpha$, *then* $\|\Phi(t,\sigma,\phi)\| < N(\alpha)$,

$\|\Phi^{-1}(t,\sigma,\phi)\| < N(\alpha)$, *for* $t \geq \sigma$

iii) For all $\alpha > 0, \ \exists$ *continuous function* $h(t, \alpha) > 0$ *with*

$\int_0^\infty h(t, \alpha)dt < \infty$ *such that if* $\|\phi\| < \alpha, g(t, \phi) \leq h(t, \alpha)$. *Then for solutions* y_t *of*

$(b), \ x_t$ *of* (a)

for all $\epsilon > 0, \ c > 0$ *there exists real* $\Theta = \Theta(\epsilon, \ c)$ *such that if* $|\phi| < c$

then the solutions of (b) *are bounded in the future and*

$|y_t(\tau, \phi) - x_t(\tau, \phi)| < \epsilon \qquad \forall \ t \geq \tau \geq \Theta.$

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