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Optimal Control of Coefficients for the Second Order Parabolic Free Boundary Problems

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ABSTRACT

Title:

Optimal Control of Coefficients for the Second Order Parabolic Free Boundary Problems

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Dissertation aims to analyze inverse Stefan type free boundary problem for the second order parabolic PDE with unknown parameters based on the additional information given in the form of the distribution of the solution of the PDE and the position of the free boundary at the final moment. This type of ill-posed inverse free boundary problems arise in many applications such as biomedical engineering problem about the laser ablation of biomedical tissues, in-flight ice accretion modeling in aerospace industry, and various phase transition processes in thermophysics and fluid mechanics. The set of unknown parameters include a space-time dependent diffusion, convection and reaction coefficients, density of the sources, time-dependent boundary flux and the free boundary. New PDE constrained optimal control framework in Hilbert-Besov spaces introduced in *U.G. Abdulla, Inverse Problems and Imaging, 7, 2(2013), 307-340; 10, 4(2016), 869-898* is employed, where the missing data and the free boundary are components of the control vector, and optimality criteria are based on the final moment measurement of the temperature and position of the free boundary, and available information on the phase transition temperature on the free boundary. The latter presents a key advantage in dealing with applications, where phase transition temperature is not known explicitly, but involve some measurement error. Another advantage of the new variational approach is based on the fact that for a given control parameter, Stefan boundary condition turns into Neumann boundary condition on the given boundary, and parabolic PDE problem is solved in a fixed domain, and therefore a perspective opens for the development of numerical meth-

ods of least computational cost. Discretization of the optimal control problem via method of finite differences is pursued and the sequence of finite-dimensional optimal control problems are introduced. The results of the dissertation are different depending on the structure of the unknown diffusion coefficient. In the case if it is only time-dependent, the well-posedness of the optimal control problem is established in Hilbert-Besov spaces. Existence of the optimal control and convergence of the sequence of the discrete optimal control problems to the continuous optimal control problem both with respect to functional and control is proved. The methods of the proof are based on uniform H^1-energy estimates in discrete Sobolev-Hilbert norms, weak compactness argument, Weierstrass theorem in weak topology and weak convergence of the bilinear interpolations of the solutions of the discrete PDE problems to the solution of the optimal PDE problem in the class of weakly differentiable functions. To prove similar results in the case when unknown diffusion coefficient is space-time dependent, a new Banach space is introduced. The motivation for the new space is dictated with the optimal result on the convergence of the bilinear interpolations of the grid functions in the class of weakly differentiable functions, and establishment of the discrete H^1-energy estimate under minimal assumptions on the diffusion coefficient. Existence of the optimal control and convergence of the sequence of discrete optimal control problems to the continuous optimal control problem both with respect to functional and control is proved in the setting of the new Banach space.

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List of Notations

Let U be open subset of the real line $\mathbb{R}$.

- The Sobolev-Besov space $B_2^k(U)$, for $k = 1, 2, \ldots$ is the Banach space of $L_2(U)$ functions whose weak derivatives up to order k exist and are in $L_2(U)$. The norm in $B_2^k(U)$ is
$$\|u\|^2_{B_2^k(U)} := \sum_{i=0}^{k} \left\| \frac{d^i u}{dx^i} \right\|^2_{L_2(U)}.$$

- For $\ell \notin \mathbf{Z}_+$, $B_2^\ell(U)$ is the Banach space of measurable functions with finite norm
$$\|u\|_{B_2^\ell(U)} := \|u\|_{B_2^{(\ell)}(U)} + [u]_{B_2^\ell(U)}$$
where
$$[u]^2_{B_2^\ell(U)} := \int_U \int_U \frac{\left| \frac{\partial^{[\ell]} u(x)}{\partial x^{[\ell]}} - \frac{\partial^{[\ell]} u(y)}{\partial x^{[\ell]}} \right|^2}{|x-y|^{1+2(\ell-[\ell])}} \, dx\, dy.$$

- Let $\ell_1, \ell_2 > 0$ and $D = U \times (0,T)$. The Besov space $B_2^{\ell_1,\ell_2}(D)$ is defined as the closure of the set of smooth functions under the norm
$$\|u\|_{B_2^{\ell_1,\ell_2}(D)} := \left(\int_0^T \|u(x,t)\|^2_{B_2^{\ell_1}(U)} \, dt \right)^{1/2} + \left(\int_U \|u(x,t)\|^2_{B_2^{\ell_2}(0,T)} \, dx \right)^{1/2}.$$
When $\ell_1 = \ell_2 \equiv \ell$, the corresponding Besov space is denoted by $B_2^\ell(D)$. $\mathring{B}_2^{\ell_1,\ell_2}(D)$ denotes the closure of the set of smooth functions with compact support with respect

to x in U under the $B_2^{\ell_1,\ell_2}$-norm.

- $V_2(\Omega)$ is the subspace of $B_2^{1,0}(\Omega)$ for which the norm

$$\|u\|^2_{V_2(\Omega)} = \operatorname*{ess\,sup}_{0\le t\le T} \|u(\cdot,t)\|^2_{L_2\left(0,s(t)\right)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)} < \infty.$$

- $V_2^{1,0}(\Omega)$ is the completion of $B_2^{1,1}(\Omega)$ in the $V_2(\Omega)$ norm. $V_2^{1,0}(\Omega)$ is a Banach space with norm

$$\|u\|^2_{V_2^{1,0}(\Omega)} = \max_{0\le t\le T} \|u(\cdot,t)\|^2_{L_2\left(0,s(t)\right)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)}.$$

In the next section we describe the new variational formulation of this inverse problem.

Let U be open subset of the real line $\mathbb{R}$ and and $D = U \times (0,T)$.

$L_\infty(D)$ - Banach space of essentially bounded real-valued measurable functions on D with norm

$$\|u\|_{L_\infty(D)} = \operatorname*{ess\,sup}_{(x,t)\in D} |u(x,t)| < +\infty$$

$L_{\infty,\gamma}(D)$ - Banach Space with the following norm (here $\gamma > 1$):

$$\|u\|_{L_{\infty,\gamma}(D)} = \Big(\int_0^T \underset{0\le x\le l}{esssup}\; |u|^\gamma\; dt\Big)^{\frac{1}{\gamma}}$$

$L_2[0,T]$ - Hilbert space with scalar product

$$(u,v) = \int_0^T uvdt$$

$W_2^k[0,T], k = 1,2,...$ - Hilbert space of all elements of $L_2[0,T]$ whose weak derivatives up to order k belongs to $L_2[0,T]$ and scalar product is defined as

$$(u,v) = \int_0^T \sum_{s=0}^{k} \frac{d^s u}{dt^s}\frac{d^s v}{dt^s} dt$$

$W_2^{\frac{1}{4}}[0,T]$ - Banach space of all elements of $L_2[0,T]$ with finite norm

$$\|u\|_{W_2^{\frac{1}{4}}[0,T]} = \Big(\|u\|^2_{L_2[0,T]} + \int_0^T dt \int_0^T \frac{|u(t)-u(\tau)|^2}{|t-\tau|^{\frac{3}{2}}} d\tau\Big)^{\frac{1}{2}}$$

$L_2(\Omega)$ - Hilbert space with scalar product

$$(u,v) = \int_\Omega uv dx dt$$

$W_2^{1,0}(\Omega)$ - Hilbert space of all elements of $L_2(\Omega)$ whose weak derivative $\frac{\partial u}{\partial x}$ belongs to $L_2(\Omega)$, and scalar product is defined as

$$(u,v) = \int_\Omega \Big(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\Big) dx dt$$

$W_2^{1,1}(\Omega)$ - Hilbert space of all elements of $L_2(\Omega)$ whose weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ belong to $L_2(\Omega)$, and scalar product is defined as

$$(u,v) = \int_\Omega \Big(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t}\Big) dx dt$$

$V_2(\Omega)$ - Banach space of all elements of $W_2^{1,0}(\Omega)$ with finite norm

$$\|u\|_{V_2(\Omega)} = \Big(esssup_{0\le t\le T} \|u(x,t)\|^2_{L_2[0,s(t)]} + \Big\|\frac{\partial u}{\partial x}\Big\|^2_{L_2(\Omega)}\Big)^{\frac{1}{2}}$$

$V_2^{1,0}(\Omega)$ - Banach space which is the completion of $W_2^{1,1}(\Omega)$ in the norm of $V_2(\Omega)$. It consists of all elements of $V_2(\Omega)$, continuous with respect to t in norm of $L_2[0,s(t)]$ and with finite norm

$$\|u\|_{V_2^{1,0}(\Omega)} = \Big(\max_{0\le t\le T} \|u(x,t)\|^2_{L_2[0,s(t)]} + \Big\|\frac{\partial u}{\partial x}\Big\|^2_{L_2(\Omega)}\Big)^{\frac{1}{2}}$$

$W_2^{2,1}(\Omega)$ - Hilbert space of all elements of $L_2(\Omega)$ whose weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$ belong to $L_2(\Omega)$, and scalar product is defined as

$$(u,v) = \int_\Omega \Big(uv + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\Big)dxdt$$

$\tilde{W}_2^{1,1}(D) = \{u \in W_2^{1,1}(D) : u_{xt} \in L_2(D)\}$

$$\|u\|^2_{\tilde{W}_2^{1,1}(D)} = \|u\|^2_{L^2(D)} + \|u_x\|^2_{L^2(D)} + \|u_t\|^2_{L^2(D)} + \|u_{xt}\|^2_{L^2(D)}.$$

$\tilde{W}^{1,1}_{\infty,\gamma}(D) = \{\, u \mid u,\ u_x \in L_\infty(D),\ u_t \in L_{\infty,\gamma}(D)\}$ is a Banach space with the norm

$$\|u\|_{\tilde{W}^{1,1}_{\infty,\gamma}(D)} = \|u\|_{L_\infty(D)} + \|u_x\|_{L_\infty(D)} + \|u_t\|_{L_{\infty,\gamma}(D)}$$

where $\gamma > 1$.

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Sense of community is very important and you realize this even better studying in a graduate school. Faculty and staff did an excellent job in addressing students questions and worries in an effective and timely manner. I was fortunate enough to be a part of this amazing community and hope to stay in touch even after I graduate and become a part of evergrowing list of graduates of Florida Institute Of Technology.

Dedication

I would like to dedicate this work to my beloved grandmother Sanubar! Her prayers and support played an enormous role in this achievement!

Chapter 1

Introduction

1.1 Free Boundary Problems

In many important applications arising in science, engineering and economy some partial differential equation (PDE) or system of PDEs must be solved in a domain which is apriori unknown. This type of problems are called free boundary problems, where the term free boundary is associated with unknown portion of the boundary of the domain. Some examples of free boundary problems are various phase transition processes in fluid mechanics and thermophysics; growth of cancerous tumor or laser ablation of tissues in medicine; evolution of the price of American option with random payoff chosen by holder in stock market; in-flight ice accretion modeling in aerospace industry etc. The classical example of a free boundary problem in mathematical physics is the so-called Stefan problem. The Stefan problem is a boundary value problem for the heat/diffusion equation, where the unknown/free phase transition boundary between two or several phases is changing as a function of time. Few examples of Stefan problem in applications are the melting of ice, or freezing of water, the formations of crystals from liquid, or the laser ablation of skin tissue. The key mathematical feature of the Stefan problem is expressed via additional condition on the free boundary expressing dynamic of the movement of the free boundary in time, in terms of the conservation of energy during the phase transition process. As an example, consider the classical one-phase Stefan problem about the melting of the ice [34]: let a semi-infinite block of ice, initially at melting temperature of 0 degrees, starts melting under the time dependent heat flux $f(t)$ applied on the left end. Then the unknown temperature profile $u(x,t)$ and unknown boundary curve $s(t)$ between water and ice satisfy the following system of equations:

$$u_t = u_{xx}, \qquad 0 < x < s(t), t > 0 \tag{1.1}$$

$$u_x(0,t) = f(t), \qquad t > 0 \tag{1.2}$$

$$u(s(t),t) = 0, \qquad t > 0 \tag{1.3}$$

$$\frac{ds}{dt} = -u_x(s(t),t), \qquad t > 0 \tag{1.4}$$

$$u(x,0) = 0, \qquad x \geq 0 \tag{1.5}$$

$$s(0) = 0 \tag{1.6}$$

Stefan condition (1.4) expresses the fact that the free boundary is pushed forward due to jump of the flux on the free boundary during the phase transition. Generalization of the Stefan problem presents a relevant mathematical model for phase transition phenomenon in biomedical engineering applications. For example, consider bioengineering problem about the laser ablation of biomedical tissues in a simplified one-dimensional case where space variable x is measuring the ablation depth of the tissue. Then the temperature function $u(x,t)$ and the free boundary $x = s(t)$, measuring the ablation depth at the moment

t, satisfy the general one-phase Stefan problem [42, 62]:

$$Lu \equiv (a(x,t)u_x)_x + c(x,t)u_x + d(x,t)u - u_t = f(x,t) - \frac{\partial p(x,t)}{\partial x}, \quad \text{in } \Omega \tag{1.7}$$

$$u(x,0) = \phi(x), \quad 0 \le x \le s(0) =: s_0 \tag{1.8}$$

$$a(0,t)u_x(0,t) = g(t), \quad 0 \le t \le T \tag{1.9}$$

$$a(s(t),t))u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \quad 0 \le t \le T \tag{1.10}$$

$$u(s(t),t) = \mu(t), \quad 0 \le t \le T, \tag{1.11}$$

where a, b, c, d, f, p, ϕ, g, γ, χ, μ are known functions with $a(x,t) \ge \underline{a} > 0$, $s_0 > 0$, and

$$\Omega = \{(x,t) : 0 < x < s(t),\ 0 < t \le T\}$$

In the context of heat conduction, γ represents latent heat released by the melting at the boundary, χ a heat source or sink on the boundary, f and p characterize the density of the sources, ϕ is the initial temperature, g is the heat flux on the fixed boundary $x = 0$, and μ is the phase transition temperature. The coefficients a, c, and d represent the diffusive, convective, and reactive properties, respectively, in the domain Ω.

The 1D Stefan problem has a well-established mathematical theory, and an extensive list of works on it can be found in [76]. The existence and uniqueness of the classical solution of the one dimensional Stefan problem is a well known result [34, 40]. For example, it can be established by reduction to Volterra type integral equations by using Green's functions [40]. In the multidimensional case, classical, i.e. smooth solution of the Stefan problem in general exists in the short time interval only [62]. Local existence and uniqueness of a classical solution to the multidimensional Stefan problem is established in [61]. In general, the solution may develop singularities and thus a global solution will only exist in the weak sense. In the one phase case, in [41, 55], the Stefan problem is transformed to an obstacle problem and existence and uniqueness of a global weak solution is proved through the method of variational inequalities. Significant progress in regularity of free boundaries of the weak solution in the one-phase case are proved in celebrated papers [30, 31]. A very powerful method of solving the multidimensional multiphase Stefan problem is based on the transformation introduced in [67, 53]. The method transforms the free boundary problem into a nonlinear PDE in a fixed domain with discontinuous coefficients.

1.2 Inverse Free Boundary Problems

Inverse free boundary problems arise in applications where some or several parameters of the system are not known, and must be identified along with the solution of the PDE and the free boundary. Motivation for the inverse free boundary problems in applications is twofold. First motivation is associated with the development of the mathematical models of free boundary systems. Identification of various input parameters of such models is pursued via series of experiments in which some accessible measurements of the system are taken, and inverse problem on the identification of input parameters which produce the observed measurements is analyzed. Second motivation arise in problems of control or design of free boundary systems. It is required to identify input control parameter which develops the system to desired state, or provides particular design with desirable features. This type of inverse problems are equivalent to optimal control problems for free boundary systems with distributed parameters.

Relevance of inverse Stefan type free boundary problems (ISP) appears in two different contexts: ISP with given or unknown free boundary. Consider typical example of ISP with given free boundary arising in Aerospace Industry for in-flight ice accretion modeling [45]. Protection of new aircrafts from atmospheric hazards is one of the most challenging problems in aerospace industry. During the flight aerocraft is exposed to supercooled droplets forming the ice layer on its surface which may significantly compromise aerodynamic performances [45]. In a simplified one-dimensional situation mathematical model of in-flight ice accretion is described with Stefan problem (1.7)-(1.11), where $u(x,t)$ is a temperature and the free boundary $x = s(t)$ is ice layer depth on the surface of the plane. It is required to select heat flux on the fixed boundary in such a way that to achieve a desired ice accretion front. The following is the mathematical formulation of the inverse Stefan problem with given free boundary:

Inverse Stefan Problem (ISP) with known free boundary : given the free boundary $s(t)$, find the temperature function $u(x,t)$ and the boundary heat flux $g(t)$, which satisfy the conditions (1.7)-(1.11).

ISP with known free boundary is similar to the non-characteristic Cauchy problem for the heat equation. It is not a well-posed problem in the sense of Hadamard. Existence of the solution is conditioned on the coordination of the data. Even if solution exists, it may be non-unique, and there is no continuous dependence of the solution on the input data. Historically, ISP with known free boundary was first considered in the paper [33]. A posteriori estimate of the error in the position of the free boundary is determined without the assumption of the existence of a solution of the stated problem, and an a priori bound is derived in the case a solution exists. The most popular methods for solving ISP with known free boundary is based on variational formulation in an optimal control framework. Historically, variational method for the solution of the ISP with known free boundary was first developed in [26, 27, 28]. Optimal control problem for the minimization of the cost functional

$$\mathcal{J}(g) = \int_0^T \int_0^{s(t)} |u_1(x,t) - u_2(x,t)|^2 \, dx dt \tag{1.12}$$

over the control set $g \in G$ is analyzed, where the state vectors $u_1 = u_1(x,t;g)$ and $u_2 = u_2(x,t;g)$ are weak solutions of the parabolic initial bondary value problems (1.7)–(1.10) and (1.7),(1.8),(1.9),(1.11) respectively. Hence, Stefan condition (1.10), and phase transition temperature boundary condition (1.9) are assigned to two different initial boundary value problems as a Neumann and Dirichlet boundary conditions imposed on a given free boundary $x = s(t)$, respectively. Optimal control parameter $g(t)$ is searched via minimization of the mismatch between two solutions u_1 and u_2. In [27] existence of the optimal control is proved and gradient method based on Frechet differentiability is suggested. Uniqueness of the solution is proved in [28] under various smoothness assumptions on the data.

Alternative variational formulation in an optimal control framework for solving ISP with known free boundary can be posed as a minimization of the cost functional

$$\mathcal{J}(g) = \int_0^T \left|u(s(t),t) - \mu(t)\right|^2 dt \tag{1.13}$$

over the control set $g \in G$, where the state vector $u = u(x,t;g)$ solves parabolic Neumann boundary value problem (1.7)–(1.10). In this formulation, Stefan condition is assigned as a Neumann boundary condition on the given free boundary, and optimal boundary heat flux on the fixed boundary $x = 0$ is searched through minimization of the mismatch of the trace of the solution of the parabolic Neumann problem on the given free boundary from the phase transition temperature. This approach turned to be the most popular approach for solving ISP with known free boundary. The main methods were functional-analytic methods for proving existence and uniqueness of solutions, necessary conditions of optimality, Frechet differentiability in functional setting, Tikhonov regularization and development of numerical methods based on gradient type methods in functional spaces were developed and successfully implemented in [21, 23, 35, 37, 38, 48, 72, 43, 68, 69, 36, 82, 54, 83, 45].

ISP with unknown free boundary is more relevant in many applications. For example, consider the modeling of bioengineering problem on the laser ablation of biological tissues through the Stefan problem (1.7)–(1.14), where $s(t)$ is the ablation depth at the moment t. Assume now that some of the data is not available, or involves some measurement error. For example, assume that the coefficients a, c and d, heat flux g on the fixed boundary $x = 0$ and the "regular part" of the density of heat sources, f are unknown and must be found along with the temperature u and the free boundary s. The unknown parameters of the model are very difficult to measure directly through experiments. In order to find the unknown parameters inverse problem must be solved based on the available measurements taken in the lab experiments. For example, assume that this information is provided in the form of a measurement of temperature and the position of the free boundary at the final time $t = T$,

$$u(x, T) = w(x), \quad 0 \leq x \leq s(T) =: \bar{s} \tag{1.14}$$

and consequently, ISP must be solved for the identification of some, or all, of the unknown parameters a, b, c, g, f, etc. Alternatively, measurement can be taken on a fixed boundary $x = 0$:

$$u(0, t) = \nu(t), \quad 0 \leq t \leq T \tag{1.15}$$

Still another important motivation arises from the optimal control of the laser ablation process. A typical control problem arises when unknown control parameters, such as the intensity of the laser source f, heat flux g on the known boundary, and the coefficients a, c and d, must be chosen with the purpose of achieving a desired ablation depth and temperature distribution at the end of the time interval.

The following is the mathematical formulation of the general **Inverse Stefan Problem (ISP) with unknown free boundary**: find functions $u(x, t)$ and $s(t)$, the boundary heat flux $g(t)$, density of sources $f(x, t)$, and coefficients $a(t)$, $c(x, t)$, $d(x, t)$ satisfying conditions (1.7)–(1.14) (or (1.7)–(1.11), (1.15)).

Furthermore across the dissertation ISP with unknown free boundary will be referred simply as ISP. ISP is heavily ill-posed problem, and as such the properties of existence and uniqueness of solution, and property of continuous dependence of solution on the data may be violated. Historically, the first result on ISP appeared in [78] in optimal control framework. Precisely, it analyzed the problem on finding external temperature on the fixed boundary $x = 0$, to ensure that the solution of the one-phase Stefan problem for the heat equation are close to measurements taken at the final moment. In [78] existence of the optimal control was established. For the same problem in [81] the Frechet differentiability and the convergence of the difference schemes was proved and Tikhonov regularization is implemented.

Research on the optimal control of Stefan type free boundary problems, or equivalently ISP with unknown free boundaries was pursued in [22, 39, 48, 49, 51, 52, 56, 60, 66, 64, 70, 71, 75, 43, 44, 24, 46, 47]. Summarizing the research development up to 2012 one can observe that the main methods used to solve the ISP are based on variational formulation, method of quasisolutions or Tikhonov regularization which takes into account ill-posedness in terms of the dependence of the solution on the inaccuracy involved in the measurement, Frechet differentiability and iterative gradient type methods for numerical solution. For example, typical variational formulation of the ISP with unknown flux on the known boundary $x = 0$, or equivalently optimal control of Stefan problem arising in bioengineering problem on the laser ablation of tissues, would be minimization of the cost functional

$$\mathcal{I}(g) = \beta_1 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx + \beta_2 \, |s(T) - \bar{s}|^2 \tag{1.16}$$

in certain control set G, where the state vector $u = u(x,t;g)$ is a solution of the Stefan problem (1.7)-(1.11). As it is mentioned in [1, 2], despite effectiveness, this approach has some deficiencies in many practical applications:

- Solution of the inverse Stefan problem is not continuously dependent on the phase transition temperature $\mu(t)$: small perturbation of the phase transition temperature may imply significant change of the solution to the inverse Stefan problem. Accordingly, any regularization which equally takes into account instability with respect to both $w(x)$ and $s(T)$ from measurement (3.10), and the phase transition temperature $\mu(t)$ from (3.5) will be preferred. It should be also mentioned that in many applications the phase transition temperature is not known explicitly. In many processes the melting temperature of pure material at a given external action depends on the process evolution. For example, gallium (Ga, atomic number 31) may remain in the liquid phase at temperatures well below its mean melting temperature ([62]).

- Numerical implementation of iterative gradient type methods within the existing approach requires solving the full free boundary problem at every step of the iteration, and accordingly has quite a high computational cost. An iterative gradient method which requires solution of the boundary value problem in a fixed region at every step would definitely be much more effective in terms of the computational cost.

In recent papers [1, 2] a new variational approach is developed based on the optimal control theory which is capable of addressing both of the mentioned issues and allows the inverse Stefan problem to be solved numerically with least computational cost by using gradient methods in Hilbert spaces. The main idea of the new variational formulation is that the unknown free boundary $x = s(t)$ is treated as a control parameter. Having the free boundary s as one of the components of the control vector, for any given control, state vector is taken as a solution of the Neumann initial boundary value problem for the parabolic PDE in a fixed domain. Moreover, Stefan condition turns into Neumann boundary condition on the fixed boundary, and the second condition on the free boundary expressing phase transition temperature is added to the cost functional to express the criterion for the mismatch of the temperature on the free boundary vs phase transition temperature. The latter addresses the first concern raised above, while the former settles down a framework for the development of the iterative methods with least computational cost.

In [1] the new variational framework is introduced for the ISP with unknown boundary flux function $g(t)$. The existence of the optimal control and convergence of the family of time-discretized optimal control problems to the continuous problem is proved. In [2] full discretization through finite differences is implemented and convergence of the discrete optimal control problems to the continuous problem both with respect to cost functional and control is established. The new variational formulation introduced in [1, 2] employs a Sobolev spaces framework which allows to reduce the regularity and structural requirements on the data. In [8] the new variational formulation and results of [1, 2] is extended to solve ISP with unknown parameters $a(t), c(x,t), d(x,t), f(x,t), g(t)$. Frechet differentiability and derivation of the necessary condition for optimality in a new variational formulation of the ISP is addressed in [3, 4]. In [3] ISP with unknown boundary heat flux g and the unknown density of the sources f is analyzed in the optimal control framework introduced in [1, 2]. Frechet differentiability in Besov spaces is proved, and the formula for the Frechet differential, expressed in terms of the adjoined PDE problem, is derived under minimal regularity assumptions on the data. The result of [3] implied a necessary condition for optimal control and opened the way to the application of projective gradient methods in Besov spaces for the numerical solution of the ISP. In [4] the results of [3] are extended to ISP with unknown parameters $a(x,t), c(x,t), d(x,t), g(t), f(x,t), g(t)$. In [5] computational analysis of ISP with unknown boundary flux g is pursued via gradient descent algorithm in Hilbert-Besov spaces based on Frechet differentiability results and Frechet gradient formula derived in [3]. Primarily by applying the Frechet differentiabilty result of [4], in [8] computational analysis based on gradient descent method is performed for ISP with unknown time-dependent diffusion coefficient $a(t)$.

The new variational approach developed in [1, 2] is not applicable to the inverse multiphase Stefan problem. The reason is that the Stefan condition on the phase transition boundary includes the flux calculated from both phases. Therefore, it can't be treated as a Neumann condition, even if we include the free boundary as one of the control components. In [6] a new approach was developed based on the weak formulation of the multiphase Stefan problem as a boundary value problem for the nonlinear PDE with discontinuous coefficient. The optimal control framework was applied to the inverse multiphase Stefan problem with non-homogeneous Neumann conditions on the fixed boundaries in the case when the space dimension is one. Existence of the optimal control is proved. Optimal control problem is discretized and convergence of the sequence of finite-dimensional discrete optimal control problems to original optimal control problem is established both with respect to functional and control. In [9], the results of [6] are extended to the case of general second order parabolic PDEs. Multiphase ISP with unknown boundary flux g is transformed to boundary optimal control of singular parabolic PDE problem with time-derivative term being a distributional derivative of the maximal monotone graph. Existence of optimal control and convergence of sequence of discrete optimal control problems is proved. In a recent paper [7] the new method of [6] is developed to solve multi-dimensional and multiphase inverse Stefan problem with unknown source density function $f(x,t)$. The method transforms the problem to boundary optimal control problem for the singular PDE with measure coefficient in the time derivative term. Existence of the optimal control and convergence of the sequence of discrete optimal control problems to the continuous problem both with respect to the functional and control is proved. The proof is based on establishing a uniform L_∞ bound, and $W_2^{1,1}$-energy estimate for the discrete multiphase Stefan problem, and delicate results on the convergence of suitable interpolations of the discrete solutions.

1.3 Formulation of the Open Problem and Outline of Main Results

Consider optimal control problem on the minimization of the cost functional

$$\mathcal{J}(v) = \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx + \beta_1 \int_0^T \left|u(s(t),t) - \mu(t)\right|^2 dt$$
$$+\beta_2 |s(T) - \bar{s}|^2, \tag{1.17}$$

on the control set

$$V_R = \left\{ v = (s, g, f, a, c, d) \in H : \ \delta \le s(t), \ s(0) = s_0, \ s'(0) = 0, \ a(x,t) \ge \underline{a}, \ ||v||_H \le R \right\}, \tag{1.18}$$

where $u = u(x,t;v)$ is a solution of the following Neumann problem for the second order parabolic PDE

$$(a(x,t)u_x)_x + c(x,t)u_x + d(x,t)u - u_t = f(x,t) - \frac{\partial p(x,t)}{\partial x}, \ 0 < x < s(t), 0 < t \le T \tag{1.19}$$

$$u(x,0) = \phi(x), \quad 0 \le x \le s(0) \tag{1.20}$$

$$a(0,t)u_x(0,t) = g(t), \quad 0 \le t \le T \tag{1.21}$$

$$a(s(t),t))u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \quad 0 \le t \le T \tag{1.22}$$

where p, ϕ, γ, χ, μ, w are known functions, $\beta_i \in \mathbb{R}, i = 1,2,3$ and H is some suitably chosen Banach space of vector-function v.

Optimal control problem (1.17)-(1.22) is a variational formulation of the general **Inverse Stefan Problem** with unknown parameters g, f, a, c, d. It was introduced in [1, 2] as a new variational formulation of ISP with unknown boundary flux g. It is essential to note that optimal control problem (1.17)-(1.22) can be adjusted to be a variational formulation of the ISP with known free boundary $x = s(t)$. In this case one have to remove $s(t)$ from the control set V_R, and set up $\beta_0 = \beta_2 = 0$, $\beta_1 = 1$ in (1.17). The aim of the dissertation is to generalize the results of [1, 2] to the case of the optimal control problem (1.17)-(1.22). The main goal is the following:

- Prove existence of the optimal control under minimal conditions on the input data and control parameters;
- Introduce discretization via method of finite differences and prove the convergence of the sequence of finite-dimensional optimal control problems to the original optimal control problem both with respect to functional and control;

In Chapter 2 the general Inverse Stefan Problem with unknown parameters such as time-dependent diffusion coefficient $a(t)$, space-time dependent convection coefficient $c(x,t)$, reaction coefficient $d(x,t)$ and density of sources $f(x,t)$, boundary heat flux $g(t)$ and a free boundary $s(t)$ is considered. Optimal control problem for the free boundary system with distributed parameters for the second order parabolic equation in Hilbert-Besov space

$$H = B_2^1(0,T) \times B_2^{1+\epsilon}(D) \times B_2^{1+\epsilon}(D) \times L_2(D) \times B_2^1(0,T) \times B_2^2(0,T)$$

is introduced, where unknown parameters and the free boundary are components of the control vector, and the state vector is the weak solution of the parabolic Neumann problem in Sobolev-Hilbert space $B_2^1(D)$. Optimality criteria are based on the final moment measurement of the temperature and the position of the free boundary, and the temperature on the phase transition boundary.

- Existence of the optimal control is proved. The methods of proof are based on energy estimates in Sobolev-Hilbert spaces, weak continuity of the cost functional and Weierstrass theorem in weak topology of the Hilbert-Besov spaces.
- Method of finite differences is implemented and space-time discretization of the

optimal control problem is introduced. Convergence of the sequence of the finite-dimensional discrete optimal control problems to the original optimal control problem both with respect to functional and control is proved. Namely,

- it is proved that the sequence of infima of the discrete optimal control problems converge to the infimum of the original optimal control problem,
- It is proved that sequence of interpolations of the discrete optimal controls converge to the optimal control in a weak topology of H, and the sequence of multi-linear interpolations of the discrete PDE problems associated with discrete minimizers converge weakly in the class of weakly differentiable functions to the solution of the PDE problem associated with optimal control. The methods of the proof are based on establishing two energy estimates in discrete Sobolev-Hilbert spaces, use of weak compactness criteria, and delicate interpolation results in Sobolev spaces.

The results of Chapter 2 are published in [8]. It presents solution of the general ISP or equivalently optimal control of parameters (a,c,d,f,g) for the second order Stefan type parabolic free boundary problems. One of the most challenging problems in optimal control of systems with distributed parameters described by elliptic and parabolic PDEs is the problem where control parameter is a diffusion or heat conduction coefficient. Mathematical difficulties are associated with the fact that the diffusion coefficient is embedded in terms with second order spatial derivatives in state PDEs, and therefore they provide high sensitivity and ill-posedness with respect to small errors in measurements. The results of Chapter 2 are valid when diffusion coefficient component a of the control vector is only time dependent. The methods developed in Chapter 2 are not applicable to the case when diffusion coefficient depends on both time variable t and spatial variable x.

In Chapter 3 we consider the Inverse Stefan Problem with unknown space-time dependent diffusion coefficient $a(x,t)$. Dissertation introduces a new Banach space

$$(\tilde{W}_2^{1,1}(D) \cap \tilde{W}_{\infty,\gamma}^{1,1}(D)) \times B_2^2(0,T).$$

Following the new variational formulation introduced in [1, 2], ISP is formulated as a parabolic PDE constrained optimal control problem with control vector $(a(x,t), s(t))$ in a Banach space H. The following are the main results of the Chapter 3:

- Finite difference discretization of the optimal control problem is carried out and sequence of finite-dimensional optimal control problems is introduced. Convergence of the sequence of discrete optimal control problems to continuous optimal control problem both with respect to functional and control is proved.

- Convergence of the sequence of multi-linear interpolations of the minimizing discrete optimal control parameters to optimal diffusion coefficient $a(x,t)$ in a weak topology of $\tilde{W}_2^{1,1}(D)$ is proved. Convergence of the multi-linear interpolations of the associated discrete PDE problems to the optimal state PDE problem in a weak topology of the space of weakly differentiable functions is established.

- H^1-energy estimates are proved for the solutions of the discrete and continuous PDE problems under the minimal assumption $a \in \tilde{W}_{\infty,\gamma}^{1,1}(D)$. Primarily by applying energy estimate, and new interpolation results, existence of the optimal control is proved.

The results of Chapters 2 and 3 can be extended to ISP, and associated optimal control problem with unknown parameter vector $(a(x,t), c(x,t), d(x,t), f(x,t), g(t), s(t)$ in the Banach space

$$H = (\tilde{W}_2^{1,1}(D) \cap \tilde{W}_{\infty,\gamma}^{1,1}(D)) \times B_2^{1+\epsilon}(D) \times B_2^{1+\epsilon}(D) \times L_2(D) \times B_2^1(0,T) \times B_2^2(0,T)$$

Dissertation is outlined as follows. In Section 2.1 variational formulation of ISP with unknown parameter vector $(a(t), c(x,t), d(x,t), f(x,t), g(t)$ is presented. Section 2.2 introduces space-time discretization of the optimal control problem via method of finite differences, and formulates finite-dimensional discrete optimal control problems. Main results of the Chapter 2 and respective assumptions on the data are presented in Section 2.3. Some essential preliminary results are summarized in Section 2.4. In Section 2.5 proofs of the main results are carried out. In Section 2.5.1 energy estimates and compactness results are proved. Finally, in Section 2.5.2 proof of the existence of the optimal control and convergence of the discrete optimal control problems to the original optimal control problem are completed. In Section 3.1 of Chapter 3 general ISP with unknown space-time dependent diffusion coefficient is formulated. Section 3.1 presents variational formulation of the ISP with unknown vector $(a(x,t), s(t)$ as an optimal control problem in the new Banach space. Discretization via finite differences is pursued in Section 3.3, where sequence of finite-dimensional optimal control problems are introduced. Main results of Chapter 3 and respective assumptions are formulated in Section 3.4. Section 3.5 describes some preliminary results. Proofs of the main results are described in Section 3.6. Section 3.6.1 proves the first energy estimation and $V_2^{1,0}$-approximation theorem. In Section 3.6.2 the second energy estimate and the existence of the optimal control is proved. Section 3.6.3 presents the proof of the main convergence theorem. Finally, main conclusions of the dissertation are outlined in Section 4.1 of Chapter 4. Section 4.2 lists research publication and conference presentations.

Chapter 2

Optimal Control of Coefficients in Parabolic Free Boundary Problems

Results of this Chapter are published in [8].

2.1 Variational Formulation of ISP

Let $R, \delta > 0$, $0 < \epsilon \ll 1$. We introduce the following control set:

$$V_R = \Big\{ v = (s, g, f, a, c, d) \in H : \ \delta \le s(t), \ s(0) = s_0, \ s'(0) = 0, \ a(t) \ge \underline{a}, \\ \|v\|_H \le R \Big\}, \tag{2.1}$$

$$H = B_2^2(0,T) \times B_2^1(0,T) \times L_2(D) \times B_2^1(0,T) \times B_2^{1+\epsilon}(D) \times B_2^{1+\epsilon}(D),$$

$$\|v\|_H := \max\Big(\|s\|_{B_2^2(0,T)}; \|g\|_{B_2^1(0,T)}; \|f\|_{L_2(D)}; \|a\|_{B_2^1(0,T)}; \|c\|_{B_2^{1+\epsilon}(D)}; \\ \|d\|_{B_2^{1+\epsilon}(D)} \Big) \tag{2.2}$$

where we define D as follows

$$D := \{(x,t) : 0 \le x \le \ell, \ 0 \le t \le T\},$$

here $\ell = \ell(R) > 0$ and it is chosen so that $v \in V_R$, s satisfies $s(t) \leq \ell$. Such ℓ exists due to Morrey's inequality [58, 25]. We extend $f \in L_2(D)$ to $L_2(\mathbb{R}^2)$ by zero.

Definition 2.1.1. Fix $\beta_i \geq 0$, $i = 0, 1, 2$. Define by problem $\mathcal{I}$ the minimization of the following functional

$$\mathcal{J}(v) = \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx + \beta_1 \int_0^T \left| u(s(t),t) - \mu(t) \right|^2 dt$$
$$+\beta_2 |s(T) - \bar{s}|^2 \tag{2.3}$$

over the control set V_R, where the state vector $u = u(x,t;v)$ solves (1.7)–(1.10).

Definition 2.1.2. We call $u \in B_2^{1,1}(\Omega)$ to be a weak solution of the problem (1.7)–(1.10) if $u(x,0) = \phi(x) \in B_2^1(0, s_0)$ and

$$0 = \int_0^T \int_0^{s(t)} [abu_x\Phi_x - cu_x\Phi - du\Phi + u_t\Phi + f\Phi + p\Phi_x] \, dx\, dt$$
$$+ \int_0^T [\gamma(s(t),t)s'(t) - \chi(s(t),t)] \, \Phi(s(t),t) \, dt + \int_0^T g(t)\Phi(0,t) \, dt \tag{2.4}$$

for any $\Phi \in B_2^{1,1}(\Omega)$.

Definition 2.1.3. $u \in V_2(\Omega)$ is called a weak solution of the problem (1.7)–(1.10) if $u(x,0) = \phi(x) \in B_2^1(0, s_0)$ and

$$0 = \int_0^T \int_0^{s(t)} [abu_x\Phi_x - cu_x\Phi - du\Phi + u_t\Phi + f\Phi + p\Phi_x] \, dx\, dt$$
$$- \int_0^{s(0)} \phi(x)\Phi(x,0) \, dx + \int_0^T g(t)\Phi(0,t) \, dt$$
$$+ \int_0^T [\gamma(s(t),t)s'(t) - u(s(t),t)s'(t) - \chi(s(t),t)] \, \Phi(s(t),t) \, dt \tag{2.5}$$

for any $\Phi \in B_2^{1,1}(\Omega)$ with $\Phi(x,T) = 0$.

2.2 Discrete Optimal Control Problem

Let

$$\omega_\tau = \left\{ t_j = j\tau,\ j = 0, 1, \ldots, n \right\}$$

be a grid on $[0, T]$ and $\tau = \frac{T}{n}$. Following standard notations will be used for $\{d_i\}$,

$$d_{k,\bar{t}} = \frac{d_k - d_{k-1}}{\tau}, \quad d_{kt} = d_{k+1,\bar{t}}, \quad d_{k,\bar{t}t} = \frac{d_{k+1,\bar{t}} - d_{k,\bar{t}}}{\tau} \tag{2.6}$$

Next we introduce the spatial grid in a following way. Let $[s]_n \in \mathbb{R}^{n+1}$ be a discrete boundary and $(p_0, p_1, \ldots, p_n)$ be a permutation of $(0, 1, \ldots, n)$ corresponding to the order $s_{p_0} \le s_{p_1} \le \cdots \le s_{p_n}$. For arbitrary k there exists a unique j_k such that $s_k = s_{p_{j_k}}$. Instead of subscript j_k we are going to use subscript j. Let

$$\omega_{p_0} = \{x_i : x_i = ih,\ i = 0, 1, \ldots, m_0^{(n)}\}$$

be a grid on $[0, s_{p_0}]$ and $h = \frac{s_{p_0}}{m_0^{(n)}}$. We will impose the following assumption on the grid

$$h = O(\sqrt{\tau}), \quad \text{as } \tau \to 0. \tag{2.7}$$

Construction of the spatial grid is carried out by induction. Based on $\omega_{p_{k-1}}$ on $[0, s_{p_{k-1}}]$ we construct

$$\omega_{p_k} = \{x_i :\ i = 0, 1, \ldots, m_k^{(n)}\}$$

on $[0, s_{p_k}]$, where $m_k^{(n)} \ge m_{k-1}^n$, and inequality is strict if and only if $s_{p_k} > s_{p_{k-1}}$; for $i \le m_{k-1}^{(n)}$ points x_i are the same as in grid $\omega_{p_{k-1}}$. If $s_{p_n} < \ell$, then we define a grid on $[s_{p_n}, \ell]$

$$\overline{\omega} = \{x_i : x_i = s_{p_n} + (i - m_n^{(n)})\overline{h},\ i = m_n^{(n)}, \ldots, N\}$$

with the stepsize of order h, i.e. $\overline{h} = O(h)$ as $h \to 0$. For simplicity we will write $m_k^{(n)} \equiv m_k$. Let

$$h_i = x_{i+1} - x_i,\ i = 0, 1, \ldots, N-1;$$

We denote ω_h to be the space grid on $[0, \ell]$ and set

$$\Delta = \max_{i=0,\ldots,N-1} h_i$$

Assume that $m_k \to +\infty$, as $n \to \infty$. We define Steklov averages of $c, d, p, f, b, w, a, \nu, \mu, g$ as follows

$$h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt, \quad \bar{b}_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} \bar{b}(x)\,dx,$$
$$\bar{c}_{ik} = \frac{1}{h_i\tau}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} \bar{c}(x,t)\,dt\,dx,$$

where $i = 0, 1, \ldots, N-1$; $k = 1, \ldots, n$; $\bar{c}$ stands for any of the functions c, d, p, or f; $\bar{b}$ stands for b ,w; and h stands for a, ν, μ, g, etc. We define the discrete control set V_R^n in a following way

$$V_R^n = \Big\{[v]_n = ([s]_n, [g]_n, [f]_{nN}, [a]_n, [c]_n, [d]_n) \in \bar{H} :\ \delta \le s_k;\ \underline{a} \le a_k;$$
$$\|[v]_n\|_{\bar{H}} \le R\Big\} \tag{2.8}$$

where

$$\bar{H} := \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{nN} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$
$$\|[v]_n\|_{\bar{H}} := \max\Big(\|[s]_n\|_{b_2^2}; \|[g]_n\|_{b_2^1}; \|[f]_{nN}\|_{\ell_2}; \|[a]_n\|_{b_2^1}; \|[c]_n\|_{b_2}; \|[d]_n\|_{b_2}\Big),$$

and

$$\|[\bar{g}]_n\|^2_{b_2^1} = \sum_{k=0}^{n-1} \tau \bar{g}_k^2 + \sum_{k=1}^{n} \tau \bar{g}^2_{k,\bar{t}}, \quad \|[s]_n\|^2_{b_2^2} = \|[s]_n\|^2_{b_2^1} + \sum_{k=1}^{n-1} \tau s^2_{k,\bar{t}t},$$
$$\|[f]_{nN}\|^2_{\ell_2} = \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i f_{ik}^2, \quad \|[\bar{c}]_n\|^2_{b_2} = \sum_{k=0}^{n} |\bar{c}_k|^2,$$

where $s_k \equiv s_0$ for $k \le 1$, $\bar{g}$ represents a or g, and $\bar{c}$ represents c or d. Let $\{\psi_k,\ k = 0, 1, \dots\}$ be an orthonormal set in $B_2^{1+\epsilon}(D)$. We denote the inner product on the Hilbert space $B_2^{1+\epsilon}(D)$ by $\langle \cdot, \cdot \rangle_{B_2^{1+\epsilon}}$. Next, we are introducing the following mappings $\mathcal{Q}_n$ and $\mathcal{P}_n$ between continuous and discrete control sets: Define $\mathcal{Q}_n(v)$ for $v \in V_R$ by $s_k = s(t_k)$ for $k = 2, \dots, n$, $g_k = g(t_k)$ and $a_k = a(t_k)$ for $k = 0, 1, \dots, n$, and

$$f_{ik} = \frac{1}{h_i \tau} \int_{t_{k-1}}^{t_k} \int_{x_i}^{x_{i+1}} f(x,t)\, dx\, dt, \quad k = 0, \dots, n,\ i = 0, \dots, N,$$

$\bar{c}_k = \langle \bar{c}, \psi_k \rangle_{B_2^{1+\epsilon}}$ for $k = 0, 1, \dots$ where $\bar{c}$ represents c or d. Define $\mathcal{P}_n([v]_n) = v^n = (s^n, g^n, f^n, a^n, c^n, d^n) \in H$ for $[v]_n \in V_R^n$ by

$$s^n(t) = s_{k-1} + \left(t - t_{k-1} - \frac{\tau}{2}\right) s_{k-1,\bar{t}} + \frac{1}{2}(t - t_{k-1})^2 s_{k-1,\bar{t}t},\ t_{k-1} \le t \le t_k, \tag{2.9}$$
$$\bar{a}^n(t) = \bar{a}_{k-1} + \bar{a}_{k,\bar{t}}(t - t_{k-1}),\ t_{k-1} \le t \le t_k,$$
$$f^n(x,t) = f_{ik},\ x_i \le x < x_{i+1},\ t_{k-1} \le t < t_k,\ i = \overline{0, N-1}$$
$$\bar{c}^n(x,t) = \sum_{k=0}^{n} \bar{c}_k \psi_k(x,t)$$

for any $k = \overline{1, n}$ where $\bar{a}$ represents a or g, and $\bar{c}$ represents c or d. Given $v = (s, g, f, a, c, d) \in V_R$. We also introduce the Steklov averages of traces by

$$\chi_s^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s(t), t)\, dt, \quad (\gamma_s s')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s(t), t) s'(t)\, dt, \quad k = 1, 2, \dots, n \tag{2.10}$$

For any $[v]_n \in V_R^n$ we introduce Steklov averages $\chi_{s^n}^k$ and $(\gamma_{s^n}(s^n)')^k$ by (2.10) with s replaced by s^n. The Steklov averages c_{ik}, and d_{ik} are defined by

$$\bar{c}_{ik} = \frac{1}{h_i \tau} \int_{t_{k-1}}^{t_k} \int_{x_i}^{x_{i+1}} \bar{c}^n(x,t)\, dx\, dt. \tag{2.11}$$

Here $\bar{c}$ represents c or d. Definition of a discrete state vector is given through discretization of the integral identity (2.4)

Definition 2.2.1. Given discrete control vector $[v]_n \in V_R^n$, the vector function

$$[u([v]_n)]_n = (u(0), u(1), \ldots, u(n)),\ u(k) = (u_0, \ldots, u_N) \in \mathrm{Re}^{N+1},\ k = 0, \ldots, n$$

is called a discrete state vector if

(a) The first $m_0 + 1$ components of the vector $u(0)$ satisfy $u_i(0) = \phi_i := \phi(x_i),\ i = 0, 1, \ldots, m_0$;

(b) For arbitrary $k = 1, \ldots, n$ first $m_j + 1$ components of the vector $u(k)$ solve the following system of $m_j + 1$ linear algebraic equations:

$$\begin{gathered}
\Big[b_0 a_k + h c_{0k} - h^2 d_{0k} + \frac{h^2}{\tau}\Big] u_0(k) - \Big[b_0 a_k + h c_{0k}\Big] u_1(k) = \frac{h^2}{\tau} u_0(k-1) \\
-h^2 f_{0k} - h g_k^n - h p_{0k}, \\
\Big[b_{i-1} a_k h_i + b_i a_k h_{i-1} + c_{ik} h_i h_{i-1} - d_{ik} h_i^2 h_{i-1} + \frac{h_i^2 h_{i-1}}{\tau}\Big] u_i(k) \\
-b_{i-1} a_k h_i u_{i-1}(k) - \Big[b_i a_k h_{i-1} + c_{ik} h_i h_{i-1}\Big] u_{i+1}(k) = -h_i^2 h_{i-1} f_{ik} \\
+h_i h_{i-1} p_{ik,\bar{x}} + \frac{h_i^2 h_{i-1}}{\tau} u_i(k-1),\ i = 1, \ldots, m_j - 1 \\
-b_{m_j-1} a_k u_{m_j-1}(k) + b_{m_j-1} a_k u_{m_j}(k) = -h_{m_j-1}\Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\Big].
\end{gathered} \tag{2.12}$$

(c) For arbitrary $k = 0, 1, \ldots, n$, we define the remaining components of $u(k)$ as $u_i(k) = \hat{u}(x_i; k)$ for $m_j \le i \le N$. Here $\hat{u}(x; k) \in B_2^1(0, \ell)$ is a piecewise linear interpolation of

$\{u_i(k): \ i = 0, \ldots, m_j\}$, i.e

$$\hat{u}(x;k) = u_i(k) + u_{ix}(k)(x - x_i), \ x_i \le x \le x_{i+1}, i = 0, \ldots, m_j - 1,$$

iteratively continued for $0 \le x < \infty$ as

$$\hat{u}(x;k) = \hat{u}(2^n s_k - x;k), \ 2^{n-1} s_k \le x \le 2^n s_k, n = 1, 2, \ldots \tag{2.13}$$

It is enough to consider $n^* = 1 + \log_2 \left[\frac{\ell}{\delta}\right]$ reflections to cover $[0, \ell]$. Note that for any $k = 1, 2, \ldots, n$ system (3.19) is equivalent to the following summation identity

$$\sum_{i=0}^{m_j-1} h_i \Big[b_i a_k u_{ix}(k) \eta_{ix} - c_{ik} u_{ix}(k) \eta_i - d_{ik} u_i(k) \eta_i + f_{ik} \eta_i + p_{ik} \eta_{ix} + u_{i\bar{t}}(k) \eta_i \Big]$$
$$+ \Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big] \eta_{m_j} + g_k^n \eta_0 = 0, \tag{2.14}$$

for arbitrary numbers η_i, $i = 0, 1, \ldots, m_j$.

Definition 2.2.2. Denote by problem $\mathcal{I}_n$ the minimization of the functional

$$I_n([v]_n) = \beta_0 \sum_{i=0}^{m_n-1} h_i \Big(u_i(n) - w_i \Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big(u_{m_k}(k) - \mu_k \Big)^2 + \beta_2 |s_n - \bar{s}|^2 \tag{2.15}$$

on the set V_R^n subject where the state vector $[u([v]_n)]_n$ satisfies Definition 3.3.1.

From now on we are going to use piecewise constant and piecewise linear interpolations of the discrete state vector: given discrete state vector $[u([v]_n)]_n$, let

$$u^\tau(x,t) = \hat{u}(x;k), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le \ell, \ k = \overline{0,n},$$
$$\hat{u}^\tau(x,t) = \hat{u}(x;k-1) + \hat{u}_{\bar{t}}(x;k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \le t_k, \ 0 \le x \le \ell,$$

for $k = \overline{1,n}$,

$$\hat{u}^{\tau}(x,t) = \hat{u}(x;n), \quad \text{if } t \geq T,\ 0 \leq x \leq \ell.$$

$$\tilde{u}^{\tau}(x,t) = u_i(k), \quad \text{if } t_{k-1} < t \leq t_k,\ x_i \leq x < x_{i+1},\ k = \overline{1,n},\ i = \overline{0,N-1}.$$

Standard notations for difference quotients of the discrete state vector are employed:

$$u_{ix}(k) = \frac{u_{i+1}(k) - u_i(k)}{h_i},\ u_{i\bar{t}} = \frac{u_i(k) - u_i(k-1)}{\tau}, \quad \text{etc.}$$

Let ϕ^n be a piecewise constant approximation to ϕ:

$$\phi^n(x) = \phi_i,\ x_i < x \leq x_{i+1},\ i = 0, \ldots, N-1$$

2.3 Main Results

Here are our assumptions on functions $b, w, \phi, \mu, p, \chi, \gamma$

$$b \in B^1_\infty(0,l)$$

$$w \in L_2(0,\ell),\ \chi, \gamma \in B_2^{1,1}(D),\ \phi \in B_2^1(0,s_0),\ \mu \in L_2(0,T),\ p \in \mathring{B}_2^{0,1}(D_\delta).$$

where $D_\delta = (0,\delta) \times (0,T)$. Note that the distributional derivative $\frac{\partial p}{\partial x}$ is understood in the sense of measures. Extend arbitrary $\mu \in L_2(0,T)$ to $L_2(\mathbb{R})$ by zero. Now we are going to state the main result of Chapter 2:

Theorem 2.3.1 (Existence of an Optimal Control)**.** *Problem $\mathcal{I}$ has a solution. That is,*

$$V_* := \left\{ v \in V_R : \mathcal{J}(v) = J_* =: \inf_{v \in V_R} \mathcal{J}(v) \right\} \neq \emptyset$$

Theorem 2.3.2. *$\mathcal{I}_n$ approximate the continuous problem $\mathcal{I}$* with respect to functional *in* *the sense that*

$$\lim_{n\to\infty} I_n^* = J_*, \quad \text{where } I_n^* = \inf_{V_R^n} I_n, \text{ and } J_* = \inf_{V_R} \mathcal{J}$$

Moreover, the sequence $\mathcal{I}_n$ *approximates* $\mathcal{I}$ with respect to control *in the sense that if* $[u]_{n,\epsilon} \in V_R^n$ *is chosen such that*

$$I_n^* \le I_n([v]_{n,\epsilon}) \le I_n^* + \epsilon_n, \quad \text{where } \epsilon_n \downarrow 0$$

then the sequence $v^n = (s^n, g^n, f^n, a^n, c^n, d^n) = \mathcal{P}_n([v]_{n,\epsilon})$ *converges to an element* $v_* = (s_*, g_*, f_*, a_*, c_*, d_*) \in V_*$ *weakly in* $B_2^2(0,T) \times B_2^1(0,T) \times L_2(D) \times B_2^1(0,T) \times B_2^{1+\epsilon}(D) \times B_2^{1+\epsilon}(D)$, *and* $(s^n, g^n, a^n, b^n, c^n)$ *converge strongly in* $B_2^1(0,T) \times L_2(0,T) \times L_2(0,T) \times L_2(D) \times L_2(D)$. *Moreover,* s^n *converges to* s_* *uniformly on* $[0,T]$. *For any* $\delta > 0$*, define*

$$\Omega' = \Omega \cap \{x < s(t) - \delta,\ 0 < t < T\}$$

Then the piecewise linear interpolations $\hat{u}^\tau$ *of the corresponding discrete state vectors* $[[v]_{n,\epsilon}]_n$ *converge to the solution* $u(x,t;v_*) \in B_2^{1,1}(\Omega_*)$ *of the Neumann problem* (1.7)–(1.10) *weakly in* $B_2^{1,1}(\Omega')$.

2.4 Preliminary Results

Lemma 2.4.1. *For arbitrary sufficiently small* $\epsilon > 0$*, there exists* n_ϵ *such that*

$$\mathcal{Q}_n(v) \in V_R^n, \text{ for all } v \in V_{R-\epsilon},\ n > n_\epsilon \tag{2.16}$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \text{ for all } [v]_n \in V_R^n,\ n > n_\epsilon \tag{2.17}$$

Proof. The first two entries of either $\mathcal{Q}_n(v)$ for $v \in V_{R-\epsilon}$ or $\mathcal{P}_n([v]_n)$ for $[v]_n \in V_R^n$ are estimated as in [2, Lem. 2.2]; therefore we are going to focus on estimating the components corresponding to f, a, c, and d in both. We will carry out estimation for d

component and will omit the details for c component due to similarity. Fix $v \in V_{R-\epsilon}$ and let $([s]_n, [g]_n, [f]_{nN}, [a]_n, [c]_n, [d]_n) = Q_n(v)$. Estimation for $a(t)$ term is performed as for $g(t)$ term in [2, Lem. 2.2] since both $a(t)$ and $g(t)$ terms belong to B_2^1. By Cauchy-Bunyakovski-Schwarz (CBS) inequality,

$$\|[f]_{nN}\|_{\ell_2}^2 \leq \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} |f(x,t)|^2 \, dt \, dx$$
$$= \int_0^T \int_0^\ell |f(x,t)|^2 \, dx \, dt = \|f\|_{L_2(D)}^2 \leq (R-\epsilon)^2 \tag{2.18}$$

By Bessel's inequality,

$$\|[d]_n\|_{b_2}^2 = \sum_{k=0}^{n} |d_k|^2 \leq \sum_{k=0}^{\infty} |d_k|^2 \leq \|d\|_{B_2^{1+\epsilon}(D)}^2 \leq (R-\epsilon)^2 \tag{2.19}$$

By (2.18), (2.19), and the proof of [2, Lem. 2.1], it follows that

$$\|Q_n(v)\|_{V_R^n}^2 \leq R^2$$

for τ sufficiently small, which implies (2.16). Next, consider $[v]_n \in V_R^n$ and let $(s, g, f, a, c, d) = \mathcal{P}_n([v]_n)$. Since a is a piecewise-linear interpolation of the values $[a]_n$, on each interval $[t_{k-1}, t_k]$ the interpolation attains its maximum on the boundary, and in particular, $a_k \geq \underline{a}$. The estimate of the norm of $a(t)$ follows from estimate for g as in [2, Lem. 2.2].

Lets consider the estimate for the term f, we get

$$\|f\|_{L_2(D)}^2 = \int_0^T \int_0^\ell |f(x,t)|^2 \, dx \, dt = \sum_{k=1}^{n} \tau \sum_{i=0}^{N-1} h_i |f_{ik}|^2 = \|[f]_n\|_{\ell_2(0,\ell)}^2 \leq R^2 \tag{2.20}$$

By definition,

$$\left\|d^n\right\|^2_{B_2^{1+\epsilon}(D)} = \langle d^n, d^n\rangle_{B_2^{1+\epsilon}} = \left\langle \sum_{k=0}^{n} d_k\psi_k(x,t), \sum_{j=0}^{n} d_j\psi_j(x,t)\right\rangle_{B_2^{1+\epsilon}}$$
$$= \sum_{k=0}^{n}\sum_{j=0}^{n} d_k d_j \left\langle \psi_k, \psi_j\right\rangle_{B_2^{1+\epsilon}} = \|[d]_n\|^2_{b_2(D)} \le R^2 \tag{2.21}$$

By (2.20), (2.21), and the proof of [2, Lem. 2.1], it follows that

$$\|\mathcal{P}_n([v]_n)\|^2_H \le (R+\epsilon)^2$$

for τ sufficiently small, which implies (2.17). Lemma is proved. □

As in [1], it follows from Theorem 2.4.1 that

Corollary 2.4.2. *Let either $[v]_n \in V_R^n$ or $[v]_n = \mathcal{Q}_n(v)$ for $v \in V_R$. Then for large n,*

$$|s_k - s_{k-1}| \le C'\tau,\ k = 1,2,\ldots,n \tag{2.22}$$

where C' is independent of n.

Notice that we might only have one of the following: $h_i = h$, or $h_i = \overline{h}$, or $h_i \le |s_k - s_{k-1}|$ for some k. Hence, from (3.15) and (2.22), it follows that

$$\Delta = O(\sqrt{\tau}), \quad \text{as } \tau \to 0. \tag{2.23}$$

Using Lemma 2.4.1, we derive

Corollary 2.4.3. *For a given discrete control vectors $[d]_n \in b_2$, the coefficients $\{d_{ik}\}$ defined by* (2.11) *satisfy the estimate*

$$\max_{ik} |d_{ik}| \le C\|[d]_n\|_{b_2} \tag{2.24}$$

for C independent of n and $[d]_n$. *In particular, when* $\|[d]_n\|_{b_2}$ *are bounded, then* $\{d_{ik}\}$ *are uniformly bounded. Using similar argument, we can show that coefficients* $\{c_{ik}\}$ *are uniformly bounded.*

Proof. By embedding of $B_2^{1+\epsilon}(D)$ in $L_\infty(D)$ [25, 65, 73, 74],

$$\max_{ik}|d_{ik}| = \max_{ik}\frac{1}{h_i\tau}\left|\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}} d^n(x,t)\,dx\,dt\right| \le \left\|d^n\right\|_{L_\infty(D)}$$
$$\le C\left\|d^n\right\|_{B_2^{1+\epsilon}(D)} \le C\|[d]_n\|_{b_2(D)} \qquad \square$$

Lemma 2.4.4. *For given* $[v]_n \in V_R^n$*, the discrete state vector* $\left[u([v]_n)\right]_n$ *exists and is unique for all sufficiently small* $\tau > 0$.

As the regularity of the coefficients is proven to be sufficient, and the form of the corresponding homogeneous equations to (3.19) are the same after renaming, we can prove Lemma 2.4.4 as in [2, Lem. 2.1].

2.5 Proof of Main Results

2.5.1 Energy Estimates and their Consequences

Theorem 2.5.1. *For* τ *sufficiently small, and for any discrete control* $[v]_n \in V_R^n$*, the corresponding discrete state vector satisfies the estimate*

$$\max_{0\le k\le n}\sum_{i=0}^{N-1} h_i u_i^2(k) + \sum_{k=1}^{n}\tau\sum_{i=0}^{N-1} h_i u_{ix}^2(k) \le C\Big(\left\|\phi^n\right\|_{L_2(0,s_0)}^2 + \left\|g^n\right\|_{L_2(0,T)}^2$$
$$+\left\|f^n\right\|_{L_2(D)}^2 + \left\|\gamma(s^n(t),t)(s^n)'(t)\right\|_{L_2(0,T)}^2 + \left\|\chi(s^n(t),t)\right\|_{L_2(0,T)}^2$$
$$+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\sum_{i=m_j}^{m_{j_{k+1}}-1} h_i u_i^2(k)\Big), \qquad (2.25)$$

Theorem 2.5.1 is an extension of [2, Thm. 3.1]. As in [1, Thm. 3.4], from Theorem 2.5.1 we have

Theorem 2.5.2. *Let* $[v]_n \in V_R^n$ *for* $n = 1, 2, \ldots$ *be a sequence of discrete controls with* $\{\mathcal{P}_n([v]_n)\}$ *converging weakly in* $B_2^2(0,T) \times B_2^1(0,T) \times L_2(D) \times B_2^1(0,T) \times B_2^{1+\epsilon}(D) \times B_2^{1+\epsilon}(D)$ *to an element* $v = (s, g, f, a, c, d)$*. Then* $\{u^\tau\}$ *converges as* $\tau \to 0$ *weakly in* $B_2^{1,0}(D)$ *to a weak solution* $u \in V_2^{1,0}(\Omega)$ *of* (1.7)–(1.10)*. Moreover, u satisfies the energy estimate*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big[\|\phi\|^2_{L_2(0,s_0)} + \sup_n \big\|f^n\big\|^2_{L_2(D)} + \|p\|^2_{L_2(D)} + \|\gamma\|^2_{B_2^{1,0}(D)} + \|\chi\|^2_{B_2^{1,0}(D)} + \|g\|^2_{L_2(0,T)}\Big] \tag{2.26}$$

From application of CBS inequality, equivalence of the piecewise constant interpolations c_{ik} and $c^n(x,t)$ in $L_2(D)$ follows. We get the following Corollary from Theorem 2.5.2

Corollary 2.5.3. *For any* $v = (s, g, f, a, c, d) \in V_R$*, there exists a weak solution* $u \in V_2^{1,0}(\Omega)$ *of the Neumann problem* (1.7)–(1.10) *satisfying the energy estimate* (2.26)*.*

Given any discrete control vector $[v]_n$ and the corresponding discrete state vector $\big[u([v]_n)\big]_n$, define the constant continuation $\big[\tilde{u}([v]_n)\big]_n$ by $\tilde{u}_i(k) = u_i(k)$ for $0 \leq i \leq m_j$ and $\tilde{u}_i(k) = u_{m_j}(k)$ for $m_j < i$ for $k = 0, \ldots, n$.

Theorem 2.5.4 (Second Energy Estimate)**.** *For* τ *sufficiently small, and for any discrete control* $[v]_n \in V_R^n$*, the modified discrete state vector* $[\tilde{u}([v]_n)]$ *satisfies*

$$\max_{1\leq k\leq n} \sum_{i=0}^{m_j-1} h_i u_{ix}^2(k) + \tau \sum_{k=1}^{n} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}(k)^2 + \tau^2 \sum_{k=1}^{n} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{ix\bar{t}}^2(k)$$
$$\leq C\Big[\big\|\phi^n\big\|^2_{L_2(0,s_0)} + \|\phi\|^2_{B_2^1(0,s_0)} + \big\|f^n\big\|^2_{L_2(D)} + \big\|g^n\big\|^2_{B_2^{1/4}(0,T)} + \|p\|^2_{\mathring{B}_2^{0,1}(D_\delta)} + \big\|\gamma(s^n(t),t)(s^n)'(t)\big\|^2_{B_2^{1/4}(0,T)} + \big\|\chi(s^n(t),t)\big\|^2_{B_2^{1/4}(0,T)}\Big] \tag{2.27}$$

Proof. In (3.21), take $\eta = 2\tau\tilde{u}_{i\bar{t}}(k)$ to derive

$$\sum_{i=0}^{m_j-1} 2\tau h_i\Big[b_i a_k u_{ix}(k)\tilde{u}_{ix\bar{t}}(k) - c_{ik}u_{ix}(k)\tilde{u}_{i\bar{t}}(k) - d_{ik}u_i(k)\tilde{u}_{i\bar{t}}(k) + f_{ik}\tilde{u}_{i\bar{t}}(k)$$

$$+p_{ik}\tilde{u}_{ix\bar{t}}(k) + u_{i\bar{t}}(k)\tilde{u}_{i\bar{t}}(k)\Big] + 2\tau\Big[\big(\gamma_{s^n}(s^n)'\big)^k - \chi^k_{s^n}\Big]\tilde{u}_{m_j\bar{t}}(k) + 2\tau g^n_k\tilde{u}_{0\bar{t}}(k) = 0 \tag{2.28}$$

Arguing as in [2, Thm. 3.3] (in particular, Eq. 3.46), it follows that

$$\sum_{i=0}^{m_{j_q}-1} h_i\tilde{u}^2_{ix}(q) + \tau^2\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}^2_{ix\bar{t}}(k) + \tau\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}^2_{i\bar{t}}(k) \le C\sum_{i=0}^{m_{j_0}-1} h_i\phi^2_{ix}$$

$$+C\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i\tilde{u}^2_{ix}(k) + C\max_{1\le k\le q}\sum_{i=0}^{m_j-1} h_i\tilde{u}^2_i(k) + C\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i f^2_{ik}$$

$$-\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i p_{ik}\tilde{u}_{ix\bar{t}}(k) - 2\tau\sum_{k=1}^{q}\Big[\big(\gamma_{s^n}(s^n)'\big)^k - \chi^k_{s^n}\Big]\tilde{u}_{m_j\bar{t}}(k) - 2\tau\sum_{k=1}^{q} g^n_k\tilde{u}_{0\bar{t}}(k) \tag{2.29}$$

for any $1 \le q \le n$. We estimate the second and third terms on the right-hand side of (2.29) using the first energy estimate; To handle the term with p_{ik} we use summation by parts; since p has a compact support with respect to x in $(0,\delta)$, there exists i_δ with $i_\delta < m_{j_k} - 1$ for all k such that $p_{ik} \equiv 0$ for $i > i_\delta$, and hence

$$\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i p_{ik}\tilde{u}_{ix\bar{t}}(k) = \sum_{i=0}^{i_\delta} h_i p_{iq}\tilde{u}_{ix}(q) - \sum_{i=0}^{i_\delta} h_i p_{i,1}\phi_{ix}$$

$$-\sum_{k=1}^{q-1}\sum_{i=0}^{i_\delta}\tau h_i p_{i,k+1,\bar{t}}\tilde{u}_{ix}(k) \tag{2.30}$$

Therefore, from (2.29), (2.30), Corollary 2.4.3, and Cauchy inequality with ϵ, it follows that

$$\frac{a_0}{2}\sum_{i=0}^{m_{j_q}-1} h_i\tilde{u}_{ix}^2(p)+\tau^2 a_0\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)+\frac{\tau}{2}\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)$$
$$\le C\Big\{\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2+\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i\tilde{u}_{ix}^2(k)+\max_{1\le k\le p}\sum_{i=0}^{m_j-1} h_i\tilde{u}_i^2(k)+\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i f_{ik}^2$$
$$+\sum_{i=0}^{i_\delta} h_i p_{iq}^2+\sum_{i=0}^{i_\delta} h_i p_{i1}^2+\sum_{k=1}^{q-1}\sum_{i=0}^{i_\delta}\tau h_i p_{i,k+1,\bar{t}}^2+\tau\sum_{k=1}^{q}\Big[\big(\gamma_{s^n}(s^n)'\big)^k-\chi_{s^n}^k\Big]\tilde{u}_{m_j\bar{t}}(k)$$
$$+\tau\sum_{k=1}^{q} g_k^n\tilde{u}_{0\bar{t}}(k)\Big\} \qquad (2.31)$$

for some C independent of τ. By CBS inequality and Fubini's theorem we have

$$\tau\sum_{k=1}^{m-1}\sum_{i=0}^{m_j-1} h_i p_{i,k+1,t}^2\le\frac{1}{\tau^2}\sum_{k=1}^{m-1}\int_0^{s_k}\int_{t_{k-1}}^{t_k}|p(x,t+\tau)-p(x,t)|^2\,dt\,dx$$
$$\le\|p_t\|_{L_2(D)}^2 \qquad (2.32)$$

By CBS inequality and Sobolev embedding theorem [65, 25]

$$\sum_{i=0}^{m_j-1} h_i p_{ik}^2=\sum_{i=0}^{m_j-1}\frac{1}{h_i\tau^2}\left(\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} p(x,t)\,dt\,dx\right)^2$$
$$\le\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_0^{\delta} p^2(x,t)\,dx\,dt\le C\|p\|_{\mathring{B}_2^{0,1}(D_\delta)}^2 \qquad (2.33)$$

Having (2.33) and (2.32), from (2.31) it follows that

$$\frac{a_0}{2}\sum_{i=0}^{m_{j_q}-1} h_i\tilde{u}_{ix}^2(q)+\tau^2 a_0\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)+\frac{\tau}{2}\sum_{k=1}^{q}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)$$
$$\le C\Big\{\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2+\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i\tilde{u}_{ix}^2(k)+\max_{1\le k\le q}\sum_{i=0}^{m_j-1} h_i\tilde{u}_i^2(k)+\sum_{k=1}^{q}\sum_{i=0}^{m_j-1}\tau h_i f_{ik}^2$$
$$+\|p\|_{\mathring{B}_2^{0,1}(D_\delta)}^2+\tau\sum_{k=1}^{q}\Big[\big(\gamma_{s^n}(s^n)'\big)^k-\chi_{s^n}^k\Big]\tilde{u}_{m_j\bar{t}}(k)+\tau\sum_{k=1}^{q} g_k^n\tilde{u}_{0\bar{t}}(k)\Big\} \qquad (2.34)$$

Since this inequality holds for all $1 \leq q \leq n$, it follows that

$$\frac{a_0}{2}\max_{1\leq k\leq n}\sum_{i=0}^{m_{j_k}-1} h_i\tilde{u}_{ix}^2(k)+\tau^2 a_0\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)+\frac{\tau}{2}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)$$
$$\leq C\Big\{\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2+\sum_{k=1}^{n}\sum_{i=0}^{m_j-1}\tau h_i\tilde{u}_{ix}^2(k)+\max_{1\leq k\leq n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_i^2(k)+\|f\|_{L_2(D)}^2$$
$$+\|p\|_{\mathring{B}_2^{0,1}(D_\delta)}^2+\tau\sum_{k=1}^{n}\Big[\big(\gamma_{s^n}(s^n)'\big)^k-\chi_{s^n}^k\Big]\tilde{u}_{m_j\bar{t}}(k)+\tau\sum_{k=1}^{n} g_k^n\tilde{u}_{0\bar{t}}(k)\Big\} \tag{2.35}$$

Applying the method of [1] allows the estimation of the last two terms of (2.35); if $\gamma,\chi \in B_2^{1,1}(D)$ and $[v]_n = ([s]_n,[g]_n,[f]_{nN},[a]_n,[c]_n,[d]_n) \in V_R^n$, then for n large enough, $\mathcal{P}_n([v]_n) \in V_{R+1}$ by Theorem 2.4.1, and hence the traces of χ and $\gamma\cdot(s^n)'$ on the curves $x = s^n(t)$ are in $B_2^{1/4}(0,T)$ [65, 25] and

$$\Big\|\gamma(s^n(t),t)(s^n)'(t)\Big\|_{B_2^{1/4}(0,T)} \leq C\|\gamma\|_{B_2^{1,1}(D)},$$
$$\Big\|\chi(s^n(t),t)\Big\|_{B_2^{1/4}(0,T)} \leq C\|\chi\|_{B_2^{1,1}(D)} \tag{2.36}$$

Let $\Psi(x,t) \in B_2^{2,1}(D)$ be a solution of the heat equation satisfying

$$\Psi(x,0) = \phi(x),\ \text{for } x\in[0,s_0],\quad b(0)a(t)\Psi_x(0,t) = g^n(t),\ \text{for a.e. } t\in[0,T],$$
$$b(s^n(t))a(t)\Psi_x(s^n(t),t) = \chi(s^n(t),t)-\gamma(s^n(t),t)(s^n)'(t),\ \text{for a.e. } t\in[0,T]$$

and

$$\|\Psi\|_{B_2^{2,1}(D)} \leq C\Big[\big\|g^n\big\|_{B_2^{1/4}(0,T)}+\|\phi\|_{B_2^1(0,s_0)}$$
$$+\Big\|\chi(s^n(t),t)-\gamma(s^n(t),t)(s^n)'(t)\Big\|_{B_2^{1/4}(0,T)}\Big] \tag{2.37}$$

Existence of such Ψ follows from e.g. [58, Ch. 3, Thm. 6.1]. Then replacing u, s and g

with $u-\Psi$, s^n, and g^n in (2.35) with $f(x)$ replaced by $f(x)-L\Psi(x)\in L_2(D)$, we derive

$$\frac{a}{2}\max_{1\le k\le n}\sum_{i=0}^{m_{j_k}-1} h_i\tilde{u}_{ix}^2(k)+\tau^2\underline{a}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)+\frac{\tau}{2}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)$$
$$\le C\Big\{\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2+\sum_{k=1}^{n}\sum_{i=0}^{m_j-1}\tau h_i\tilde{u}_{ix}^2(k)+\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_i^2(k)+\|f\|^2_{L_2(D)}$$
$$+\|L\Psi\|^2_{L_2(D)}+\|p\|^2_{\mathring{B}_2^{0,1}(D_\delta)}\Big\} \tag{2.38}$$

Using first energy estimate (2.25), together with (2.37), and (2.36), from (2.38) it follows that for τ sufficiently small, u satisfies

$$\frac{a}{2}\max_{1\le k\le n}\sum_{i=0}^{m_{j_k}-1} h_i\tilde{u}_{ix}^2(k)+\tau^2\underline{a}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)+\frac{\tau}{2}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)$$
$$\le C\Big\{\|\phi\|_{B_2^1(0,s_0)}+\left\|\phi^n\right\|^2_{L_2(0,s_0)}+\|f\|^2_{L_2(D)}+\left\|g^n\right\|_{B_2^{1/4}(0,T)}+\|\phi\|_{B_2^1(0,s_0)}$$
$$+\left\|\chi(s^n(t),t)-\gamma(s^n(t),t)(s^n)'(t)\right\|_{B_2^{1/4}(0,T)}+\|p\|^2_{\mathring{B}_2^{0,1}(D_\delta)}$$
$$+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\sum_{i=m_j}^{m_{j_{k+1}}-1} h_iu_i^2(k)\Big\} \tag{2.39}$$

where C independent of τ has been used to absorb the constants on the left-hand side, and τ is sufficiently small as in the hypotheses of Theorem 2.5.1, which implies (2.27). □

As in [2, Thm. 3.4], from Theorem 2.5.4 we have

Theorem 2.5.5. *Let $[v]_n\in V_R^n$ for $n=1,2,\dots$ be a sequence of discrete controls with $\{\mathcal{P}_n([v]_n)\}$ converging weakly to an element $v=(s,g,f,a,c,d)$ in H (with (s^n,g^n,a^n,b^n,c^n) converging strongly in $B_2^1(0,T)\times L_2(0,T)\times L_2(0,T)\times L_2(D)\times L_2(D)$ to (s,g,a,c,d)) and, for any $\delta>0$, define*

$$\Omega'=\Omega\cap\{x<s(t)-\delta,\ 0<t<T\}.$$

Then $\{\hat{u}^\tau(x,t;v_n)\}$ converges as $\tau\to 0$ weakly in $B_2^{1,1}(\Omega')$ to a weak solution $u\in B_2^{1,1}(\Omega)$

of (1.7)–(1.10). *Moreover, u satisfies the energy estimate*

$$\|u\|^2_{B_2^{1,1}(\Omega)} \le C\Big[\|\phi\|^2_{B_2^1(0,s_0)} + \sup_n \big\|f^n\big\|^2_{L_2(D)} + \|p\|^2_{B_2^{0,1}(D)} + \|\gamma\|^2_{B_2^{1,1}(D)} + \|\chi\|^2_{B_2^{1,1}(D)} + \|g\|^2_{B_2^{1/4}(0,T)}\Big] \tag{2.40}$$

Theorem 2.5.5 implies the following

Corollary 2.5.6. *For any $v \in V_R$, there exists a weak solution $u \in B_2^{1,1}(\Omega)$ of the Neumann problem* (1.7)–(1.10) *satisfying the energy estimate* (2.40)*. By Sobolev extension theorem, u may be extended to a $B_2^{1,1}(D)$ function with norm preservation, so it satisfies the energy estimate*

$$\|u\|^2_{B_2^{1,1}(D)} \le C\Big[\|\phi\|^2_{B_2^1(0,s_0)} + \|f\|^2_{L_2(D)} + \|p\|^2_{\mathring{B}_2^{0,1}(D_\delta)} + \|\gamma\|^2_{B_2^{1,1}(D)} + \|\chi\|^2_{B_2^{1,1}(D)} + \|g\|^2_{B_2^{1/4}(0,T)}\Big]$$

2.5.2 Proof of the Existence and Convergence Results

Compactness results of Theorems 2.5.2 and 2.5.5 imply the weak continuity of the functional $\mathcal{J}$, so Theorem 2.3.1 follows from Weierstrass Theorem in the weak topology of $B_2^2(0,T)\times B_2^1(0,T)\times L_2(D)\times B_2^1(0,T)\times B_2^{1+\epsilon}(D)\times B_2^{1+\epsilon}(D)$. The necessary results to complete Theorem 2.3.2 will be given in three Lemmas.

Lemma 2.5.7. *For $\epsilon > 0$ define $J_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} \mathcal{J}(v)$ Then*

$$\lim_{\epsilon\to 0} J_*(\epsilon) = J_* = \lim_{\epsilon\to 0} J_*(-\epsilon) \tag{2.41}$$

Lemma 2.5.7 is established as in [1, Lem. 3.9]

Lemma 2.5.8. *For $v \in V_R$,*

$$\lim_{n\to\infty} I_n(Q_n(v)) = \mathcal{J}(v) \tag{2.42}$$

Proof. Fix $v \in V_R$ and let $[v]_n = ([s]_n, [g]_n, [f]_{nN}, [a]_n, [c]_n, [d]_n) = \mathcal{Q}_n(v)$. Let $u = u(x,t;v)$ and $[u([v]_n)]_n$ be the corresponding continuous and discrete state vector, respectively, and denote by $v^n = (s^n, g^n, f^n, a^n, c^n, d^n) = \mathcal{P}_n([v]_n)$. By Sobolev embedding theorem, $s^n(t) \to s(t)$ uniformly on $[0,T]$. Let $\epsilon_m \downarrow 0$ be an arbitrary sequence, and define

$$\Omega_m = \{(x,t) : 0 < x < s(t) - \epsilon_m, 0 < t \le T\}$$

and fix $m > 0$.

In Theorem 2.5.5 it was shown that $\{\hat{u}^\tau\}$ converges to u weakly in $B_2^{1,1}(\Omega_m)$ for any fixed m; by the embeddings of traces, it follows that $\{\hat{u}^\tau(s(t) - \epsilon_m, t)\}$ and $\{\hat{u}^\tau(x,T)\}$ converge to the corresponding traces $u(s(t)-\epsilon_m, t)$ and $u(x,T)$ weakly in $L_2(0,T)$ and $L_2(0, s(t) - \epsilon_m)$, respectively. We shall prove that the corresponding traces of u^τ satisfy the same property.

By Sobolev embedding theorem, it is enough to show that $\{u^\tau\}$ and $\{\hat{u}^\tau\}$ are equivalent in $B_2^{1,0}(\Omega_m)$.

Denote by $s_k^m = x_{\hat{\imath}}$ where

$$\hat{\imath} = \max\left\{ i \le N : \ -\epsilon_m \le x_i - \max_{t_{k-1} \le t \le t_k} s(t) \le -\frac{\epsilon_m}{2} \right\}.$$

Arguing as in [2, Eqs. 101–104] it follows that there exists $N = N(\epsilon_m)$ such that $n > N$ implies

$$s_k^m < \min(s_k, s_{k-1}), \ k = 1, \ldots, n \tag{2.43}$$

and accordingly

$$\left\| \frac{\partial \hat{u}^\tau}{\partial x} - \frac{\partial u^\tau}{\partial x} \right\|^2_{L_2(\Omega_m)} = \frac{\tau^3}{3} \sum_{k=1}^{n} \sum_{i=0}^{\hat{\imath}-1} h_i u_{ix\bar{t}}^2(k) \le \frac{\tau^3}{3} \sum_{k=1}^{n} \sum_{i=0}^{m_j - 1} h_i \tilde{u}_{ix\bar{t}}^2(k) = O(\tau). \tag{2.44}$$

Estimate the first term in $I_n(\mathcal{Q}_n(v)) - \mathcal{J}(v)$ as

$$\left|\beta_0 \sum_{i=0}^{m_n-1} h_i \,|u_i(n) - w_i|^2 \, dx - \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx\right|$$
$$\leq \beta_0 \left\{ \left| \sum_{i=0}^{\hat{\imath}-1} \left[h_i \,|u_i(n) - w_i|^2 - \int_{x_i}^{x_{i+1}} |u(x,T) - w(x)|^2 \, dx \right] \right| + I_{n,m} + \tilde{I}_m \right\} \tag{2.45}$$

where

$$I_{n,m} = \left| \sum_{\hat{\imath}}^{m_n-1} h_i \,|u_i(n) - w_i|^2 \right|, \quad \tilde{I}_m = \left| \int_{s_n^m}^{s(T)} |u(x,T) - w(x)|^2 \, dx \right| \tag{2.46}$$

By absolute continuity of the integral, $\tilde{I}_m \to 0$ as $m \to \infty$. Considering $I_{n,m}$,

$$I_{n,m} \leq 2 \left| \sum_{\hat{\imath}}^{m_n-1} h_i \,|u_i(n)|^2 \right| + \left| \sum_{\hat{\imath}}^{m_n-1} h_i \,|w_i|^2 \right|$$

By Morrey's inequality,

$$\left| \sum_{\hat{\imath}}^{m_n-1} h_i \,|u_i(n)|^2 \right| \leq C \left| s^n(T) - s(T) + \epsilon_m \right| ||\hat{u}(x;n)||^2_{B_2^1(0,\ell)}$$

From (2.25) and (2.27), it follows that

$$||\hat{u}(x;n)||^2_{B_2^1(0,\ell)} \leq C_1 \tag{2.47}$$

For a constant C_1 depending on the given data ϕ, f, etc. but not τ (or m). Now, considering the second term in $I_{n,m}$, by CBS inequality,

$$\left| \sum_{\hat{\imath}}^{m_n-1} h_i \,|w_i|^2 \right| = \left| \sum_{\hat{\imath}}^{m_n-1} \frac{1}{h_i} \left| \int_{x_i}^{x_{i+1}} w(x) \, dx \right|^2 \right| \leq \left| \int_{s(T)}^{s^n(T)} |w(x)|^2 \, dx \right|$$
$$+ \left| \int_{s(T)-\epsilon_m}^{s(T)} |w(x)|^2 \, dx \right|$$

From convergence $s^n(T) \to s(T)$ and absolute continuity of the integral we get that there is some $N_1 = N_1(m)$ such that for $n > N_1$,

$$\left| \sum_{\hat{\imath}}^{m_n - 1} h_i |w_i|^2 \right| \le 2 \int_{s(T)-\epsilon_m}^{s(T)} |w(x)|^2 \, dx + \frac{1}{m} \tag{2.48}$$

By (2.47) and (2.48), it follows that for $n > N_1$

$$0 \le I_{n,m} \le CC_1 \left(\epsilon_m + \left| s^n(T) - s(T) \right| \right) + 2 \int_{s(T)-\epsilon_m}^{s(T)} |w(x)|^2 \, dx + \frac{1}{m} \tag{2.49}$$

By (2.45) and (2.49), it follows that

$$\begin{aligned} 0 \le \limsup_{n\to\infty} &\left| \beta_0 \sum_{i=0}^{m_n-1} h_i |u_i(n) - w_i|^2 \, dx - \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx \right| \\ &\le CC_1 \epsilon_m + 2 \int_{s(T)-\epsilon_m}^{s(T)} |w(x)|^2 \, dx + \frac{1}{m} + \tilde{I}_m \end{aligned}$$

for all m. Passing to the limit as $m \to \infty$ it follows that

$$\lim_{n\to\infty} \beta_0 \sum_{i=0}^{m_n-1} h_i |u_i(n) - w_i|^2 = \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2 \, dx$$

The convergence of the second and third terms of I_n to corresponding terms in $\mathcal{J}$ is established in a similar way. Lemma is proved. □

Lemma 2.5.9. *For arbitrary* $[v]_n \in V_R^n$, $\lim_{n\to\infty} (\mathcal{J}(\mathcal{P}_n([v]_n)) - I_n([v]_n)) = 0$

Proof. Let $[v]_n \in V_R^n$ and $v^n = (s^n, g^n, f^n, a^n, c^n, d^n) = \mathcal{P}_n([v]_n)$. Then $\{v^n\}$ is weakly precompact in H; assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{g}, \tilde{f}, \tilde{a}, \tilde{c}, \tilde{d})$. Then $\tilde{v} \in V_R$, and moreover, Rellich-Kondrachov compactness theorem implies that $(s^n, g^n, a^n, b^n, c^n) \to (\tilde{s}, \tilde{g}, \tilde{a}, \tilde{b}, \tilde{c})$ strongly in $B_2^1(0,T) \times L_2(0,T) \times L_2(0,T) \times L_2(D) \times L_2(D)$; in particular, $s^n \to \tilde{s}$ uniformly on $[0,T]$. Write the difference $\mathcal{J}(\mathcal{P}_n([v]_n)) - I_n([v]_n)$ in the preceding notation, as

$$I_n([v]_n) - \mathcal{J}(\mathcal{P}_n([v]_n)) = I_n([v]_n) - \mathcal{J}(v^n) = I_n([v]_n) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)$$

By weak continuity of $\mathcal{J}$, we have $\lim_{n\to\infty}(\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)) = 0$. Lastly, we need to show

$$\lim_{n\to\infty}(I_n([v]_n) - \mathcal{J}(\tilde{v})) = 0$$

Since $\tilde{v} \in V_{R+\epsilon}$ for some $\epsilon > 0$, and by strong convergence of $\mathcal{P}_n([v]_n) \to \tilde{v}$, similar argument as in the proof of Lemma 2.5.8 completes the result. □

By Lemmas 2.5.7–2.5.9 and [2, Lem. 2.2], Theorem 2.3.2 is proved.

Chapter 3

Optimal Control of Diffusion Coefficient in Parabolic Free Boundary Problems

3.1 Inverse Stefan Problem with Unknown Diffusion Coefficient

Consider the general one-phase Stefan problem ([41, 62]): find the temperature function $u(x,t)$ and the free boundary $x = s(t)$ from the following conditions

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \quad \text{for } (x,t) \in \Omega \tag{3.1}$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le s(0) = s_0 \tag{3.2}$$

$$a(0,t)u_x(0,t) = g(t), \qquad 0 \le t \le T \tag{3.3}$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \qquad 0 \le t \le T \tag{3.4}$$

$$u(s(t),t) = \mu(t), \qquad 0 \le t \le T \tag{3.5}$$

$$\tag{3.6}$$

$$\text{where } a, b, c, f, \phi, g, \gamma, \chi, \mu \text{ are known functions and} \tag{3.7}$$

$$a(x,t) \geq \underline{a} > 0, \quad s_0 > 0 \tag{3.8}$$

$$\Omega = \{(x,t):\ 0 < x < s(t),\ 0 < t \leq T\} \tag{3.9}$$

Function $f(x,t)$ can be characterized as the density of the sources, function $\phi(x)$ is the initial temperature, $g(t)$ is the heat flux on the fixed boundary, μ is the phase transition temperature. We are going to make the following assumption: some of the data is not available, or has some measurement error. For istance, assume that the diffusion coefficint $a(x,t)$ is not known and we need to find it along with the temperature function $u(x,t)$ and the free boundary $s(t)$. For that, we need to introduce some new information. One way of doing this is by adding a new measurement of temperature at the final moment $t = T$:

$$u(x,T) = w(x), \quad \text{for } 0 \leq x \leq s(T) =: \bar{s}, \tag{3.10}$$

Inverse Stefan Problem (ISP): *Find the functions $u(x,t)$, $s(t)$ and $a(x,t)$ satisfying conditions (3.1)-(3.10).*

3.2 Optimal Control Problem

Consider a minimization of the cost functional

$$\mathcal{J}(v) = \beta_0 \int_0^{s(T)} |u(x,T) - w(x)|^2\, dx + \beta_1 \int_0^T \left|u(s(t),t) - \mu(t)\right|^2\, dt + \beta_2 |s(T) - \bar{s}|^2$$

on the control set

$$V_R = \{v = (s,a) \in H : \delta \le s(t) \le l, s(0) = s_0, s'(0) = 0, a(x,t) \ge \underline{a}$$

$$\|v\|_H \le R\}$$

$$H = B_2^2(0,T) \times (\tilde{W}_2^{1,1}(D) \cap \tilde{W}_{\infty,\gamma}^{1,1}(D))$$

$$\|v\|_H := \max(\|s\|_{B_2^2(0,T)}; \|a\|_{\tilde{W}_2^{1,1}(D)}; \|a\|_{\tilde{W}_{\infty,\gamma}^{1,1}(D)})$$

where $\delta, l, R, \beta_0, \beta_1$ are given positive numbers, and $u = u(x,t;v)$ be a solution of the Neumann problem (3.1)-(3.4). By employing standard Sobolev extension results [25, 74], we are going to assume throughout the paper that any control function $a \in V_R$ is continued to $(0,l) \times (-1,0)$ as an element of $\tilde{W}_2^{1,1}(D) \cap \tilde{W}_{\infty,\gamma}^{1,1}(D)$

Definition 3.2.1. The function $u \in W_2^{1,1}(\Omega)$ is called a weak solution of the problem (3.1)-(3.4) if $u(x,0) = \phi(x) \in W_2^1[0, s_0]$ and

$$0 = \int_0^T \int_0^{s(t)} [au_x\Phi_x - bu_x\Phi - cu\Phi + u_t\Phi + f\Phi]\,dx\,dt$$
$$+ \int_0^T [\gamma(s(t),t)s'(t) - \chi(s(t),t)]\Phi(s(t),t)\,dt + \int_0^T g(t)\Phi(0,t)\,dt \qquad (3.11)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$

Definition 3.2.2. The function $u \in V_2(\Omega)$ is called a weak solution of (3.1)-(3.4) if

$$0 = \int_0^T \int_0^{s(t)} [au_x\Phi_x - bu_x\Phi - cu\Phi - u\Phi_t + f\Phi]\,dx\,dt - \int_0^{s_0} \phi(x)\Phi(x,0)\,dx +$$
$$\int_0^T g(t)\Phi(0,t)\,dt + \int_0^T [\gamma(s(t),t)s'(t) - u(s(t),t)s'(t) - \chi(s(t),t)]\Phi(s(t),t)\,dt \qquad (3.12)$$

for arbitrary $\Phi \in W_2^{1,1}(\Omega)$ such that $\Phi|_{t=T} = 0$.

3.3 Discrete Optimal Control Problem

Consider the grid

$$\omega_\tau = \left\{t_j = j\tau,\ j = 0, 1, \ldots, n\right\}$$

on $[0, T]$ where $\tau = \frac{T}{n}$. We are going to use the following notation for $\{d_i\}$,

$$d_{k,\bar{t}} = \frac{d_k - d_{k-1}}{\tau}, \quad d_{kt} = d_{k+1,\bar{t}}, \quad d_{k,\bar{t}t} = \frac{d_{k+1} - 2d_k + d_{k-1}}{\tau^2} \tag{3.13}$$

Next we introduce the spatial grid. Given a discrete boundary $[s]_n \in \mathbb{R}^{n+1}$, let $(p_0, p_1, \ldots, p_n)$ be a permutation of $(0, 1, \ldots, n)$ according to the order $s_{p_0} \leq s_{p_1} \leq \cdots \leq s_{p_n}$.

Note that, for every k there exists a unique j_k so that

$$s_k = s_{p_{j_k}} \tag{3.14}$$

Throughout the work instead of subscript j_k we will use j. Let

$$\omega_{p_0} = \{x_i : x_i = ih,\ i = 0, 1, \ldots, m_0^{(n)}\}$$

be a grid on $[0, s_{p_0}]$ and $h = \frac{s_{p_0}}{m_0^{(n)}}$. We will assume

$$h = O(\sqrt{\tau}), \quad \text{as } \tau \to 0. \tag{3.15}$$

By induction once $\omega_{p_{k-1}}$ is constructed on $[0, s_{p_{k-1}}]$ we construct

$$\omega_{p_k} = \{x_i : \ i = 0, 1, \ldots, m_k^{(n)}\}$$

on $[0, s_{p_k}]$, where $m_k^{(n)} \geq m_{k-1}^n$, where we have a strict inequality if and only if $s_{p_k} > s_{p_{k-1}}$; for $i \leq m_{k-1}^{(n)}$ points x_i are the same as in grid $\omega_{p_{k-1}}$. Finally, if $s_{p_n} < \ell$, then we introduce a grid on $[s_{p_n}, \ell]$

$$\overline{\omega} = \{x_i : x_i = s_{p_n} + (i - m_n^{(n)})\overline{h}, \ i = m_n^{(n)}, \ldots, N\}$$

of stepsize order h, i.e. $\overline{h} = O(h)$ as $h \to 0$. Furthermore we simplify the notation and write $m_k^{(n)} \equiv m_k$. Let

$$h_i = x_{i+1} - x_i, \ i = 0, 1, \ldots, N-1;$$

and denote the space grid on $[0, \ell]$ by ω_h and set

$$\Delta = \max_{i=0,\ldots,N-1} h_i$$

Discretized control set have the following form:

$$V_R^n = \{[v]_n = ([s]_n, [a]_{nN}) \in \bar{H} : \ 0 < \delta \leq s_k \leq l, \ a_{ik} \geq \underline{a}, \ \|[v]_n\|_{\bar{H}} \leq R\}$$

$$\|[v]_n\|_{\bar{H}} = \max\big(\|[s]_n\|_{b_2^2}; \|[a]_{nN}\|_{\tilde{w}_2^{1,1}}; \|[a]_{nN}\|_{\tilde{w}_{\infty,\gamma}^{1,1}}\big)$$

where,

$$\bar{H} = \mathbb{R}^{n+1} \times \mathbb{R}^{nN}$$

$$[s]_n = (s_0, s_1, ..., s_n) \in \mathbb{R}^{n+1}, \ [a]_{nN} = \{a_{ik}\} \ i = 0, 1, \ldots, N; k = 0, 1, \ldots, n$$

$$\|[s]_n\|_{b_2^2}^2 = \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 + \sum_{k=0}^{n-1} \tau s_{\bar{t}t,k}^2,$$

$$\|[a]_{nN}\|^2_{\tilde{w}_2^{1,1}} = \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ik}^2 + \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2 + \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ik\bar{t}}^2 + \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx\bar{t}}^2$$

$$\|[a]_{nN}\|_{\tilde{w}_{\infty,\gamma}^{1,1}} = \max_{\substack{0\le i\le N-1\\1\le k\le n}} |a_{ik}| + \max_{\substack{0\le i\le N-1\\0\le k\le n}} |a_{ik,x}| + \Big(\sum_{k=1}^{n}\tau \max_{0\le i\le N} |a_{ik,\bar{t}}|^{\gamma}\Big)^{\frac{1}{\gamma}}$$

where we assign $s_{-1} = s_0$ and use the standard notation for the finite differences:

$$s_{\bar{t},k} = \frac{s_k - s_{k-1}}{\tau},\; s_{t,k} = \frac{s_{k+1} - s_k}{\tau},\; s_{\bar{t}t,k} = \frac{s_{k+1} - 2s_k + s_{k-1}}{\tau^2}.$$

Next we define two mappings $\mathcal{Q}_n$ and $\mathcal{P}_n$ between continuous and discrete control sets as follows:

$$\mathcal{Q}_n(v) = [v]_n = ([s]_n, [a]_{nN}), \quad \text{for } v \in V_R$$

where $s_k = s(t_k),\ k = 0, 1, ..., n.$

$$a_{ik} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} a(x_i, t)dt,\ i = 0, 1, \ldots, N;\ k = 0, 1, \ldots, n$$

$$\mathcal{P}_n([v]_n) = v^n = (s^n, a^n) \in H \quad \text{for } [v]_n \in V_R^n,$$

where

$$s^n(t) = \begin{cases} s_0 + \frac{t^2}{2\tau}s_{\bar{t},1} & 0 \le t \le \tau, \\ s_{k-1} + (t - t_{k-1} - \frac{\tau}{2})s_{\bar{t},k-1} + \frac{1}{2}(t - t_{k-1})^2 s_{\bar{t}t,k-1} & t_{k-1} \le t \le t_k, k = \overline{2,n}. \end{cases} \tag{3.16}$$

$$a^n(x,t) = a_{ik} + a_{ikx}(x - x_i) + a_{ik\bar{t}}(t - t_k) + a_{ikx\bar{t}}(x - x_i)(t - t_k), \tag{3.17}$$

$$t_{k-1} \le t < t_k,\ x_i \le x < x_{i+1} \text{ for } i = 0, 1, \ldots, N-1;\ k = 1, \ldots, n$$

Define Steklov averages

$$d_k(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} d(x,t)\,dt,\ h_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} h(t)\,dt,\ d_{ik} = \frac{1}{h_i\tau}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} d(x,t)\,dt\,dx,$$

where $i = 0, 1, \cdots, N-1$; $k = 1, \cdots, n$; d represents functions b, c, f, and h represents functions ν, μ, g. For $v = (s, a) \in V_R$ we introduce Steklov averages of traces as follows

$$\chi_s^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \chi(s(t), t)\, dt, \ (\gamma_s s')^k = \frac{1}{\tau} \int_{t_{k-1}}^{t_k} \gamma(s(t), t) s'(t)\, dt. \tag{3.18}$$

Given $[v]_n = ([s]_n, [a]_{nN}) \in V_R^n$ we define Steklov averages $\chi_{s^n}^k$ and $(\gamma_{s^n}(s^n)')^k$ through (3.18) with s replaced by s^n from (3.16).

Let ϕ^n be a piecewise constant approximation of ϕ:

$$\phi^n(x) = \phi_i := \phi(x_i), \quad \text{for } x_i < x \le x_{i+1}, i = 0, .., N-1$$

Next we define a discrete state vector through discretization of the integral identity (3.11)

Definition 3.3.1. For a given discrete control vector $[v]_n$, the vector function

$$[u([v]_n)]_n = (u(0), u(1), ..., u(n)),\ u(k) \in \mathbb{R}^{N+1},\ k = 0, \cdots, n$$

is called a discrete state vector if

(a) First $m_0 + 1$ components of the vector $u(0) \in \mathbb{R}^{N+1}$ satisfy

$$u_i(0) = \phi_i := \phi(x_i),\ i = 0, 1, \cdots, m_0;$$

(b) Recalling (3.14), for arbitrary $k = 1, \cdots, n$ first $m_j + 1$ components of the vector $u(k) \in$

$\mathbb{R}^{N+1}$ solve the following system of m_j+1 linear algebraic equations:

$$\Big[a_{0k}+hb_{0k}-h^2c_{0k}+\frac{h^2}{\tau}\Big]u_0(k)-\Big[a_{0k}+hb_{0k}\Big]u_1(k)=\frac{h^2}{\tau}u_0(k-1)-h^2f_{0k}-hg_k,$$
$$-a_{i-1,k}h_iu_{i-1}(k)+\Big[a_{i-1,k}h_i+a_{ik}h_{i-1}+b_{ik}h_ih_{i-1}-c_{ik}h_i^2h_{i-1}+\frac{h_i^2h_{i-1}}{\tau}\Big]u_i(k)-$$
$$\Big[a_{ik}h_{i-1}+b_{ik}h_ih_{i-1}\Big]u_{i+1}(k)=-h_i^2h_{i-1}f_{ik}+\frac{h_i^2h_{i-1}}{\tau}u_i(k-1),\ i=1,\cdots,m_j-1$$
$$-a_{m_j-1,k}u_{m_j-1}(k)+a_{m_j-1,k}u_{m_j}(k)=-h_{m_j-1}\Big[(\gamma_{s^n}(s^n)')^k-\chi^k_{s^n}\Big]. \tag{3.19}$$

(c) For arbitrary $k=0,1,...,n$, the remaining components of $u(k)\in\mathbb{R}^{N+1}$ are calculated as

$$u_i(k)=\hat{u}(x_i;k),\ m_j\le i\le N$$

where $\hat{u}(x;k)\in W_2^1[0,l]$ is a piecewise linear interpolation of $\{u_i(k):\ i=0,\cdots,m_j\}$, that is to say

$$\hat{u}(x;k)=u_i(k)+\frac{u_{i+1}(k)-u_i(k)}{h_i}(x-x_i),\ x_i\le x\le x_{i+1}, i=0,\cdots,m_j-1,$$

iteratively continued to $[0,l]$ as

$$\hat{u}(x;k)=\hat{u}(2^ns_k-x;k),\ 2^{n-1}s_k\le x\le 2^ns_k, n=\overline{1,n_k},\ n_k\le n_*=1+\log_2\Big[\frac{l}{\delta}\Big] \tag{3.20}$$

where $[r]$ means integer part of the real number r.

It is worth to note that for any $k=1,2,\cdots,n$ system (3.19) is equivalent to the following summation identity

$$\sum_{i=0}^{m_j-1}h_i\Big[a_{ik}u_{ix}(k)\eta_{ix}-b_{ik}u_{ix}(k)\eta_i-c_{ik}u_i(k)\eta_i+f_{ik}\eta_i+u_{i\bar{t}}(k)\eta_i\Big]+$$
$$\Big[(\gamma_{s^n}(s^n)')^k-\chi^k_{s^n}\Big]\eta_{m_j}+g_k\eta_0=0, \tag{3.21}$$

for any numbers $\eta_i, i = 0, 1, \cdots, m_j$.

Next we introduce a discrete optimal control problem of minimization of the following cost functional

$$\mathcal{I}_n([v]_n) = \beta_0 \sum_{i=0}^{m_n-1} h_i\Big(u_i(n) - w_i\Big)^2 + \beta_1 \tau \sum_{k=1}^{n} \Big(u_{m_k}(k) - \mu_k\Big)^2 + \beta_2 |s_n - \bar{s}|^2 \tag{3.22}$$

over a set V_R^n subject to the state vector which was described in Definition 1.3. Throughout the paper we will call the discrete optimal control problem by Problem $\mathcal{I}_n$.

Several interpolations of the discrete state vector will be used among which we are going to take advantage of piecewise constant and piecewise linear interpolations. Let $[u([v]_n)]_n = (u(0), u(1), ..., u(n))$ be a discrete state vector, the we define

$$u^\tau(x,t) = \hat{u}(x;k), \quad \text{if } t_{k-1} < t \le t_k,\ 0 \le x \le l,\ k = \overline{0,n},$$

$$\hat{u}^\tau(x,t) = \hat{u}(x;k-1) + \hat{u}_{\bar{t}}(x;k)(t - t_{k-1}), \quad \text{if } t_{k-1} < t \le t_k,\ 0 \le x \le l,\ k = \overline{1,n},$$

$$\hat{u}^\tau(x,t) = \hat{u}(x;n), \quad \text{if } t \ge T,\ 0 \le x \le l.$$

$$\tilde{u}^\tau(x,t) = u_i(k), \quad \text{if } t_{k-1} < t \le t_k,\ x_i \le x < x_{i+1},\ k = \overline{1,n},\ i = \overline{0,N-1}.$$

We have

$$u^\tau \in V_2(D),\ \ \hat{u}^\tau \in W_2^{1,1}(D),\ \ \tilde{u}^\tau \in L_2(D).$$

Here are some standard notations for difference quotients of the discrete state vector:

$$u_{ix}(k) = \frac{u_{i+1}(k) - u_i(k)}{h_i},\ u_{i\bar{t}} = \frac{u_i(k) - u_i(k-1)}{\tau}, \quad \text{etc.}$$

3.4 Main Result

In this section we are going to formulate the main results of the chapter 3.

We denote

$$D = \{(x,t): \ 0 < x < l, \ 0 < t \leq T\}$$

We impose the following assumption on the data:

$$b, c \in L_\infty(D), \ f \in L_2(D), \ g \in W_2^{\frac{1}{4}}(0,T), \ w \in L_2(0,l)$$

$$\phi \in W_2^1[0, s_0], \ \gamma, \chi \in W_2^{1,1}(D), \ \mu \in L_2[0,T],$$

Main theorems:

Theorem 3.4.1. *The Problem I has a solution, i.e.*

$$V_* = \{v \in V_R : \ \mathcal{J}(v) = \mathcal{J}_* \equiv \inf_{v \in V_R} \mathcal{J}(v)\} \neq \emptyset$$

Theorem 3.4.2. *Sequence of discrete optimal control problems I_n approximates the optimal control problem I with respect to functional, i.e.*

$$\lim_{n \to +\infty} \mathcal{I}_{n_*} = \mathcal{J}_*, \tag{3.23}$$

where

$$\mathcal{I}_{n_*} = \inf_{V_R^n} \mathcal{I}_n([v]_n), \ n = 1, 2, ...$$

If $[v]_{n_\epsilon} \in V_R^n$ is chosen such that

$$\mathcal{I}_{n_*} \leq \mathcal{I}_n([v]_{n_\epsilon}) \leq \mathcal{I}_{n_*} + \epsilon_n, \ \epsilon_n \downarrow 0,$$

then the sequence $v_n = (s_n, a_n) = \mathcal{P}_n([v]_{n_\epsilon})$ converges to some element $v_* = (s_*, a_*) \in V_*$

weakly in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$*, and strongly in* $W_2^1[0,T] \times L_2[0,T]$*. In particular* s_n *converges to* s_* *uniformly on* $[0,T]$*. For any* $\delta > 0$*, define*

$$\Omega_*^{`} = \Omega_* \cap \{x < s_*(t) - \delta,\ 0 < t < T\}$$

Then the piecewise linear interpolation $\hat{u}^\tau$ *of the discrete state vector* $[u[v]_{n_\epsilon}]_n$ *converges to the solution* $u(x,t;v_*) \in W_2^{1,1}(\Omega_*)$ *of the Neumann problem (3.1)-(3.4) weakly in* $W_2^{1,1}(\Omega_*^{`})$.

3.5 Preliminary Results

The proof of the existence and uniqueness of the discrete state vector r $[u([v]_n)]_n$ (see Definition 3.3.1) for arbitrary discrete control vector $[v]_n \in V_R^n$ is carried out in Lemma 3.5.1. Then in Lemma 3.5.2 we present general approximation criteria for the optimal control problems from ([79]). Lemma 3.6.8 highlights some properties of the $\mathcal{Q}_n$ and $\mathcal{P}_n$ mappings.

Lemma 3.5.1. *For sufficiently small time step* τ*, there exists a unique discrete state vector* $[u([v]_n)]_n$ *for arbitrary discrete control vector* $[v]_n \in V_R^n$*.*

Proof. We are going to use the fact that for any $k = 1, 2, \cdots, n$ system (3.19) is equivalent to the summation identity (3.21) where $\eta_i, i = 0, 1, \cdots, m_j$ can be chosen arbitrarily. Next, let $\{\tilde{u}_i(k)\}$ be a solution of the homogeneous system corresponding to (3.19). In other words let $\{\tilde{u}_i(k)\}$ be a solution of the system (3.19) where

$$g_k = (\gamma_{s^n}(s^n)')^k = \chi_{s^n}^k = f_{ik} = u_i(k-1) = 0.$$

Since η_i can be chosen arbitrarily, let us choose $\eta_i = \tilde{u}_i(k)$ in (3.21). Thus we get

$$\sum_{i=0}^{m_j-1} h_i a_{ik} \tilde{u}_{ix}^2(k) + \frac{1}{\tau} \sum_{i=0}^{m_j-1} h_i \tilde{u}_i^2(k) = \sum_{i=0}^{m_j-1} h_i \Big[b_{ik} \tilde{u}_{ix}(k) \tilde{u}_i(k) + c_{ik} \tilde{u}_i^2(k) \Big] \tag{3.24}$$

Since a is bounded from below by $\underline{a}$ and applying Cauchy inequality with $\epsilon > 0$ we derive the following inequality

$$\underline{a} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{ix}^2(k) + \frac{1}{\tau} \sum_{i=0}^{m_j-1} h_i \tilde{u}_i^2(k) \leq \frac{\epsilon M}{2} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{ix}^2(k) + \Big(\frac{M}{2\epsilon} + M \Big) \sum_{i=0}^{m_j-1} h_i \tilde{u}_i^2(k). \tag{3.25}$$

Here

$$M = \max\Big(||b||_{L_\infty(D)}; ||c||_{L_\infty(D)} \Big).$$

If we choose $\epsilon = \underline{a}/M$ in (3.25) we get

$$\frac{\underline{a}}{2} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{ix}^2(k) + \Big(\frac{1}{\tau} - \frac{1}{\tau_0} \Big) \sum_{i=0}^{m_j-1} h_i \tilde{u}_i^2(k) \leq 0, \tag{3.26}$$

where

$$\tau_0 = \Big(\frac{M^2}{2\underline{a}} + M \Big)^{-1}.$$

Therefore if $\tau < \tau_0$, (3.26) implies $\tilde{u}_i(k) = 0$, $i = 0, 1, \cdots, m_j$ i.e the homogeneous system only has a trivial solution. Therefore, system has a unique solution thus for any given discrete control vector $[v]_n$ there exists a unique discrete state vector defined by Definition 1.3. This completes the proof of the Lemma. □

Next we are going to state very important approximation criteria. It will be used to prove Theorem 3.4.2.

Lemma 3.5.2. *[78] Sequence of discrete optimal control problems I_n approximates the continuous optimal control problem I if and only if the following conditions are satisfied:*

(1) *for arbitrary sufficiently small $\epsilon > 0$ there exists number $N_1 = N_1(\epsilon)$ such that $Q_N(v) \in$*

V_R^n for all $v \in V_{R-\epsilon}$ and $N \geq N_1$; and for any fixed $\epsilon > 0$ and for all $v \in V_{R-\epsilon}$ the following inequality is satisfied:

$$\limsup_{N\to\infty}\Big(\mathcal{I}_N(\mathcal{Q}_N(v)) - \mathcal{J}(v)\Big) \leq 0. \tag{3.27}$$

(2) *for arbitrary sufficiently small $\epsilon > 0$ there exists number $N_2 = N_2(\epsilon)$ such that $\mathcal{P}_N([v]_N) \in V_{R+\epsilon}$ for all $[v]_N \in V_R^N$ and $N \geq N_2$; and for all $[v]_N \in V_R^N$, $N \geq 1$ the following inequality is satisfied:*

$$\limsup_{N\to\infty}\Big(\mathcal{J}(\mathcal{P}_N([v]_N)) - \mathcal{I}_N([v]_N)\Big) \leq 0. \tag{3.28}$$

(3) *the following inequalities are satisfied:*

$$\limsup_{\epsilon\to 0}\mathcal{J}_*(\epsilon) \geq \mathcal{J}_*, \quad \liminf_{\epsilon\to 0}\mathcal{J}_*(-\epsilon) \leq \mathcal{J}_*, \tag{3.29}$$

where $\mathcal{J}_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} \mathcal{J}(u)$.

Corollary 3.5.3. *([1]) Let either $[v]_n \in V_R^n$ or $[v]_n = \mathcal{Q}_n(v)$ for $v \in V_R$. Then*

$$|s_k - s_{k-1}| \leq C'\tau, \quad k = 1, 2, \cdots, n \tag{3.30}$$

where C' is independent of n.

Let $v \in V_R$, then $s' \in W_2^1[0,T]$. Applying Morrey inequality we get

$$\|s'\|_{C[0,T]} \leq C_1 \|s'\|_{W_2^1[0,T]} \leq C_1 R \tag{3.31}$$

Therefore for the first component $[s]_n$ of $[v]_n = \mathcal{Q}_n(v)$ we conclude (3.30). Now, let $[v]_n \in V_R^n$, then the sequence $v^n = \mathcal{P}_n([v]_n)$ belongs to V_{R+1} by Lemma 3.6.8 and the component s^n of v^n satisfies (3.31). Since, $(s^n)'(t_k) = s_{\bar{t},k}, k = 1, ..., n$, from (3.31), (3.30) follows.

Notice that a given step size h_i can satisfy eather one of the following cases $h_i = h$, $h_i = \overline{h}$, or $h_i \leq |s_k - s_{k-1}$ for some k. By employing (3.15) and (3.30) we get

$$\max_{0 \leq i \leq N-1} h_i = O(\sqrt{\tau}), \quad \text{as } \tau \to 0. \tag{3.32}$$

3.6 Proofs of the Main Results

3.6.1 First Energy Estimate and its Consequences

Next we prove the following energy estimation for the discrete state vector.

Theorem 3.6.1. *For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies the following stability estimations:*

$$\max_{0 \leq k \leq n} \sum_{i=0}^{N-1} h_i u_i^2(k) + \sum_{k=1}^{n} \tau \sum_{i=0}^{N-1} h_i u_{ix}^2(k) \leq$$
$$C\Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$
$$+ \|\chi(s^n(t),t)\|^2_{L_2(0,T)} + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \sum_{i=m_j}^{m_{j_{k+1}}-1} h_i u_i^2(k) \Big), \tag{3.33}$$

where C is independent of τ and $\mathbf{1}_+$ be an indicator function of the positive semiaxis.

We start by proving the Lemma (3.6.2)

Lemma 3.6.2. *For all sufficiently small τ, discrete state vector $[u([v]_n)]_n$ satisfies the following estimation:*

$$\max_{1 \leq k \leq n} \sum_{i=0}^{m_j - 1} h_i u_i^2(k) + \sum_{k=1}^{n} \tau \sum_{i=0}^{m_j-1} h_i u_{ix}^2(k) + \sum_{k=1}^{n} \tau^2 \sum_{i=0}^{m_j-1} h_i u_{i\bar{t}}^2(k) \leq$$
$$C\Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$
$$+ \|\chi(s^n(t),t)\|^2_{L_2(0,T)} + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \sum_{i=m_j}^{m_{j_{k+1}}-1} h_i u_i^2(k) \Big), \tag{3.34}$$

where C is independent of τ.

Proof. . If we choose $\eta_i = 2\tau u_i(k)$ in (3.21) and make use of the following equality

$$2\tau u_{i\bar{t}}(k)u_i(k) = u_i^2(k) - u_i^2(k-1) + \tau^2 u_{i\bar{t}}^2(k)$$

we obtain the equality

$$\sum_{i=0}^{m_j-1} h_i u_i^2(k) - \sum_{i=0}^{m_j-1} h_i u_i^2(k-1) + \tau^2 \sum_{i=0}^{m_j-1} h_i u_{i\bar{t}}^2(k) + 2\tau \sum_{i=0}^{m_j-1} h_i a_{ik} u_{ix}^2(k) =$$
$$2\tau \sum_{i=0}^{m_j-1} h_i \Big[b_{ik} u_{ix}(k) u_i(k) + c_{ik} u_i^2(k) - f_{ik} u_i(k)\Big] -$$
$$2\tau \Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\Big] u_{m_j}(k) - 2\tau g_k u_0(k). \tag{3.35}$$

Employing (3.8), Morrey inequality and Cauchy inequalities with $\epsilon > 0$ we derive

$$\max_{0\le i\le m_j} u_i^2(k) \le C_* \|\hat{u}(x;k)\|^2_{W_2^1[0,s_k]} \le C \sum_{i=0}^{m_j-1} h_i (u_i^2(k) + u_{ix}^2(k)) \tag{3.36}$$

Here C_*, C don't depend on τ and $[u([v]_n)]_n$. Using (3.35) we obtain

$$\sum_{i=0}^{m_j-1} h_i u_i^2(k) - \sum_{i=0}^{m_j-1} h_i u_i^2(k-1) + \underline{a}\tau \sum_{i=0}^{m_j-1} h_i u_{ix}^2(k) + \tau^2 \sum_{i=0}^{m_j-1} h_i u_{i\bar{t}}^2(k) \le$$
$$C_1 \tau \Big[|(\gamma_{s^n}(s^n)')^k|^2 + |\chi_{s^n}^k|^2 + |g_k|^2 + \sum_{i=0}^{m_j-1} h_i f_{ik}^2 + \sum_{i=0}^{m_j-1} h_i u_i^2(k)\Big]. \tag{3.37}$$

where C_1 does not depend on τ. If we assume $\tau < C_1$, then from (3.37) we derive

$$(1 - C_1\tau) \sum_{i=0}^{m_j-1} h_i u_i^2(k) \le \sum_{i=0}^{m_{j_{k-1}}-1} h_i u_i^2(k-1) + \mathbf{1}_+(s_k - s_{k-1}) \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i u_i^2(k-1) +$$
$$C_1 \tau \left[|(\gamma_{s^n}(s^n)')^k|^2 + |\chi_{s^n}^k|^2 + |g_k|^2 + \sum_{i=0}^{m_j-1} h_i f_{ik}^2\right], \tag{3.38}$$

Using induction we obtain

$$\sum_{i=0}^{m_j-1} h_i u_i^2(k) \le (1-C_1\tau)^{-k} \sum_{i=0}^{m_{j_0}-1} h_i u_i^2(0) + \sum_{l=1}^{k} (1-C_1\tau)^{-k+l-1} \Big\{ C_1\tau \Big[|(\gamma_{s^n}(s^n)')^l|^2 + |\chi_{s^n}^l|^2 + |g_l|^2 + \sum_{i=0}^{m_{j_l}-1} h_i f_{il}^2 \Big] + \mathbf{1}_+(s_l - s_{l-1}) \sum_{i=m_{j_{l-1}}}^{m_{j_l}-1} h_i u_i^2(l-1) \Big\}. \tag{3.39}$$

For any $1 \le l \le k \le n$ we get

$$(1-C_1\tau)^{-k+l-1} \le (1-C_1\tau)^{-k} \le (1-C_1\tau)^{-n} = \Big(1 - \frac{C_1 T}{n}\Big)^{-n} \to e^{C_1 T}, \tag{3.40}$$

as $\tau \to 0$. Therefore for sufficiently small τ we get

$$(1-C_1\tau)^{-k+l-1} \le 2e^{C_1 T} \quad \text{for } 1 \le l \le k \le n, \tag{3.41}$$

Using (CBS) inequality from (3.39)-(3.41) it derive that

$$\max_{1\le k\le n} \sum_{i=0}^{m_j-1} h_i u_i^2(k) \le C_2 \Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)} + \|\chi(s^n(t),t)\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \sum_{l=1}^{n-1} \mathbf{1}_+(s_{l+1} - s_l) \sum_{i=m_{j_l}}^{m_{j_{l+1}}-1} h_i u_i^2(l) \Big). \tag{3.42}$$

where C_2 does not depend on τ. Using (3.42), and after performing summation of (3.37) with respect to k from 1 to n we obtain

$$\sum_{i=0}^{m_{j_n}-1} h_i u_i^2(n) + \underline{a} \sum_{k=1}^{n} \tau \sum_{i=0}^{m_j-1} h_i u_{ix}^2(k) + \sum_{k=1}^{n} \tau^2 \sum_{i=0}^{m_j-1} h_i u_{i\bar{t}}^2(k) \le \|\phi^n\|^2_{L_2(0,s_0)} + C_3 \Big(\|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)} + \|\chi(s^n(t),t)\|^2_{L_2(0,T)} + \sum_{k=1}^{n} \tau \sum_{i=0}^{m_j-1} h_i u_i^2(k) \Big) + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \sum_{i=m_{j_k}}^{m_{j_{k+1}}-1} h_i u_i^2(k) \tag{3.43}$$

From (3.42) and (3.43), (3.34) follows. This concludes the proof of the lemma. □

Next we are going to prove Theorem 3.6.1:

Proof. Using (3.34), it is enough to show that the LHS of (3.33) is bounded by the LHS of (3.34). Reflective continuation property of $\hat{u}(x;k)$ helps us to obtain

$$\sum_{k=1}^{n}\tau\sum_{i=0}^{N-1}h_i u_{ix}^2(k) = \tau\sum_{k=1}^{n}\int_0^l \Big|\frac{d\hat{u}(x;k)}{dx}\Big|^2 dx \le$$
$$2^{n_*}\tau\sum_{k=1}^{n}\int_0^{s_k}\Big|\frac{du(x;k)}{dx}\Big|^2 dx = 2^{n_*}\sum_{k=1}^{n}\tau\sum_{i=0}^{m_j-1}h_i u_{ix}^2(k). \tag{3.44}$$

By using (3.15) and (3.31) we have

$$\sum_{i=0}^{N-1}h_i u_i^2(k) \le 2\int_0^l \hat{u}^2(x;k)\,dx + \frac{2}{3}\sum_{i=0}^{N-1}h_i^3 u_{ix}^2(k) \le 2^{n_*+1}\int_0^{s_k}\hat{u}^2(x;k)\,dx +$$
$$C_1\tau\sum_{i=0}^{N-1}h_i u_{ix}^2(k) \le 2^{n_*+2}\sum_{i=0}^{m_j-1}h_i u_i^2(k) + 2^{n_*+2}\sum_{i=0}^{m_j-1}\frac{1}{3}h_i^3 u_{ix}^2(k) +$$
$$C_1\tau\sum_{i=0}^{N-1}h_i u_{ix}^2(k) \le 2^{n_*+2}\sum_{i=0}^{m_j-1}h_i u_i^2(k) + C_2\tau\sum_{i=0}^{N-1}h_i u_{ix}^2(k). \tag{3.45}$$

and (3.33) follows from (3.44), (3.45) and (3.34). Theorem is proved. □

Take $[v]_n \in V_R^n, n = 1,2,...$ to be a sequence of discrete controls. Then using Lemma 3.6.8 we conclude that the sequence $\{\mathcal{P}_n([v]_n)\}$ is weakly precompact in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$. Assume that the whole sequence converges to $v = (s,a)$ weakly in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$. This implies strong convergence in $W_2^1[0,T] \times L_2(D)$. Since $\|a^n\|_{\tilde{W}^{1,1}_{\infty,\gamma}(D)} \le R$, from Mazur's Theorem [80] it follows that $a \in \tilde{W}_2^{1,1}(D) \cap \tilde{W}^{1,1}_{\infty,\gamma}(D)$. On the other hand, given control $v = (s,a) \in V_R$ we can select a sequence of discrete controls $[v]_n = Q_n(v)$. After using Lemma 3.6.8 twice one can easily deduce that the sequence $\{\mathcal{P}_n([v]_n\}$ converges to $v = (s,a)$ weakly in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$, and strongly in $W_2^1[0,T] \times L_2(D)$. Next we are going to show the continuous dependence of the family of interpolarions $\{u^\tau\}$ on this convergence.

Theorem 3.6.3. *Let $[v]_n \in V_R^n, n = 1,2,...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges weakly in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$ (and strongly in $W_2^1[0,T] \times L_2(D)$ to $v = (s,a) \in V_R$). Then the sequence $\{u^\tau\}$ converges as $\tau \to 0$ weakly in $W_2^{1,0}(\Omega)$ to weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (3.1)-(3.4), i.e. to the solution of the integral identity (3.12). Moreover, u satisfies the energy estimate*

$$\|u\|^2_{V_2^{1,0}(D)} \leq C\Big(\|\phi\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big) \qquad (3.46)$$

Proof. In (3.16) we considered quadratic interpolation of $[s]_n$. We introduce the following two linear interpolations:

$$\tilde{s}^n(t) = s_{k-1} + \frac{s_k - s_{k-1}}{\tau}(t - t_{k-1}), \ \ t_{k-1} \leq t \leq t_k, k = \overline{1,n}; \ \tilde{s}^n(t) \equiv s_n, \ \ t \geq T;$$

$$\tilde{s}_1^n(t) = \tilde{s}^n(t+\tau), \ \ 0 \leq t \leq T.$$

It can be easily shown that sequences $\tilde{s}^n$ and $\tilde{s}_1^n$ are equivalent to the sequence s^n in $W_2^1[0,T]$ and they converge strongly to s in $W_2^1[0,T]$. Moreover,

$$\sup_n \|\tilde{s}_1^n\|_{W_2^1[0,T]} < C_* \qquad (3.47)$$

Here C_* does not depend n.

Next we are going to absorb the last term on the RHS of (3.33) into the LHS. We have

$$\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \sum_{i=m_j}^{m_{j_{k+1}}-1} h_i u_i^2(k) \leq$$
$$2\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \int_{s_k}^{s_{k+1}} \hat{u}^2(x;k)dx + \frac{2}{3}\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1} - s_k) \sum_{i=m_j}^{m_{j_{k+1}}-1} h_i^3 u_{ix}^2(k) \qquad (3.48)$$

If $s_{k+1} > s_k$, then all the factors h_i in the second term are bounded by $s_{k+1} - s_k$ then by employing (3.30) we get

$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\sum_{i=m_j}^{m_{j_{k+1}}-1}h_i^3u_{ix}^2(k)\le(C^{`})^2\tau\sum_{k=1}^{n-1}\tau\int_{s_k}^{s_{k+1}}\Big|\frac{d\hat{u}}{dx}\Big|^2dx \tag{3.49}$$

Using reflective continuation property of $\hat{u}(x;k)$ we obtain

$$\int_{s_k}^{s_{k+1}}\Big|\frac{d\hat{u}}{dx}\Big|^2dx\le 2^{n_*-1}\int_0^{s_k}\Big|\frac{d\hat{u}}{dx}\Big|^2dx=2^{n_*-1}\sum_{i=0}^{m_j-1}h_iu_{ix}^2(k). \tag{3.50}$$

Applying (3.49) and (3.50) leads to

$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\sum_{i=m_j}^{m_{j_{k+1}}-1}h_i^3u_{ix}^2(k)\le 2^{n_*-1}(C^{`})^2\tau\sum_{k=1}^{n-1}\tau\sum_{i=0}^{m_j-1}h_iu_{ix}^2(k) \tag{3.51}$$

For sufficiently small τ and from (3.48) - (3.51) in (3.33), last term on the RHS of (3.51) can be absorbed into the LHS of (3.33) which lets us to get modified (3.33) with a new constant C:

$$\max_{0\le k\le n}\sum_{i=0}^{N-1}h_iu_i^2(k)+\sum_{k=1}^{n}\tau\sum_{i=0}^{N-1}h_iu_{ix}^2(k)\le$$
$$C\Big(\|\phi^n\|^2_{L_2(0,s_0)}+\|g\|^2_{L_2(0,T)}+\|f\|^2_{L_2(D)}+\|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)}$$
$$+\|\chi(s^n(t),t)\|^2_{L_2(0,T)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}\hat{u}^2(x;k)dx\Big), \tag{3.52}$$

Next we will estimate the last term on the RHS of (3.52) as in [1]:

$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}}\hat{u}^2(x;k)dx=\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_k}^{t_{k+1}}(\tilde{s}^n)'(t)\hat{u}^2(\tilde{s}^n(t);k)dt=$$
$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_k}^{t_{k+1}}(\tilde{s}^n)'(t)\Big(u^\tau(\tilde{s}^n(t),t-\tau)\Big)^2dt=$$
$$\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\int_{t_{k-1}}^{t_k}(\tilde{s}_1^n)'(t)\Big(u^\tau(\tilde{s}_1^n(t),t)\Big)^2dt. \tag{3.53}$$

Applying CBS inequality we get

$$\Big|\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} \hat{u}^2(x;k)dx\Big| \le \|(\tilde{s}_1^n)^{`}\|_{L_2[0,T]}\|u^\tau(\tilde{s}_1^n(t),t)\|^2_{L_4[0,T]}. \tag{3.54}$$

For arbitrary $u \in V_2(D)$ results on traces of the elements of space $V_2(D)$ ([58, 25, 65]) implies

$$\|u(\tilde{s}_1^n(t),t)\|_{L_4[0,T]} \le \tilde{C}\|u\|_{V_2(D)}, \tag{3.55}$$

here the constant $\tilde{C}$ doesn't depend on u and n. From (3.47),(3.54) and (3.55) we derive

$$\Big|\sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1}-s_k)\int_{s_k}^{s_{k+1}} u^2(x;k)dx\Big| \le C_*\tilde{C}\|u^\tau\|^2_{V_2(D)}. \tag{3.56}$$

If C_* from (3.47) satisfies

$$C_* < (C\tilde{C})^{-1} \tag{3.57}$$

then from (3.52) and (3.56) we derive

$$\|u^\tau\|^2_{V_2^{1,0}(D)} \le C\Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma(s^n(t),t)(s^n)'(t)\|^2_{L_2(0,T)} + \|\chi(s^n(t),t)\|^2_{L_2(0,T)}\Big), \tag{3.58}$$

where C is a new constant which does not depend on n. Applying traces theorems in $W_2^{1,0}(D)$ ([25, 65]) to the $x = s^n(t)$ along with Morrey inequality for $(s^n)'$ and (3.105) we get

$$\|\gamma(s^n(t),t)(s^n)'(t)\|_{L_2(0,T)} \le \|(s^n)^{`}\|_{C[0,T]}\|\gamma(s^n(t),t)\|_{L_2[0,T]} \le C_3\|\gamma\|_{W_2^{1,0}(D)}$$
$$\|\chi(s^n(t),t)\|_{L_2[0,T]} \le C_3\|\chi\|_{W_2^{1,0}(D)}, \tag{3.59}$$

where C_3 does not depend on γ,χ and n. Thus, from (3.58), (3.59) we receive the following estimate

$$\|u^\tau\|^2_{V_2^{1,0}(D)} \le C\Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|g\|^2_{L_2(0,T)} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,0}(D)} + \|\chi\|^2_{W_2^{1,0}(D)}\Big), \qquad (3.60)$$

where C does not depend on n.

If the condition (3.57) is not satisfied, then using (3.30) we can partition the interval $[0,T]$ into finitely many closed intervals $[t_{n_{j-1}}, t_{n_j}]$, $j = \overline{1,q}$ with $t_{n_0} = 0$, $t_{n_q} = T$ so that when replacing interval $[0,T]$ with any of the intervals $[t_{n_{j-1}}, t_{n_j}]$ (3.47) is satisfied with C_* small enough to satisfy (3.57). Therefore, we partition D into finitely many sets of the form

$$D^j = D \cap \{t_{n_{j-1}} < t \le t_{n_j}\}$$

so that norms $\|u^\tau\|^2_{V_2(D^j)}$ are uniformly bounded by the RHS of (3.60). After performing summation with over $j = 1, \dots, q$ we get (3.60).

Due to the fact that ϕ^n converge to ϕ strongly in $L_2[0,s_0]$, from (3.60) it follows that the sequence $\{u^\tau\}$ is weakly precompact in $W_2^{1,0}(D)$. Now, let $u \in W_2^{1,0}(D)$ be a weak limit of u^τ in $W_2^{1,0}(D)$, and assume that whole sequence $\{u^\tau\}$ converges to u weakly in $W_2^{1,0}(D)$. Let us show that u satisfies the integral identity (3.12) for any test function $\Phi \in W_2^{1,1}(\Omega)$ so that $\Phi|_{t=T} = 0$. Since $C^1(\overline{\Omega})$ is dense in $W_2^{1,1}(\Omega)$ it suffices to consider $\Phi \in C^1(\overline{\Omega})$. Without loss of generality we consider $\Phi \in C^1(\overline{D}_{T+\tau})$, $\Phi \equiv 0, \quad$ for $T \le t \le T+\tau$, where

$$D_{T+\tau} = \{(x,t): \ 0 < x < l+1, \ 0 < t \le T+\tau\}$$

Otherwise, we can extend Φ to $D_{T+\tau}$ with the same properties. Let

$$\Phi_i(k) = \Phi(x_i, t_k), \ k = 0, \cdots, n+1, \ i = 0, \cdots, N$$

$$\Phi^\tau(x,t)=\Phi_i(k),\Phi^\tau_x(x,t)=\Phi_{ix}(k),\Phi^\tau_t(x,t)=\Phi_{i\bar t}(k+1),\quad \text{for } t_{k-1}<t\le t_k, x_i\le x<x_{i+1}.$$

and

$$\tilde a^\tau(x,t)=a_{ik},\ \ if\ \ t_{k-1}<t\le t_k,\ x_i\le x<x_{i+1},\ k=\overline{1,n},\ i=\overline{0,N-1} \tag{3.61}$$

It is easy to show that the sequences $\{\Phi^\tau\}$, $\{\Phi^\tau_x\}$ and $\{\Phi^\tau_t\}$ converge as $\tau\to 0$ uniformly in $\overline{D}$ to Φ, $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial\Phi}{\partial t}$ respectively. By taking $\eta_i=\tau\Phi_i(k)$ in (3.21) and after summation with respect to $k=\overline{1,n}$ and transforming the time difference term in the following form

$$\sum_{k=1}^{n}\tau\sum_{i=0}^{m_j-1}h_iu_{i\bar t}(k)\Phi_i(k)=-\sum_{k=1}^{n-1}\tau\sum_{i=0}^{m_{j_{k+1}}-1}h_iu_i(k)\Phi_{i\bar t}(k+1)-\sum_{i=0}^{m_{j_1}-1}h_iu_i(0)\Phi_i(1)+$$
$$\sum_{k=1}^{n-1}sign(s_k-s_{k+1})\sum_{i=\alpha_k}^{\beta_k-1}h_iu_i(k)\Phi_i(k)=-\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_0^{s_{k+1}}\tilde u^\tau\Phi^\tau_t\,dx\,dt-$$
$$\int_0^{s_1}\phi^n(x)\Phi^\tau(x,\tau)\,dx+\sum_{k=1}^{n-1}sign(s_k-s_{k+1})\sum_{i=\alpha_k}^{\beta_k-1}\int_{x_i}^{x_{i+1}}\Big(\hat u(x;k)-u_{ix}(k)(x-x_i)\Big)\Phi_i(k)dx=$$
$$-\int_0^T\int_0^{s(t)}\tilde u^\tau\Phi^\tau_t\,dx\,dt-\int_0^{s_1}\phi^n(x)\Phi^\tau(x,\tau)\,dx-\int_0^{T-\tau}(\tilde s_1^n)^{'}(t)u^\tau((\tilde s_1^n)(t),t)\Phi^\tau((\tilde s_1^n)(t),t)\,dt$$
$$-\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_{k+1}}\tilde u^\tau\Phi^\tau_t\,dx\,dt-\frac{1}{2}\sum_{k=1}^{n-1}sign(s_k-s_{k+1})\sum_{i=\alpha_k}^{\beta_k-1}h_i^2u_{ix}(k)\Phi_i(k), \tag{3.62}$$

where

$$\alpha_k=\min(m_{j_k},m_{j_{k+1}}),\ \beta_k=\max(m_{j_k},m_{j_{k+1}}),$$

we obtain

$$\int_0^T \int_0^{s(t)} \Big\{\tilde{a}^\tau \frac{\partial u^\tau}{\partial x}\Phi_x^\tau - b\frac{\partial u^\tau}{\partial x}\Phi^\tau - c\tilde{u}^\tau\Phi^\tau + f\Phi^\tau - \tilde{u}^\tau\Phi_t^\tau\Big\}\,dx\,dt - \int_0^{s_0} \phi^n(x)\Phi^\tau(x,\tau)\,dx$$
$$- \int_0^{T-\tau} (\tilde{s}_1^n)^{`}(t)u^\tau((\tilde{s}_1^n)(t),t)\Phi^\tau((\tilde{s}_1^n)(t),t)\,dt + \int_0^T g(t)\Phi^\tau(0,t)\,dt$$
$$+ \int_0^T \Big[\gamma(s^n(t),t)(s^n)^{`}(t) - \chi(s^n(t),t))\Big]\Phi^\tau(s^n(t),t)\,dt + R = 0 \tag{3.63}$$

where

$$R = \sum_{k=1}^n \int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k} \Big\{\tilde{a}^\tau \frac{\partial u^\tau}{\partial x}\Phi_x^\tau - b\frac{\partial u^\tau}{\partial x}\Phi^\tau - c\tilde{u}^\tau\Phi^\tau + f\Phi^\tau\Big\}\,dx\,dt - \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_{k+1}} \tilde{u}^\tau\Phi_t^\tau\,dx\,dt$$
$$+ \sum_{k=1}^n \int_{t_{k-1}}^{t_k}\int_{s^n(t)}^{s_k} \Big[\gamma(s^n(t),t)(s^n)^{`}(t) - \chi(s^n(t),t))\Big]\frac{\partial \Phi^\tau}{\partial x}\,dx\,dt + \int_{s_0}^{s_1}\phi^n(x)\Phi^\tau(x,\tau)\,dx$$
$$+ \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_{k+1}} \tilde{u}^\tau\Phi_t^\tau\,dx\,dt - \frac{1}{2}\sum_{k=1}^{n-1} sign(s_k - s_{k+1})\sum_{i=\alpha_k}^{\beta_k-1} h_i^2 u_{ix}(k)\Phi_i(k)$$

Notice that the sequences $\{\tilde{u}^\tau\}$, $\{u^\tau\}$ are equivalent in strong and weak topology of $L_2(D)$, and therefore $\{\tilde{u}^\tau\}$ converges to u weakly in $L_2(D)$. In fact from (3.33) we get

$$\|\tilde{u}^\tau - u^\tau\|^2_{L_2(D)} = \frac{1}{3}\sum_{k=1}^n \tau \sum_{i=0}^{N-1} h_i u_{ix}^2(k)\max_i h_i^2 \to 0, \quad \text{as } n \to \infty. \tag{3.64}$$

Denote

$$\tilde{\Delta} = \bigcup_{k=1}^n \{(x,t) : t_{k-1} < t < t_k,\ \min(s(t),s_k) < x < \max(s(t),s_k)\}$$

where $|\tilde{\Delta}|$ represents the Lebesgue measure of $\tilde{\Delta}$. Since $\tilde{s}^n(t_k) = s_k$, we get

$$|\tilde{\Delta}| \le \sum_{k=1}^n \int_{t_{k-1}}^{t_k}\int_t^{t_k} |s'(\tau)|\,d\tau\,dt + \sum_{k=1}^n \tau|s(t_k) - \tilde{s}^n(t_k)| \le$$
$$\sqrt{T}\|s'\|_{L^2(0,T)}\tau + T\|s - \tilde{s}^n\|_{C[0,T]} \to 0 \quad \text{as } \tau \to 0$$

and all of the integrands are uniformly bounded in $L^1(D)$. It follows that the first term in R tends to zero as $\tau \to 0$. Similarly second, third and fifth terms also tends to zero as $\tau \to 0$. The fourth term in R tends to zero by the Corollary 3.5.3 and uniform convergence of $\{\Phi^\tau\}$ in $\overline{D}$. In order to prove convergence to zero of the last term of R, consider

$$\tilde{\Delta} = \bigcup_{k=1}^{n-1} \{(x,t) : t_{k-1} < t < t_k, d_k \equiv \min(s_k, s_{k+1}) < x < d_{k+1} \equiv \max(s_k, s_{k+1})\}$$

Then using Corollary 3.5.3 we obtain

$$|\tilde{\Delta}| \leq C\tau \to 0, \quad \text{as } \tau \to 0.$$

Since

$$\sum_{i=\alpha_k}^{\beta_k - 1} h_i = |s_k - s_{k+1}|$$

we derive

$$\Big|\sum_{k=1}^{n-1} sign(s_k - s_{k+1}) \sum_{i=\alpha_k}^{\beta_k - 1} h_i^2 u_{ix}(k)\Phi_i(k)\Big| \leq \sum_{k=1}^{n-1} |s_k - s_{k+1}| \int_{d_k}^{d_{k+1}} \Big|\frac{\partial u^\tau}{\partial x}\Big| |\Phi^\tau| dx \leq$$
$$C \sum_{k=1}^{n-1} \tau \int_{d_k}^{d_{k+1}} \Big|\frac{\partial u^\tau}{\partial x}\Big| |\Phi^\tau| dx \leq \Big\|\frac{\partial u^\tau}{\partial x}\Big\|_{L_2(\tilde{\Delta})} \|\Phi^\tau\|_{L_2(\tilde{\Delta})} \tag{3.65}$$

By the uniform boundedness of the integrands in $L_2(D)$ RHS of (3.65) tends to zero as $\tau \to 0$.

We are going to show calculations for the first term in R:

$$
\begin{aligned}
\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\tilde{a}^{\tau}\frac{\partial u^{\tau}}{\partial x}\Phi_x^{\tau}dxdt &\le \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}|\tilde{a}^{\tau}\frac{\partial u^{\tau}}{\partial x}\Phi_x^{\tau}|dxdt \\
&\le \|\tilde{a}^{\tau}\|_{L_\infty}\|\Phi_x\|_{C(D)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\Big|\frac{\partial u^{\tau}}{\partial x}\Big|dxdt \\
\le \|\tilde{a}^{\tau}\|_{L_\infty}\|\Phi_x\|_{C(D)}\Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}1\ dxdt\Big)^{\frac{1}{2}} & \Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\Big|\frac{\partial u^{\tau}}{\partial x}\Big|^2 dxdt\Big)^{\frac{1}{2}} \\
&\le \|\tilde{a}^{\tau}\|_{L_\infty}\|\Phi_x\|_{C(D)}\Big\|\frac{\partial u^{\tau}}{\partial x}\Big\|_{L_2(D)}\sqrt{\tilde{\Delta}} \to 0
\end{aligned} \tag{3.66}
$$

(latter tends to zero since $\|\tilde{a}^{\tau}\|_{L_\infty}$ is bounded by R, L_2 norm of $\frac{\partial u^{\tau}}{\partial x}$ is bounded by the energy estimate, and $\|\Phi_x\|_{C(D)}$ is bounded)

Thus we get

$$\lim_{\tau\to 0} R = 0 \tag{3.67}$$

Using the weak convergence of u^{τ} to u in $W_2^{1,0}(D)$, weak convergence of $\tilde{u}^{\tau}$ to u in $L_2(D)$ and uniform convergence of the sequences $\{\Phi^{\tau}\}$, $\{\frac{\partial \Phi^{\tau}}{\partial x}\}$ and $\{\Phi_t^{\tau}\}$ to Φ, $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial t}$ respectively, and taking the limit as $\tau \to 0$, we obtain that the first, second and fourth integrals on the left-hand side of (3.63) converge to respective integrals with u^{τ} (or $\tilde{u}^{\tau}$), Φ^{τ}, Φ_t^{τ}, $\Phi^{\tau}(x,\tau)$, $\phi^n(x)$ and $\Phi^{\tau}(0,t)$ replaced by u,Φ, $\frac{\partial \Phi}{\partial t}$, $\Phi(x,0)$, $\phi(x)$ and $\Phi(0,t)$ respectively. Due to strong convergence of s^n to s strongly in $W_2^1[0,T]$, the traces $\gamma(s^n(t),(t))$, $\chi(s^n(t),t)$ converge strongly in $L_2[0,T]$ to traces $\gamma(s(t),(t))$, $\chi(s(t),t)$ respectively. Due to the uniform convergence of $\Phi^{\tau}(s^n(t),t)$ to $\Phi(s(t),t)$ on $[0,T]$ by taking limit as $\tau\to 0$, the last integral on the LHS of (3.63) converge to respective integral with s^n and Φ^{τ} replaced by s and Φ.

We are going to show calculations for the first term of (3.63):

$$\int_0^T\int_0^{s(t)} \tilde{a}^\tau \frac{\partial u^\tau}{\partial x}\Phi_x^\tau dxdt = \int_0^T\int_0^{s(t)} a\frac{\partial u^\tau}{\partial x}\Phi_x^\tau dxdt + \int_0^T\int_0^{s(t)} (\tilde{a}^\tau - a)\frac{\partial u^\tau}{\partial x}\Phi_x^\tau dxdt \tag{3.68}$$

$$\left|\int_0^T\int_0^{s(t)} \tilde{a}^\tau \frac{\partial u^\tau}{\partial x}\Phi_x^\tau dxdt - \int_0^T\int_0^{s(t)} a\frac{\partial u^\tau}{\partial x}\Phi_x^\tau dxdt\right| \le \int_0^T\int_0^{s(t)} \left|(\tilde{a}^\tau - a)\frac{\partial u^\tau}{\partial x}\Phi_x^\tau\right| dxdt \tag{3.69}$$

$$\int_0^T\int_0^{s(t)} \left|(\tilde{a}^\tau - a)\frac{\partial u^\tau}{\partial x}\Phi_x^\tau\right| dxdt \le C\left\|\tilde{a}^\tau - a\right\|_{L_2(D)}\left\|\frac{\partial u^\tau}{\partial x}\right\|_{L_2(D)} \to 0, \ \ as \ \ n\to\infty \tag{3.70}$$

since $\tilde{a}^\tau$ converges to a strongly in $L_2(D)$, and $\frac{\partial u^\tau}{\partial x}$ is bounded in $L_2(D)$

It only remains to show the following equality

$$\lim_{\tau\to 0}\int_0^{T-\tau} (\tilde{s}_1^n)^{`}(t) u^\tau(\tilde{s}_1^n(t),t)\Phi^\tau(\tilde{s}_1^n(t),t)\,dt = \int_0^T s'(t)u(s(t),t)\Phi(s(t),t) \tag{3.71}$$

Due to strong convergence of $\{\tilde{s}_1^n\}$ to s in $W_2^1[0,T]$, from (3.56) we get $\{u^\tau(\tilde{s}_1^n(t),t)\}$ is uniformly bounded in $L_2[0,T]$ and

$$\|u^\tau(\tilde{s}_1^n(t),t) - u^\tau(s(t),t)\|_{L_2[0,T]} \to 0 \quad \text{as } \tau\to 0 \tag{3.72}$$

Due to weak convergence of $\{u^\tau\}$ to u in $W_2^{1,0}(D)$, we get

$$u^\tau(s(t),t) \to u(s(t),t), \quad \text{weakly in } L_2[0,T] \tag{3.73}$$

Since $\{\Phi^\tau(\tilde{s}_1^n(t),t)\}$ converges to $\Phi(s(t),t)$ uniformly in $[0,T]$, from (3.72),(3.73), we can conclude (3.71).

Taking the limit as $\tau\to 0$, from (3.63) we get that u satisfies integral identity (3.12), i.e it is a weak solution of the problem (3.1)-(3.4). Due to uniqueness of the solution ([58]) it follows that the whole sequence $\{u^\tau\}$ converges to $u \in V_2^{1,0}(\Omega)$ weakly in $W_2^{1,0}(\Omega)$. By the property of weak convergence and (3.59), we conclude (3.46). Theorem is proved.

Theorem 3.6.3 implies the following existence result ([58]):

Corollary 3.6.4. *For arbitrary $v = (s,a) \in V_R$ there exists a weak solution $u \in V_2^{1,0}(\Omega)$ of the problem (3.1)-(3.4) that satisfy the energy estimate (3.46)*

Remark: We can use the following weaker assumptions in this section to carry out the proves

$$\phi \in L_2[0,l],\ \gamma,\chi \in W_2^{1,0}(D),$$

and (3.8) instead of conditions given in Section 3.4. We would only need to define ϕ_i as a Steklov average

$$\phi_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} \phi(x)dx,\ i = 0,...,N-1$$

and substitute the norm of ϕ^n in the first energy estimate by norm of ϕ.

3.6.2 Second Energy Estimate and Existence of the Optimal Control

Let $[v]_n$ to be a given discrete control vector and $[u([v]_n)]_n$ discrete state vector, we deine the vector

$$[\tilde{u}([v]_n)]_n = (\tilde{u}(0), \tilde{u}(1), ..., \tilde{u}(n))$$

as

$$\tilde{u}_i(k) = \begin{cases} u_i(k) \ \ 0 \le i \le m_j, \\ u_{m_j}(k) \ \ m_j < i \le N, k = \overline{0,n}. \end{cases}$$

We are going to prove the energy estimation for the vector $\tilde{u}([v]_n)$.

Theorem 3.6.5. *For all sufficiently small τ discrete state vector $[u([v]_n)]_n$ satisfies the* *following stability estimation:*

$$\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(k)+\tau\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)+\tau^2\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)\le$$

$$C\Big[\left\|\phi^n\right\|^2_{L_2[0,s_0]}+\|\phi\|^2_{W_2^1[0,s_0]}+\|g\|^2_{W_2^{\frac{1}{4}}[0,T]}+\left\|\gamma(s^n(t),t)(s^n)'(t)\right\|^2_{W_2^{\frac{1}{4}}[0,T]}$$

$$+\left\|\chi(s^n(t),t)\right\|^2_{W_2^{\frac{1}{4}}[0,T]}+\|f\|^2_{L_2(D)}\Big], \tag{3.74}$$

Proof: If $s_{k-1}\ge s_k$ then we can substitute $u_i(k)$ by $\tilde{u}_i(k)$ in all terms of (3.21). Choose $\eta_i=2\tau\tilde{u}_{i\bar{t}}(k)$ in (3.21) thus from equality

$$2\tau a_{ik}^n\tilde{u}_{ix}(k)\tilde{u}_{ix\bar{t}}(k)=a_{ik}^n\tilde{u}_{ix}^2(k)-a_{i,k-1}^n\tilde{u}_{ix}^2(k-1)-\tau a_{ik\bar{t}}^n\tilde{u}_{ix}^2(k-1)+\tau^2a_{ik}^n\tilde{u}_{ik\bar{t}}^2(k), \tag{3.75}$$

we get

$$\sum_{i=0}^{m_j-1} h_ia_{ik}^n\tilde{u}_{ix}^2(k)-\sum_{i=0}^{m_j-1} h_ia_{i,k-1}^n\tilde{u}_{ix}^2(k-1)+2\tau\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)+\tau^2\sum_{i=0}^{m_j-1} h_ia_{ik}^n\tilde{u}_{ix\bar{t}}^2(k)$$

$$=\tau\sum_{i=0}^{m_j-1} h_ia_{ik\bar{t}}^n\tilde{u}_{ix}^2(k-1)+2\tau\sum_{i=0}^{m_j-1} h_ib_{ik}\tilde{u}_{ix}(k)\tilde{u}_{i\bar{t}}(k)+2\tau\sum_{i=0}^{m_j-1} h_ic_{ik}\tilde{u}_i(k)\tilde{u}_{i\bar{t}}(k)$$

$$-2\tau\sum_{i=0}^{m_j-1} h_if_{ik}\tilde{u}_{i\bar{t}}(k)-2\tau\Big[(\gamma_{s^n}(s^n)')^k-\chi_{s^n}^k\Big]\tilde{u}_{m_j,\bar{t}}(k)-2\tau g_ku_{0,\bar{t}}(k) \tag{3.76}$$

If $s_{k-1}<s_k$, then we can substitute $u_i(k)$ by $\tilde{u}_i(k)$ in all terms of (3.21) except in the term including backward discrete time derivative. The latter will be estimated using the following inequality:

$$2\tau\sum_{i=0}^{m_j-1} h_iu_{i\bar{t}}(k)\tilde{u}_{i\bar{t}}(k)\ge\tau\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)-(C^{`})^2\tau\sum_{i=m_{j_{k-1}}}^{m_j-1} h_iu_{ix}^2(k-1) \tag{3.77}$$

In order to show (3.77), we transform the LHS employing CBS and Cauchy inequalty with $\epsilon = \tau$ to derive

$$2\tau \sum_{i=0}^{m_j-1} h_i u_{i\bar{t}}(k)\tilde{u}_{i\bar{t}}(k) = 2\tau \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) - 2 \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i \tilde{u}_{i\bar{t}}(k) \sum_{p=m_{j_{k-1}}}^{i-1} h_p u_{px}(k-1)$$
$$\geq \tau \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) - \frac{1}{\tau} \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i \Big(\sum_{p=m_{j_{k-1}}}^{i-1} h_p u_{px}(k-1) \Big)^2$$
$$\geq \tau \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) - \frac{1}{\tau} |s_k - s_{k-1}|^2 \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i u_{ix}^2(k-1) \tag{3.78}$$

which gives us (3.77) due to (3.30). Thus (3.76) is replaced with the inequality

$$\sum_{i=0}^{m_j-1} h_i a_{ik}^n \tilde{u}_{ix}^2(k) - \sum_{i=0}^{m_j-1} h_i a_{i,k-1}^n \tilde{u}_{ix}^2(k-1) + \tau \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) + \tau^2 \sum_{i=0}^{m_j-1} h_i a_{ik}^n \tilde{u}_{ix\bar{t}}^2(k)$$
$$\leq \tau \sum_{i=0}^{m_j-1} h_i a_{ik\bar{t}}^n \tilde{u}_{ix}^2(k-1) + 2\tau \sum_{i=0}^{m_j-1} h_i b_{ik} \tilde{u}_{ix}(k) \tilde{u}_{i\bar{t}}(k) + 2\tau \sum_{i=0}^{m_j-1} h_i c_{ik} \tilde{u}_i(k) \tilde{u}_{i\bar{t}}(k)$$
$$+(C^{`})^2 \tau \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i u_{ix}^2(k-1) - 2\tau \sum_{i=0}^{m_j-1} h_i f_{ik} \tilde{u}_{i\bar{t}}(k)$$
$$-2\tau \Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big] \tilde{u}_{m_j,\bar{t}}(k) - 2\tau g_k \tilde{u}_{0,\bar{t}}(k) \tag{3.79}$$

After summing inequalities (3.79) w.r.t k from 1 to arbitrary $p \leq n$ we obtain

$$\sum_{i=0}^{m_{j_p}-1} h_i a_{ip}^n \tilde{u}_{ix}^2(p) + \tau \sum_{k=1}^{p} \sum_{i=0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) + \tau^2 \sum_{k=1}^{p} \sum_{i=0}^{m_j-1} h_i a_{ik}^n \tilde{u}_{ix\bar{t}}^2(k)$$
$$\leq (C^{`})^2 \tau \sum_{k=1}^{p} 1_+(s_k - s_{k-1}) \sum_{i=m_{j_{k-1}}}^{m_j-1} h_i u_{ix}^2(k-1) + \tau \sum_{k=1}^{p} \sum_{i=0}^{m_j-1} h_i a_{ik\bar{t}}^n \tilde{u}_{ix}^2(k-1)$$
$$+2\tau \sum_{k=1}^{p} \Big[\sum_{i=0}^{m_j-1} h_i b_{ik} \tilde{u}_{ix}(k) \tilde{u}_{i\bar{t}}(k) + 2 \sum_{i=0}^{m_j-1} h_i c_{ik} \tilde{u}_i(k) \tilde{u}_{i\bar{t}}(k) - 2 \sum_{i=0}^{m_j-1} h_i f_{ik} \tilde{u}_{i\bar{t}}(k) \Big]$$
$$+ \sum_{i=0}^{m_{j_0}-1} h_i a_{i0}^n \phi_{ix}^2 - 2\tau \sum_{k=1}^{p} \Big[(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k \Big] \tilde{u}_{m_j,\bar{t}}(k) - 2\tau \sum_{k=1}^{p} g_k \tilde{u}_{0,\bar{t}}(k) \tag{3.80}$$

Using (3.8) and by employing Cauchy inequalities with appropriately chosen $\epsilon > 0$, from

(3.80) we get

$$\underline{a}\sum_{i=0}^{m_{j_p}-1} h_i\tilde{u}_{ix}^2(p)+\frac{\tau}{2}\sum_{k=1}^{p}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)+\underline{a}\tau^2\sum_{k=1}^{p}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)\le\tau\sum_{k=1}^{p}\sum_{i=0}^{m_j-1} h_i a_{ik\bar{t}}^n\tilde{u}_{ix}^2(k-1)$$
$$+C\tau\sum_{k=1}^{n}\Big[\sum_{i=0}^{m_j-1} h_i u_i^2(k)+\sum_{i=0}^{m_j-1} h_i u_{ix}^2(k)+\sum_{i=0}^{m_j-1} h_i f_{ik}^2\Big]+C\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2$$
$$+2\tau\sum_{k=1}^{n}\left|(\gamma_{s^n}(s^n)')^k-\chi_{s^n}^k\right|\left|\tilde{u}_{m_j,\bar{t}}(k)\right|+2\tau\sum_{k=1}^{n}|g_k|\left|\tilde{u}_{0,\bar{t}}(k)\right| \tag{3.81}$$

where C does not depend n. First term on the RHS will be estimated as follows:

$$\tau\sum_{k=1}^{p}\sum_{i=0}^{m_j-1} h_i a_{ik\bar{t}}^n\tilde{u}_{ix}^2(k-1)=\sum_{k=1}^{n}\sum_{i=0}^{m_j-1}\frac{1}{\tau}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}\int_{t-\tau}^{t}\frac{\partial a^n(x,\xi)}{\partial\xi}d\xi dt dx u_{ix}^2(k-1)$$
$$\le 2\int_0^T esssup_{0\le x\le l}\left|\frac{\partial a^n(x,t)}{\partial t}\right|dt\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(k)+C\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2 \tag{3.82}$$

Since p is chosen arbitrary, from (3.81) we derive

$$\underline{a}\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(p)+\frac{\tau}{2}\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)+a_0\tau^2\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)$$
$$\le 2\int_0^T esssup_{0\le x\le l}\left|\frac{\partial a^n(x,t)}{\partial t}\right|dt\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(k)+C\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2$$
$$+C\tau\sum_{k=1}^{n}\Big[\sum_{i=0}^{m_j-1} h_i u_i^2(k)+\sum_{i=0}^{m_j-1} h_i u_{ix}^2(k)+\sum_{i=0}^{m_j-1} h_i f_{ik}^2\Big]$$
$$+2\tau\sum_{k=1}^{n}\left|(\gamma_{s^n}(s^n)')^k-\chi_{s^n}^k\right|\left|\tilde{u}_{m_j,\bar{t}}(k)\right|+2\tau\sum_{k=1}^{n}|g_k|\left|\tilde{u}_{0,\bar{t}}(k)\right| \tag{3.83}$$

If

$$2\int_0^T \text{esssup}_{0\le x\le l}\Big|\frac{\partial a^n(x,t)}{\partial t}\Big|dt$$
$$\le 2\Big(\int_0^T \text{esssup}_{0\le x\le l}\Big|\frac{\partial a^n(x,t)}{\partial t}\Big|^{\gamma}dt\Big)^{\frac{1}{\gamma}}T^{\frac{\gamma-1}{\gamma}}$$
$$\le 2RT^{\frac{\gamma-1}{\gamma}} < \underline{a} \tag{3.84}$$

or, analogously

$$T < \Big(\frac{\underline{a}}{2R}\Big)^{\frac{\gamma}{\gamma-1}} \tag{3.85}$$

then the first term on the RHS of (3.83) is absorbed into the first term on the LHS. If (3.84) is not satisfied, then $[0,T]$ can be partitioned into finitely many closed subintervals which satisfy (3.84), absorb first term on the RHS into the LHS in each such subintervals and using summation obtain the same for (3.83) in general. Therefore we get

$$\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(p) + \tau\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k) + \tau^2\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)$$
$$\le C\tau\sum_{k=1}^{n}\Big[\sum_{i=0}^{m_j-1} h_iu_i^2(k) + \sum_{i=0}^{m_j-1} h_iu_{ix}^2(k) + \sum_{i=0}^{m_j-1} h_if_{ik}^2\Big] + C\sum_{i=0}^{m_{j_0}-1} h_i\phi_{ix}^2$$
$$+C\tau\sum_{k=1}^{n}\Big|(\gamma_{s^n}(s^n)')^k - \chi_{s^n}^k\Big|\Big|\tilde{u}_{m_j,\bar{t}}(k)\Big| + C\tau\sum_{k=1}^{n}|g_k|\Big|\tilde{u}_{0,\bar{t}}(k)\Big| \tag{3.86}$$

with constant C which does not depend on n.

Due to the fact that $\gamma,\chi \in W_2^{1,1}(D)$ we get $\gamma(s^n(t),t), \chi(s^n(t),t) \in W_2^{\frac{1}{4}}[0,T]$ ([65, 25, 58]) and

$$||\gamma(s^n(t),t)||_{W_2^{\frac{1}{4}}[0,T]} \le C||\gamma||_{W_2^{1,1}(D)},\ ||\chi(s^n(t),t)||_{W_2^{\frac{1}{4}}[0,T]} \le C||\chi||_{W_2^{1,1}(D)}, \tag{3.87}$$

where C is independent of n. By Lemma 3.6.8 $\mathcal{P}_n([v]_n) \in V_{R+1}$. Applying Morrey inequality to $(s^n)'$ we can show that $\gamma(s^n(t),t)(s^n)'(t) \in W_2^{\frac{1}{4}}[0,T]$. Furthermore

$$\|\gamma(s^n(t),t)(s^n)'(t)\|_{W_2^{\frac{1}{4}}[0,T]} \leq C_1\|\gamma(s^n(t),t)\|_{W_2^{\frac{1}{4}}[0,T]}\|s^n\|_{W_2^2[0,T]} \leq C\|\gamma\|_{W_2^{1,1}(D)}, \tag{3.88}$$

where C does not depend on n.

Consider $w(x,t)$ to be a function in $W_2^{2,1}(D)$ so that

$$w(x,0) = \phi(x) \quad \text{for } x \in [0,s_0],\ a(0,t)w_x(0,t) = g(t), \quad \text{for a.e. } t \in [0,T] \tag{3.89}$$

$$a(s^n(t),t)w_x(s^n(t),t) = \gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t) \quad \text{for a.e. } t \in [0,T] \tag{3.90}$$

and

$$\begin{aligned}\|w\|_{W_2^{2,1}(D)} \leq C\Big[&\|g\|_{W_2^{\frac{1}{4}}[0,T]} + \|\phi(x)\|_{W_2^1[0,s_0]}\\ &+ \Big\|\gamma(s^n(t),t)(s^n)'(t) - \chi(s^n(t),t)\Big\|_{W_2^{\frac{1}{4}}[0,T]}\Big]\end{aligned} \tag{3.91}$$

Function w exists due to the result on traces of Sobolev functions [25, 65]. For instance, w can be constructed as a solution from $W_2^{2,1}(\Omega^n)$ of the heat equation in

$$\Omega^n = \{0 < x < s^n(t), 0 < t < T\}$$

with initial-boundary conditions (3.89),(3.90) and subsequent continuation to $W_2^{2,1}(D)$ with norm preservation [73, 74].

Therefore if we substitute in the original problem (3.1)-(3.4) u by $u - w$ we can obtain modified (3.86) without the last three terms on the RHS and with f, substituted by

$$F = f + w_t - (aw_x)_x - bw_x - cw \in L_2(D). \tag{3.92}$$

After using the stability estimation (3.34), from modified (3.86),(3.91) and (3.92), we have the following estimate:

$$\max_{1\le k\le n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix}^2(k)+\tau\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{i\bar{t}}^2(k)+\tau^2\sum_{k=1}^{n}\sum_{i=0}^{m_j-1} h_i\tilde{u}_{ix\bar{t}}^2(k)\le$$
$$C\Big[\left\|\phi^n\right\|^2_{L_2[0,s_0]}+\|\phi\|^2_{W_2^1[0,s_0]}+\|g\|^2_{W_2^{\frac{1}{4}}[0,T]}+\left\|\gamma(s^n(t),t)(s^n)'(t)\right\|^2_{W_2^{\frac{1}{4}}[0,T]}+$$
$$\left\|\chi(s^n(t),t)\right\|^2_{W_2^{\frac{1}{4}}[0,T]}+\|f\|^2_{L_2(D)}+\sum_{k=1}^{n-1}\mathbf{1}_+(s_{k+1}-s_k)\sum_{i=m_j}^{m_{j_{k+1}}-1} h_iu_i^2(k)\Big],\qquad(3.93)$$

After estimating the last term on the RHS of 3.93 as in the proof of Theorem 3.6.3, we obtain (3.74). Theorem is proved.

Result of Theorem 3.6.3 is strenghten by Second energy estimate (3.74).

Theorem 3.6.6. *Let $[v]_n=([s]_n,[a]_{nN})\in V_R^n, n=1,2,...$ be a sequence of discrete controls and the sequence $\{\mathcal{P}_n([v]_n\}$ converges weakly in $W_2^2[0,T]\times\tilde{W}_2^{1,1}(D)$ to $v=(s,a)\in H$ (i.e. strongly in $W_2^1[0,T]\times L_2(D)$) to $v=(s,a)$ for any $\delta>0$,*

$$\Omega'=\Omega\cap\{x<s(t)-\delta,\ 0<t<T\}$$

Then the sequence $\{\hat{u}^\tau\}$ converges as $\tau\to 0$ weakly in $W_2^{1,1}(\Omega^‘)$ to weak solution $u\in W_2^{1,1}(\Omega)$ of the problem (3.1)-(3.4), i.e. to the solution of the integral identity (3.11). Moreover, u satisfies the energy estimate

$$\|u\|^2_{W_2^{1,1}(\Omega)}\le C\Big(\|\phi\|^2_{W_2^1(0,s_0)}+\|g\|^2_{W_2^{\frac{1}{4}}[0,T]}+\|f\|^2_{L_2(D)}+\|\gamma\|^2_{W_2^{1,1}(D)}+\|\chi\|^2_{W_2^{1,1}(D)}\Big)\qquad(3.94)$$

Proof: Take $\epsilon_m\downarrow 0$ to be an arbitrary sequence and

$$\Omega_m=\{(x,t):0<x<s(t)-\epsilon_m,0<t\le T\}$$

Notice that the sequence s^n converges to s uniformly in $[0,T]$. Compute

$$||\hat{u}^\tau||^2_{W_2^{1,1}(\Omega_m)} = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_0^{s(t)-\epsilon_m} |\hat{u}^\tau|^2 + \Big|\frac{\partial \hat{u}^\tau}{\partial x}\Big|^2 + \Big|\frac{\partial \hat{u}^\tau}{\partial t}\Big|^2\,dx\,dt$$

We define $s_k^m = x_{\hat{i}}$, where

$$\hat{i} = \max\Big\{i \le N : \max_{t_{k-1}\le t\le t_k} s(t) - \epsilon_m \le x_i \le \max_{t_{k-1}\le t\le t_k} s(t) - \frac{\epsilon_m}{2}\Big\}$$

and substitute to obtain

$$\begin{aligned}||\hat{u}^\tau||^2_{W_2^{1,1}(\Omega_m)} \le & \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_0^{s_k^m} |\hat{u}(x;k-1)+\hat{u}_{\bar{t}}(x;k)(t-t_{k-1})|^2\,dx\,dt+\\ &+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_0^{s_k^m} \Big|\frac{d\hat{u}(x;k-1)}{dx}+\frac{d\hat{u}_{\bar{t}}(x;k)}{dx}(t-t_{k-1})\Big|^2\,dx\,dt+\\ &+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_0^{s_k^m} |\hat{u}_{\bar{t}}(x;k)|^2\,dx\,dt\end{aligned}$$

then by using (3.15) we derive

$$\begin{aligned}||\hat{u}^\tau||^2_{W_2^{1,1}(\Omega_m)} \le C\Big\{&\tau\sum_{k=1}^{n}\sum_{0}^{\hat{i}-1} h_i u_i^2(k-1) + \tau\sum_{k=1}^{n}\sum_{0}^{\hat{i}-1} h_i u_{ix}^2(k-1)+\\ &+\tau\sum_{k=1}^{n}\sum_{0}^{\hat{i}-1} h_i u_{i\bar{t}}^2(k) + \tau^2\sum_{k=1}^{n}\sum_{0}^{\hat{i}-1} h_i u_{ix\bar{t}}^2(k)\Big\}\end{aligned} \tag{3.95}$$

where C does not depend on n or τ. We need to prove that the RHS of (3.95) is bounded by the LHS of (3.74) for sufficiently large n. It is enough to prove that: *for fixed ϵ_m, there exists $N = N(\epsilon_m)$ so that for $\forall n > N$*

$$s_k^m < \min(s_k, s_{k-1}),\ k = 1,\dots,n \tag{3.96}$$

From (3.96) we get

$$\tau \sum_{k=1}^{n} \sum_{0}^{\hat{i}-1} h_i u_{ix}^2(k-1) + \tau \sum_{k=1}^{n} \sum_{0}^{\hat{i}-1} h_i u_{i\bar{t}}^2(k) + \tau^2 \sum_{k=1}^{n} \sum_{0}^{\hat{i}-1} h_i u_{ix\bar{t}}^2(k)$$
$$\leq \tau \sum_{k=1}^{n} \sum_{0}^{m_j-1} h_i \tilde{u}_{ix}^2(k-1) + \tau \sum_{k=1}^{n} \sum_{0}^{m_j-1} h_i \tilde{u}_{i\bar{t}}^2(k) + \tau^2 \sum_{k=1}^{n} \sum_{0}^{m_j-1} h_i \tilde{u}_{ix\bar{t}}^2(k) \tag{3.97}$$

In order to prove (3.96), first we prove that for sufficiently large n and all $t_{k-1} \leq t \leq t_k$

$$s(t) - s_k < \frac{\epsilon_m}{2} \tag{3.98}$$

We have

$$s(t) - s_k = s(t) - s(t_k) + s(t_k) - s^n(t_k) + s^n(t_k) - s_k$$
$$\leq \|s'\|_{C[0,T]}\tau + \|s - s^n\|_{C[0,T]} + s^n(t_k) - s_k$$

Note that

$$s^n(t_k) = \frac{s_k + s_{k-1}}{2}$$

so

$$|s(t) - s_k| \leq \|s'\|_{C[0,T]}\tau + \|s - s^n\|_{C[0,T]} + \frac{|s_{k-1} - s_k|}{2}$$

Next, employing Morrey's inequality and (3.5.3) we derive

$$|s(t) - s_k| \leq \left(C\|s'\|_{W_2^1[0,T]} + \frac{C'}{2} \right)\tau + \|s - s^n\|_{C[0,T]} \tag{3.99}$$

Since $s^n \to s$ uniformly on $[0,T]$ then there exists $N_1 = N_1(\epsilon_m)$ for which (3.98) holds. In a similar manner, there exists some $N_2 = N_2(\epsilon_m)$, such that for $n > N_2$ and for all $t_{k-1} \leq t \leq t_k$, (3.98) is true with s_k replaced by s_{k-1}. Therefore, (3.96) holds with $N = \max(N_1, N_2)$.

Using (3.95), (3.97), second energy estimate (3.74), and the first energy estimate (3.33), we derive

$$\|\hat{u}^\tau\|^2_{W_2^{1,1}(\Omega_m)} \le C\Big[\,\big\|\phi^n\big\|^2_{L_2[0,s_0]} + \|\phi\|^2_{W_2^1[0,s_0]} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \\ + \|\chi\|^2_{W_2^{1,1}(D)} + \|g\|^2_{W_2^{1/4}[0,T]} + \sum_{k=1}^{n-1} \mathbf{1}_+(s_{k+1}-s_k) \sum_{i=m_j}^{m_{j_{k+1}}-1} h_i u_i^2(k)\Big] \qquad (3.100)$$

After estimating the last term on the RHS of (3.100) like in the proof of Theorem 3.6.3, we obtain

$$\|\hat{u}^\tau\|^2_{W_2^{1,1}(\Omega_m)} \le C\Big(\|\phi^n\|^2_{L_2(0,s_0)} + \|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}[0,T]} + \|f\|^2_{L_2(D)} + \\ \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big), \qquad (3.101)$$

Due to the fact that $\phi^n \to \phi$ strongly in $L_2[0,s_0]$, the RHS is uniformly bounded independent of n. Thus, $\{\hat{u}^\tau\}$ is weakly precompact in $W_2^{1,1}(\Omega_m)$. Therefore it is strongly precompact in $L_2(\Omega_m)$. Let u be a weak limit point of $\{\hat{u}^\tau\}$ in $W_2^{1,1}(\Omega_m)$, and thus a strong limit point in $L_2(\Omega_m)$. On the other hand the sequences $\{\hat{u}^\tau\}$ and $\{u^\tau\}$ are equivalent in strong topology of $L_2(\Omega_m)$. In fact, we have that for all $n > N(m)$

$$\|\hat{u}^\tau - u^\tau\|^2_{L_2(\Omega_m)} \le 2\tau^3 \sum_{k=1}^{n} \sum_{i=0}^{m_j-1} \Big[h_i \tilde{u}^2_{i\bar{t}}(k) + \frac{1}{3} h_i^3 \tilde{u}^2_{ix\bar{t}}(k)\Big] = O(\tau), \quad \text{as } \tau \to 0, \qquad (3.102)$$

because of the second energy estimate (3.74). Which implies that u is a strong limit point of the sequence $\{u^\tau\}$ in $L_2(\Omega_m)$. By Theorem 3.6.3 whole sequence $\{u^\tau\}$ converges weakly in $W_2^{1,0}(\Omega)$ to the unique weak solution from $V_2^{1,0}(\Omega)$ of the problem (3.1)-(3.4). Thus, u is a weak solution of the problem (3.1)-(3.4) and we obtain that the whole sequence $\{\hat{u}^\tau\}$ converges weakly in $W_2^{1,1}(\Omega_m)$ to $u \in W_2^{1,1}(\Omega_m)$ which is a weak solution of the problem (3.1)-(3.4) from $V_2^{1,0}(\Omega)$. Therefore, u_t exists in Ω_m and $\|u_t\|_{L_2(\Omega_m)}$ is uniformly bounded by the RHS of (3.101). Consequently the weak derivative u_t exists in Ω, and $u \in W_2^{1,1}(\Omega)$.

Employing the property of the weak convergence, after passing to limit first as $n \to$

$+\infty$, and then as $m \to +\infty$, from (3.101), we derive (3.94). Theorem is proved.

Furthermore, Theorem 3.6.6 implies the following existence result:

Corollary 3.6.7. *For arbitrary $v = (s,a) \in V_R$ there exists a weak solution $u \in W_2^{1,1}(\Omega)$ of the problem (3.1)-(3.4) which satisfy the energy estimate (3.94). By Sobolev extension theorem u can be continued to $W_2^{1,1}(D)$ with the norm preservation:*

$$\|u\|^2_{W_2^{1,1}(D)} \leq C\Big(\|\phi\|^2_{W_2^1(0,s_0)} + \|g\|^2_{W_2^{\frac{1}{4}}[0,T]} + \|f\|^2_{L_2(D)} + \|\gamma\|^2_{W_2^{1,1}(D)} + \|\chi\|^2_{W_2^{1,1}(D)}\Big) \qquad (3.103)$$

Remark: Slightly higher regularity of u was proved and both in Theorem 3.6.6 and Corollary 3.6.7 $W_2^{1,1}(\Omega)$ or $W_2^{1,1}(D)$-norm on the left-hand sides of (3.94) or (3.103) can be substituted with

$$\|u\|^2 = \max_{0\leq t\leq T} \|u(x,t)\|^2_{W_2^1[0,s(t)]} + \|u_t\|^2_{L_2(\Omega)} \quad \text{or } \|u\|^2 = \max_{0\leq t\leq T} \|u(x,t)\|^2_{W_2^1[0,l]} + \|u_t\|^2_{L_2(D)}$$

The proof of the Theorem 3.4.1 is the same as the proof ofl Theorem 3.4.1 in [1]. The main idea is that first and second energy estimates imply weak continuity of the functional $\mathcal{J}(v)$ in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$. Using Weierstrass theorem in weak topology and weak compactness of V_R existence of the optimal control follows.

3.6.3 Proof of Convergence Theorem 3.4.2

We divide the remainder of the proof of Theorem 3.4.2 into four lemmas. Lemma (3.6.8) shows that the mappings $\mathcal{Q}_n$ and $\mathcal{P}_n$ introduced in Section 3.3 satisfy the conditions of Lemma 3.5.2.

Lemma 3.6.8. *For arbitrary sufficiently small $\epsilon > 0$ there exists n_ϵ such that*

$$\mathcal{Q}_n(v) \in V_R^n, \quad \textit{for all } v \in V_{R-\epsilon} \quad \textit{and } n > n_\epsilon. \qquad (3.104)$$

$$\mathcal{P}_n([v]_n) \in V_{R+\epsilon}, \quad \textit{for all } [v]_n \in V_R^n \quad \textit{and } n > n_\epsilon. \tag{3.105}$$

Proof. Let $0 < \epsilon << R$, $v \in V_{R-\epsilon}$ and $\mathcal{Q}(v) = [v]_n = ([s]_n, [a]_{nN})$. The estimates for the first component of the control vectors is handled as in ([1]). By applying Cauchy-Bunyakovski-Schwarz (CBS) inequality and Fubini's theorem we have

$$\sum_{k=1}^{n-1} \tau s_{\bar{t}t,k}^2 = \sum_{k=1}^{n-1} \frac{1}{\tau^3}\Big[\int_{t_k}^{t_{k+1}} (s'(t) - s'(t-\tau))dt\Big]^2 \leq \frac{1}{\tau^2}\int_\tau^T |s'(t) - s'(t-\tau)|^2 dt$$
$$\leq \frac{1}{\tau}\int_\tau^T dt \int_{t-\tau}^t |s''(\xi)|^2 d\xi \leq \int_0^T |s''(t)|^2 dt, \ \sum_{k=1}^{n} \tau s_{\bar{t},k}^2 \leq \int_0^T |s'(t)|^2 dt, \tag{3.106}$$

$$\tau s_{\bar{t}t,0}^2 = \frac{1}{\tau^3}\Big[\int_0^\tau (s'(t) - s'(0))dt\Big]^2 \leq \frac{1}{2}\int_0^\tau |s''(t)|^2 dt, \tag{3.107}$$

$$\Big|\sum_{k=0}^{n-1} \tau s_k^2 - \int_0^T s^2(t)dt\Big| = \Big|\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_t^{t_k} (s^2(\xi))' d\xi dt\Big| \leq$$
$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} [s^2(\xi) + (s'(\xi))^2 \ ] d\xi dt \leq \tau \int_0^T [s^2(t) + (s'(t))^2 \ ]dt \leq (R-\epsilon)^2\tau, \tag{3.108}$$

Thus,

$$\|[s]_n\|_{w_2^2}^2 \leq \int_0^T |s''(t)|^2 dt + \int_0^T |s'(t)|^2 dt + \frac{1}{2}\int_0^\tau |s''(t)|^2 dt + (R-\epsilon)^2\tau$$
$$\leq \|s\|_{W_2^2[0,T]}^2 + O(\tau) \tag{3.109}$$

Using triangle inequality to estimate the difference between corresponding norms, we get

$$\left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i\,|a_{ik}|^2\right]^{1/2}-\left[\iint_D |a(x,t)|^2\,dx\,dt\right]^{1/2}$$
$$\leq \left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}|a_{ik}-a(x,t)|^2\,dx\,dt\right]^{1/2} \tag{3.110}$$

We are going to show that RHS of (3.110) tends to zero

$$\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}|a_{ik}-a(x,t)|^2\,dx\,dt$$

Calculate

$$\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}|a_{ik}-a(x,t)|^2\,dx\,dt$$
$$=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau^2}\left|\int_{t_{k-1}}^{t_k}[a(x_i,z)-a(x,t)]\,dz\right|^2 dx\,dt$$
$$=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau^2}\left|\int_{t_{k-1}}^{t_k}[a(x_i,z)-a(x,z)+a(x,z)-a(x,t)]\,dz\right|^2 dx\,dt$$
$$=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau^2}\left|\int_{t_{k-1}}^{t_k}\Big[\int_{x}^{x_i}\frac{\partial a}{\partial y}(y,z)\,dy+\int_t^z\frac{\partial a}{\partial \xi}(x,\xi)\,d\xi\Big]dz\right|^2 dx\,dt$$
$$\overset{\text{CBS}}{\leq} 2\Bigg[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\left|\int_{x_i}^{x}\frac{\partial a}{\partial x}(y,z)\,dy\right|^2 dz\,dx\,dt,$$
$$+\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\left|\int_t^z\frac{\partial a}{\partial t}(x,\xi)\,d\xi\right|^2 dz\,dx\,dt\Bigg]\leq I_1+I_2$$

Let's denote terms on the right hand side by I_1, I_2 then

$$I_1 = \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\left|\frac{\partial a}{\partial x}(y,z)\right|^2 dy(x-x_i)\,dz\,dx\,dt$$
$$\leq \frac{\Delta^2}{2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\left|\frac{\partial a}{\partial x}(y,z)\right|^2 dy\,dz\,dx\,dt$$
$$\leq \frac{\Delta^2}{2}\|a_x\|^2_{L_2(D)} \leq \frac{\Delta^2}{2}(R-\epsilon)^2 \tag{3.111}$$

Similarly,

$$I_2 = \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k}\left|\frac{\partial a}{\partial t}(x,\xi)\right|^2 d\xi\,dz\,dx\,dt$$
$$= \tau^2\|a_t\|^2_{L_2(D)} \leq \tau^2(R-\epsilon)^2 \tag{3.112}$$

By (3.110), (3.111), (3.112) it follows that

$$\left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i\,|a_{ik}|^2\right]^{1/2} \leq \|a\|_{L_2(D)} + \sqrt{\frac{\Delta^2}{2}+\tau^2}(R-\epsilon)$$

Since

$$\Delta \leq C\sqrt{\tau} \tag{3.113}$$

and hence

$$\left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i\,|a_{ik}|^2\right]^{1/2} \leq \|a\|_{L_2(D)} + \sqrt{C\tau+\tau^2}(R-\epsilon) \tag{3.114}$$

Next, consider the second term in $\|[a]_{nN}\|_{b_2}$:

$$\left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i\left|a_{ik,x}\right|^2\right]^{1/2} = \left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{1}{h_i\tau}\left|\int_{t_{k-1}}^{t_k} a(x_{i+1},t)-a(x_i,t)\,dt\right|^2\right]^{1/2}$$
$$= \left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{1}{h_i\tau}\left|\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{\partial a}{\partial x}(x,t)\,dx\,dt\right|^2\right]^{1/2}$$
$$\leq \left[\iint_D\left|\frac{\partial a}{\partial x}(x,t)\right|^2 dx\,dt\right]^{1/2} = \left\|\frac{\partial a}{\partial x}\right\|_{L_2(D)} \tag{3.115}$$

Lets consider the third term in $\left\|[a]_{nN}\right\|_{b_2}$:

By triangle inequality,

$$\left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i\left|a_{ik,\bar{t}}\right|^2\right]^{1/2} - \left[\iint_D\left|\frac{\partial a}{\partial t}(x,t)\right|^2 dx\,dt\right]^{1/2}$$
$$\leq \left[\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\left|a_{ik,\bar{t}}-\frac{\partial a}{\partial t}(x,t)\right|^2 dx\,dt\right]^{1/2}$$

It is enough then to estimate

$$J_1 := \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\left|a_{ik,\bar{t}}-\frac{\partial a}{\partial t}(x,t)\right|^2 dx\,dt \tag{3.116}$$

Calculate

$$
\begin{aligned}
J_1 &= \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\left|\frac{1}{\tau^2}\left[\int_{t_{k-1}}^{t_k}[a(x_i,\xi)-a(x_i,\xi-\tau)]\,d\xi\right]-\frac{\partial a}{\partial t}(x,t)\right|^2 dx\,dt \\
&= \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{1}{\tau^4}\left|\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}[\frac{\partial a}{\partial t}(x_i,\mu)-\frac{\partial a}{\partial t}(x,t)]\,d\mu\,d\xi\right|^2 dx\,dt \\
&\le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}\left|\frac{\partial a(x_i,\mu)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\mu d\xi \le K_1+K_2 \\
K_1 &= \frac{2}{\tau^2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}\left|\frac{\partial a(x_i,\mu)}{\partial t}-\frac{\partial a(x,\mu)}{\partial t}\right|^2 d\mu d\xi \\
&= \frac{2}{\tau^2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}\left|\int_{x}^{x_i}\frac{\partial^2 a(y,\mu)}{\partial x\partial t}dy\right|^2 d\mu d\xi \\
&\le \frac{2\Delta}{\tau^2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}\int_{x_i}^{x_{i+1}}\left|\frac{\partial^2 a(y,\mu)}{\partial x\partial t}\right|^2 dy d\mu d\xi
\end{aligned}
$$

By using Fubinis theorem:

$$
\begin{aligned}
K_1 &\le \frac{2\Delta}{\tau^2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}dt\left(\int_{x_i}^{x_{i+1}}\int_{t_{k-2}}^{t_{k-1}}d\xi\int_{t_{k-1}}^{\xi+\tau}\left|\frac{\partial^2 a(y,\mu)}{\partial x\partial t}\right|^2 d\mu dy+\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}d\xi\int_{\xi}^{t_k}\left|\frac{\partial^2 a(y,\mu)}{\partial x\partial t}\right|^2 d\mu dy\right) \\
&\le 2\Delta^2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\left(\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}\left|\frac{\partial^2 a(y,\mu)}{\partial x\partial t}\right|^2 d\mu dy+\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}\left|\frac{\partial^2 a(y,\mu)}{\partial x\partial t}\right|^2 d\mu dy\right) \\
&\le 4\Delta^2\left\|\frac{\partial^2 a}{\partial x\partial t}\right\|^2_{L^2(D)} \qquad (3.117)
\end{aligned}
$$

$$
\begin{aligned}
K_2 &= \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{2}{\tau^2}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\int_{\xi-\tau}^{\xi}\left|\frac{\partial a(x,\mu)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\mu d\xi \\
&= \frac{2}{\tau^2}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{t_{k-1}}^{t_k}dt\int_{x_i}^{x_{i+1}}dx\int_{t_{k-1}}^{t_k}\left|\frac{\partial a(x,\mu)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\mu\cdot\tau \\
&= \frac{2}{\tau}\int_0^l\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}dt\int_{t_{k-1}}^{t_k}\left|\frac{\partial a(x,\mu)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\mu \qquad (3.118)\\
&= \frac{2}{\tau}\int_0^l\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}dt\int_{t_{k-1}-t}^{t_k-t}\left|\frac{\partial a(x,t+\sigma)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\sigma \\
&\le \frac{2}{\tau}\int_0^l\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}dt\int_{-\tau}^{\tau}\left|\frac{\partial a(x,t+\sigma)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 d\sigma \\
&\le \frac{2}{\tau}\int_{-\tau}^{\tau}d\sigma\int_0^l\int_0^T\left|\frac{\partial a(x,t+\sigma)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right|^2 dtdx \\
&\le 4\max_{|\sigma|\le\tau}\left\|\frac{\partial a(x,t+\sigma)}{\partial t}-\frac{\partial a(x,t)}{\partial t}\right\|_{L^2(D)}^2 \qquad (3.119)
\end{aligned}
$$

Lastly,

$$
\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}h_i\tau\left|a_{ik,x\bar{t}}\right|^2=\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}\frac{1}{h_i\tau}\left|(a_{i+1,k}-a_{ik})-(a_{i+1,k-1}-a_{i,k-1})\right|^2 \qquad (3.120)
$$

$$
=\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}\frac{1}{h_i\tau^3}\left|\int_{t_{k-1}}^{t_k}(a(x_{i+1},t)-a(x_i,t))-(a(x_{i+1},t-\tau)-a(x_i,t-\tau)\,dt\right|^2 \qquad (3.121)
$$

$$
=\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}\frac{1}{h_i\tau^3}\left|\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\frac{\partial a}{\partial x}(z,t)-\frac{\partial a}{\partial x}(z,t-\tau)\,dz\,dt\right|^2 \qquad (3.122)
$$

$$
=\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}\frac{1}{h_i\tau^3}\left|\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\int_{t-\tau}^{t}\frac{\partial^2 a}{\partial x\partial t}(z,\xi)\,d\xi\,dz\,dt\right|^2 \qquad (3.123)
$$

By CBS inequality,

$$
\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}h_i\tau\left|a_{ik,x\bar{t}}\right|^2\le\sum_{k=1}^{n-1}\sum_{i=0}^{N-1}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_{x_i}^{x_{i+1}}\int_{t-\tau}^{t}\left|\frac{\partial^2 a}{\partial x\partial t}(z,\xi)\right|^2 d\xi\,dz\,dt \qquad (3.124)
$$

$$
=\frac{1}{\tau}\int_0^T\int_0^\ell\int_{t-\tau}^{t}\left|\frac{\partial^2 a}{\partial x\partial t}(z,\xi)\right|^2 d\xi\,dz\,dt \qquad (3.125)
$$

By Fubini's theorem, it follows that

$$\sum_{k=1}^{n-1}\sum_{i=0}^{N-1} h_i\tau \left|a_{ik,x\bar{t}}\right|^2 \leq \int_{-\tau}^{0}\int_0^{\ell}\left|\frac{\partial^2 a}{\partial x\partial t}(x,t)\right|^2 dx\,dt + \int_0^{T}\int_0^{\ell}\left|\frac{\partial^2 a}{\partial x\partial t}(x,t)\right|^2 dx\,dt \tag{3.126}$$

Thus,

$$\begin{aligned} \|[a]_{nN}\|_{\tilde{w}_2^{1,1}} &\leq \|a\|_{\tilde{W}_2^{1,1}(D)} + 2\Delta(R-\epsilon) + \sqrt{\tau}(R-\epsilon) \\ &+2\Delta\left\|\frac{\partial^2 a}{\partial x\partial t}\right\|_{L^2(D)} + 2\sqrt{\max_{|\sigma|\leq\tau}\left\|\frac{\partial a(x,t+\sigma)}{\partial t} - \frac{\partial a(x,t)}{\partial t}\right\|^2_{L^2(D)}} \\ +\Big(\int_{-\tau}^{0}\int_0^{\ell}\left|\frac{\partial^2 a}{\partial x\partial t}(x,t)\right|^2 dx\,dt\Big)^{\frac{1}{2}} &\leq \|a\|_{\tilde{W}_2^{1,1}(D)} + O(\sqrt{\tau}) + o(1), \quad \text{as} \quad \tau\downarrow 0 \end{aligned} \tag{3.127}$$

Next,

$$|a_{ik}| = \left|\frac{1}{\tau}\int_{t_{k-1}}^{t_k} a(x_i,t)dt\right| \leq \|a\|_{L_\infty(D)} \tag{3.128}$$

$$\sum_{k=1}^{n}\tau|a_{ik,\bar{t}}|^{\gamma}=\sum_{k=1}^{n}\tau\left|\frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\int_{t-\tau}^{t}|a_\xi(x_i,\xi)|d\xi dt\right|^{\gamma}$$

$$\le\sum_{k=1}^{n}\tau^{1-2\gamma}\left[\left(\int_{t_{k-1}}^{t_k}\int_{t-\tau}^{t}|a_\xi(x_i,\xi)|^{\gamma}d\xi dt\right)^{\frac{1}{\gamma}}\left(\int_{t_{k-1}}^{t_k}\int_{t-\tau}^{t}1\ d\xi dt\right)^{\frac{\gamma-1}{\gamma}}\right]^{\gamma}$$

$$\le\sum_{k=1}^{n}\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\int_{t-\tau}^{t}|a_\xi(x_i,\xi)|^{\gamma}d\xi dt$$

$$\le\sum_{k=1}^{n}\frac{1}{\tau}\left[\int_{t_{k-2}}^{t_{k-1}}\int_{t_{k-1}}^{\xi+\tau}|a_\xi(x_i,\xi)|^{\gamma}dtd\xi+\int_{t_{k-1}}^{t_k}\int_{\xi}^{t_k}|a_\xi(x_i,\xi)|^{\gamma}dtd\xi\right]$$

$$\le\sum_{k=1}^{n}\frac{1}{\tau}\left[\int_{t_{k-2}}^{t_{k-1}}|a_\xi(x_i,\xi)|^{\gamma}(\xi+\tau-t_{k-1})d\xi+\int_{t_{k-1}}^{t_k}|a_\xi(x_i,\xi)|^{\gamma}(t_k-\xi)d\xi\right]$$

$$=\sum_{k=1}^{n}\frac{1}{\tau}\left[\int_{t_{k-1}}^{t_k}|a_\xi(x_i,\xi-\tau)|^{\gamma}(\xi-t_{k-1})d\xi+\int_{t_{k-1}}^{t_k}|a_\xi(x_i,\xi)|^{\gamma}(t_k-\xi)d\xi\right]$$

$$=\sum_{k=1}^{n}\frac{1}{\tau}\left[\int_{t_{k-1}}^{t_k}|a_\xi(x_i,\xi)|^{\gamma}(t_k-t_{k-1})d\xi+\int_{t_{k-1}}^{t_k}\left(|a_\xi(x_i,\xi-\tau)|^{\gamma}-|a_\xi(x_i,\xi)|^{\gamma}\right)(\xi-t_{k-1})d\xi\right]$$

$$\le\int_0^T \underset{0\le x\le l}{esssup}\ |a_\xi(x,\xi)|^{\gamma}d\xi+\int_0^T\left||a_\xi(x_i,\xi-\tau)|^{\gamma}-|a_\xi(x_i,\xi)|^{\gamma}\right|d\xi \tag{3.129}$$

We are going to show that the second term on the RHS tends to zero as $\tau\to 0$

We have:

$$\int_0^T\left||a_\xi(x_i,\xi-\tau)|^{\gamma}-|a_\xi(x_i,\xi)|^{\gamma}\right|d\xi$$

$$=\int_0^T\left|\int_0^1\frac{d}{d\theta}\left(\theta|a_\xi(x_i,\xi-\tau)|+(1-\theta)|a_\xi(x_i,\xi)|\right)^{\gamma}d\theta\right|d\xi$$

$$=\int_0^T\gamma\int_0^1\left(\theta|a_\xi(x_i,\xi-\tau)|+(1-\theta)|a_\xi(x_i,\xi)|\right)^{\gamma-1}d\theta\cdot\left||a_\xi(x_i,\xi-\tau)|-|a_\xi(x_i,\xi)|\right|d\xi$$

$$
\begin{aligned}
&\le \gamma \int_0^T \Big(|a_t(x_i,t-\tau)| + |a_t(x_i,t)|\Big)^{\gamma-1} \cdot |a_t(x_i,t-\tau) - a_t(x_i,t)|dt \\
&\le \gamma \Big(\int_0^T |a_t(x_i,t-\tau) - a_t(x_i,t)|^\gamma dt\Big)^{\frac{1}{\gamma}} \Big(\int_0^T (|a_t(x_i,t-\tau)| + |a_t(x_i,t)|)^\gamma dt\Big)^{\frac{\gamma-1}{\gamma}} \\
&\le \gamma \Bigg[\Bigg(\int_0^T \underset{0\le x\le l}{esssup}\, |a_t(x,t-\tau)|^\gamma dt \Bigg)^{\frac{1}{\gamma}} + \Bigg(\int_0^T \underset{0\le x\le l}{esssup}\, |a_t(x,t)|^\gamma dt \Bigg)^{\frac{1}{\gamma}} \Bigg]^{\gamma-1} \\
&\qquad \times \Bigg(\int_0^T \underset{0\le x\le l}{esssup}\, |a_t(x,t-\tau) - a_t(x,t)|^\gamma dt \Bigg)^{\frac{1}{\gamma}} \\
&\le \gamma 2^{\gamma-1} \Bigg[\Bigg(\int_{-\tau}^T \underset{0\le x\le l}{esssup}\, |a_t(x,t)|^\gamma dt \Bigg)^{\frac{1}{\gamma}} \Bigg]^{\gamma-1} \cdot \Bigg(\int_0^T \underset{0\le x\le l}{esssup}\, |a_t(x,t-\tau) - a_t(x,t)|^\gamma dt \Bigg)^{\frac{1}{\gamma}} \\
&= \gamma 2^{\gamma-1} \|a_t\|^{\gamma-1}_{L_{\infty,\gamma}(D)} \|a_t(x,t-\tau) - a_t(x,t)\|_{L_{\infty,\gamma}(D)} = o(1), \quad \tau \downarrow 0 \qquad (3.130)
\end{aligned}
$$

$$
\begin{aligned}
|a_{ik,x}| &= \frac{\Big| \int_{t_{k-1}}^{t_k} a(x_{i+1},t)dt - \int_{t_{k-1}}^{t_k} a(x_i,t)dt \Big|}{\tau h_i} \\
&\le \frac{\int_{t_{k-1}}^{t_k} \int_{x_i}^{x_{i+1}} |a_\xi(\xi,t)| d\xi dt}{\tau h_i} \le \Big\| \frac{\partial a}{\partial x} \Big\|_{L_\infty(D)} \qquad (3.131)
\end{aligned}
$$

Thus,

$$
\|[a]_{nN}\|_{w^{1,1}_{\infty,\gamma}} \le \|a\|_{W^{1,1}_{\infty,\gamma}} + o(1) \qquad (3.132)
$$

From (3.106)-(3.132) it follows that

$$
\begin{aligned}
&\max(\|[s]_n\|_{b_2^2}; \|[a]_{nN}\|_{\tilde{w}_2^{1,1}}; \|[a]_{nN}\|_{w^{1,1}_{\infty,\gamma}}) \\
&\le \max(\|s\|_{B_2^2(0,T)}; \|a\|_{\tilde{W}_2^{1,1}(D)}; \|a\|_{W^{1,1}_{\infty,\gamma}(D)}) + O(\sqrt{\tau}) + o(1), \quad \text{as } \tau \downarrow 0 \qquad (3.133)
\end{aligned}
$$

From (3.159), (3.104) follows. Now let $[v]_n \in V_R^n$ and let $(s,a) = \mathcal{P}_n([v]_n)$

$$\int_0^l \int_0^T |a^n(x,t)|^2 dxdt$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} [a_{ik} + a_{ikx}(x-x_i) + a_{ik\bar{t}}(t-t_k) + a_{ikx\bar{t}}(x-x_i)(t-t_k)]^2 dtdx$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ik}^2 dtdx + \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ikx}^2 (x-x_i)^2 dtdx$$

$$+ \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ik\bar{t}}^2 (t-t_k)^2 dtdx + \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ikx\bar{t}}^2 (x-x_i)^2 (t-t_k)^2 dtdx$$

$$+2 \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ik} a_{ikx} (x-x_i) dtdx + 2 \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a_{ik} a_{ik\bar{t}} (t-t_k) dtdx$$

$$+2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ik}a_{ikx\bar{t}}(x-x_i)(t-t_k)dtdx$$

$$+2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ikx}(x-x_i)a_{ik\bar{t}}(t-t_k)dtdx$$

$$+2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ikx}a_{ikx\bar{t}}(x-x_i)^2(t-t_k)dtdx$$

$$+2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ik\bar{t}}a_{ikx\bar{t}}(x-x_i)(t-t_k)^2dtdx$$

$$=\sum_{i=1}^{10}K_i \tag{3.134}$$

$$K_1=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ik}^2dtdx=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ik}^2=\|[a]_n\|_{l^2}^2$$

$$K_2=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ikx}^2(x-x_i)^2dtdx=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{\tau h_i^3}{3}a_{ikx}^2\le\frac{\Delta^2}{3}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{\tau h_i}{3}a_{ikx}^2$$

$$\le\frac{\Delta^2}{3}R^2 \tag{3.135}$$

$$K_3=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ik\bar{t}}^2(t-t_k)^2dtdx\le\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{h_i\tau^3}{3}a_{ik\bar{t}}^2=\tau^2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{h_i\tau}{3}a_{ik\bar{t}}^2\le\tau^2R^2 \tag{3.136}$$

$$K_4=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}a_{ikx\bar{t}}^2(x-x_i)^2(t-t_k)^2dtdx\le\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{h_i^3\tau^3}{9}a_{ikx\bar{t}}^2\le\frac{\tau^2\Delta^2}{9}\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ikx\bar{t}}^2$$

$$\le\frac{\tau^2\Delta^2}{9}\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ikx\bar{t}}^2\le\frac{\tau^2\Delta^2R^2}{9} \quad (3.137)$$

$$K_5=2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}a_{ik}a_{ikx}(x-x_i)dtdx=\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i^2a_{ik}a_{ikx}$$

$$\le\Delta\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_ia_{ik}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_ia_{ikx}^2\Big)^{\frac{1}{2}}\le\Delta R^2 \quad (3.138)$$

$$K_6=2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}a_{ik}a_{ik\bar{t}}(t-t_k)dtdx\le\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau^2a_{ik}a_{ik\bar{t}}$$

$$\le\tau\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ik}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ik\bar{t}}^2\Big)^{\frac{1}{2}}\le\tau R^2 \quad (3.139)$$

$$K_7=2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}a_{ik}a_{ikx\bar{t}}(x-x_i)(t-t_k)dtdx\le\sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{\tau^2h_i^2}{2}a_{ik}a_{ikx\bar{t}}$$

$$\le\frac{\tau\Delta}{2}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_ia_{ik}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_ia_{ikx\bar{t}}^2\Big)^{\frac{1}{2}}\le\frac{\tau\Delta R^2}{2} \quad (3.140)$$

$$K_8 = 2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ikx}(x-x_i)a_{ik\bar{t}}(t-t_k)dtdx \le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{\tau^2 h_i^2}{2}a_{ikx}a_{ik\bar{t}}$$

$$\le \frac{\tau\Delta}{2}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ik\bar{t}}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2\Big)^{\frac{1}{2}} \le \frac{\tau\Delta R^2}{2} \quad (3.141)$$

$$K_9 = 2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ikx}a_{ikx\bar{t}}(x-x_i)^2(t-t_k)dtdx \le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{h_i^3\tau^2}{3}a_{ikx}a_{ikx\bar{t}}$$

$$\le \frac{\Delta^2\tau}{3}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ikx}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ikx\bar{t}}^2\Big)^{\frac{1}{2}} \le \frac{\Delta^2\tau R^2}{3} \quad (3.142)$$

$$K_{10} = 2\sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k} a_{ik\bar{t}}a_{ikx\bar{t}}(x-x_i)(t-t_k)^2dtdx \le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{h_i^2\tau^3}{3}a_{ik\bar{t}}a_{ikx\bar{t}}$$

$$\le \frac{\Delta\tau^2}{3}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ikx\bar{t}}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}h_i\tau a_{ik\bar{t}}^2\Big)^{\frac{1}{2}} \le \frac{\Delta\tau^2 R^2}{3} \quad (3.143)$$

Using CBS inequality and $h = O(\sqrt{\tau})$ it follows that:

$$\int_0^l\int_0^T |a^n(x,t)|^2dxdt \le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ik}^2 + O(\sqrt{\tau}) \quad (3.144)$$

$$\int_0^l\int_0^T |\frac{\partial a^n}{\partial x}(x,t)|^2dxdt = \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}[a_{ikx}+a_{ikx\bar{t}}(t-t_k)]^2dtdx$$

$$= \sum_{k=1}^{n}\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}}\int_{t_{k-1}}^{t_k}[a_{ikx}^2+a_{ikx\bar{t}}^2(t-t_k)^2+2a_{ikx}a_{ikx\bar{t}}(t-t_k)]dtdx$$

$$\le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2 + \sum_{k=1}^{n}\sum_{i=0}^{N-1}\frac{\tau^3 h_i}{3}a_{ikx\bar{t}}^2 + \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau^2 h_i a_{ikx}a_{ikx\bar{t}}$$

$$\le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2 + \frac{\tau^2}{3}\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx\bar{t}}^2 + \tau\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx\bar{t}}^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2\Big)^{\frac{1}{2}}$$

$$\le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2 + \frac{\tau^2R^2}{3} + \tau R^2$$

$$\le \sum_{k=1}^{n}\sum_{i=0}^{N-1}\tau h_i a_{ikx}^2 + O(\tau) \quad (3.145)$$

$$\int_0^l \int_0^T |\frac{\partial a^n}{\partial t}(x,t)|^2 dxdt = \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} [a_{ik\bar{t}} + a_{ikx\bar{t}}(x-x_i)]^2 dtdx$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} [a^2_{ik\bar{t}} + a^2_{ikx\bar{t}}(x-x_i)^2 + 2a_{ik\bar{t}}a_{ikx\bar{t}}(x-x_i)]dtdx$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ik\bar{t}} + \sum_{k=1}^{n} \sum_{i=0}^{N-1} \frac{\tau h_i^3}{3} a^2_{ikx\bar{t}} + \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i^2 a_{ik\bar{t}} a_{ikx\bar{t}}$$

$$\leq \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ik\bar{t}} + \frac{\Delta^2}{3} \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ikx\bar{t}} + \Delta \Big(\sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ikx\bar{t}} \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ik\bar{t}} \Big)^{\frac{1}{2}}$$

$$\leq \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ik\bar{t}} + \frac{\Delta^2 R}{3} + \Delta R^2$$

$$\leq \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ik\bar{t}} + O(\sqrt{\tau}) \qquad (3.146)$$

$$(3.147)$$

$$\int_0^l \int_0^T \Big|\frac{\partial^2 a^n}{\partial x \partial t}(x,t)\Big|^2 dxdt = \sum_{k=1}^{n} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \int_{t_{k-1}}^{t_k} a^2_{ikx\bar{t}}\, dtdx = \sum_{k=1}^{n} \sum_{i=0}^{N-1} \tau h_i a^2_{ikx\bar{t}} \qquad (3.148)$$

Thus we get $||a^n||^2_{\tilde{W}_2^{1,1}(D)} \leq ||[a]_n||^2_{\tilde{w}_2^{1,1}} + O(\sqrt{\tau})$ (3.149)

$$
\begin{aligned}
\|a^n(x,t)\|_{L_\infty(D)} &= \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} \max_{\substack{x_i\le x\le x_{i+1}\\ t_{k-1}\le t\le t_k}} \left| a_{ik} + a_{ikx}(x-x_i) + a_{ik\bar{t}}(t-t_k) + a_{ikx\bar{t}}(x-x_i)(t-t_k) \right| \\
&= \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} \max_{\substack{x_i\le x\le x_{i+1}\\ t_{k-1}\le t\le t_k}} \left[|a_{ik}| + |a_{ikx}|\cdot|x-x_i| + |a_{ik\bar{t}}|\cdot|t-t_k| + |a_{ikx\bar{t}}|\cdot|x-x_i|\cdot|t-t_k| \right] \\
&\le \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik}| + \Delta \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ikx}| \\
&\quad + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik\bar{t}}| \\
&\quad + \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} \max_{t_{k-1}\le t\le t_k} \left|\frac{a_{i+1k\bar{t}} - a_{ik\bar{t}}}{h_i}\right| \cdot h_i \cdot |t-t_k| \\
&\le \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik}| + \Delta \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ikx}| \\
&\quad + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik\bar{t}}| \\
&\quad + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{i+1k\bar{t}} - a_{ik\bar{t}}| \\
&\le \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik}| + \Delta \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ikx}| \\
&\quad + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik\bar{t}}| \\
&\quad + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{i+1k\bar{t}}| + \tau \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik\bar{t}}| \\
&\le \max_{\substack{0\le i\le N-1\\ 1\le k\le n}} |a_{ik}| + C_1\tau + C_2\sqrt{\tau} \\
&\le \|[a]_{nN}\|_{l_\infty} + C_1\tau + C_2\sqrt{\tau} \qquad (3.150)
\end{aligned}
$$

$$\|\frac{\partial a^n}{\partial x}(x,t)\|_{L_\infty(D)} = \max_{\substack{0\le i\le N-1\\1\le k\le n}} \max_{t_{k-1}\le t\le t_k} |a_{ik,x} + a_{ikx\bar{t}}(t-t_k)|$$

$$= \max_{\substack{0\le i\le N-1\\1\le k\le n}} \max_{t_{k-1}\le t\le t_k} |a_{ik,x} + \frac{a_{ik,x} - a_{ik-1,x}}{\tau}(t-t_k)|$$

$$\le \max_{\substack{0\le i\le N-1\\1\le k\le n}} |a_{ik,x}| \tag{3.151}$$

$$\left\|\frac{\partial a^n}{\partial t}(x,t)\right\|^\gamma_{L_{\infty,\gamma}(D)} = \int_0^T \operatorname*{esssup}_{0\le x\le l} \left|\frac{\partial a^n(x,t)}{\partial t}\right|^\gamma dt$$

$$= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \max_{0\le i\le N-1} \max_{x_i\le x\le x_{i+1}} |a_{ik\bar{t}} + a_{ikx\bar{t}}(x-x_i)|^\gamma \, dt$$

$$= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \max_{0\le i\le N-1} \max_{x_i\le x\le x_{i+1}} |a_{ik\bar{t}} + \frac{a_{i+1k\bar{t}} - a_{ik\bar{t}}}{h_i}(x-x_i)|^\gamma \, dt$$

$$= \sum_{k=1}^n \tau \max_{0\le i\le N-1} \max_{x_i\le x\le x_{i+1}} |a_{ik\bar{t}} + \frac{a_{i+1k\bar{t}} - a_{ik\bar{t}}}{h_i}(x-x_i)|^\gamma \, dt$$

$$\le \sum_{k=1}^n \tau \max_{0\le i\le N-1} \{|a_{i+1k\bar{t}}|^\gamma; |a_{ik\bar{t}}|^\gamma\}$$

$$\le \sum_{k=1}^n \tau \max_{0\le i\le N} |a_{ik\bar{t}}|^\gamma \tag{3.152}$$

Thus,

$$\|a^n\|_{\tilde{W}^{1,1}_{\infty,\gamma}(D)} \le \|[a]_{nN}\|_{\tilde{w}^{1,1}_{\infty,\gamma}} + O(\sqrt{\tau}) \tag{3.153}$$

Direct calculations lead to

$$\|s\|^2_{W^2_2[0,T]} \le \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n-1} \tau s^2_{\bar{t},k} + \sum_{k=0}^{n-1} \tau s^2_{\bar{t}t,k} + \frac{1}{3}\tau s^2_{\bar{t},1} + C\tau, \tag{3.154}$$

where C does not depend of τ. We use C for all constants which are independent of τ. By applying CBS we get

$$\frac{1}{3}\tau s_{\bar{t},1}^2 = \frac{4}{3\tau}(s(\tau) - s(0))^2 \leq \frac{4}{3}\int\limits_0^\tau |s'(t)|^2 dt \tag{3.155}$$

After applying Morrey inequality to $s'(t)$ in (3.155) we get the following estimate

$$\frac{1}{3}\tau s_{\bar{t},1}^2 \leq C\tau \|s\|_{W_2^2[0,T]}^2 \tag{3.156}$$

Since $[v]_n \in V_R^n$, from (3.154),(3.156) it follows that for all $\tau \leq (2C)^{-1}$

$$\|s\|_{W_2^2[0,T]}^2 \leq C, \tag{3.157}$$

Thus from (3.154),(3.156),(3.157) it follows that for sufficiently small τ

$$\|s\|_{W_2^2[0,T]}^2 \leq \sum_{k=0}^{n-1} \tau s_k^2 + \sum_{k=1}^{n-1} \tau s_{\bar{t},k}^2 + \sum_{k=0}^{n-1} \tau s_{\bar{t}t,k}^2 + C\tau \tag{3.158}$$

From (3.134) – (3.158) we conclude that for sufficiently small τ

$$\begin{aligned} &\max(\|s\|_{B_2^2(0,T)}; \|a\|_{\tilde{W}_2^{1,1}(D)}; \|a\|_{W_{\infty,\gamma}^{1,1}(D)}) \\ &\leq \max(\|[s]_n\|_{b_2^2}; \|[a]_{nN}\|_{\tilde{w}_2^{1,1}}; \|[a]_{nN}\|_{w_{\infty,\gamma}^{1,1}}) + O(\sqrt{\tau}) + o(1) \end{aligned} \tag{3.159}$$

From (3.133) and (3.159) lemma follows. □

Lemma 3.6.9. *Let* $\mathcal{J}_*(\pm\epsilon) = \inf_{V_{R\pm\epsilon}} \mathcal{J}(v)$, $\epsilon > 0$. *Then*

$$\lim_{\epsilon\to 0} \mathcal{J}_*(\epsilon) = \mathcal{J}_* = \lim_{\epsilon\to 0} \mathcal{J}_*(-\epsilon) \tag{3.160}$$

The proof of Lemma 3.6.9 is the same as the proof of lemma 3.9 from [1].

Lemma 3.6.10. *For arbitrary $v = (s, g) \in V_R$,*

$$\lim_{n\to\infty} \mathcal{I}_n(Q_n(v)) = \mathcal{J}(v) \tag{3.161}$$

Proof: Let $v \in V_R$, $u = u(x,t;v)$, $Q_n(v) = [v]_n$ and $[u([v]_n)]_n$ be a corresponding discrete state vector. In Theorem 3.6.6 it is proved that the sequence $\{\hat{u}^\tau\}$ converges to u weakly in $W_2^{1,1}(\Omega_m)$ for any fixed m. Therefore the sequences of traces $\{\hat{u}^\tau(0,t)\}$ and $\{\hat{u}^\tau(s(t)-\epsilon_m, t)\}$ converge strongly in $L_2[0,T]$ to corresponding traces $u(0,t)$ and $u(s(t)-\epsilon_m,t)$. We will show that the sequences of traces $\{u^\tau(0,t)\}$ and $\{u^\tau(s(t)-\epsilon_m,t)\}$ converge strongly in $L^2[0,T]$ to traces $u(0,t)$ and $u(s(t)-\epsilon_m,t)$ respectively. Using Sobolev embedding theorem ([25, 65]) it suffices to show that the sequences $\{u^\tau\}$ and $\{\hat{u}^\tau\}$ are equivalent in strong topology of $W_2^{1,0}(\Omega_m)$. In Theorem 3.6.6 it is demonstrated that they are equivalent in strong topology of $L_2(\Omega_m)$, therefore we only need to prove that the sequences of derivatives $\frac{\partial u^\tau}{\partial x}$ and $\frac{\partial \hat{u}^\tau}{\partial x}$ are equivalent in strong topology of $L_2(\Omega_m)$. Using the proof of the Theorem 3.6.6, from the second energy estimate (3.74) we conclude that for all $n > N(m)$

$$\left\| \frac{\partial u^\tau}{\partial x} - \frac{\partial \hat{u}^\tau}{\partial x} \right\|^2_{L_2(\Omega_m)} \le \frac{1}{3} \sum_{k=1}^{n} \tau^3 \sum_{i=0}^{m_j-1} h_i \tilde{u}^2_{ix\bar{t}}(k) = O(\tau), \quad \text{as } \tau \to 0. \tag{3.162}$$

Let $\tilde{\omega}^\tau(x) = \omega_i,\ \mu^\tau(t) = \mu^k$, if $t_{k-1} < t \le t_k,\ k = 1,\dots,n,\ x_i \le x < x_{i+1}, i = \overline{0, m-1}$.
We have

$$\|\mu^\tau - \mu\|_{L^2[0,T]} \to 0 \text{ as } \tau \to 0 \tag{3.163}$$

Estimation of the first term in $\mathcal{I}_n(Q_n(v))$ derived as follows:

We have

$$\|\tilde{\omega}^\tau(x) - \omega(x)\|_{L_2(0,l)} \to 0, \quad as \quad n \to 0 \tag{3.164}$$

$$\sum_{i=1}^{m} h(u_i(n)-\omega_i)^2 = \sum_{i=1}^{m}\int_{x_{i-1}}^{x_i} |u_i(n)-\omega_i|^2 dx = \int_0^l |\tilde{u}^\tau(x,T)-\tilde{\omega}^\tau(x)|^2 \tag{3.165}$$

$$= \|\tilde{u}^\tau(x,T)-\tilde{\omega}^\tau(x)\|_{L^2(0,l)} \tag{3.166}$$

Now we are going to use backward triangle inequality

$$\begin{aligned}
&\Big|\|\tilde{u}^\tau(x,T)-\tilde{\omega}^\tau(x)\|_{L^2(0,l)} - \|u(x,T)-\omega(x)\|_{L^2(0,l)}\Big| \\
&\le \|\tilde{u}^\tau(x,T)-\tilde{\omega}^\tau(x)-u(x,T)+\omega(x)\|_{L^2(0,l)} \\
&\le \|\tilde{u}^\tau(x,T)-u(x,T)\|_{L^2(0,l)} + \|\tilde{\omega}^\tau(x)-\omega(x)\|_{L^2(0,l)} \\
&= \|\tilde{u}^\tau(x,T)-u^\tau(x,T)+u^\tau(x,T)-u(x,T)\|_{L^2(0,l)} + \|\tilde{\omega}^\tau(x)-\omega(x)\|_{L^2(0,l)} \\
&\le \|\tilde{u}^\tau(x,T)-u^\tau(x,T)\|_{L^2(0,l)} + \|u^\tau(x,T)-u(x,T)\|_{L^2(0,l)} + \|\tilde{\omega}^\tau(x)-\omega(x)\|_{L^2(0,l)}
\end{aligned} \tag{3.167}$$

Here 2nd term on the right hand side goes to zero as $\tau \to 0$ by the similar argument stated in the [1].

To prove that the first term goes to zero, lets consider the following:

$$\begin{aligned}
\|\tilde{u}^\tau(x,T)-u^\tau(x,T)\|^2_{L^2(0,l)} &= \sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}} \Big[u_i(n)-\hat{u}(x,n)\Big]^2 dx \\
= \sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}} \Big[\frac{u_{i+1}(n)-u_i(n)}{h}(x-x_i)\Big]^2 dx &= \frac{h}{3}\sum_{i=0}^{m-1}[u_{i+1}(n)-u_i(n)]^2 \\
= \frac{h^3}{3}\sum_{i=0}^{m-1}\frac{[u_{i+1}(n)-u_i(n)]^2}{h^2} &= \frac{h^3}{3}\sum_{i=0}^{m-1} u_{ix}^2(n)
\end{aligned} \tag{3.168}$$

Here is the estimation for the second term in $\mathcal{I}_n(Q_n(v))$

$$\beta_1\tau\sum_{k=1}^{n}|u_{m_k}(k)-\mu^k|^2 = 2\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\left(u^\tau(s(t),t)-\mu^\tau(t)\right)dx\,dt$$

$$+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}|u(s(t);k)-\mu^k|^2\,dt+\beta_1\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\int_{s(t)}^{s_k}\frac{\partial u^\tau}{\partial x}\,dx\right)^2 dt = I_1+I_2+I_3 \qquad (3.169)$$

Since $\left\|\frac{\partial u^\tau}{\partial x}\right\|_{L_2(D)}$ and $\|u^\tau(s(t),t)-\mu^\tau\|_{L_2[0,T]}$ are uniformly bounded, $\{s^n\}\to s$ uniformly on $[0,T]$, by using CBS inequality and (3.99) we obtain

$$\lim_{n\to\infty} I_1 = 0,\ \lim_{n\to\infty} I_3 = 0 \qquad (3.170)$$

Lastly we need to show that

$$\lim_{n\to\infty} I_2 = \lim_{n\to\infty}\beta_1\int_0^T |u^\tau(s(t),t)-\mu^\tau(t)|^2\,dt = \beta_1\int_0^T |u(s(t),t)-\mu(t)|^2\,dt \qquad (3.171)$$

Due to strong convergence of μ^τ to μ in $L_2[0,T]$ it suffices to demonstrate

$$\|u^\tau(s(t),t)-u(s(t),t)\|_{L_2[0,T]} \to 0 \qquad (3.172)$$

as $\tau\to 0$. For any fixed $m>0$,

$$\|u^\tau(s(t),t)-u(s(t),t)\|_{L_2[0,T]} \le \|u^\tau(s(t),t)-u^\tau(s(t)-\epsilon_m,t)\|_{L_2[0,T]}+$$

$$+\|u^\tau(s(t)-\epsilon_m,t)-\hat{u}^\tau(s(t)-\epsilon_m,t)\|_{L_2[0,T]}+$$

$$+\|\hat{u}^\tau(s(t)-\epsilon_m,t)-u(s(t)-\epsilon_m,t)\|_{L_2[0,T]}+\|u(s(t)-\epsilon_m,t)-u(s(t),t)\|_{L_2[0,T]} \qquad (3.173)$$

Next we estimate the first term on the RHS of (3.173) as

$$\|u^\tau(s(t),t)-u^\tau(s(t)-\epsilon_m,t)\|_{L_2[0,T]} = \left(\int_0^T\left|\int_{s(t)-\epsilon_m}^{s(t)}\frac{\partial u^\tau(x,t)}{\partial x}\,dx\right|^2 dt\right)^{1/2} \qquad (3.174)$$

Using CBS inequality and the first energy estimate

$$\|u^\tau(s(t),t) - u^\tau(s(t)-\epsilon_m,t)\|_{L_2[0,T]} \leq \sqrt{\epsilon_m}\left\|\frac{\partial u^\tau}{\partial x}\right\|_{L_2(D)} \leq C\sqrt{\epsilon_m} \tag{3.175}$$

where C does not depend on n. In a similar manner we estimate the last term in (3.173) by employing CBS and energy estimate (3.46):

$$\|u(s(t)-\epsilon_m,t) - u(s(t),t)\|_{L_2[0,T]} \leq C\sqrt{\epsilon_m} \tag{3.176}$$

Fix $\epsilon > 0$ and find M such that for all $m \geq M$, $C\sqrt{\epsilon_m} \leq \epsilon/4$. Taking $m = M$, it follows from (3.173)–(3.176)

$$\begin{aligned} \|u^\tau(s(t),t) - u(s(t),t)\|_{L_2[0,T]} &\leq \frac{\epsilon}{2} + \|u^\tau(s(t)-\epsilon_M,t) - \hat{u}^\tau(s(t)-\epsilon_M,t)\|_{L_2[0,T]} \\ &+ \|\hat{u}^\tau(s(t)-\epsilon_M,t) - u(s(t)-\epsilon_M,t)\|_{L_2[0,T]} \end{aligned} \tag{3.177}$$

We estimate the second term in (3.177) by using Sobolev embedding of traces

$$\|u^\tau(s(t)-\epsilon_M,t) - \hat{u}^\tau(s(t)-\epsilon_M,t)\|_{L_2[0,T]} \leq C\|\hat{u}^\tau - u^\tau\|_{W_2^{1,0}(\Omega_M)} \tag{3.178}$$

By (3.102), (3.162) there exists $\tau_0(M) > 0$ such that $\forall \tau < \tau_0$

$$\|u^\tau(s(t)-\epsilon_M,t) - \hat{u}^\tau(s(t)-\epsilon_M,t)\|_{L_2[0,T]} \leq \frac{\epsilon}{4} \tag{3.179}$$

Using well-known compact embedding theorem weak convergence of $\hat{u}^\tau \to u$ in $W_2^{1,1}(\Omega_M)$ implies strong convergence of traces $\hat{u}^\tau|_{x=s(t)-\epsilon_M}$ to $u|_{x=s(t)-\epsilon_M}$ in $L_2[0,T]$ [58]; i.e there exists $\tau_1(M)$ such that for all $\tau < \tau_1$ we get

$$\|\hat{u}^\tau(s(t)-\epsilon_M,t) - u(s(t)-\epsilon_M,t)\|_{L_2[0,T]} < \frac{\epsilon}{4} \tag{3.180}$$

Therefore by (3.177), (3.179), (3.180), for arbitrary $\epsilon > 0$ we can choose $\tau_2 = \min(\tau_0;\tau_1)$ such that for all $\tau < \tau_2$ we get

$$\|u^\tau(s(t),t) - u(s(t),t)\|_{L_2[0,T]} < \epsilon. \tag{3.181}$$

This proves (3.171) and completes the proof of the Lemma.

Lemma 3.6.11. *For arbitrary* $[v]_n \in V_R^n$

$$\lim_{n\to\infty} \Big(\mathcal{J}(\mathcal{P}_n([v]_n)) - \mathcal{I}_n([v]_n)\Big) = 0 \tag{3.182}$$

Proof: Let $[v]_n \in V_R^n$ and $v^n = (s^n, a^n) = \mathcal{P}_n([v]_n)$. Lemma 3.6.8 implies that the sequence $\{\mathcal{P}_n([v]_n\}$ is weakly precompact in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$. Next, assume that the whole sequence converges to $\tilde{v} = (\tilde{s}, \tilde{a})$ weakly in $W_2^2[0,T] \times \tilde{W}_2^{1,1}(D)$. This implies the strong convegence in $W_2^1[0,T] \times L_2(D)$. Using the property of weak convergence we deduce that $\tilde{v} \in V_R$. Furthermore $s^n \to \tilde{s}$ uniformly on $[0,T]$ and we obtain

$$\lim_{n\to\infty} \max_{0\le i\le n} \left| s^n(t_i) - \tilde{s}(t_i) \right| = 0 \tag{3.183}$$

Consider the following equality

$$\mathcal{I}_n([v]_n) - \mathcal{J}(v^n) = \mathcal{I}_n([v]_n) - \mathcal{J}(\tilde{v}) + \mathcal{J}(\tilde{v}) - \mathcal{J}(v^n) \tag{3.184}$$

Due to weak continuity of $\mathcal{J}(v)$ in $W_2^2[0,T] \times W_2^1[0,T]$ we get

$$\lim_{n\to\infty} (\mathcal{J}(\tilde{v}) - \mathcal{J}(v^n)) = 0.$$

Thus, it remains to prove

$$\lim_{n\to\infty} \mathcal{I}_n([v]_n) = \mathcal{J}(\tilde{v}) \tag{3.185}$$

But the proof of (3.185) is almost the same as the proof of Lemma 3.6.10. This completes the proof of the Lemma.

Having Lemmas 3.6.9, 3.6.10 and 3.6.11, Theorem 3.4.2 follows from Lemma 3.5.2.

Chapter 4

Conclusions

4.1 Conclusions

Dissertation analyzes inverse Stefan type free boundary problem for the second order parabolic PDE with unknown parameters based on the additional information given in the form of the distribution of the solution of the PDE and the position of the free boundary at the final moment. This type of ill-posed inverse free boundary problems arise in many applications such as biomedical engineering problem about the laser ablation of biomedical tissues, in-flight ice accretion modeling in aerospace industry, and various phase transition processes in thermophysics and fluid mechanics. The set of unknown parameters include a space-time dependent diffusion, convection and reaction coefficients, density of the sources, time-dependent boundary flux and the free boundary. New PDE constrained optimal control framework in Hilbert-Besov spaces introduced in *U.G. Abdulla, Inverse Problems and Imaging, 7, 2(2013), 307-340; 10, 4(2016), 869-898* is employed, where the missing data and the free boundary are components of the control vector, and optimality criteria are based on the final moment measurement of the temperature and position of the free boundary, and available information on the phase transition temperature on the free boundary. The latter presents a key advantage in dealing with applications, where phase transition temperature is not known explicitly, but involve some measurement error. Another advantage of the new variational approach is based on the fact that for a given control parameter, Stefan boundary condition turns into Neumann boundary condition on the given boundary, and parabolic PDE problem is solved in a fixed domain, and therefore a perspective opens for the development of numerical methods of least computational cost.

In Chapter 2 the general Inverse Stefan Problem with unknown parameters such as time-dependent diffusion coefficient, space-time dependent convection coefficient, reaction coefficient and density of sources, boundary heat flux and a free boundary is analyzed. Optimal control problem for the free boundary system with distributed parameters for the second order parabolic equation in Hilbert-Besov space is introduced, where unknown parameters and the free boundary are components of the control vector, and the state vector is the weak solution of the parabolic Neumann problem in Sobolev-Hilbert space. Optimality criteria are based on the final moment measurement of the temperature and the position of the free boundary, and the temperature on the phase transition boundary.

- Existence of the optimal control is proved. The methods of proof are based on energy estimates in Sobolev-Hilbert spaces, weak continuity of the cost functional and Weierstrass theorem in weak topology of the Hilbert-Besov spaces.

- Method of finite differences is implemented and space-time discretization of the optimal control problem is introduced. Convergence of the sequence of the finite-dimensional discrete optimal control problems to the original optimal control problem both with respect to functional and control is proved. Namely,

  – it is proved that the sequence of infima of the discrete optimal control problems converge to the infimum of the original optimal control problem,

  – It is proved that sequence of interpolations of the discrete optimal controls converge to the optimal control in a weak topology of Hilbert-Besov space, and the sequence of multi-linear interpolations of the discrete PDE problems associated with discrete minimizers converge weakly in the class of weakly differentiable functions to the solution of the PDE problem associated with optimal control. The methods of the proof are based on establishing two energy estimates in discrete Sobolev-Hilbert spaces, use of weak compactness criteria, and delicate interpolation results in Sobolev spaces.

Chapter 3 analyzes the Inverse Stefan Problem with unknown space-time dependent diffusion coefficient. Dissertation introduces a new Banach space, and formulates an inverse problem as a parabolic PDE constrained optimal control problem in a new Banach space with control parameters being space-time dependent diffusion coefficient and a free boundary. The motivation for the new space is dictated with the optimal result on the convergence of the bilinear interpolations of the grid functions in the class of weakly differentiable functions, and establishment of the discrete H^1-energy estimate under minimal assumptions on the diffusion coefficient.The following are the main results of Chapter 3:

- Finite difference discretization of the optimal control problem is carried out and sequence of finite-dimensional optimal control problems is introduced. Convergence of the sequence of discrete optimal control problems to continuous optimal control problem both with respect to functional and control is proved.

- Convergence of the sequence of multi-linear interpolations of the minimizing discrete optimal control parameters to optimal diffusion coefficient in a weak topology of the new space is proved. Convergence of the multi-linear interpolations of the associated discrete PDE problems to the optimal state PDE problem in a weak topology of the space of weakly differentiable functions is established.

- H^1-energy estimates are proved for the solutions of the discrete and continuous PDE problems under the minimal assumption on the diffusion coefficient. Primarily by applying energy estimate, and new interpolation results, existence of the optimal control is proved.

4.2 Publications and Conference Presentations

The results of the Chapter 2 of the dissertation is published in the following paper:

- U. G. Abdulla, J. Goldfarb, A. Hagverdiyev, Optimal Control of Coefficients in Parabolic Free Boundary Problems Modeling laser Ablation, *Journal of Computational and Applied Mathematics, Volume 372, July 2020, 112736*

The research paper on the results of Chapter 3 are in process of submission.

The results of the dissertation are presented in the following conferences:

- U.G. Abdulla, J. Goldfarb, **A. Hagverdiyev,** Optimal Control of Coefficients in Parabolic Free Boundary Problems Modeling Laser Ablation, Joint Mathematics Meetings (JMM), Denver, January 15 - 18, 2020

- U.G. Abdulla, J. Goldfarb, **A. Hagverdiyev**, Optimal Control of Coefficients in Parabolic Free Boundary Problems Modeling Laser Ablation, AMS Fall Southeastern Meeting, University of Florida, Gainesville, November 2, 2019.

- U.G. Abdulla, J. Goldfarb, **A. Hagverdiyev**, Optimal Control of Coefficients in Parabolic Free Boundary Problems Modeling Laser Ablation, 39th Southeastern-Atlantic Regional Conference on Differential Equations (SEARCDE), Sat, Oct 26, 2019 – Sun, Oct 27, 2019. Embry-Riddle Aeronautical University, Daytona Beach, Florida.

4.3 Future Research

The results of the dissertation motivate the development of the implemented methods to different open problems in the field. For example, it would be interesting to analyze optimal control of free boundary problem for the nonlinear non-homogeneous reaction-diffusion-convection equation of the type

$$u_t = \left(a(x)u^m\right)_{xx} + b(x)\left(u^\gamma\right)_x + c(x)u^\beta = 0,$$

by exploiting mathematical theory in non-cylindrical domains ([10, 11, 12, 13]), and properties of the interfaces of nonlinear degenerate parabolic PDEs [14, 15].

Another important class of problems are optimal control or free boundary problems for the elliptic and parabolic PDEs in domains with non-compact boundaries, and in particular with non-compact free boundaries. Generalization of the methods of [1, 2] to this class of problems requires delicate use potential-theoretic results on the well-posedness of the elliptic and parabolic boundary value problems in domains with non-compact boundaries [16, 17, 18, 19, 20].

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