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# Discrete and Continuous Operational Calculus in Stochastic Games

Kenneth Ibe Iwezulu

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# Discrete and Continuous Operational Calculus in Stochastic Games

by

Kenneth Ibe Iwezulu

A dissertation submitted to the College of Science at
Florida Institute of Technology
in partial fulfillment of the requirements
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in
Operations Research

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We the undersigned committee,
having examined the attached dissertation,
"Discrete and Continuous Operational Calculus in Stochastic Games"
by Kenneth Ibe Iwezulu.

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J.H Dshalalow, Ph.D. Major Advisor
Professor, Mathematical Sciences

---

Nezamoddin Nezamoddini-Kachouie, Ph.D.
Assistant Professor, Mathematical Sciences.

---

B. Webster, Ph.D.
Assistant Professor, Engineering Systems

---

M.C Silaghi, Ph.D.
Associate Professor, Computer Sciences and Cybersecurity

---

U.G Abdulla, Ph.D Habil.
Professor and Head, Mathematical Sciences

# Abstract


Title: Discrete and Continuous Operational Calculus in Stochastic Games.

Author: Kenneth Ibe Iwezulu

Advisor: J.H. Dshalalow, Ph.D.

First, we consider a class of antagonistic stochastic games between two players A and B. The game is specified in terms of two "hostile" stochastic processes representing mutual attacks upon random times exerting casualties of random magnitudes. This game is observed upon random epochs of time and the outcome of the game is not known in real time. The game ends at the time when the underlying fixed threshold of either player is crossed (referred to as the first passage time). The first passage time is then shifted to an epoch, i.e. upon one of the observation instants of time. Thus, the narrative of the game is delayed allowing the players to continue fighting each other beyond their assumed merits of endurance. We target the first passage time of the defeat and the amount of casualties to either player upon the end of the game. Here we validate our claim of analytic tractability of the general formulas obtained in [1] under various transforms.

We also consider a class of antagonistic stochastic games in real time between two players A and B formalized by two marked point processes. The players attack each other at random times with random impacts. Either player can sustain casualties up to a fixed threshold. A player is defeated when its underlying threshold is crossed. Upon that time (referred to as the first passage time), the game is over. We introduce a joint functional of the first passage, along with the status of each player upon this time, meaning the cumulative magnitude of casualties to each player upon the end of the game, obtained in an analytically tractable form. We then use discrete and continuous operational calculus for the transform inversion. We demonstrate that in a special case that the discrete operational calculus is more efficient allowing us to avoid numerical inversion. It leads to totally explicit formulas for the joint distribution of associated random variables (first passage time and the status of cumulative casualties to the players upon the end of the game).

# Contents

# List of Keywords.

Exit time.

First passage time.

Fluctuation theory.

Marked point processes.

Modified Bessel functions.

Noncooperative stochastic games.

Poisson process.

Ruin time.

# List of Abbreviations.

**a.s** Almost surely (i.e with probability of 1).

**PGF** Probability generating function.

**LST** Laplace-Stieltjes transforms.

**R.V** Random variables.

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Finally, I thank Dr. Christian Happi, who initiated the idea of me starting a Ph.D., you will be forever remembered.

# Dedication

I dedicate this dissertation to my wife and our nine (9) month old son.

## Chapter 1

# Introduction.

We shall consider an antagonistic stochastic game where two players A and B attack each other at some random times with strikes of random magnitude say $X$ and $Y$ respectively while the game is being observed at random times (or not observed if the attacks are simultaneous, as in the case of a real time game). At some point when either player A or B can no longer stand any more strikes because they have attained some predefined threshold say $M$ and $N$ respectively. The threshold represent the tolerance/endurance level of the players and at these thresholds, either player is said to have been defeated, overpowered or ruined by the other. At a time when it takes place (called the *first passage time*), i.e. when one of the players loses the game, the game should formally stop. The game which we describe above can be specified as a random walk process.

In Chapter 3, we shall discuss three delayed-time games where in one case the attacks of either players is distributed geometrically, the second follows an ordinary Poisson process. The third case we consider is a mixed game. In all three cases, we determine the first passage time upon one of the observation epochs and find the collateral damage to both players and the damage to either player. In Chapter 4 we consider real time games (Discrete and Continuous versions), where the attacks are simultaneous and independent of each other and proceed to obtain the first passage time. In Chapter 5, we look at a slightly different game where we have five components (as against three in Chapters 3 and 4), two of which are active and three are passive. Unlike the games in the Chapter 3, the casualties are non-monotone. The non-monotonicity imply that upon a strike by either one player, there is a chance that he is likely to sustain some damages while inflicting damages to the other player. This will lead to fluctuation towards or about his threshold till it crosses it. To this end we shall employ the key Fluctuation Theorem to define such functional (since our game exhibit the stationary increment and independent increment properties which are necessary for fluctuation analysis) and utilize the discrete and continuous operational calculus for the transform inversion of the joint functional of the first passage time.

## 1.1 Motivation and Application.

The class of antagonistic games which we study occur almost in every sphere of life. The following are some examples of games pertinent to our model which will be described and examined in Chapter 3.

**Cancer Treatment**. Some cancers are curable while others are not. Most metastatic cancer (which spreads from a primary site to other parts of the body over lymph nodes and blood vessels) are incurable but can be managed to some extent using radiation alone or with other forms of treatment like chemotherapy. In relation to the antagonistic games, an oncologist, along with his/her treatment, can be regarded as player A while the tumor - as player B. The oncologist attacks the tumor cells with radiation and/or chemotherapy. While the tumor can shrink under the treatment, it may also continue spreading to other parts of the body (metastasize). Notice that any treatment by itself always has side effects (such as weakening immune system) that can be regarded as a collateral damage. At some point, when the cancer continues to spread and thus the body does not respond to the treatment, unless there are alternative options, player A is defeated. On the other hand, if the body responds well to the treatment and the tumor vanishes (the state of remission), we declare that player B is defeated. In a more modest form of a defeat, the tumor can shrink or significantly shrink instead of disappearing entirely.

Note that cancer cells like bacteria cells typically divide in two progeny and they initially evolve as a deterministic branching process. However, some cancer cells are eliminated by T-killer cells, and at some point, when cancer matures, it evolves not from a single but many cells. If we also take into consideration mutations exhibiting an increase of the number of chromosomes (beginning in 46 to 64 and further), on an early stage, the general tumor development becomes rather chaotic allowing us to model it by an independent and stationary increment process.

In Chapter 3, we work on entirely discrete operational calculus in which case, the “nature” of attacks goes to integer-valued increments, hence, we would like to emphasize why some applications can contain entirely discrete components (or at worst they can be approximated by units made arbitrarily small).

**Global Military Warfare**. This is a situation where a country or group of countries are at war with one another under military operations. One classical example is the war between the United States and Japan during the WWII which consisted of multiple phases [22]. Phase 1 began with economic sanctions imposed on Japan by the US in 1940-1941 due to Japan's aggression in Manchuria. Japan tried to negotiate with the US (apparently until November 26 of 1941), but the concessions offered by the Japanese were not satisfactory to the US, and Japan not wishing to give in had no other choice as to strike on December 7, 1941. This corresponds to the beginning of Phase 2. Undoubtedly, Japan was not ruined economically, but it was significantly crippled (being deprived of steel and oil, to name a few). At the same time Japan did not want to stop her campaign in China, which the US chose not to tolerate, also fearing Japan's further expansion. The Japanese Pearl Harbor attack followed by their Pacific campaign is yet another intermediate phase prior to a full scale war, because Japan believed the US will be deterred from further actions under the inflicted casualties and loss of territories in the Pacific.

**Global Economic Warfare**. A recent economic confrontation between the US/Europe and Russia is an antagonistic game. Here player A will be the US/Europe while player B is Russia. US and Europe stroke Russia with numerous sanctions in an attempt to weaken its economy and to drive Russia out of Ukraine, while Russia reciprocated with their own sanctions (such as forbidding US and Canada officials from entering Russia and adopting a ban on fruit, vegetables, fish, meat, and dairy products from the US and Europe) to counter such attacks.

**Corporate Economic Hostilities**. Here we refer to a hostile relationship between two or more corporations which have a similar goal or offer similar services. In particular, we consider ride sharing companies (examples include *Uber* and *Lyft*) and taxi cabs (such as the *Yellow Cab* which is a sole licensed taxi cab company in Long Beach city). *Uber* in recent times have had to reduce their fares for their riders and this has brought about a drift of riders from *Yellow Cab* to *Uber* while they also make use of recent and flashy cars to attract its riders and make them feel more comfortable compared to *Yellow Cab*. This move by *Uber* is some form of attack on *Yellow Cab* which seems to be working as *Uber* gains more riders defecting from *Yellow Cab* riders. In turn, *Yellow Cab* attacks *Uber* now by calling on the authorities to make ride sharing companies face the same regulatory burdens as they do. While at the same time they are working with city councils to remove taxi's fare floor, discount fares as condition warrants, provide an ordering applications as well as getting a new branding identity.

The following example is pertinent to the game which we consider in Chapter 4.

**Politics (Elections).** In the recent race of two contestants, Hillary Clinton and Donald Trump, either side exercised sharp attacks against each other during the three presidential debates and outside the debates, such as twitters, media, public speeches, and rallies to name a few. Notice that many exchanges were almost simultaneous, especially when one candidate interrupted the other. We tend to believe that attacks were rehearsed and then embedded at a right time. For this reason, they have been essentially independent. Most responses were instigated by questions posed by moderators. That is the closest scenario to a real-time game. Of course, opinions on who won the debates were partisan and subjective, but they were calibrated by the polls.

As a consequence of the election we can relate the stock market with the delayed game of Chapter 3.

**Stock Market**. We all noticed how the course elections affected the stock market that welcomed Hillary Clinton and acted in fear about Donald Trump. Reactions on either candidate moving up and down were almost simultaneous.

## 1.2 Related Literature

The literature on game-theoretic models is very rich [4,7-10,16,22,24-25, 28-29,33-34,37,39-40,43,45]. A majority of work includes cooperative games embellished by the renowned Nash equilibrium. However, many relate to antagonistic games [13-15,42,44], in the book by Hajek [30]and in Leitman's book [38] stemmed from World War II models and then economics. War games are still of interest [8,22,28,33,43,45] as well as games modeling economic corporate competitions [9,10,24,29,34,37] and biology [40]. Some other game-theoretic techniques are integrated in stochastic methods used in queueing [4,7] and cancer research [24].

Techniques used in this dissertation pertain to the theory of fluctuations of stochastic processes, namely the behavior of underlying processes about specified thresholds or upon exits from specified sets at some time referred to as the first passage time [3] and in the book by Redner [41]. Although never implied as games, papers [17,18] appearing in 1994 and 1997, respectively, laid a foundation for antagonistic games of two players through closed-form functionals of multivariate marked processes further improved and generalized in [1] of 2005. However, not until 2016 [24] did we begin to explore the results in their fullness and relate them to various real-world situations occurring in economics and medical sciences.

Fluctuations of stochastic processes stem from random walk of an erratically moving particle along a multi-dimensional lattice in a set which the particle tries to escape. In the later years, some major research on fluctuations was focused on renewal, recurrent, and semi-Markov processes as per a seminal paper by Takács [46] and others such as Kadankov and Kadankova [35] and Kadankova [36] applied to stationary and independent increment processes as well as a compound Poisson process, respectively. Other fluctuations are found in Agarwal, Dshalalow, and O'Regan in the context of Cox processes and in the papers [5,6,11,12,31,32].

Of those application-oriented models, two major types can be singled out. In one case, there are real-time games, in which the information on an underlying game is probabilistically available upon attacks without delays. We impose this condition in our model in Chapter 4. The other type of the games we considered in the past articles [22,25,28] was with games in which the status of the game and the players was only available upon a sequence of random epochs of time referred to as observation epochs. While this approach to modeling seems to be more realistic, it would not apply to all models. Besides, the resulting expressions are more bulky in the latter case.

# Chapter 2

# Formalism and Basic Model of the Game.

In this chapter, we shall formulate a basic model for two different games. The first we shall consider will be a game that is observed upon random epochs of time say $\mathcal{T} = \{\tau_1, \tau_2, \dots\}$ and the outcome of the game is not known in real time. In the second game, we consider real-time games, in which the information on an underlying game is probabilistically available upon attacks without delays.

## 2.1 Formalism of the Delayed Game

We model purely antagonistic stochastic games of two players, A and B, who periodically attack each other according to two independent marked random measures

$$\mathcal{A} : = \sum_{j\geq 1} w_j \varepsilon_{s_j}, \text{ and } \mathcal{B} : = \sum_{k\geq 1} z_k \varepsilon_{t_k}, \ s_1, \ t_1 > 0. \tag{2.1.1}$$

The game evolves as a mutual conflict involving two players A and B hitting each other at random until one of the players is "exhausted." In short, the players attack each other in accordance with two independent marked point processes $\mathcal{A}$ and $\mathcal{B}$ of (2.1.1) on a probability space $(\Omega, \mathcal{F}, P)$, where $\varepsilon_a$ is the Dirac point mass at point $a \in \mathbb{R}$, $\sum_{j \geq 1} \varepsilon_{s_j}$, and $\sum_{k \geq 1} \varepsilon_{t_k}$ are underlying point random measures representing the times of attacks, and the marks $w_j$'s and $z_k$'s (nonnegative random variables) represent respective damages to players A and B. Players A and B can sustain the attacks until their respective cumulative casualties cross thresholds $M$ and $N$ (positive real numbers). At a time when it takes place (called the *first passage time*), i.e. when one of the players loses the game, the game should formally stop.

In this model, we assume that mutual attacks of the players are rendered at different times. The status of the game is being updated upon a third party point process, $\mathcal{T} = \{\tau_1, \tau_2, \ldots\}$ thus making the information on an underlying game *delayed* and allowing the players to continue fighting each other beyond their assumed merits of endurance, thereby letting the game to follow the path of a more realistic scenario. The first passage time is then shifted to an epoch $\tau_\rho$, i.e. upon one of the observation instants of time. See the diagram below.

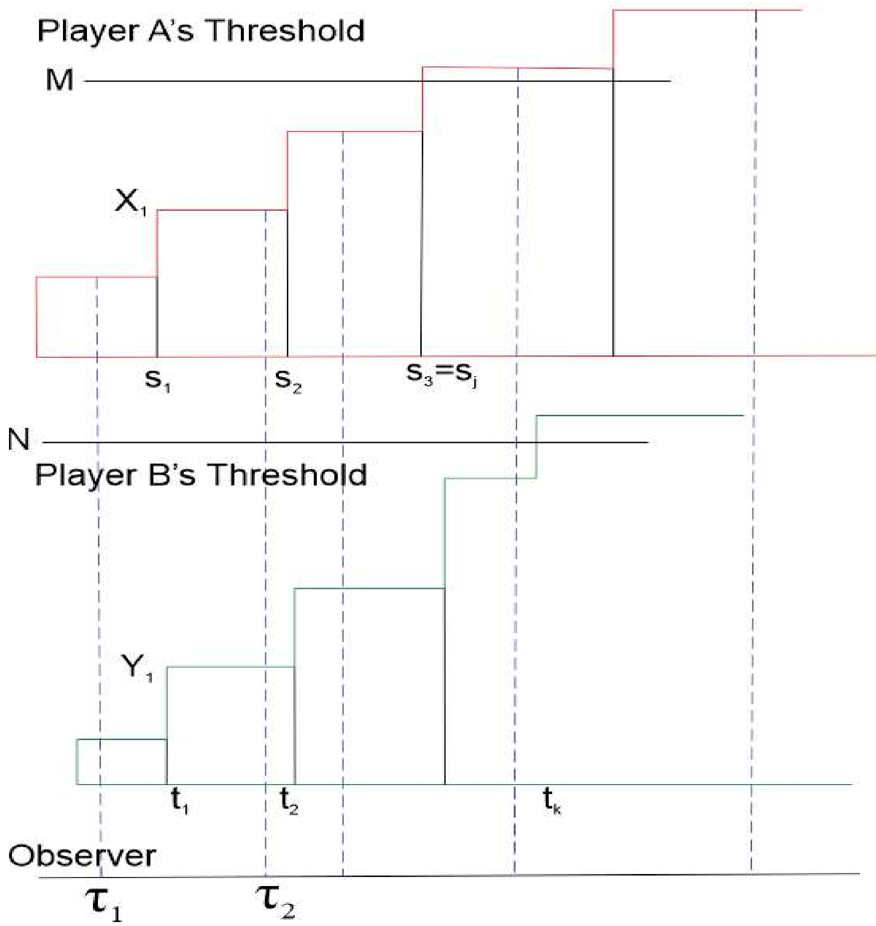


Figure 2.1: Delayed time game being observed by a third party.

In the sequel, we make assumptions on $\mathcal{A}$, $\mathcal{B}$, as being Poisson marked processes and $\mathcal{T}$ being a Poisson point process. If $X_i$ and $Y_i$ are casualties to players A and B over the interval $(\tau_{i-1}, \tau_i]$, and observed upon $\tau_i$, then

$$A_k = X_0 + X_1 + \ldots + X_k,\; B_k = Y_0 + Y_1 + \ldots + Y_k \tag{2.1.2}$$

are the cumulative damages to players A and B by time $\tau_k$. With the *exit indices*

$$\nu_1 := \inf\{j \geq 0 : A_j = X_0 + X_1 + \ldots + X_j \geq M\} \tag{2.1.3}$$

and

$$\nu_2 := \inf\{k \geq 0 : B_k = Y_0 + Y_1 + \ldots + Y_k \geq N\}, \tag{2.1.4}$$

and

$$\rho := \min\{\nu_1, \nu_2\}, \tag{2.1.5}$$

the random time $\tau_\rho$ is the *observed first passage time* or the *observed ruin time* or the *observed exit time* from the game. We recall that the real ruin time is unknown and it takes place anywhere between $\tau_{\rho-1}$ (*observed pre-exit time*) and $\tau_\rho$. Obviously, the finer are the observation times, the shorter is a delay of the end of the game. The other information of interest are $A_\rho$ and $B_\rho$ being the total damages to players A and B upon the ruin time. Clearly, $A_\rho \geq M$ or $B_\rho \geq N$, whereas $A_{\rho-1} < M$ and $B_{\rho-1} \leq N$.

In this case, we target the joint transforms

$$\Phi(u, v, \theta) = Eu^{A_\rho} v^{B_\rho} e^{-\theta\tau_\rho},\; \|u\| \leq 1, \|v\| \leq 1, \mathrm{Re}\theta \geq 0, \tag{2.1.6}$$

or

$$\Phi(u, v, \theta) = Eu^{A_\rho} e^{-vB_\rho} e^{-\theta\tau_\rho},\; \mathrm{Re}u \geq 0, \|v\| \leq 1, \mathrm{Re}\theta \geq 0. \tag{2.1.7}$$

The first transform is suited for discrete-valued components due to integer-valued marks $w_j$'s and $z_k$'s, whereas the second transform accounts to the mixed case.

A method of finding $\Phi$ was suggested by Agarwal and Dshalalow [1] in which the authors treated a multivariate marked point process with mutually dependent marks of which exactly two were so-called *active*. The latter means that the cumulative marks identified as active are to cross thresholds (such as $M$ and $N$ previously introduced) which bring the entire process to a hold upon *crossing* at the first passage time, whereas the rest of the marks identified as *passive* just assumes their respective values. One of them is the first passage time $\tau_\rho$. Although the functional $\Phi$ is a special case of a more general functional in [1] (that was not related to a game), we want to demonstrate the actual use of some discrete operators proposed in [1] and not only that. We also want to show that the mathematical outcome of the game is analytically tractable and numerically tame.

## 2.2 Formalism of the Real-Time Game

Unlike the delayed game, here we consider classes of antagonistic stochastic games of two players A and B, attacking each other simultaneously and at random epochs. As a results of these attacks, there are respective casualties of random magnitudes to each player. Players A and B can only endure a certain amount of losses (expressed through two fixed thresholds, say $M$ and $N$) before their respective "capitulations." The game is over when the cumulative losses to one of the players exceeds his associated endurance threshold.

A pertinent version of such game can be modeled by a bivariate marked random measure

$$\sum_{k\geq 1}(X_k, Y_k)\varepsilon_{\tau_k}, \tag{2.2.1}$$

where $\varepsilon_a$ is the Dirac point mass at $a$, $(X_k, Y_k)$ is the amount of losses to players $(\mathrm{A}, \mathrm{B})$ exerted at time $\tau_k$. Therefore,

$$\mathcal{T} = \sum_{k\geq 1}\varepsilon_{\tau_k} \tag{2.2.2}$$

is the underlying point process of attacks.

We assume that all processes and random variables (r.v.'s) are defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $A_k$ and $B_k$ be the cumulative damages to players A and B by time $\tau_k$ and be defined as in (2.1.2) and the indices as in (2.1.3-2.1.5).

Therefore, the time $\tau_\rho$, referred to as the *first passage time*, is when the game ends with one or both players being ruined. The random quantities $A_\rho$ and $B_\rho$ give the status of the players by the end of the game. It is clear from the definition that $A_\rho$ or $B_\rho$ will hit or arbitrarily exceed $M$ or $N$, respectively. See the diagram below:

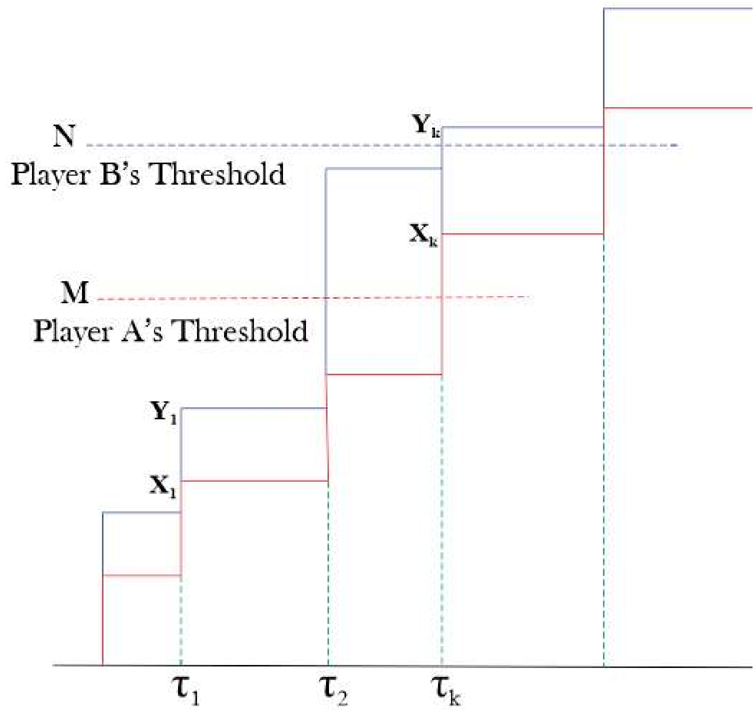

Figure 2.2: Real time game of simultaneous attacks.

As we shall see in Chapter 4, related to this game are two versions: discrete and continuous. In the former case, we assume that the marks $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ (successive increments of mutual attacks) are integer-valued and so are their associated thresholds, $M$ and $N$. In the latter case, we allow the marks $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be real-valued r.v.'s, with the thresholds $M$ and $N$ also be real-valued. For either case, we consider two different joint transforms

$$\Phi(u, v, \theta) = Eu^{A_\rho} v^{B_\rho} e^{-\theta\tau_\rho}, \|u\| \leq 1, \|v\| \leq 1, \text{Re}\theta \geq 0,$$

which is same as in (2.1.6) and

$$\Psi(u, v, \theta) = Ee^{-uA_\rho} e^{-vB_\rho} e^{-\theta\tau_\rho}, \text{Re}(\theta) \geq 0, \text{Re}(u) \geq 0, \text{Re}(v) \geq 0. \quad (2.2.3)$$

Our focus after now will be on obtaining closed forms for these functionals and their inverses and compare the results. As we will see it, we end up comparing discrete and continuous operational calculus on two very congruent special cases. In the former case, we use a fairly new technique of $\mathcal{D}$-operators developed in [3,4,11,16], and in the latter case, we apply a more familiar Laplace-Carson inverse previously used in other versions of stochastic games [13,14,16,17].

Either formula for $\Phi$ and $\Psi$ in (2.1.6) and (2.2.3) was introduced in Dshalalow [9-13] and then in Agarwal and Dshalalow [1]. We state either one as Theorems 1 and 2 for the "discrete" and "continuous" cases, respectively.

**Theorem 1** (Agarwal-Dshalalow [1], discrete case). *Under the assumptions* (2.1.1-2.1.6), *the functional* $\Phi$ *of the process can be expressed through* $\gamma(u, v, \theta) = Eu^{X_1}v^{Y_1}e^{-\theta\tau_1}$ *and it satisfies the following formula*:

$$\Phi(u, v, \theta) = Eu^{A_\rho}v^{B_\rho}e^{-\theta\tau_\rho}$$

$$= 1 - [1 - \gamma(u, v, \theta)]\mathcal{D}_{xy}^{M-1,\, N-1}\left\{\frac{1}{1-\gamma(ux, vy, \theta)}\right\}, \qquad (2.2.4)$$

*where the operator* $\mathcal{D}$ *applied to a function* $\varphi : \mathbb{C}^3 \to \mathbb{C}$ *analytic at* $x = y = 0$ *is defined as*

$$\mathcal{D}_{x,y}^{k,m}\varphi(x, y, z)$$

$$= \begin{cases} \lim_{x\to 0, y\to 0} \frac{1}{k!m!}\frac{\partial^{k+m}}{\partial x^k \partial y^m}\left[\frac{1}{(1-x)(1-y)}\varphi(x, y, z)\right], & k, m \geq 0 \\ 0, & k < 0 \text{ or } m < 0. \end{cases} \qquad (2.2.5)$$

□

Note that the transform $\gamma$ is assumed to be known or obtained in a closed form pertinent to the underlying model.

**Theorem 2** (Agarwal-Dshalalow [1], continuous case). *Under the assumptions* (2.1.1-2.1.4 and 2.2.3), *the functional* $\Psi$ *can be expressed through* $\gamma(u, v, \theta) = Ee^{-uX_1 - vY_1 - \theta\tau_1}$ and it *satisfies the following formula*:

$$\Psi(u, v, \theta) = Ee^{-uA_\rho}e^{-vB_\rho}e^{-\theta\tau_\rho} = 1 - \mathcal{L}C_{xy}^{-1}\left\{\frac{1-\gamma(u, v, \theta)}{1-\gamma(u+x, v+y, \theta)}\right\}(q, s) \quad (2.2.6)$$

*with the double Laplace-Carson transform,* $\mathcal{L}C_{pq}(\,\cdot\,)(x, y)$ *defined as*

$$\mathcal{L}C_{qs}(\,\cdot\,)(x, y) := xy\int_{q=0}^{\infty}\int_{s=0}^{\infty}e^{-xq-ys}(\,\cdot\,)d(q, s), Re(x) > 0, Re(y) > 0$$

*and the inverse Laplace-Carson transform*, $\mathcal{L}C_{xy}^{-1}$ *is defined as*

$$\mathcal{L}C_{xy}^{-1}(\,\cdot\,)(q, s) = \mathcal{L}^{-1}(\,\cdot\,\tfrac{1}{x}\tfrac{1}{y})C_{xy}^{-1}. \qquad \square$$

Here for notational convenience, we denoted by $q$ and $s$ the respective endurance thresholds of players A and B.

The two special cases we are going to compare will be as follows. We assume that the mutual attacks follow bivariate marked Poisson processes with geometrically distributed marks for the discrete game and with exponentially distributed marks in the continuous game. These two versions are congruent due to the similarities between the discrete and continuous distributions, namely either one is a unique memoryless distribution in their respective classes. Furthermore, their associated transforms, the probability generating function (pgf) and Laplace-Stieltjes transform (LST), are simple rational functions. It would thus be fair to say that neither case has stronger assumptions over the other.

# Chapter 3

# Discrete Operational Calculus in Delayed Games.

In this chapter, we shall consider three delayed games. The attacks in the first case follows in accordance with and ordinary Poisson Processes, in the second game, the attacks are distributed geometrically while the third game is a mixed case of a discrete and continuous component.

## 3.1 Delayed Game of Two Independent Ordinary Poisson Process.

In this game, we assume that the mutual attacks on players A and B follow in accordance with two independent ordinary Poisson processes $\mathcal{A}$ and $\mathcal{B}$ [specified in (2.1.1)] of intensities $\lambda$ and $\mu$.

Notice that in most applications, the functional

$$\gamma(u, v, \theta) = Eu^{X_1}v^{Y_1}e^{-\theta\tau_1}$$

can be readily found, as it is in our case.

Since $\mathcal{A}$ and $\mathcal{B}$ are ordinary, the respective marks $w_j$'s and $z_k$'s are 1 a.s. Furthermore, the observations take place at times $\tau_1, \tau_2, \ldots$ that forms a renewal process, with interrenewal times $\Delta_1 = \tau_1, \Delta_2 = \tau_2 - \tau_1, \ldots \in [\Delta]$, i.e., being identically distributed with the common Laplace-Stieltjes transform

$$\gamma(\theta) = Ee^{-\theta\Delta}. \tag{3.1.1}$$

In this case, since $X$ and $Y$ are conditionally independent given $\Delta$,

$$\begin{aligned} \gamma(u, v, \theta) &= Eu^{X_1}v^{Y_1}e^{-\theta\tau_1} \\ &= E\big[E\big[u^{X_1}v^{Y_1}e^{-\theta\tau_1}|\Delta\big]\big] = E\big[e^{-\theta\Delta}E[u^{X_1}|\Delta]E[v^{Y_1}|\Delta]\big] \\ &= E\big[e^{-\theta\Delta}e^{\lambda\Delta(u-1)}e^{\mu\Delta(v-1)}\big] \\ &= \gamma[\theta + \lambda - \lambda u + \mu - \mu v]. \end{aligned} \tag{3.1.2}$$

In the special case when $\Delta \in [Exp(\gamma)]$ (exponentially distributed with parameter $\gamma$), from (3.1.2) we have

$$\gamma(u, v, \theta) = \frac{\gamma}{\gamma+\lambda+\mu+\theta-\lambda u-\mu v}, \tag{3.1.3}$$

and thus,

$$\frac{1}{1-\gamma(u,v,\theta)} = 1 + \frac{\gamma}{p - \lambda u - \mu v}, \text{ where } p = \lambda + \mu + \theta. \tag{3.1.4}$$

Now we are going to use the following properties of the $\mathcal{D}$-operator [2,13].

**Theorem 3**. (Cf. Dshalalow [13].)

$(i)$ $\mathcal{D}_{x,y}^{k,m} = \mathcal{D}_x^k \circ \mathcal{D}_y^m = \mathcal{D}_y^m \circ \mathcal{D}_x^k$.

$(ii)$ *$\mathcal{D}$ is a linear functional with $\mathcal{D}_x \mathbf{1}(x) = 1$, where $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}$.*

$(iii)$ $\mathcal{D}_x^k(x^j g(x)) = \mathcal{D}_x^{k-j} g(x)$.

$(iv)$ *For any real number $b$ it holds true that*

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\right\} = \begin{cases} \frac{1-b^{k+1}}{1-b}, & b \neq 1 \\ k+1, & b = 1 \end{cases}.$$

$(v)$ *For any real number $a$ and for a positive integer $n$*

$$\mathcal{D}_x^k\left\{\frac{1}{(1-ax)^n}\right\} = \begin{cases} \sum_{j=0}^{k} \binom{n+j-1}{j} a^j, & \text{except for } a = n = 1 \\ k+1, & a = n = 1 \end{cases}.$$

$(vi)$ *For two real numbers $a$ and $b$ it holds*

$$\mathcal{D}_x^k\left\{\frac{1}{1-bx}\frac{1}{(1-ax)^n}\right\}$$

$$= \begin{cases} \frac{1}{1-b} \sum_{j=0}^{k} \binom{n+j-1}{j} \left(a^j - b^{k+1}\left(\frac{a}{b}\right)^j\right), & b \neq 1 \\ \sum_{j=0}^{k} \binom{n+j-1}{j} a^j (k-j+1), & b = 1 \end{cases} \qquad \square$$

$(vii)$ Let $a(x) = \sum_{i=0}^{\infty} a_i x^i$, then

$$\mathcal{D}_x^k(a(x)) = \sum_{i=0}^{k} a_i$$

**Theorem 4**. *For the special case of a discrete antagonistic game of two players, the joint functional* $\Phi$ *satisfies the following formulas*:

$$\Phi(u,v,\theta)=\frac{\gamma}{\gamma+\lambda(1-u)+\mu(1-v)+\theta}\left(1-\frac{\lambda(1-u)+\mu(1-v)+\theta}{\lambda+\mu(1-v)+\theta}\psi\right) \tag{3.1.5}$$

where,

$$\psi=\frac{1-b^{M}}{1-b}-\frac{1}{1-b}\,C^{N}\sum_{j=0}^{M-1}\binom{N+j-1}{j}\left[a^{j}-b^{M}\left(\frac{a}{b}\right)^{j}\right]. \tag{3.1.6}$$

$$a=\frac{\lambda u}{p},\ b=\frac{\lambda u}{p-\mu v},\ C=\frac{\mu v}{p},p=\lambda+\mu+\theta. \tag{3.1.7}$$

**Proof**. From Theorem 3$(i)$,

$$\mathcal{D}_{xy}^{M-1,\,N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}=\mathcal{D}_{x}^{M-1}\left\{\mathcal{D}_{y}^{N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}\right\}.$$

From (3.1.4), Theorem 3$(ii)$ and $(iv)$, and using notation (3.1.7),

$$\mathcal{D}_{y}^{N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}=\mathcal{D}_{y}^{N-1}\left\{1+\frac{\gamma}{p-\lambda ux}\,\frac{1}{1-\frac{\mu v}{p-\lambda ux}.y}\right\}$$

$$=1+\frac{\gamma}{p-\lambda ux}\cdot\mathcal{D}_{y}^{N-1}\left\{\frac{1}{1-\frac{\mu v}{p-\lambda ux}\cdot y}\right\}$$

$$=1+\frac{\gamma}{p-\lambda ux}\,\frac{1-\left(\frac{\mu v}{p-\lambda ux}\right)^{N}}{1-\frac{\mu v}{p-\lambda ux}}$$

$$=1+\frac{\gamma}{p-\mu v}\cdot\frac{1}{1-\frac{\lambda u}{p-\mu v}x}\left[1-\left(\frac{\mu v}{p}\right)^{N}\left(\frac{1}{1-\frac{\lambda u}{p}x}\right)^{N}\right].$$

After some algebra,

$$\mathcal{D}_{y}^{N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}=1+\frac{\gamma}{p-\mu v}\left[\frac{1}{1-bx}-\frac{1}{1-bx}\,C^{N}\,\frac{1}{(1-ax)^{N}}\right].$$

Then,

$$\mathcal{D}_{xy}^{M-1,\,N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\Big\}$$

$$= \mathcal{D}_{x}^{M-1}\Big\{1+\tfrac{\gamma}{p-\mu v}\Big[\tfrac{1}{1-bx}-C^N\tfrac{1}{1-bx}\,\tfrac{1}{(1-ax)^N}\Big]\Big\}.$$

From $(iv)$, and $(vi)$ with $b \neq 1$ as in case 1 of Theorem $3(vi)$,

$$\mathcal{D}_{xy}^{M-1,\,N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\Big\}$$

$$= 1+\tfrac{\gamma}{p-\mu v}\Bigg[\tfrac{1-b^M}{1-b}-\tfrac{1}{1-b}\,C^N\sum_{j=0}^{M-1}\binom{N+j-1}{j}\Big[a^j-b^M\big(\tfrac{a}{b}\big)^j\Big]\Bigg].$$

By (2.2.4),

$$\Phi(u,v,\theta)$$

$$= 1-[1-\gamma(u,v,\theta)\,]\Bigg\{1+\tfrac{\gamma}{p-\mu v}\Bigg[\tfrac{1-b^M}{1-b}-\tfrac{1}{1-b}\,C^N\sum_{j=0}^{M-1}\binom{N+j-1}{j}$$

$$\times\Big(a^j-b^M\big(\tfrac{a}{b}\big)^j\Big)\Bigg]\Bigg\}.$$

But from (3.1.3) and (3.1.6), we have that

$$1-\gamma(u,v,\theta)=1-\tfrac{\gamma}{\gamma+p-\lambda u-\mu v}=\tfrac{p-\lambda u-\mu v}{\gamma+p-\lambda u-\mu v}.$$

Therefore,

$$\Phi(u,v,\theta)=1-\tfrac{p-\lambda u-\mu v}{\gamma+p-\lambda u-\mu v}\Big(1+\tfrac{\gamma}{p-\mu v}\psi\Big)$$

$$=\tfrac{\gamma}{\gamma+p-\lambda u-\mu v}\Big(1-\tfrac{p-\mu v-\lambda u}{p-\mu v}\psi\Big). \qquad (3.1.8)$$

Where,

$$\psi = \frac{1-b^M}{1-b} - \frac{1}{1-b}\, C^N \sum_{j=0}^{M-1} \binom{N+j-1}{j} \left[ a^j - b^M \left(\frac{a}{b}\right)^j \right]. \tag{3.1.9}$$

$$a = \frac{\lambda u}{\lambda+\mu+\theta}, \qquad b = \frac{\lambda u}{\lambda+\mu+\theta-\mu v}. \tag{3.1.10}$$

The statement now follows from Theorem 1 and (3.1.3-3.1.10). □

### 3.1.1 The Marginal Transform of $B_\rho$.

The status of the game (total casualty of player B) which is called the marginal transform at the end of the game is known if we let $u = 1$ and $\theta = 0$ in $\Phi(u, v, \theta)$ of Theorem 4 we arrive at

$$Ev^{B_\rho} = \frac{\gamma}{\gamma+\mu(1-v)} \left[ b^M + \left(\frac{\mu v}{\lambda+\mu}\right)^N \sum_{j=0}^{M-1} \binom{N+j-1}{j} \left(\frac{\lambda}{\lambda+\mu}\right)^j (1 - b^{M-j}) \right].$$

Where,

$$b = \frac{\lambda}{\lambda+\mu(1-v)}$$ □

### 3.1.2 The Marginal Transform of $A_\rho$.

The marginal transform of player A, $A_\rho$ which is the total casualty of player A is given by $\Phi(u, 1, 0)$ of Theorem 4 and after some algebra, we arrive at

$$Eu^{A_\rho} = \frac{\gamma}{\gamma+\lambda(1-u)} \left[ u^M + \left(\frac{\mu}{\mu+\lambda}\right)^N \sum_{j=0}^{M-1} \binom{N+j-1}{j} \left(\frac{\lambda u}{\lambda+\mu}\right)^j (1 - u^{M-j}) \right]$$ □

### 3.1.3 The Marginal Transform upon $\tau_\rho$, $Ee^{-\theta\tau_\rho}$.

The marginal transform of $\tau_\rho$, $Ee^{-\theta\tau_\rho}$ can be obtained similarly by letting $u = v = 1$ in Theorem 4 and after some algebra, we obtain

$$Ee^{-\theta\tau_\rho} = \frac{\gamma}{\gamma+\theta}\left\{\left(\frac{\lambda}{\lambda+\theta}\right)^M + \left(\frac{\mu}{\lambda+\mu+\theta}\right)\sum_{j=0}^{NM-1}\binom{N+j-1}{j}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^j\left[1-\left(\frac{\lambda}{\lambda+\theta}\right)^{M-j}\right]\right\}\square$$

### 3.1.4 Probability Density Function of the First Passage Time.

To find the density of the first passage time, $\tau_\rho$ of the LST-inverse in the marginal functional of $\Phi$, we will need Lemma 5 below whose proof can be found in [15] except for $(vi)$ and $(vii)$ which we shall prove here. Some of the results here will also be used in Chapter 4.

**Lemma 5**. *Let* $\gamma, \lambda, a, b, c$ *and* $\alpha$ *be some fixed parameters and* $\mathcal{L}^{-1}$ *be the inverse of the Laplace transform. Then,*

$(i)$ $\quad \mathcal{L}_\theta^{-1}\{\frac{1}{\theta}\}(t) = 1.$

$(ii)$ $\quad \mathcal{L}_\theta^{-1}\left(\frac{1}{\gamma+\theta}\right)(t) = e^{-\gamma t}.$

$(iii)$ $\quad \mathcal{L}_y^{-1}\left(\frac{1}{c+y}e^{\frac{a}{b+y}}\right)(s)$

$$= e^{-bs}I_0\left(2\sqrt{as}\right) + (b-c)e^{-cs}\int_{u=0}^{s}e^{-(b-c)r}I_0\left(2\sqrt{au}\right)du.$$

*where $I_0$ is the modified Bessel function of the first kind defined as*

$$I_0(x) = \sum_{m\geq 0}(m!)^{-2}\left(\frac{x}{2}\right)^{2m}.$$

$(iv)\quad \mathcal{L}_\theta^{-1}\Big(\frac{1}{(\lambda+\theta)^n}\Big)(t) = e^{-\lambda t}\frac{t^{n-1}}{(n-1)!}.$

$(v)\quad \mathcal{L}_\theta^{-1}\Big(\frac{1}{\gamma+\theta}\frac{1}{(\lambda+\theta)^n}\Big)(t)$

$$= e^{-\gamma t}\frac{1}{(\lambda-\gamma)^n}\Big[1 - e^{(\gamma-\lambda)t}\sum_{i=0}^{n-1}\frac{(\lambda-\gamma)^i}{i!}t^i\Big]$$

$$= \frac{e^{-\gamma t}}{(\lambda-\gamma)^n}P(n-1,(\lambda-\gamma)t).$$

$(vi)\quad \mathcal{L}_\theta^{-1}\Big(\frac{1}{(\lambda+\theta)^m}\frac{1}{(\alpha+\theta)^n}\Big)(t)$

$$= \frac{1}{(n-1)!(\alpha-\lambda)^n}\sum_{j=0}^{m-1}\frac{(n+j-1)!}{j!(m-j-1)!}t^{m-j-1}\frac{1}{(\lambda-\alpha)^j}$$

$$\times\Big[e^{-\lambda t} - e^{-\alpha t}\sum_{i=0}^{n+j-1}\frac{(\alpha-\lambda)^i}{i!}t^i\Big].$$

$(vii)\quad \mathcal{L}_\theta^{-1}\Big(\frac{1}{\gamma+\theta}\frac{1}{(\lambda+\theta)^m}\frac{1}{(\alpha+\theta)^n}\Big)(t)$

$$= e^{-\gamma t}g(\alpha,\lambda)\Big\{\frac{1}{(\lambda-\gamma)^{m-k}}\Big[1 - e^{-(\lambda-\gamma)t}\sum_{r=0}^{m-k-1}\frac{(\lambda-\gamma)^r}{r!}t^r\Big]$$

$$- \frac{1}{(\lambda-\alpha)^{m-k}}\sum_{i=0}^{n+k-1}(-1)^i\binom{m+i-k-1}{i}$$

$$\times\Big(1 - e^{-(\lambda-\alpha)t}\sum_{s=0}^{m+i-k-1}\frac{(\lambda-\alpha)^s}{s!}t^s\Big)\Big\}.$$

Also in the form,

$$= e^{-\gamma t}(-1)^n\sum_{k=0}^{m-1}\binom{n-k-1}{k}\frac{1}{(\lambda-\alpha)^{n+k}}$$

$$\times\Big\{\frac{1}{(\lambda-\gamma)^{m-k}}P(m-k+1,(\lambda-\gamma)t)$$

$$- \frac{1}{(\lambda-\alpha)^{m-k}}\sum_{i=0}^{n+k-1}(-1)^i\binom{m+i-k-1}{i}$$

$$\times P(m+i-k-1,(\lambda-\alpha)t)\Bigg\}.$$

Where,

$$g(\alpha,\lambda) = \frac{1}{(n-1)!(\alpha-\lambda)^n}\sum_{k=0}^{m-1}\frac{(n+k-1)!}{k!}\frac{1}{(\lambda-\alpha)^k}$$

$$= (-1)^n\sum_{k=0}^{m-1}\binom{n+k-1}{k}\frac{1}{(\lambda-\alpha)^{n+k}}$$

and

$$P(n,\alpha t) = 1 - e^{-\alpha t}\sum_{j=0}^{n}\frac{(\alpha t)^j}{j!}$$

is the regularized gamma function. □

**Proof of $(vi)$**

$$\mathcal{L}_\theta^{-1}\left(\frac{1}{(\lambda+\theta)^{m+1}}\frac{1}{(\alpha+\theta)^{n+1}}\right)(t) = \int_{u=0}^{t}e^{-\lambda(t-u)}\frac{(t-u)^m}{m!}e^{-\alpha u}\frac{u^n}{n!}du$$

$$= \frac{1}{m!n!}e^{-\lambda t}\int_{u=0}^{t}e^{(\lambda-\alpha)u}(t-u)^m u^n du$$

$$= \frac{1}{m!n!}e^{-\lambda t}\sum_{j=0}^{m}\binom{m}{j}t^{m-j}(-1)^j\int_{u=0}^{t}e^{(\lambda-\alpha)u}u^{n+j}du$$

$$= \frac{1}{m!n!(\alpha-\lambda)^{n+1}}\sum_{j=0}^{m}\binom{m}{j}(n+j)!t^{m-j}\frac{1}{(\lambda-\alpha)^j}$$

$$\times\left(e^{-\lambda t} - e^{-\alpha t}\sum_{i=0}^{n+j}\frac{(\alpha-\lambda)^i}{i!}t^i\right)$$

$$= \frac{1}{n!(\alpha-\lambda)^{n+1}}\sum_{j=0}^{m}\frac{(n+j)!}{j!(m-j)!}t^{m-j}\frac{1}{(\lambda-\alpha)^j}$$

$$\times\left(e^{-\lambda t} - e^{-\alpha t}\sum_{i=0}^{n+j}\frac{(\alpha-\lambda)^i}{i!}t^i\right).$$

Therefore,

$$\mathcal{L}_\theta^{-1}\left(\tfrac{1}{(\lambda+\theta)^m}\tfrac{1}{(\alpha+\theta)^n}\right)(t)$$

$$= \tfrac{1}{(n-1)!(\alpha-\lambda)^n}\textstyle\sum_{j=0}^{m-1}\tfrac{(n+j-1)!}{j!(m-j-1)!}t^{m-j-1}\tfrac{1}{(\lambda-\alpha)^j}$$

$$\times\left(e^{-\lambda t}-e^{-\alpha t}\textstyle\sum_{i=0}^{n+j-1}\tfrac{(\alpha-\lambda)^i}{i!}t^i\right). \qquad \square$$

**Proof of $(vii)$**

$$\mathcal{L}_\theta^{-1}\left(\tfrac{1}{\gamma+\theta}\tfrac{1}{(\lambda+\theta)^m}\tfrac{1}{(\alpha+\theta)^n}\right)(t)$$

$$= \textstyle\int_{u=0}^{t} e^{-\gamma(t-u)}\tfrac{1}{(n-1)!(\alpha-\lambda)^n}\textstyle\sum_{j=0}^{m-1}\tfrac{(n+j-1)!}{j!(m-j-1)!}u^{m-j-1}\tfrac{1}{(\lambda-\alpha)^j}$$

$$\times\left(e^{-\lambda u}-e^{-\alpha u}\textstyle\sum_{i=0}^{n+j-1}\tfrac{(\alpha-\lambda)^i}{i!}u^i\right)$$

$$= \underbrace{\tfrac{1}{(n-1)!(\alpha-\lambda)^n}\textstyle\sum_{j=0}^{m-1}\tfrac{(n+j-1)!}{j!(m-j-1)!}\tfrac{1}{(\lambda-\alpha)^j}e^{-\gamma t}}_{f(t)}$$

$$\times\left[\textstyle\int_{u=0}^{t}\left(u^{m-j-1}e^{(\gamma-\lambda)u}-\textstyle\sum_{i=0}^{n+j-1}\tfrac{(\alpha-\lambda)^i}{i!}e^{-(\alpha-\gamma)u}u^{m+i-j-1}\right)du\right]$$

$$= f(t)\Bigg[\textstyle\int_{u=0}^{t}e^{(\gamma-\lambda)u}u^{m-j-1}du$$

$$-\textstyle\sum_{i=0}^{n+j-1}\tfrac{(\alpha-\lambda)^i}{i!}\textstyle\int_{u=0}^{t}e^{-(\gamma-\alpha)u}u^{m+i-j-1}du\Bigg].$$

After some algebra, the result follows from Lemma 5 $(vii)$. $\square$

**The Probability Density Function $f_{\tau_\rho}$ of $\tau_\rho$.** Revisiting (3.1.3), we go further on obtaining the probability density function of the first observed passage time of the game end. Since $Ee^{-\theta\tau_\rho}$ is the Laplace-Stieltjes transform, we need to take the Laplace inverse to obtain the density function.

$$f_{\tau_\rho}(t) = \mathcal{L}_\theta^{-1}\{Ee^{-\theta\tau_\rho}\}(t)$$

$$= \mathcal{L}_\theta^{-1}\Bigg\{\frac{\gamma\lambda^M}{(\gamma+\theta)(\lambda+\theta)^M} + \sum_{j=0}^{M-1}\binom{N+j-1}{j}\frac{\gamma\lambda^j\mu^N}{(\gamma+\theta)(\lambda+\mu+\theta)^{N+j}}$$

$$- \sum_{j=0}^{M-1}\binom{N+j-1}{j}\frac{\gamma\lambda^M\mu^N}{(\gamma+\theta)(\lambda+\theta)^{M-j}(\lambda+\mu+\theta)^{N+j}}\Bigg\}(t).$$

By Lemma 5 $(vi$-$vii)$ and after some algebra we arrive at

$$f_{\tau_\rho}(t) = \frac{\gamma\lambda^M e^{-\gamma t}}{(\lambda-\gamma)^M}P(M-1,\lambda-\gamma)$$

$$+ \gamma\mu^N e^{-\gamma t}\sum_{j=0}^{M-1}\binom{N+j-1}{j}\frac{\lambda^j}{(\lambda+\mu-\gamma)^{N+j}}P(N+j-1,\lambda+\mu-\gamma)$$

$$- \gamma\mu^N e^{-\gamma t}\sum_{j=0}^{M-1}\binom{N+j-1}{j}\lambda^{M-N-j}\sum_{k=0}^{M-j-1}\binom{N+j-k-1}{k}\frac{(-1)^k}{\lambda^k}$$

$$\times\Bigg\{\frac{1}{(\lambda-\gamma)^{M-j-k}}P(M-j-k+1,(\lambda-\gamma)t)$$

$$- \frac{(-1)^{M-j-k}}{\mu^{M-j-k}} \sum_{l=0}^{N+j+k-1} (-1)^l \binom{M-j+l-k-1}{l}$$

$$\times P(M-j+l-k-1, -\mu t) \Big\}. \qquad \square$$

## 3.2 Delayed Game with Discrete Components distributed Geometrically.

In this section, we consider a game that is observed upon random epoch of time and forms a renewal process, which have interrenewal times $\Delta_1 = \tau_1, \Delta_2 = \tau_2 - \tau_1, \ldots \in [\Delta]$, i.e., being identically distributed with the common Laplace-Stieltjes transform

$$\gamma(\theta) = Ee^{-\theta\Delta} = \frac{\gamma}{\gamma+\theta}. \tag{3.2.1}$$

Suppose that the attacks of players A and B are $X$ and $Y$ which are geometrically distributed (type 1) with parameter $p$ and $p_1$ respectively, Then under conditions (3.1.1)-(3.1.3),

$$\gamma(u, v, \theta) = \gamma\Big(\theta + \lambda - \lambda \frac{pu}{1-qu} + \mu - \mu \frac{p_1 v}{1-q_1 v}\Big)$$

$$= \gamma\Big(\theta + \lambda \frac{1-u}{1-qu} + \mu \frac{1-v}{1-q_1 v}\Big). \tag{3.2.2}$$

In the special case where $\Delta \in [Exp(\gamma)]$ (exponentially distributed with parameter $\gamma$), from (3.2.1) and (3.2.2) we have,

$$\gamma(u, v, \theta) = \frac{\gamma}{\gamma + \theta + \lambda \frac{1-u}{1-qu} + \mu \frac{1-v}{1-q_1 v}}. \tag{3.2.3}$$

Let,

$$\vartheta = \theta + \lambda\frac{1-u}{1-qu} + \mu\frac{1-v}{1-q_1 v},$$

then,

$$1 - \gamma(\vartheta) = \frac{\vartheta}{\gamma+\vartheta} \text{ and } \frac{1}{1-\gamma(\vartheta)} = 1 + \gamma \cdot \frac{1}{\vartheta}.$$

$$\frac{1}{\vartheta} = \frac{(1-qu)(1-q_1 v)}{\theta(1-qu)(1-q_1 v)+\lambda(1-u)(1-q_1 v)+\mu(1-v)(1-qu)}.$$

Similarly,

$$\frac{1}{\vartheta(x,y)} = \frac{(1-qux)+(qq_1uvx-q_1v)y}{\theta+\lambda+\mu-qu\theta x-\lambda ux-q\mu ux-(q_1v\theta-qq_1uv\theta x+\lambda q_1 v-\lambda q_1 uvx+\mu v-q\mu uvx)y}$$

$$= \frac{a}{c-dy} + \frac{by}{c-dy}.$$

Hence,

$$\frac{1}{1-\gamma(ux,vy,\theta)} = \frac{1}{1-\gamma(\vartheta)} = 1 + \gamma \cdot \frac{1}{\vartheta(x,y)}$$

$$= 1 + \gamma \cdot \left(\frac{a}{c-dy} + \frac{by}{c-dy}\right), \qquad (3.2.4)$$

where,

$$a = 1 - qux, \qquad b = qq_1uvx - q_1 v,$$

$$c = \theta + \lambda + \mu - qu\theta x - \lambda ux - q\mu ux,$$

$$d = q_1 v\theta - qq_1uv\theta x + \lambda q_1 v - \lambda q_1 uvx + \mu v - q\mu uvx.$$

### 3.2.1 Special Case of a Geometrically Distributed Attack with $M = 1$ and any $N$.

**Theorem 6**. *For the special case where* $M = 1$ *and* $N$ *is any integer, then under the conditions and assumptions of (3.2.1)-(3.2.3), the joint functional,* $\Phi_g$ *of the first passage time of the game satisfies the following formulas*:

$$\Phi_g(u, v, \theta) = \tfrac{\gamma}{C}(1 - H^N)(F - 1) + \tfrac{\gamma}{C} q_1 v(1 - H^{N-1})(1 - F) + F. \tag{3.2.5}$$

Where,

$$F = \tfrac{\gamma(1-qu)(1-q_1v)}{(\gamma+\theta)(1-qu)(1-q_1v)+\lambda(1-u)(1-q_1v)+\mu(1-v)(1-qu)}, \tag{3.2.6}$$

$$C = (1 - q_1 v)(\theta + \lambda) + \mu(1 - v), \tag{3.2.7}$$

$$H = \tfrac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu}. \tag{3.2.8}$$

**Proof**. From Theorem 3$(i)$, with $M = 1$,

$$\mathcal{D}_{xy}^{M-1,N-1}\left\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\right\} = \mathcal{D}_x^0\left\{\mathcal{D}_y^{N-1}\left\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\right\}\right\}.$$

From (3.2.4) and by Theorem 3 $(iii)$ and $(iv)$,

$$\mathcal{D}_y^{N-1}\left\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\right\} = \mathcal{D}_y^{N-1}\left\{1 + \gamma\cdot\left(\tfrac{a}{c-dy} + \tfrac{by}{c-dy}\right)\right\}$$

$$= 1 + \gamma\cdot\tfrac{a}{c}\mathcal{D}_y^{N-1}\left\{\tfrac{1}{1-\frac{d}{c}y}\right\} + \gamma\cdot\tfrac{b}{c}\mathcal{D}_y^{N-1}\left\{\tfrac{y}{1-\frac{d}{c}y}\right\}$$

$$= 1 + \gamma\cdot\tfrac{a}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^N}{1-\frac{d}{c}}\right\} + \gamma\cdot\tfrac{b}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^{N-1}}{1-\frac{d}{c}}\right\}.$$

Subsequently,

$$\mathcal{D}_{xy}^{0,N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}$$

$$= \mathcal{D}_x^0\left\{1+\gamma\cdot\frac{a}{c}\left\{\frac{1-\left(\frac{d}{c}\right)^N}{1-\frac{d}{c}}\right\}+\gamma\cdot\frac{b}{c}\left\{\frac{1-\left(\frac{d}{c}\right)^{N-1}}{1-\frac{d}{c}}\right\}\right\}$$

$$= 1+\gamma\mathcal{D}_x^0\left\{\frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$-\gamma q_1 v\mathcal{D}_x^0\left\{\frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\} \tag{3.2.9}$$

where,

$$\alpha_1=\theta+\lambda+\mu, \qquad \beta_1=qu\theta+\lambda u+q\mu u,$$

$$c=\alpha_1-\beta_1 x, \qquad d=\alpha-\beta x,$$

$$\alpha=q_1v\theta+\lambda q_1 v+\mu v, \qquad \beta=qq_1uv\theta+\lambda q_1uv+q\mu uv,$$

$$a_1=\theta+\lambda+\mu-\lambda q_1v-q_1v\theta-\mu v,$$

$$b_1=qu\theta+\lambda u+q\mu u-qq_1uv\theta-\lambda q_1uv-q\mu uv,$$

$$c-d=a_1-b_1x.$$

The first part in (3.2.9) is given as,

$$\mathcal{D}_x^0\left\{\frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$= \mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$-\mathcal{D}_x^0\left\{\frac{qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_{11}x}\right)^N\right]\right\} \tag{3.2.10}$$

and

$$\mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$=\frac{1}{a_1}\left[\frac{1-\left(\frac{b_1}{a_1}\right)}{1-\frac{b_1}{a_1}}\right]$$

$$-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N\mathcal{D}_x^0\left\{\frac{1}{1-\frac{b_1}{a_1}x}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^N}\sum_{k=0}^{N}\binom{N}{k}\left(\frac{-\beta}{\alpha}\right)^k x^k\right\}.$$

By Theorem 3 $(iv)$ with $\frac{b_1}{a_1}\neq 1$ which implies $a_1\neq b_1$,

$$=\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N\sum_{k=0}^{N}\binom{N}{k}\left(\frac{-\beta}{\alpha}\right)^k\mathcal{D}_x^{-k}\left\{\frac{1}{1-\frac{b_1}{a_1}x}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^N}\right\},$$

which by Theorem 1 (2.2.5) gives,

$$\mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}=\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N. \tag{3.2.11}$$

The second part of (3.2.10) is calculated as

$$\mathcal{D}_x^0\left\{\frac{qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$=\frac{qu}{a_1}\left[\mathcal{D}_x^0\left\{\frac{x}{1-\frac{b_1}{a_1}x}\right\}-\mathcal{D}_x^0\left\{\left(\frac{\alpha}{\alpha_1}\right)^N\frac{x}{1-\frac{b_1}{a_1}x}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^N}\left(1-\frac{\beta}{\alpha}x\right)^N\right\}\right]$$

$$=0. \qquad \text{(by Theorem 1 (2.2.5)} \tag{3.2.12}$$

Next we find the second part of (3.2.9),

$$\mathcal{D}_x^0\left\{\frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}$$

$$=\mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}$$

$$-\mathcal{D}_x^0\left\{\frac{qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}. \qquad (3.2.13)$$

Calculating the first part of (3.2.13), we have,

$$\mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}$$

$$=\frac{1}{a_1}\mathcal{D}_x^0\left\{\frac{1}{1-\frac{b_1}{a_1}x}\left[1-\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\left(\frac{1-\frac{\beta}{\alpha}x}{1-\frac{\beta_1}{\alpha_1}x}\right)^{N-1}\right]\right\}.$$

Which after some algebra reduce to,

$$=\frac{1}{a_1}\left[\mathcal{D}_x^0\left\{\frac{1}{1-\frac{b_1}{a_1}x}\right\}\right.$$

$$\left.-\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\sum_{s=0}^{N-1}\binom{N-1}{s}\left(\frac{-\beta}{\alpha}\right)^s\mathcal{D}_x^{-s}\left\{\frac{1}{1-\frac{b_1}{a_1}x}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^{N-1}}\right\}\right]$$

$$=\frac{1}{a_1}\left[\frac{1-\frac{b_1}{a_1}}{1-\frac{b_1}{a_1}}\right]-\frac{1}{a_1}\left[\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\sum_{s=0}^{N-1}\binom{N-1}{s}\left(\frac{-\beta}{\alpha}\right)^s\mathcal{D}_x^{-s}\left\{\frac{1}{1-\frac{b_1}{a_1}x}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^{N-1}}\right\}\right]$$

which again by Theorem 1 (2.2.5) and Theorem 3 $(vi)$ gives

$$\mathcal{D}_x^0\left\{\frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}\qquad=\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^{N-1} \qquad (3.2.14)$$

Similarly,

$$\mathcal{D}_x^0\left\{\frac{qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}$$

$$=\frac{qu}{a_1}\mathcal{D}_x^0\left\{\frac{x}{1-\frac{b_1}{a_1}x}\left[1-\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\left(\frac{1-\frac{\beta}{\alpha}x}{1-\frac{\beta_1}{\alpha_1}x}\right)^{N-1}\right]\right\}$$

$$=\frac{qu}{a_1}\left[\mathcal{D}_x^{-1}\left\{\frac{1}{1-\frac{b_1}{a_1}x}\right\}\right.$$

$$\left.-\mathcal{D}_x^{-1}\left\{\frac{1}{1-\frac{b_1}{a_1}x}\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\frac{1}{\left(1-\frac{\beta_1}{\alpha_1}x\right)^{N-1}}\sum_{n=0}^{N-1}\binom{N-1}{n}\left(\frac{-\beta}{\alpha}\right)^n x^n\right\}\right]$$

$$=\frac{qu}{a_1}[0-0]=0. \tag{3.2.15}$$

Then using (3.2.11), (3.2.12), (3.2.14) and (3.2.15) in (3.2.9) and simplifying we obtain,

$$\mathcal{D}_{xy}^{0,N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}$$

$$=1+\gamma\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N\right\}-\gamma q_1 v\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\right\}$$

By (3.2.4),

$$\Phi_g(u,v,\theta)=1-\left[1+\gamma\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N\right\}-\gamma q_1 v\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\right\}\right]$$

$$+\gamma(u,v,\theta)\left[1+\gamma\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^N\right\}\right.$$

$$\left.-\gamma q_1 v\left\{\frac{1}{a_1}-\frac{1}{a_1}\left(\frac{\alpha}{\alpha_1}\right)^{N-1}\right\}\right].$$

After some algebra we arrive at

$$\Phi(u,v,\theta) = \tfrac{\gamma}{C}(1-H^N)(F-1) + \tfrac{\gamma}{C}q_1 v(1-H^{N-1})(1-F) + F.$$

Where,

$$C = a_1 = (1-q_1 v)(\theta+\lambda) + \mu(1-v),$$

$$H = \tfrac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu},$$

$$F = \gamma(u,v,\theta) = \tfrac{\gamma(1-qu)(1-q_1 v)}{(\gamma+\theta)(1-qu)(1-q_1 v)+\lambda(1-u)(1-q_1 v)+\mu(1-v)(1-qu)}. \qquad \square$$

### 3.2.2 The Marginal Transform of $\tau_\rho$.

Letting $u = v = 1$ in (3.2.6) to (3.2.8) and using this in (3.2.5) of Theorem 6, and after some algebra, we arrive at

$$Ee^{-\theta\tau_\rho} = \tfrac{\gamma}{(1-q_1)(\theta+\lambda)}\left[1-\left(\tfrac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu}\right)^N\right]\left[\tfrac{\gamma}{(\gamma+\theta)}-1\right]$$

$$+\tfrac{\gamma q_1}{(1-q_1)(\theta+\lambda)}\left[1-\left(\tfrac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu}\right)^{N-1}\right]\left[1-\tfrac{\gamma}{(\gamma+\theta)}\right]+\tfrac{\gamma}{(\gamma+\theta)}. \qquad \square$$

<u>Check</u>

We run a simple check to see if

$$\Phi_g(u,v,\theta)|_{u=v=1,\theta=0} = Eu^{A_\rho}v^{B_\rho}e^{-\theta\tau_\rho}|_{u=v=1,\theta=0} = 1.$$

From 3.2.2,

$$Ee^{-\theta\tau_\rho}|_{\theta=0} = \tfrac{\gamma}{(1-q_1)\lambda}\left[1-\left(\tfrac{\lambda q_1+\mu}{\lambda+\mu}\right)^N\right]\left[\tfrac{\gamma}{\gamma}-1\right]$$

$$+ \frac{\gamma q_1}{(1-q_1)\lambda}\left[1-\left(\frac{\lambda q_1+\mu}{\lambda+\mu}\right)^{N-1}\right]\left[1-\frac{\gamma}{\gamma}\right]+\frac{\gamma}{\gamma}$$

$$= \frac{\gamma}{(1-q_1)\lambda}\left[1-\left(\frac{\lambda q_1+\mu}{\lambda+\mu}\right)^{N}\right]0$$

$$+ \frac{\gamma q_1}{(1-q_1)\lambda}\left[1-\left(\frac{\lambda q_1+\mu}{\lambda+\mu}\right)^{N-1}\right]0+1$$

$$= 1. \qquad \square$$

### 3.2.3 The Probability Density Function $f_{\tau_\rho}$ of $\tau_\rho$.

Revisiting 3.2.1, we go further in obtaining the probability density function of the first observed passage time of the game end. As before, since $Ee^{-\theta\tau_\rho}$ is the Laplace-Stieltjes transform, we need to take the Laplace inverse to obtain the density function.

$$f_{\tau_\rho}(t) = \mathcal{L}_\theta^{-1}\{Ee^{-\theta\tau_\rho}\}(t)$$

$$= \mathcal{L}_\theta^{-1}\left\{\frac{\gamma}{(1-q_1)(\theta+\lambda)}\left[1-\left(\frac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu}\right)^{N}\right]\left[\frac{\gamma}{(\gamma+\theta)}-1\right]\right.$$

$$\left.+ \frac{\gamma q_1}{(1-q_1)(\theta+\lambda)}\left[1-\left(\frac{q_1\theta+\lambda q_1+\mu}{\theta+\lambda+\mu}\right)^{N-1}\right]\left[1-\frac{\gamma}{(\gamma+\theta)}\right]+\frac{\gamma}{(\gamma+\theta)}\right\}(t),$$

After some algebra we obtain

$$f_{\tau_\rho}(t) = \mathcal{L}_\theta^{-1}\left\{\frac{\gamma q_1}{(1-q_1)}\frac{1}{(\theta+\lambda)} - \frac{\gamma}{(1-q_1)}\frac{1}{(\theta+\lambda)} + \frac{\gamma}{\theta+\gamma} + \frac{\gamma^2}{(1-q_1)}\frac{1}{(\theta+\lambda)(\theta+\gamma)}\right.$$

$$- \frac{\gamma^2 q_1}{(1-q_1)}\frac{1}{(\theta+\lambda)(\theta+\gamma)}$$

$$+ \frac{\gamma}{(1-q_1)} \frac{1}{(\theta+\lambda)} \sum_{k=0}^{N} \binom{N}{k} q_1^{N-k} \left( \frac{\mu(1-q_1)}{\theta+\lambda+\mu} \right)^k$$

$$- \frac{\gamma q_1}{(1-q_1)} \frac{1}{(\theta+\lambda)} \sum_{i=0}^{N-1} \binom{N-1}{i} q_1^{N-i-1} \left( \frac{\mu(1-q_1)}{\theta+\lambda+\mu} \right)^i$$

$$- \frac{\gamma^2}{(1-q_1)} \frac{1}{(\theta+\lambda)(\theta+\gamma)} \sum_{j=0}^{N} \binom{N}{j} q_1^{N-j} \left( \frac{\mu(1-q_1)}{\theta+\lambda+\mu} \right)^j$$

$$+ \frac{\gamma^2 q_1}{(1-q_1)} \frac{1}{(\theta+\lambda)(\theta+\gamma)} \sum_{s=0}^{N-1} \binom{N-1}{s} q_1^{N-s-1} \left( \frac{\mu(1-q_1)}{\theta+\lambda+\mu} \right)^s \Bigg\}(t)$$

$$= \frac{\gamma}{(1-q_1)}(q_1 - 1)e^{-\lambda t} + \gamma e^{-\gamma t} + \frac{\gamma^2}{(1-q_1)}(1-q_1)\frac{1}{\lambda-\gamma}[e^{-\gamma t} - e^{-\lambda t}]$$

$$+ \frac{\gamma}{(1-q_1)} q_1^N \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \right\}$$

$$+ \frac{\gamma}{(1-q_1)} \sum_{k=1}^{N} \binom{N}{k} q_1^{N-k} [\mu(1-q_1)]^k \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\lambda+\mu)^k} \right\}(t)$$

$$- \frac{\gamma q_1}{(1-q_1)} q_1^{N-1} \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \right\}$$

$$- \frac{\gamma q_1}{(1-q_1)} \sum_{i=1}^{N-1} \binom{N-1}{i} q_1^{N-i-1} [\mu(1-q_1)]^i \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\lambda+\mu)^i} \right\}(t)$$

$$- \frac{\gamma^2}{(1-q_1)} q_1^N \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\gamma)} \right\}(t)$$

$$- \frac{\gamma^2}{(1-q_1)} \sum_{j=1}^{N} \binom{N}{j} q_1^{N-j} [\mu(1-q_1)]^j \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\gamma)} \frac{1}{(\theta+\lambda+\mu)^j} \right\}(t)$$

$$+ \frac{\gamma^2 q_1}{(1-q_1)} q_1^{N-1} \mathcal{L}_\theta^{-1}\left\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\gamma)} \right\}(t)$$

$$+ \frac{\gamma^2 q_1}{(1-q_1)} \sum_{s=1}^{N-1} \binom{N-1}{s} q_1^{N-s-1} [\mu(1-q_1)]^s \mathcal{L}_\theta^{-1}\Big\{ \frac{1}{(\theta+\lambda)} \frac{1}{(\theta+\gamma)} \frac{1}{(\theta+\lambda+\mu)^s} \Big\}(t).$$

Applying Lemma 5 and after some algebra, we obtain the density function as

$$f_{\tau_\rho}(t) = \Big(\gamma + \frac{\gamma^2}{\lambda-\gamma}\Big)(e^{-\gamma t} - e^{-\lambda t})$$

$$+ \gamma \sum_{k=1}^{N} \binom{N}{k} q_1^{N-k}(1-q_1)^{k-1} e^{-\lambda t} P(k-1, (\mu t))$$

$$- \gamma \sum_{i=1}^{N-1} \binom{N-1}{i} q_1^{N-i}(1-q_1)^{i-1} e^{-\lambda t} P(i-1, (\mu t))$$

$$- \gamma^2 \sum_{j=1}^{N} \binom{N}{j} q_1^{N-j}(1-q_1)^{j-1} e^{-\gamma t}\Big\{ \frac{1}{(\lambda-\gamma)} \Big[1 - e^{-(\lambda-\gamma)t}\Big]$$

$$+ \frac{1}{\mu} \sum_{d=0}^{j-1} (-1)^d \Big[1 - e^{\mu t} \sum_{h=0}^{d} \frac{(-\mu)^h}{h!} t^h\Big]\Big\}$$

$$+ \gamma^2 \sum_{s=1}^{N-1} \binom{N-1}{s} q_1^{N-s}(1-q_1)^{s-1} e^{-\gamma t}\Big\{ \frac{1}{(\lambda-\gamma)} \Big[1 - e^{-(\lambda-\gamma)t}\Big]$$

$$+ \frac{1}{\mu} \sum_{l=0}^{s-1} (-1)^l \Big[1 - e^{\mu t} \sum_{r=0}^{l} \frac{(-\mu)^r}{r!} t^r\Big]\Big\}. \qquad \square$$

### 3.2.4 The Marginal Transform of $A_\rho$, $Eu^{A_\rho}$.

Letting $v = 1$ and $\theta = 0$ in (3.2.6) to (3.2.8) and using this in (3.2.5) of Theorem 6, and after some algebra, we arrive at

$$Eu^{A_\rho} = 1 - \Bigg[1 + \tfrac{\gamma}{\lambda(1-q_1)}\left\{1 - \left(\tfrac{\lambda q_1+\mu}{\lambda+\mu}\right)^N\right\}$$

$$- \tfrac{\gamma q_1}{\lambda(1-q_1)}\left\{1 - \left(\tfrac{\lambda q_1+\mu}{\lambda+\mu}\right)^{N-1}\right\}\Bigg]$$

$$+ \tfrac{\gamma(1-qu)}{[\gamma(1-qu)+\lambda(1-u)]}$$

$$\times \Bigg[1 + \tfrac{\gamma}{\lambda(1-q_1)}\left\{1 - \left(\tfrac{\lambda q_1+\mu}{\lambda+\mu}\right)^N\right\}$$

$$- \tfrac{\gamma q_1}{\lambda(1-q_1)}\left\{1 - \left(\tfrac{\lambda q_1+\mu}{\lambda+\mu}\right)^{N-1}\right\}\Bigg]. \qquad \square$$

### 3.2.5 Special Case of a Geometrically Distributed Attack with $M \geq 2$ and for any $N$.

**Theorem 7**. *For the special case where $M \geq 2$ and $N$ is any integer, then under the conditions and assumptions of (3.2.1)-(3.2.3), the joint functional, $\Phi_w$ of the first passage time of the game satisfies the following formulas*:

$$\Phi_w(u, v, \theta) = \gamma \cdot (\psi - qu\eta - q_1 v\xi + qq_1 uvB_1)$$

$$[\gamma(u, v, \theta) - 1] + \gamma(u, v, \theta). \qquad (3.2.15)$$

Where,

$$\psi = \frac{1}{A_1}\Bigg\{[1-J^M] - R^N \sum_{k=0}^{N\wedge\{M-1\}} (-1)^k \binom{M-2}{k} W^k$$

$$\times \left[\sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \{S^j - J^{M-k} G^j\}\right]\Bigg\}. \tag{3.2.16}$$

$$\eta = \frac{qu}{A_1}\frac{1}{A_1}\Bigg\{[1-J^M] - R^N \sum_{i=0}^{N\wedge\{M-2\}} (-1)^i \binom{N}{i} W^i$$

$$\times \left[\sum_{r=0}^{M-i-2} \binom{N+r-1}{r} \{S^r - J^{M-i-1} G^r\}\right]\Bigg\}, \tag{3.2.17}$$

$$\xi = \frac{1}{A_1}\Bigg\{[1-J^M] - R^{N-1} \sum_{s=0}^{N\wedge\{M-1\}} (-1)^s \binom{N-1}{s} W^s$$

$$\times \left[\sum_{g=0}^{M-s-1} \binom{N+g-2}{g} \{S^g - J^{M-s} G^g\}\right]\Bigg\}, \tag{3.2.18}$$

$$B_1 = \frac{qu}{A_1}\Bigg\{[1-J^{M-1}] - R^{N-1} \sum_{n=0}^{\{N-1\}\wedge\{M-2\}} (-1)^n \binom{N-1}{n} W^n$$

$$\times \left[\sum_{h=0}^{M-n-2} \binom{N+h-2}{h} \{S^h - J^{M-n-1} G^h\}\right]\Bigg\}, \tag{3.2.19}$$

$$\gamma(u,v,\theta) = \frac{\gamma(1-qu)(1-q_1v)}{(\gamma+\theta)(1-qu)(1-q_1v)+\lambda(1-u)(1-q_1v)+\mu(1-v)(1-qu)}, \tag{3.2.20}$$

$$a_1 = (1-q_1v)(\theta+\lambda) + \mu(1-v), \tag{3.2.21}$$

$$b_1 = qu(\theta+\mu-q_1v\theta-\mu v) + \lambda u(1-q_1v), \tag{3.2.22}$$

$$A_1 = a_1 - b_1, \qquad J = \tfrac{qu(\theta+\mu-q_1 v\theta-\mu v)+\lambda u(1-q_1 v)}{(1-q_1 v)(\theta+\lambda)+\mu(1-v)}, \tag{3.2.23}$$

$$R = \tfrac{q_1 v(\theta+\lambda)+\mu v}{\theta+\lambda+\mu}, \qquad S = \tfrac{u(q\theta+\lambda+q\mu)}{\theta+\lambda+\mu}, \tag{3.2.24}$$

$$W = \tfrac{uv(qq_1\theta+\lambda q_1+q\mu)}{q_1 v(\theta+\lambda)+\mu v} \qquad G = \tfrac{S}{J}. \tag{3.2.25}$$

**Proof**. From Theorem 3$(i)$,

$$\mathcal{D}_{xy}^{M-1,\,N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\,\theta)}\Big\} = \mathcal{D}_x^{M-1}\Big\{\mathcal{D}_y^{N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\,\theta)}\Big\}\Big\}.$$

From (3.2.4) and using Theorem 3 $(iii)$,

$$\mathcal{D}_y^{N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\,\theta)}\Big\} = \mathcal{D}_y^{N-1}\Big\{1+\gamma\cdot\Big(\tfrac{a}{c-dy}+\tfrac{by}{c-dy}\Big)\Big\}$$

$$= 1+\gamma\cdot\tfrac{a}{c}\mathcal{D}_y^{N-1}\Big\{\tfrac{1}{1-\frac{d}{c}y}\Big\}+\gamma\cdot\tfrac{b}{c}\mathcal{D}_y^{N-2}\Big\{\tfrac{1}{1-\frac{d}{c}y}\Big\}$$

$$= 1+\gamma\cdot\tfrac{a}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^N}{1-\frac{d}{c}}\right\}+\gamma\cdot\tfrac{b}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^{N-1}}{1-\frac{d}{c}}\right\}.$$

Therefore,

$$\mathcal{D}_{xy}^{M-1,\,N-1}\Big\{\tfrac{1}{1-\gamma(ux,vy,\,\theta)}\Big\}$$

$$= \mathcal{D}_x^{M-1}\left\{1+\gamma\cdot\tfrac{a}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^N}{1-\frac{d}{c}}\right\}+\gamma\cdot\tfrac{b}{c}\left\{\tfrac{1-\left(\frac{d}{c}\right)^{N-1}}{1-\frac{d}{c}}\right\}\right\}$$

$$= 1+\gamma\mathcal{D}_x^{M-1}\Big\{\tfrac{a}{c-d}\Big[1-\big(\tfrac{d}{c}\big)^N\Big]\Big\}$$

$$+\gamma\mathcal{D}_x^{M-1}\Big\{\tfrac{b}{c-d}\Big[1-\big(\tfrac{d}{c}\big)^{N-1}\Big]\Big\}$$

$$= 1+\gamma\mathcal{D}_x^{M-1}\left\{\tfrac{1-qux}{a_1-b_1x}\left[1-\left(\tfrac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^N\right]\right\}$$

$$- \gamma q_1 v \mathcal{D}_x^{M-1}\left\{ \frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N-1}\right]\right\}, \qquad (3.2.26)$$

where,

$$c = \alpha_1 - \beta_1 x, \qquad d = \alpha - \beta x,$$

$$\alpha_1 = \theta + \lambda + \mu, \qquad \beta_1 = qu\theta + \lambda u + q\mu u,$$

$$\alpha = q_1 v\theta + \lambda q_1 v + \mu v, \qquad \beta = qq_1 uv\theta + \lambda q_1 uv + q\mu uv,$$

$$a_1 = \theta + \lambda + \mu - \lambda q_1 v - q_1 v\theta - \mu v,$$

$$b_1 = qu\theta + \lambda u + q\mu u - qq_1 uv\theta - \lambda q_1 uv - q\mu uv,$$

$$c - d = a_1 - b_1 x.$$

The first part in (3.2.26) is given as

$$\mathcal{D}_x^{M-1}\left\{ \frac{1-qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N}\right]\right\}$$

$$= \mathcal{D}_x^{M-1}\left\{ \frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N}\right]\right\}$$

$$- \mathcal{D}_x^{M-1}\left\{ \frac{qux}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N}\right]\right\}. \qquad (3.2.27)$$

Proceeding like in Theorem 6 section 3.2.1, we have

$$\mathcal{D}_x^{M-1}\left\{ \frac{1}{a_1-b_1x}\left[1-\left(\frac{\alpha-\beta x}{\alpha_1-\beta_1 x}\right)^{N}\right]\right\}$$

$$= \left[\frac{1-\left(\frac{b_1}{a_1}\right)^M}{a_1-b_1}\right] - \left(\frac{\alpha}{\alpha_1}\right)^N \sum_{k=0}^{N\wedge\{M-1\}} \binom{N}{k}\left(\frac{-\beta}{\alpha}\right)^k$$

$$\times \left[ \frac{1}{a_1-b_1} \sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \left\{ \left(\frac{\beta_1}{\alpha_1}\right)^j - \left(\frac{b_1}{a_1}\right)^{M-k} \left(\frac{\beta_1 a_1}{\alpha_1 b_1}\right)^j \right\} \right]. \tag{3.2.28}$$

While the second part of (3.2.27) is calculated as

$$\mathcal{D}_x^{M-1} \left\{ \frac{qux}{a_1-b_1x} \left[ 1 - \left( \frac{\alpha-\beta x}{\alpha_1-\beta_1 x} \right)^N \right] \right\}$$

$$= qu \left[ \frac{1-\left(\frac{b_1}{a_1}\right)^{M-1}}{a_1-b_1} - \left(\frac{\alpha}{\alpha_1}\right)^N \sum_{i=0}^{N\wedge\{M-2\}} \binom{N}{i} \left(\frac{-\beta}{\alpha}\right)^i \right.$$

$$\left. \times \left\{ \frac{1}{a_1-b_1} \sum_{r=0}^{M-i-2} \binom{N+r-1}{r} \left[ \left(\frac{\beta_1}{\alpha_1}\right)^r - \left(\frac{b_1}{a_1}\right)^{M-i-1} \left(\frac{\beta_1 a_1}{\alpha_1 b_1}\right)^r \right] \right\} \right]. \tag{3.2.29}$$

For $M-2 \geq 0$.

Next, we find the second part of (3.2.26),

$$\mathcal{D}_x^{M-1} \left\{ \frac{1-qux}{a_1-b_1x} \left[ 1 - \left( \frac{\alpha-\beta x}{\alpha_1-\beta_1 x} \right)^{N-1} \right] \right\}$$

$$= \mathcal{D}_x^{M-1} \left\{ \frac{1}{a_1-b_1x} \left[ 1 - \left( \frac{\alpha-\beta x}{\alpha_1-\beta_1 x} \right)^{N-1} \right] \right\}$$

$$- \mathcal{D}_x^{M-1} \left\{ \frac{qux}{a_1-b_1x} \left[ 1 - \left( \frac{\alpha-\beta x}{\alpha_1-\beta_1 x} \right)^{N-1} \right] \right\}. \tag{3.2.30}$$

Calculating the first and second part of (3.2.30) like before we arrive at,

$$\mathcal{D}_x^{M-1} \left\{ \frac{1}{a_1-b_1x} \left[ 1 - \left( \frac{\alpha-\beta x}{\alpha_1-\beta_1 x} \right)^{N-1} \right] \right\}$$

$$= \frac{1-\left(\frac{b_1}{a_1}\right)^M}{a_1-b_1} - \left(\frac{\alpha}{\alpha_1}\right)^{N-1} \sum_{s=0}^{N\wedge\{M-1\}} \binom{N-1}{s} \left(\frac{-\beta}{\alpha}\right)^s$$

$$\times \left[ \frac{1}{a_1 - b_1} \sum_{g=0}^{M-s-1} \binom{N+g-2}{g} \left\{ \left(\frac{\beta_1}{\alpha_1}\right)^g - \left(\frac{b_1}{a_1}\right)^{M-s} \left(\frac{\beta_1 a_1}{\alpha_1 b_1}\right)^g \right\} \right], \tag{3.2.31}$$

and

$$\mathcal{D}_x^{M-1} \left\{ \frac{qux}{a_1 - b_1 x} \left[ 1 - \left( \frac{\alpha - \beta x}{\alpha_1 - \beta_1 x} \right)^{N-1} \right] \right\}$$

$$= qu \left[ \frac{1 - \left(\frac{b_1}{a_1}\right)^{M-1}}{a_1 - b_1} - \left(\frac{\alpha}{\alpha_1}\right)^{N-1} \sum_{n=0}^{\{N-1\} \wedge \{M-2\}} \binom{N-1}{n} \left(\frac{-\beta}{\alpha}\right)^n \right.$$

$$\times \left\{ \frac{1}{a_1 - b_1} \sum_{h=0}^{M-n-2} \binom{N+h-2}{h} \right.$$

$$\left. \left. \times \left[ \left(\frac{\beta_1}{\alpha_1}\right)^h - \left(\frac{b_1}{a_1}\right)^{M-n-1} \left(\frac{\beta_1 a_1}{\alpha_1 b_1}\right)^h \right] \right\} \right]. \tag{3.2.32}$$

respectively for $M - 2 \geq 0$.

Then using (3.2.29), (3.2.30), (3.2.31) and (3.2.32) in (3.2.26) we obtain the expression for

$$\mathcal{D}_{xy}^{M-1,\, N-1} \left\{ \frac{1}{1 - \gamma(ux, vy, \theta)} \right\},$$

substituting this into 2.2.4 the statement now follows from (3.2.15)-(3.2.25). □

### 3.2.6 The Marginal Transform of $\tau_\rho$.

Letting $u = v = 1$ in (3.2.15) to (3.2.25) and simplifying, we have

$$Ee^{-\theta \tau_\rho} = \gamma \cdot (H_1 - qJ_1 - q_1 R_1 + qq_1 G_1)$$

$$+ \frac{\gamma}{\gamma + \theta} [1 + \gamma \cdot (H_1 - qJ_1 - q_1 R_1 + qq_1 G_1)].$$

Where,

$$H_1 = \frac{1}{(\theta - q\theta)(1-q_1)} \Bigg\{ \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^M \right]$$

$$- \left[\frac{q_1(\lambda+\theta)+\mu}{\theta+\lambda+\mu}\right]^N \sum_{k=0}^{N\wedge\{M-1\}} \binom{M-2}{k} \left(\frac{-q_1(\lambda+q\theta)-q\mu}{q_1(\lambda+\theta)+\mu}\right)^k$$

$$\times \left[ \sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \left(\frac{q(\theta+\mu)+\lambda}{\theta+\lambda+\mu}\right)^j \left\{ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-k-j} \right\} \right] \Bigg\},$$

$$J_1 = \frac{qu}{(\theta - q\theta)(1-q_1)} \Bigg\{ \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-1} \right]$$

$$- \left[\frac{q_1(\lambda+\theta)+\mu}{\theta+\lambda+\mu}\right]^N \sum_{i=0}^{N\wedge\{M-2\}} \binom{N}{i} \left(\frac{-q_1(\lambda+q\theta)-q\mu}{q_1(\lambda+\theta)+\mu}\right)^i$$

$$\times \sum_{r=0}^{M-i-2} \binom{N+r-1}{r} \left(\frac{q(\theta+\mu)+\lambda}{\theta+\lambda+\mu}\right)^r \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-i-r-1} \right] \Bigg\},$$

For $M - 2 \geq 0$.

$$R_1 = \frac{1}{(\theta - q\theta)(1-q_1)} \Bigg\{ \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^M \right]$$

$$- \left[\frac{q_1(\lambda+\theta)+\mu}{\theta+\lambda+\mu}\right]^{N-1} \sum_{s=0}^{N\wedge\{M-1\}} \binom{N-1}{s} \left(\frac{-q_1(\lambda+q\theta)-q\mu}{q_1(\lambda+\theta)+\mu}\right)^s$$

$$\times \sum_{g=0}^{M-s-1} \binom{N+g-2}{g} \left(\frac{q(\theta+\mu)+\lambda}{\theta+\lambda+\mu}\right)^g \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-s-g} \right] \Bigg\},$$

$$G_1 = \frac{qu}{(\theta - q\theta)(1-q_1)} \Bigg\{ \left[ 1 - \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-1} \right]$$

$$- \left[\tfrac{q_1(\lambda+\theta)+\mu}{\theta+\lambda+\mu}\right]^{N-1} \sum_{n=0}^{\{N-1\}\wedge\{M-2\}} \tbinom{N-1}{n} \left(\tfrac{-q_1(\lambda+q\theta)-q\mu}{q_1(\lambda+\theta)+\mu}\right)^n$$

$$\times \sum_{h=0}^{M-n-2} \tbinom{N+r-2}{r} \left(\tfrac{q(\theta+\mu)+\lambda}{\theta+\lambda+\mu}\right)^h \left[1 - \left(\tfrac{q\theta+\lambda}{\theta+\lambda}\right)^{M-n-h-1}\right] \Bigg\},$$

For $M - 2 \geq 0$.

$\gamma(u, v, \theta) = \frac{\gamma}{\gamma+\theta}$. □

### 3.2.7 The Marginal Transform of $A_\rho$.

Letting $v = 1$ and $\theta = 0$ in (3.2.15) to (3.2.25) we have,

$$Eu^{A_\rho} = \gamma \cdot (H_2 - quJ_2 - q_1R_2 + qq_1uG_2)\left(\tfrac{\gamma(1-q)}{\gamma(1-q)+\lambda(1-u)} - 1\right)$$

$$+ \tfrac{\gamma(1-q)}{\gamma(1-q)+\lambda(1-u)}.$$

Where,

$$H_2 = \tfrac{1}{\lambda(1-u)}\Bigg\{[1-u^M] - \left[\tfrac{q_1\lambda+\mu}{\lambda+\mu}\right]^N \sum_{k=0}^{N\wedge\{M-1\}} \tbinom{N}{k} \left(\tfrac{-\lambda q_1-q\mu}{\lambda q_1+\mu}\right)^k$$

$$\times \left[\sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \left(\tfrac{\lambda+q\mu}{\lambda+\mu}\right)^j (1-u^{M-k-j})\right]\Bigg\},$$

$$J_2 = \tfrac{qu}{\lambda(1-u)}\Bigg\{[1-u^{M-1}] - \left[\tfrac{q_1\lambda+\mu}{\lambda+\mu}\right]^N \sum_{i=0}^{N\wedge\{M-2\}} \tbinom{N}{i} \left(\tfrac{-\lambda q_1-q\mu}{\lambda q_1+\mu}\right)^i$$

$$\times \left[\sum_{r=0}^{M-i-2} \tbinom{N+r-1}{r} \left(\tfrac{\lambda+q\mu}{\lambda+\mu}\right)^r (1-u^{M-i-r-1})\right]\Bigg\},$$

$$R_2 = \frac{1}{\lambda(1-u)}\Bigg\{[1-u^M] - \left[\frac{q_1\lambda+\mu}{\lambda+\mu}\right]^{N-1} \sum_{s=0}^{N\wedge\{M-1\}} \binom{N-1}{s}\left(\frac{-\lambda q_1 - q\mu}{\lambda q_1+\mu}\right)^s$$

$$\times \left[\sum_{g=0}^{M-s-1}\binom{N+g-2}{g}\left(\frac{\lambda+q\mu}{\lambda+\mu}\right)^g (1-u^{M-s-g})\right]\Bigg\},$$

$$G_2 = \frac{qu}{\lambda(1-u)}\Bigg\{[1-u^{M-1}] - \left[\frac{q_1\lambda+\mu}{\lambda+\mu}\right]^{N-1} \sum_{n=0}^{\{N-1\}\wedge\{M-2\}} \binom{N-1}{n}\left(\frac{-\lambda q_1 - q\mu}{\lambda q_1+\mu}\right)^n$$

$$\times \left[\sum_{h=0}^{M-n-2}\binom{N+h-1}{h}\left(\frac{\lambda+q\mu}{\lambda+\mu}\right)^h (1-u^{M-n-h-1})\right]\Bigg\},$$

$$\gamma(u,v,\theta) = \frac{\gamma(1-q)}{\gamma(1-q)+\lambda(1-u)}. \qquad \square$$

### 3.2.8 The Probability Density Function, $f_{\tau_\rho}$ of $\tau_\rho$.

We state below the following Proposition, Corollary and Theorem by which we are able find some inverse Laplace transform and hence the probability density function, pdf of the first passage time.

**Proposition 8:** *Let $g$ and $h$ be analytic functions on $A \subseteq \mathbb{C}$ and $g$ have a zero at $z_0$ of order $k$ while $h$ has a zero at $z_0$ of order $k+m$. Then the function $f = g/h$ has a pole of order $m$ at $z_0$. Furthermore in the vicinity of $z_0$,*

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m},$$

such that $\varphi$ is analytic at $z_0$ and $\varphi(z_0) \neq \theta$.

**Proposition 9:** *Under the assumptions of Proposition 8,*

$$\text{Res}(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

Note that proposition 9 is practical only if we can figure out the function $\varphi$ explicitly.

**Theorem 10:** *Suppose $F$ is analytic on $\mathbb{C} - \{z_1, \ldots, z_n\}$, where $z_k$'s are isolated singularities, and let $\sigma$ be a real number such that $F$ is analytic on the semi-plane $\{z \in \mathbb{C} : \text{Re}z > \sigma\}$. Furthermore, suppose there are positive constants $M, R,$ and $\gamma$ such that $\|F(z)\| \leq M/\|z\|^{\gamma}$ for all $\|z\| \geq R$. Denote*

$$f(t) = \sum_{k=1}^{n} \{\text{Res}(e^{zt}F(z), z_k)\}.$$

*Then $\mathcal{L}f(s) = F(s)$, for all $\text{Re}s > \sigma$.*

In other words, $f = \mathcal{L}^{-1}F$.

**Corollary 11:** *Suppose under the assumptions of Theorem 10, $F(z) = P(z)/Q(z)$, where $P$ and $Q$ are polynomials such that* $\deg P < \deg Q$ *and such that $z_1, \ldots, z_n$ are poles of $P/Q$. Then,*

$$f(t) = \sum_{k=1}^{n} \text{Res}\left(e^{z_k t}\frac{P(z)}{Q(z)}, z_k\right),$$

*and* $\sigma_f = \max\{\text{Re}z_k : k = 1, \ldots, m\}$.

**Corollary 12.** *Suppose under the assumptions of Corollary* 11*, $F(z) = P(z)/Q(z)$, where $P$ and $Q$ are polynomials such that* deg$P <$ deg$Q$ *and such that $z_1, \ldots, z_n$ are simple zeros of $Q$ which are not zeros of $P$. Then*

$$f(t) = \sum_{k=1}^{n} e^{z_k t} \frac{P(z_k)}{Q'(z_k)},$$

*and* $\sigma_f = \max\{\mathrm{Re} z_k : k = 1, \ldots, m\}$.

**The Probability Density Function, $f_{\tau_\rho}$ of $\tau_\rho$.** We now return to section 3.2.7 and give an outline of obtaining the probability density function of the first observed passage time of the game end. Again like before, we need to take the Laplace inverse to obtain the density function.

$$f_{\tau_\rho} = \mathcal{L}_\theta^{-1}\{Ee^{-\theta\tau_\rho}\} = \gamma \cdot \mathcal{L}_\theta^{-1}\{(H_1 - qJ_1 - q_1R_1 + qq_1G_1)\}(t)$$

$$+ \gamma \cdot \mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}[1 + \gamma \cdot (H_1 - qJ_1 - q_1R_1 + qq_1G_1)]\right\}(t)$$

$$= \gamma \cdot [\mathcal{L}_\theta^{-1}\{H_1\} - q\mathcal{L}_\theta^{-1}\{J_1\} - q_1\mathcal{L}_\theta^{-1}\{R_1\} + qq_1\mathcal{L}_\theta^{-1}\{G_1\}](t)$$

$$+ \gamma\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}\right\}(t) + \gamma^2\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}H_1\right\}(t) - q\gamma^2\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}J_1\right\}(t)$$

$$- q_1\gamma^2\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}R_1\right\}(t) + qq_1\gamma^2\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\gamma+\theta}G_1\right\}(t).$$

Where,

$H_1, J, R_1, G_1$ are as defined in 3.2.7

$$\mathcal{L}_\theta^{-1}\{H_1\}(t) = \tfrac{1}{(1-q)(1-q_1)}\left[\mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\theta}\right\}(t) - \mathcal{L}_\theta^{-1}\left\{\tfrac{1}{\theta}\left(\tfrac{q\theta+\lambda}{\theta+\lambda}\right)^M\right\}(t)\right.$$

$$- \mathcal{L}_\theta^{-1}\Bigg\{ \frac{1}{\theta}\left[\frac{q_1\lambda+q_1\theta+\mu}{\theta+\lambda+\mu}\right]^N \sum_{k=0}^{N\wedge\{M-1\}} \binom{M-2}{k} \left(\frac{-q_1\lambda-q_1q\theta-q\mu}{q_1\lambda+q_1\theta+\mu}\right)^k$$

$$\times \sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \left(\frac{q\theta+q\mu+\lambda}{\theta+\lambda+\mu}\right)^j \Bigg\}(t)$$

$$+ \mathcal{L}_\theta^{-1}\Bigg\{ \frac{1}{\theta}\left[\frac{q_1\lambda+q_1\theta+\mu}{\theta+\lambda+\mu}\right]^N \sum_{k=0}^{N\wedge\{M-1\}} \binom{M-2}{k} \left(\frac{-q_1\lambda-q_1q\theta-q\mu}{q_1\lambda+q_1\theta+\mu}\right)^k$$

$$\times \sum_{j=0}^{M-k-1} \binom{N+j-1}{j} \left(\frac{q\theta+q\mu+\lambda}{\theta+\lambda+\mu}\right)^j \left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^{M-k-j} \Bigg\}(t)\Bigg]. \qquad (3.2.33)$$

The first term of the above is given as

$$\mathcal{L}_\theta^{-1}\left\{\frac{1}{\theta}\right\}(t) = 1.$$

With some algebra, we can rewrite the second term of (3.2.33) as

$$\mathcal{L}_\theta^{-1}\left\{\frac{1}{\theta}\left(\frac{q\theta+\lambda}{\theta+\lambda}\right)^M\right\}(t)$$

$$= 1 + M(q-1)e^{-\lambda t} + (q-1)^\delta \sum_{\delta=2}^{M} \binom{M}{\delta} \mathcal{L}_\theta^{-1}\left\{\frac{\theta^{\delta-1}}{(\theta+\lambda)^\delta}\right\}(t).$$

Now, since the degree of the numerator is less than the degree of the denominator in the inverse Laplace transform of the last expression, we can apply Corollary 12.

Similarly, we can obtain the inverse Laplace transforms of the second, third and fourth term of (3.2.33). Finally, observe that for $J_1$, $R_1$ and $G_1$ the degree in the numerator is lesser compared to that of the denominator, hence the same technique above could be used to find the inverse Laplace transforms.

## 3.3 Mixed Delayed-Time Game

In this section, we assume that one of the components is discrete while the other component is continuous and exponentially distributed with parameter $\nu$. We also assume that their respective thresholds are $M$ and $Q$. We suppose further that the game is being observed at random times according to a renewal point process, $\mathcal{T}$ with interrenewal times $\Delta_1 = \tau_1, \Delta_2 = \tau_2 - \tau_1, \ldots \in [\Delta]$, that are exponential and identically distributed with the common Laplace-Stieltjes transform.

$$\gamma(\theta) = Ee^{-\theta\Delta} = \frac{\gamma}{\gamma+\theta}. \tag{3.3.1}$$

Then we have

$$\begin{aligned} \gamma(u, v, \theta) = \ & \mathrm{E}u^{X_1} e^{-vY_1} e^{-\theta\tau_1} \\ & = \gamma(\theta + \lambda - \lambda u + \mu - \mu g(v)). \end{aligned} \tag{3.3.2}$$

Where,

$$g(v) = \frac{\nu}{\nu+v}. \tag{3.3.3}$$

Then (3.3.2) becomes

$$\gamma(u, v, \theta) = \frac{\gamma}{\gamma+\theta+\lambda+\mu-\lambda u-\mu\frac{\nu}{\nu+v}}, \tag{3.3.4}$$

and

$$\frac{1}{1-\gamma(ux, v+y, \theta)} = 1 + \gamma\frac{1}{\theta+\lambda-\lambda ux+\mu\frac{v+y}{\nu+v+y}}, \tag{3.3.5}$$

which after some algebra gives

$$\frac{1}{1-\gamma(ux,\, v+y,\, \theta)} = 1 + \gamma\left(\frac{\nu + v}{b + \eta y} + \frac{y}{b + \eta y}\right). \tag{3.3.6a}$$

With

$$b = (\theta + \lambda - \lambda ux)(\nu + v) + \mu v, \quad \eta = \theta + \lambda - \lambda ux + \mu. \tag{3.3.6b}$$

### 3.3.1 The Mixed Game.

**Theorem 13**. *Under the assumptions and conditions of (3.3.1)-(3.3.6), the joint functional $\Phi_m$ of the first passage time satisfies the following formulas:*

$$\Phi_m(u, v, \theta) = 1 - F\Bigg\{1 + L\Bigg[Z - e^{-(\nu+v)Q}\sum_{j=0}^{M-1} a^j - e^{-(\nu+v)Q} \times \sum_{n=0}^{M-1} a^n \sum_{k=0}^{n} G^k \sum_{r=1}^{\infty} (NQ)^r \binom{k+r-1}{k}\Bigg] + \gamma\delta_0 e^{-(\nu+v)Q}\left[\sum_{j=0}^{M-1} a^j + \sum_{n=0}^{M-1} a^n \sum_{k=0}^{n} G^k \times \sum_{r=1}^{\infty} (NQ)^r \binom{k+r-1}{k}\right]\Bigg\}. \tag{3.3.7}$$

Where,

$$\delta = \frac{1}{\mu v + (\theta+\lambda)(\nu + v)}, \qquad L = (\nu + v)\gamma\delta, \tag{3.3.8}$$

$$G = \frac{\mu v + (\theta+\lambda)(\nu + v)}{(\theta+\lambda+\mu)(\nu + v)}, \qquad N = \delta_0\mu\nu, \tag{3.3.9}$$

$$a = \frac{\lambda u(\nu + v)}{\mu v + (\theta+\lambda)(\nu + v)}, \qquad 1 - a = \frac{\mu v + (\theta+\lambda-\lambda\mu)(\nu + v)}{\mu v + (\theta+\lambda)(\nu + v)}, \tag{3.3.10}$$

$$F = \frac{(\theta+\lambda-\lambda u+\mu)(\nu + v)-\mu\nu}{(\gamma+\theta+\lambda-\lambda u+\mu)(\nu + v)-\mu\nu}, \quad \delta_0 = \frac{1}{\theta+\lambda+\mu}, \tag{3.3.11}$$

$$\frac{1-a^M}{1-a} = Z. \tag{3.3.12}$$

**Proof**.

From (2.1.7), (3.3.6a) and the definition of the inverse Laplace-Carson transform of Theorem 2, we have,

$$\mathcal{D}_x^{M-1}\mathcal{L}c_y^{-1}\Big\{\frac{1}{1-\gamma(ux,v+y,\theta)}\Big\}(Q)$$

$$= 1 - [1-\gamma(u,v,\theta)]$$

$$\times\, \mathcal{D}_x^{M-1}\Big(1+\gamma\Big[\tfrac{1}{b}(\nu+v)\Big(1-e^{-\frac{b}{\eta}Q}\Big)+\tfrac{1}{\eta}e^{-\frac{b}{\eta}Q}\Big]\Big). \tag{3.3.13a}$$

Where,

$$\frac{b}{\eta} = \frac{(\theta+\lambda-\lambda ux)(\nu+v)+\mu v}{\theta+\lambda-\lambda ux+\mu} = (v+\nu) - \frac{\mu\nu}{\theta+\lambda+\mu-\lambda ux}. \tag{3.3.13b}$$

Which now implies that

$$e^{-\frac{b}{\eta}Q} = e^{-(\nu+v)Q}\ e^{\delta_0\mu\nu Q\cdot\frac{1}{1-a_0x}}. \tag{3.3.14a}$$

With,

$$\delta_0 = \frac{1}{\theta+\lambda+\mu},\quad a_0 = \frac{\lambda u}{\theta+\lambda+\mu}, \tag{3.3.14b}$$

$$\frac{1}{\eta} = \delta_0\cdot\frac{1}{1-a_0x},\qquad \frac{1}{b} = \delta\cdot\frac{1}{1-a\,x}, \tag{3.3.14c}$$

$$\delta = \frac{1}{\mu v+(\theta+\lambda)(\nu+v)},\quad a = \frac{\lambda u(\nu+v)}{\mu v+(\theta+\lambda)(\nu+v)}. \tag{3.3.14d}$$

Substituting (3.3.14 a-d) into (2.1.7) and after some algebra we obtain

$$\Phi_m(u,v,\theta) = 1 - [1-\gamma(u,v,\theta)]\Big\{1 + (\nu+v)\gamma\Big[\delta \cdot \mathcal{D}_x^{M-1}\big(\tfrac{1}{1-a\,x}\big)$$

$$- \delta e^{-(\nu+v)Q}\,\mathcal{D}_x^{M-1}\Big(\tfrac{1}{1-ax} e^{\delta_0\mu\nu Q \cdot \frac{1}{1-a_0 x}}\Big)\Big]$$

$$+ \gamma\delta_0 e^{-(\nu+v)Q}\,\mathcal{D}_x^{M-1}\Big[\tfrac{1}{1-\,a_0 x}\; e^{\delta_0\mu\nu Q \cdot \frac{1}{1-a_0 x}}\Big]\Big\}. \qquad (3.3.15)$$

by the Taylor's series expansion, **T** we have,

$$\mathbf{T}e^{\delta_0\mu\nu Q \cdot \frac{1}{1-a_0 x}} = \mathbf{T}e^{\sigma \cdot \frac{1}{1-a_0 x}} = \sum_{r=0}^{\infty} \sigma^r \Big[\tfrac{1}{1-a_0 x}\Big]^r.$$

From the sum above we extract the first term, that is, when $r = 0$ to obtain

$$\mathbf{T}e^{\delta_0\mu\nu Q \cdot \frac{1}{1-a_0 x}} = 1 + \sum_{r=1}^{\infty} \sigma^r \Big[\tfrac{1}{1-a_0 x}\Big]^r,$$

for $|x| < 1$ and $|a_0| < 1$. We rewrite this as

$$\mathbf{T}e^{\delta_0\mu\nu Q \cdot \frac{1}{1-a_0 x}} = 1 + \sum_{r=1}^{\infty} \sigma^r \sum_{n=0}^{\infty} \binom{n+r-1}{n} a_0^n x^n$$

$$= 1 + \sum_{n=0}^{\infty} A_n x^n. \qquad (3.3.16)$$

Where,

$$A_n = a_0^n \sum_{r=1}^{\infty} \sigma^r \binom{n+r-1}{n}.$$

Similarly,

$$\mathbf{T}\tfrac{1}{1-ax} = \sum_{j=0}^{\infty} a^j x^j. \qquad (3.3.17)$$

Combining (3.3.17) and (3.3.16) and interchanging sums, we have

$$\mathbf{T}_{\frac{1}{1-a\,x}} e^{\delta_0 \mu\nu Q \cdot \frac{1}{1-a_0 x}} = \sum_{j=0}^{\infty} a^j x^j + \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} A_k \, a^{n-k} \right) x^n. \tag{3.3.18}$$

Therefore, using (3.3.18) in (3.3.15) and applying theorem 3$(vii)$ we obtain

$$\mathcal{D}_x^{M-1}\left[\frac{1}{1-ax} e^{\delta_0 \mu\nu Q \cdot \frac{1}{1-a_0 x}}\right]$$

$$= \sum_{j=0}^{M-1} a^j + \sum_{n=0}^{M-1} a^n \sum_{k=0}^{n} \left(\frac{a_0}{a}\right)^k \sum_{r=1}^{\infty} \sigma^r \binom{k+r-1}{k}. \tag{3.3.19}$$

Using (3.3.19) and theorem 3 $(iv)$ (for the case where $b \neq 1$) in (3.3.15) the result follows from (3.3.7)-(3.3.12) and this completes the proof. □

### 3.3.2 Marginal Transform of $A_\rho$, $Eu^{A_\rho}$.

Letting $v = \theta = 0$ in (3.3.7)-(3.3.12), we have

$$Eu^{A_\rho} = 1 - \frac{(\lambda-\lambda u)}{(\gamma+\lambda-\lambda u)} \Bigg\{ 1 + \frac{\gamma}{\lambda} \Bigg[ \frac{1-u^M}{1-u} - e^{-\nu Q} \sum_{j=0}^{M-1} u^j$$

$$- e^{-\nu Q} \sum_{n=0}^{M-1} u^n \sum_{k=0}^{n} \left(\frac{\lambda}{\lambda+\mu}\right)^k \sum_{r=1}^{\infty} \binom{k+r-1}{k} \left(\frac{\mu\nu Q}{\lambda+\mu}\right)^r \Bigg]$$

$$+ \frac{\gamma}{\lambda+\mu} e^{-\nu Q} \left[ \sum_{j=0}^{M-1} u^j + \sum_{n=0}^{M-1} u^n \sum_{k=0}^{n} \left(\frac{\lambda}{\lambda+\mu}\right)^k \sum_{r=1}^{\infty} \binom{k+r-1}{k} \left(\frac{\mu\nu Q}{\lambda+\mu}\right)^r \right] \Bigg\}. \quad \square$$

### 3.3.3 Marginal Transform of $\tau_\rho$, $Ee^{-\theta\tau_\rho}$.

Letting $v = 0$ and $u = 1$ in (3.3.7)-(3.3.12) we obtain the marginal transform as

$$Ee^{-\theta\tau_\rho} = 1 - \frac{\theta}{\gamma+\theta}\Bigg\{1 + \frac{\gamma}{\lambda}\Bigg[\frac{1-\left(\frac{\theta}{\theta+\lambda}\right)^M}{1-\left(\frac{\theta}{\theta+\lambda}\right)} - e^{-\nu Q}\sum_{j=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^j$$

$$- e^{-\nu Q}\sum_{n=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^n\sum_{k=0}^{n}\left(\frac{\theta+\lambda}{\theta+\lambda+\mu}\right)^k\sum_{r=1}^{\infty}\binom{k+r-1}{k}\left(\frac{\mu\nu}{\theta+\lambda+\mu}Q\right)^r\Bigg]$$

$$+ \frac{\gamma}{\theta+\lambda+\mu}e^{-\nu Q}\Bigg[\sum_{j=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^j + \sum_{n=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^n$$

$$\times\sum_{k=0}^{n}\left(\frac{\theta+\lambda}{\theta+\lambda+\mu}\right)^k\sum_{r=1}^{\infty}\binom{k+r-1}{k}\left(\frac{\mu\nu}{\theta+\lambda+\mu}Q\right)^r\Bigg]\Bigg\}. \qquad \square$$

#### 3.3.4 The Probability Density Function $f_{\tau_\rho}$ of $\tau_\rho$.

From 3.3.3 we can obtain the probability density function of the first observed passage time of the game end by taking the inverse Laplace transform of $Ee^{-\theta\tau_\rho}$.

$$f_{\tau_\rho}(t) = \mathcal{L}_\theta^{-1}\{Ee^{-\theta\tau_\rho}\}(t)$$

$$= \mathcal{L}_\theta^{-1}\Bigg\{1 - \frac{\theta}{\gamma+\theta}\Bigg[1 + \frac{\gamma}{\lambda}\Bigg[\frac{1-\left(\frac{\lambda}{\theta+\lambda}\right)^M}{1-\left(\frac{\lambda}{\theta+\lambda}\right)} - e^{-\nu Q}\sum_{j=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^j$$

$$- e^{-\nu Q}\sum_{n=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^n\sum_{k=0}^{n}\left(\frac{\theta+\lambda}{\theta+\lambda+\mu}\right)^k\sum_{r=1}^{\infty}\binom{k+r-1}{k}\left(\frac{\mu\nu}{\theta+\lambda+\mu}Q\right)^r\Bigg]$$

$$+ \frac{\gamma}{\theta+\lambda+\mu}e^{-\nu Q}\Bigg[\sum_{j=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^j + \sum_{n=0}^{M-1}\left(\frac{\lambda}{\theta+\lambda}\right)^n\sum_{k=0}^{n}\left(\frac{\theta+\lambda}{\theta+\lambda+\mu}\right)^k$$

$$\times \sum_{r=1}^{\infty} \binom{k+r-1}{k} \left( \frac{\mu\nu}{\theta+\lambda+\mu} Q \right)^r \Bigg] \Bigg] \Bigg\}$$

Observe from the above that we can also use the residue theorem as outlined in (3.2.8) and in Corollary 12 since after rearrangement of one of the terms, the degree of the numerator is lower as compared to that of the denominator.

See below one of the rearrangements,

$$e^{-\nu Q} \mathcal{L}_{\theta}^{-1} \left\{ \sum_{n=0}^{M-1} \left( \frac{\lambda}{\theta+\lambda} \right)^n \sum_{k=0}^{n} \left( \frac{\theta+\lambda}{\theta+\lambda+\mu} \right)^k \sum_{r=1}^{\infty} \binom{k+r-1}{k} \left( \frac{\mu\nu}{\theta+\lambda+\mu} Q \right)^r \right\}$$

$$= e^{-\nu Q} \sum_{n=0}^{M-1} \lambda^n \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \mathcal{L}_{\theta}^{-1} \left\{ \frac{\theta^k}{(\theta+\lambda)^n (\theta+\lambda+\mu-\mu\nu Q)^k} \right\}.$$

## Chapter 4

# Real Time Games.

In this chapter we shall consider two versions of an antagonistic stochastic game in real time namely; discrete and continuous games. In the real time game we consider unlike the delayed game of Chapter 3, the information on an underlying game is probabilistically available upon attacks without delays. We also state at this point that while this approach to modeling seems to be quite realistic, it would not apply to all models. Besides, the resulting expressions are more bulky in the special cases we consider here.

## 4.1 A Discrete-Marks Model in Real Time

A discrete-marks antagonistic game $\mathcal{G}_d$ will be characterized by simultaneous mutual attacks by players A and B with position-independent bivariate marks in the context of a marked Poisson process of intensity $\lambda$. We assume that the marks $X_k$ and $Y_k$ (see equation (2.2.1)) with parameters $a$ and $b$, respectively. Their respective marginal pgf's are

$Eu^{X_1} = \frac{a}{1-\alpha u}$ and $Ev^{Y_1} = \frac{b}{1-\beta v}$.    (where $a+\alpha=1$ and $b+\beta=1$)

Because the carrier process is Poisson, the times between the attacks are exponentially distributed with parameter $\lambda$. Thus its common Laplace-Stieltjes transform is

$$\gamma(\theta) = Ee^{-\theta\tau_1} = \frac{\gamma}{\gamma+\theta}. \tag{4.1.1}$$

Next, since components $X$'s and $Y$'s are independent, under the above assumptions, the joint functional $\gamma(u, v, \theta)$ of Theorem 1 is as follows

$$\gamma(u, v, \theta) = Eu^{X_1}v^{Y_1}e^{-\theta\tau_1} = \frac{a}{1-\alpha u}\frac{b}{1-\beta v}\frac{\gamma}{\gamma+\theta}. \tag{4.1.2}$$

The properties of the $\mathcal{D}$-operator (in Theorem 3) will be used in the sequel.

### 4.1.1 The Discrete Model

**Assumption 1 regarding game $\mathcal{G}_d$.** Let $\mathcal{G}_d$ be an antagonistic stochastic game of two players A and B whose mutual attacks follow a bivariate marked Poisson processes with geometrically distributed marks $X_k$ and $Y_k$ with parameters $a$ and $b$ respectively. Suppose that the player's tolerance thresholds are $M$ and $N$ and their attacks are simultaneous and independent.

**Theorem 14**. *Under Assumption* 1*, the joint functional* $\Phi$ *of game* $\mathcal{G}_d$ (*introduced in* (*2*.1.6)) *satisfies the following formula*:

$$\Phi(u,v,\theta)=1-\left[1-\tfrac{a}{1-\alpha u}\ \tfrac{b}{1-\beta v}\tfrac{\gamma}{\gamma+\theta}\right]$$

$$\times\left\{1+\left(1-(\alpha u)^M\right)\tfrac{ab\gamma}{(1-\alpha u)(1-\beta v)(\gamma+\theta)-ab\gamma}\left[1-\left(\tfrac{\beta v(1-\alpha u)(\gamma+\theta)}{(1-\alpha u)(\gamma+\theta)-ab\gamma}\right)^N\right]\right.$$

$$-\left((\alpha u)^M\sum_{k=1}^{M}\tbinom{M}{k}(ab\gamma)^{k+1}\tfrac{1}{(\gamma+\theta-ab\gamma)^k}\tfrac{1}{(1-\alpha u)(\gamma+\theta)-ab\gamma}\right.$$

$$\left.\left.\times\sum_{j=0}^{N-1}\binom{k+j-1}{j}\left(\tfrac{\beta v(\gamma+\theta)}{\gamma+\theta-ab\gamma}\right)^j\left[1-\left(\tfrac{\beta v(1-\alpha u)(\gamma+\theta)}{(1-\alpha u)(\gamma+\theta)-ab\gamma}\right)^{N-j}\right]\right)\right\}. \qquad (4.1.3)$$

**Proof**. In the sequel, for brevity we denote $\eta=\gamma(\theta)=\frac{\gamma}{\gamma+\theta}$. From (4.1.1) and (4.1.2) after a straight forward algebra, we have

$$\tfrac{1}{1-\gamma(ux,vy,\theta)}=\tfrac{1}{1-\frac{a}{1-\alpha ux}\ \frac{b}{1-\beta vy}\eta}$$

$$=1+\left(\tfrac{1-\beta vy}{ab\eta}-1\right)^{-1}\tfrac{1}{1-\xi x}.$$

where $\xi=\frac{\alpha u(1-\beta vy)}{1-\beta vy-ab\eta}$.

By Theorem 3$(i)$ and $(iv)$,

$$\mathcal{D}_{xy}^{M-1,N-1}\left\{\tfrac{1}{1-\gamma(ux,vy,\theta)}\right\}=\mathcal{D}_y^{N-1}\circ\mathcal{D}_x^{M-1}\left\{1+\left(\tfrac{1-\beta vy}{ab\eta}-1\right)^{-1}\tfrac{1}{1-\xi x}\right\}$$

$$= \mathcal{D}_y^{N-1}\left\{1 + \left(\frac{1-\beta vy}{ab\eta} - 1\right)^{-1}\frac{1-\xi^M}{1-\xi}\right\}$$

$$= \mathcal{D}_y^{N-1}\left\{1 + \left(\frac{1-\beta vy}{ab\eta} - 1\right)^{-1}\frac{1}{1-\xi} - \left(\frac{1-\beta vy}{ab\eta} - 1\right)^{-1}\frac{\xi^M}{1-\xi}\right\}. \tag{4.1.4}$$

Now from the braces above,

$$1 - \xi = \frac{1-\beta vy-ab\eta-\alpha u+\alpha\beta uvy}{1-\beta vy-ab\eta}$$

and

$$\frac{1}{1-\xi} = \frac{1-\beta vy-ab\eta}{1-ab\eta-\alpha u-(\beta v-\alpha\beta uv)y}.$$

Therefore,

$$\left(\frac{1-\beta vy}{ab\eta} - 1\right)^{-1}\frac{1}{1-\xi} = \frac{ab\eta}{1-ab\eta-\alpha u}\frac{1}{1-\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}y}. \tag{4.1.5}$$

Furthermore,

$$\xi^M = \left(\frac{\alpha u(1-\beta vy)}{1-\beta vy-ab\eta}\right)^M$$

$$= \left(\alpha u + \frac{ab\eta\alpha u}{1-\beta vy-ab\eta}\right)^M$$

$$= (\alpha u)^M \sum_{k=0}^{M}\binom{M}{k}(ab\eta)^k\frac{1}{(1-ab\eta-\beta vy)^k}.$$

This implies that,

$$\left(\frac{1-\beta vy}{ab\eta} - 1\right)^{-1}\frac{\xi^M}{1-\xi}$$

$$= \frac{ab\eta}{1-ab\eta-\alpha u-(\beta v-\alpha\beta uv)y}(\alpha u)^M \sum_{k=0}^{M}\binom{M}{k}(ab\eta)^k\frac{1}{(1-ab\eta-\beta vy)^k}$$

$$= \frac{(\alpha u)^M \sum_{k=0}^{M} \binom{M}{k} (ab\eta)^{k+1}}{(1-ab\eta-\alpha u)(1-ab\eta)^k} \frac{1}{\left(1-\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}y\right)} \frac{1}{\left(1-\frac{\beta v}{1-ab\eta}y\right)^k}. \tag{4.1.6}$$

Substituting (4.1.5) and (4.1.6) in (4.1.4) and applying Theorem 3$(iv)$ and $(vi)$, after some algebra, we get

$$\mathcal{D}_{xy}^{M-1,N-1}\left\{\frac{1}{1-\gamma(ux,vy,\theta)}\right\}$$

$$= 1 + \frac{ab\eta}{1-ab\eta-\alpha u}\mathcal{D}_{y}^{N-1}\left\{\frac{1}{1-\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}y}\right\}$$

$$-\sum_{k=0}^{M}\binom{M}{k}(ab\eta)^{k+1}\frac{(\alpha u)^M}{(1-ab\eta-\alpha u)(1-ab\eta)^k}\mathcal{D}_{y}^{N-1}\left\{\frac{1}{\left(1-\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}y\right)}\frac{1}{\left(1-\frac{\beta v}{1-ab\eta}y\right)^k}\right\}$$

$$= 1 + \left[1-(\alpha u)^M\right]\frac{ab\eta}{1-ab\eta-\alpha u-\beta v+\alpha\beta uv}\left[1-\left(\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}\right)^N\right]$$

$$-\left\{(\alpha u)^M\sum_{k=1}^{M}\binom{M}{k}(ab\eta)^{k+1}\frac{1}{(1-ab\eta)^k}\frac{1}{1-ab\eta-\alpha u-(\beta v-\alpha\beta uv)}\right.$$

$$\left.\times\sum_{j=0}^{N-1}\binom{k+j-1}{j}\left(\frac{\beta v}{1-ab\eta}\right)^j\left[1-\left(\frac{\beta v-\alpha\beta uv}{1-ab\eta-\alpha u}\right)^{N-j}\right]\right\}.$$

Using formula (2.2.4) of Theorem 1 and recalling that $\eta = \frac{\gamma}{\gamma+\theta}$ we arrive at formula (4.1.3) and herewith complete the proof. □

### 4.1.2 The Marginal Transform of the First Passage Time $\tau_\rho$ .

If we let $u = v = 1$ in formula (4.1.3) of theorem 14 then,

$$Ee^{-\theta\tau_\rho} = (1-\alpha^M)\frac{\gamma}{\gamma+\theta}\left(\frac{\beta(\gamma+\theta)}{\gamma\beta+\theta}\right)^N + \frac{\gamma}{\gamma+\theta}\alpha^M$$

$$+ b\alpha^M\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}\frac{\theta(ab\gamma)^k}{(\gamma+\theta-ab\gamma)^{k+j}}\frac{\gamma\beta^j(\gamma+\theta)^{j-1}}{(\gamma\beta+\theta)}$$

$$- b\alpha^M\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}\frac{\theta(ab\gamma)^k}{(\gamma+\theta-ab\gamma)^{k+j}}\frac{\gamma\beta^N(\gamma+\theta)^{N-1}}{(\gamma\beta+\theta)^{N-j+1}}. \qquad \square$$

### 4.1.3 The Probability Density Function of $\tau_\rho$

We apply the inverse Laplace transform of the marginal density of $\tau_\rho$ obtained in 4.1.2.

$$f_{\tau_\rho} = \mathcal{L}_\theta^{-1}\left\{Ee^{-\theta\tau_\rho}\right\}(t)$$

$$= (1-\alpha^M)\mathcal{L}_\theta^{-1}\left\{\frac{\gamma}{\gamma+\theta}\left(\frac{\beta(\gamma+\theta)}{\gamma\beta+\theta}\right)^N\right\}(t) + \mathcal{L}_\theta^{-1}\left\{\frac{\gamma}{\gamma+\theta}\alpha^M\right\}(t)$$

$$+ \mathcal{L}_\theta^{-1}\left\{b\alpha^M\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}\frac{\theta(ab\gamma)^k}{(\gamma+\theta-ab\gamma)^{k+j}}\frac{\gamma\beta^j(\gamma+\theta)^{j-1}}{(\gamma\beta+\theta)}\right\}(t)$$

$$- \mathcal{L}_\theta^{-1}\left\{b\alpha^M\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}\frac{\theta(ab\gamma)^k}{(\gamma+\theta-ab\gamma)^{k+j}}\frac{\gamma\beta^N(\gamma+\theta)^{N-1}}{(\gamma\beta+\theta)^{N-j+1}}\right\}(t). \qquad (4.1.7)$$

Next we solve and simplify (4.1.7) term by term, and using Lemma 5 $(ii)$-$(vii)$ we will arrive at

$$f_{\tau_\rho}(t) = (1-\alpha^M)\left(\gamma\beta^N e^{-\gamma t} + \gamma\beta^N \sum_{d=1}^{N} \binom{N}{d} (-1)^d e^{-\gamma t} P(d-1, (-\gamma b)t)\right)$$

$$+ \gamma\alpha^M e^{-\gamma t} + F\Bigg\{ \tfrac{t^k}{k!} e^{-\gamma t} + \sum_{h=1}^{k+j} \binom{k+j}{h} (ab\gamma)^h \Bigg( \tfrac{1}{k!(ab\gamma)^{k+1}} \sum_{p=0}^{h-1} \tfrac{1}{p!(h-p-1)!} t^{h-p-1}$$

$$\times \tfrac{1}{(-ab\gamma)^p} e^{-(\gamma - ab\gamma)t} P(k+p, (ab\gamma)t) \Bigg)$$

$$- \Bigg[ \gamma\beta e^{-\gamma\beta t} \tfrac{1}{(\gamma-\gamma\beta)^{k+1}} P(k, (\gamma-\gamma\beta)t)$$

$$+ \sum_{h=1}^{k+j} \binom{k+j}{h} g_1(\gamma)(b\alpha\gamma)^w \left(\tfrac{a}{\alpha}\right)^h P(h-w, (b\alpha\gamma)t) \Bigg]$$

$$- \gamma\beta e^{-\gamma\beta t} \sum_{h=1}^{k+j} \binom{k+j}{h} g_1(\gamma)\, (ab\gamma)^w$$

$$\times (-1)^{w+q-h} \sum_{q=0}^{k+w} \binom{h+q-w-1}{q} P(h+q-w-1, (-ab\alpha)t) \Bigg\}$$

$$- b\alpha^M \sum_{k=1}^{M} \sum_{j=0}^{N-1} \binom{M}{k} \binom{k+j-1}{j} \gamma\beta^N \sum_{s=0}^{j} \binom{j}{s} \tfrac{e^{-\gamma t}}{(-1)^{k+s}} P(k+s-1, (-ab\gamma)t)$$

$$+ F_1 \Bigg\{ \sum_{j=0}^{k+s-1} \tfrac{1}{(k+s-j-1)!} t^{k+s-j-1} \tfrac{1}{(-1)^j (ab\gamma)^{j+2}} e^{-(\gamma-ab\gamma)t} P(j+1, (ab\gamma)t)$$

$$- \sum_{r=1}^{N-j+1} \binom{N-j+1}{r} (\gamma b)^r \Bigg( e^{-\gamma t} g(\alpha,\gamma) \Big[ \tfrac{1}{(-ab\gamma)^{k+s-\omega}}$$

$$\times P(k+s-\omega-1, (-ab\gamma)t) - \tfrac{1}{(b\alpha\gamma)^{k+s-\omega}}$$

$$\times \sum_{\sigma=0}^{r+\omega-1}(-1)^{\sigma}\binom{k+s+\sigma-\omega-1}{\sigma}P(k+s+\sigma-\omega-1,(b\alpha\gamma)t)\Big]\Bigg)$$

$$+\sum_{r=1}^{N-j+1}\binom{N-j+1}{r}(\gamma b)^{r}\Bigg(te^{-\gamma t}\frac{1}{(r-1)!(-b\alpha\gamma)^{r}}\sum_{\delta=0}^{k+s-1}\frac{(r+\delta-1)!}{\delta!(k+s-\delta-1)!}\frac{1}{(b\alpha\gamma)^{\delta}}$$

$$\times t^{k+s-\delta}(ab\gamma t)^{-k-s+\delta}[P(k+s-\delta)-P(k+s-\delta,(-ab\gamma t))]\Bigg)\Bigg\}$$

$$+F_1\Bigg\{\sum_{r=1}^{N-j+1}\binom{N-j+1}{r}(\gamma b)^{r}e^{-\gamma t}\frac{1}{(r-1)!(-b\alpha\gamma)^{r}}\sum_{\delta=0}^{k+s-1}\frac{(r+\delta-1)!}{\delta!(k+s-\delta-1)!}\frac{1}{(b\alpha\gamma)^{\delta}}$$

$$\times\sum_{\varkappa=0}^{r+\delta-1}\frac{(-b\alpha\gamma)^{\varkappa}}{\varkappa!}\,t^{k+s+\varkappa-\delta+1}(-\gamma bt)^{-k-s-\varkappa+\delta-1}$$

$$\times[P(k+s+\varkappa-\delta+1)-P(k+s+\varkappa-\delta+1,(-\gamma bt))]\Bigg\}. \qquad \square$$

Where,

$$g(\alpha,\gamma)=(-1)^{r}\sum_{\omega=0}^{k+s-1}\binom{r+\omega-1}{\omega}\frac{1}{(b\alpha\gamma)^{r+\omega}},$$

$$g_1(\gamma)=\sum_{w=0}^{h-1}(-1)^{w}\binom{k+w}{k}\frac{1}{(ab\gamma)^{k+w+1}},$$

$$F=b\alpha^{M}\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}(ab\gamma)^{k}\gamma\beta^{j},$$

$$F_1=b\alpha^{M}\sum_{k=1}^{M}\sum_{j=0}^{N-1}\binom{M}{k}\binom{k+j-1}{j}\gamma\beta^{N}\sum_{s=0}^{j}\binom{j}{s}(ab\gamma)^{k+s}.$$

## 4.2 A Continuous-Marks Model in Real Time

In this model, we assume that the mutual attacks of two players A and B follow bivariate marked Poisson processes with exponentially distributed marks with parameters $\lambda$ and $\mu$. Their LST's are

$$g(u) \ = Ee^{-uX_1} = \tfrac{\lambda}{\lambda+u},$$
$$h(v) = Ee^{-vY_1} = \tfrac{\mu}{\mu+v}.$$

Furthermore, the attacks of both players are independent and occur simultaneously upon times $\mathcal{T} = \{\tau_1, \tau_2, \dots\}$ forming a Poisson point process of intensity $\lambda$. Therefore, the interrenewal times are independent and exponentially distributed with common parameter $\lambda$ and the LST

$$\gamma(\theta) = Ee^{-\theta\Delta}. \quad \text{(in notation } \Gamma)$$

Analogous to the assumptions of section 4.1 we therefore have

$$\gamma(u, v, \theta) \ = Ee^{-uX_1 - vY_1 - \theta\Delta} = \tfrac{\lambda}{\lambda+u}\tfrac{\mu}{\mu+v}\tfrac{\gamma}{\gamma+\theta}. \tag{4.2.1}$$

Next,

$$1 - \gamma(u, v, \theta) = \tfrac{(\lambda+u)(\mu+v)(\gamma+\theta) - \lambda\mu\gamma}{(\lambda+u)(\mu+v)(\gamma+\theta)} \tag{4.2.2}$$

and

$$\gamma(u + x, v + y, \theta) \ = \tfrac{\lambda}{\lambda+u+x}\tfrac{\mu}{\mu+v+y}\tfrac{\gamma}{\gamma+\theta}, \tag{4.2.3}$$

$$1 - \gamma(u + x, v + y, \theta) = 1 - \tfrac{\lambda}{\lambda+u+x}\Upsilon\Gamma, \qquad \left(\text{where } \Upsilon = \tfrac{\mu}{\mu+v+y}\right) \tag{4.2.4}$$

$$\tfrac{1}{1-\gamma(u+x,\, v+y,\, \theta)} = \tfrac{(\lambda+u+x) - \lambda\Upsilon\Gamma + \lambda\Upsilon\Gamma}{(\lambda+u+x) - \lambda\Upsilon\Gamma} = 1 + \tfrac{\lambda\Upsilon\Gamma}{\lambda+u-\lambda\Upsilon\Gamma+x}. \tag{4.2.5}$$

### 4.2.1 The Continuous Model

**Assumption 2 regarding game $\boldsymbol{\mathcal{G}_c}$**. Let $\mathcal{G}_c$ be an antagonistic stochastic game of two players A and B whose mutual attacks follow a bivariate marked Poisson processes with exponentially distributed marks $X_k$ and $Y_k$ with parameters $\lambda$ and $\mu$, respectively. Suppose that the player's tolerance thresholds are $q$ and $s$ and their attacks are simultaneous and independent.

**Theorem 15**. *Under Assumption* 2, *the joint functional* $\Psi$ *of game* $\mathcal{G}_c$ $($*introduced in* $(2.2.3))$ *satisfies the following formula*:

$$\Psi(u,v,\theta) = Ge^{-as} + He^{-(\lambda+u)q}\int_{r=0}^{s} e^{-(\mu+v)r} I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr$$

$$+ (K-H)e^{-(\lambda+u)q}e^{-as}\int_{r=0}^{s} e^{-br} I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr. \qquad (4.2.6)$$

where

$$G = \frac{\lambda\mu\gamma}{(\lambda+u)(\mu+v)(\gamma+\theta)}, \qquad H = \frac{\lambda\mu\gamma}{(\lambda+u)(\gamma+\theta)}, \qquad (4.2.7)$$

$$a = \frac{(\lambda+u)(\mu+v)(\gamma+\theta)-\lambda\mu\gamma}{(\lambda+u)(\gamma+\theta)}, \qquad b = \frac{(\lambda+u)(u-\mu)(\gamma+\theta)-\lambda\mu\gamma}{(\lambda+u)(\gamma+\theta)}, \qquad (4.2.8)$$

$$K = \frac{(\lambda\mu\gamma)(\lambda+u)(\mu+v)(\gamma+\theta)-(\lambda\mu\gamma)^2}{[(\lambda+u)(\gamma+\theta)]^2}, \qquad \Gamma = \frac{\gamma}{\gamma+\theta}. \qquad (4.2.9)$$

**Proof**. From Theorem 2,

$$\mathcal{L}C_{xy}^{-1}(\,\cdot\,)(q,s) = \mathcal{L}^{-1}(\,\cdot\,\tfrac{1}{x}\tfrac{1}{y}).$$

Using this in the functional $\Psi$ of (2.2.3) we have

$\Psi(u, v, \theta)$

$$= 1 - [1 - \gamma(u, v, \theta)]\mathcal{L}_y^{-1}\left\{\frac{1}{y}\mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{1}{1-\gamma(u+x,\, v+y,\, \theta)}\right\}(q)\right\}(s). \qquad (4.2.10)$$

Now from (4.2.5) we have,

$$\mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{1}{1-\gamma(u+x,\, v+y,\, \theta)}\right\}(q) = \mathcal{L}_x^{-1}\left\{\frac{1}{x}\left(1 + \frac{\lambda\Upsilon\Gamma}{\lambda+u-\lambda\Upsilon\Gamma+x}\right)\right\}(q)$$

$$= \mathcal{L}_x^{-1}\left\{\frac{1}{x}\right\}(q) + \mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{\lambda\Upsilon\Gamma}{\lambda+u-\lambda\Upsilon\Gamma+x}\right\}(q)$$

$$= \mathcal{L}_x^{-1}\left\{\frac{1}{x}\right\}(q) + \lambda\Upsilon\Gamma\mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{1}{b+x}\right\}(q). \quad (\text{where } b = \lambda + u - \lambda\Upsilon\Gamma)$$

After some algebra the above yields

$$= 1 + \lambda\Gamma\mu\left[\frac{1}{(\lambda+u)(\mu+v)-\lambda\Gamma\mu+(\lambda+u)y} - \frac{e^{-(\lambda+u)q}e^{\lambda\Gamma\mu q\left(\frac{1}{\mu+v+y}\right)}}{(\lambda+u)(\mu+v)-\lambda\Gamma\mu+(\lambda+u)y}\right].$$

Therefore,

$$\mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{1}{1-\gamma(u+x,\, v+y,\, \theta)}\right\}(q) = 1 + \frac{\lambda\Gamma\mu}{(\lambda+u)}\left[\frac{1}{a+y} - \frac{1}{a+y}e^{-(\lambda+u)q}e^{\lambda\Gamma\mu q\left(\frac{1}{\mu+v+y}\right)}\right].$$

Where $a = \frac{(\lambda+u)(\mu+v)-\lambda\Gamma\mu}{(\lambda+u)}$.

Furthermore,

$$\frac{1}{y}\mathcal{L}_x^{-1}\left\{\frac{1}{x}\frac{1}{1-\gamma(u+x,\, v+y,\, \theta)}\right\}(q)$$

$$= \frac{1}{y} + \frac{\lambda\Gamma\mu}{(\lambda+u)}\frac{1}{y(a+y)} - \frac{\lambda\Gamma\mu}{(\lambda+u)}\frac{1}{y(a+y)}e^{-(\lambda+u)q}e^{\lambda\Gamma\mu q\left(\frac{1}{\mu+v+y}\right)}.$$

Hence from (4.2.10) and using Lemma 5,

$$\mathcal{L}_y^{-1}\left\{\tfrac{1}{y}\mathcal{L}_x^{-1}\left\{\tfrac{1}{x}\tfrac{1}{1-\gamma(u+x,\,v+y,\,\theta)}\right\}(q)\right\}(s)$$

$$= \mathcal{L}_y^{-1}\left\{\tfrac{1}{y}\right\}(s) + \tfrac{\lambda\Gamma\mu}{(\lambda+u)}\tfrac{1}{a}\mathcal{L}_y^{-1}\left\{\tfrac{1}{y} - \tfrac{1}{a+y}\right\}(s)$$

$$- \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}\mathcal{L}_y^{-1}\left\{\frac{e^{\lambda\Gamma\mu q\left(\frac{1}{\mu+v+y}\right)}}{y}\right\}(s)$$

$$+ \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}\mathcal{L}_y^{-1}\left\{\frac{e^{\lambda\Gamma\mu q\left(\frac{1}{\mu+v+y}\right)}}{a+y}\right\}(s)$$

$$= 1 + \tfrac{\lambda\Gamma\mu}{(\lambda+u)}\tfrac{1}{a} - \tfrac{\lambda\Gamma\mu}{(\lambda+u)}\tfrac{1}{a}e^{-as}$$

$$- \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}\Big[e^{-(\mu+v)s}I_0\big(2\sqrt{\lambda\Gamma\mu qs}\big)$$

$$+ (\mu+v)\textstyle\int_{r=0}^{s}e^{-(\mu+v)r}I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr\Big]$$

$$+ \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}\Big[e^{-(\mu+v)s}I_0\big(2\sqrt{\lambda\Gamma\mu qs}\big)$$

$$+ (\mu+v-a)e^{-as}\textstyle\int_{r=0}^{s}e^{-(\mu+v-a)r}I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr\Big].$$

$$= 1 + \tfrac{\lambda\Gamma\mu}{(\lambda+u)}\tfrac{1}{a} - \tfrac{\lambda\Gamma\mu}{(\lambda+u)}\tfrac{1}{a}e^{-as}$$

$$- \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}(\mu+v)\textstyle\int_{r=0}^{s}e^{-(\mu+v)r}I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr$$

$$+ \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}\tfrac{1}{a}(\mu+v)e^{-as}\textstyle\int_{r=0}^{s}e^{-(\mu+v-a)r}I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr$$

$$- \tfrac{\lambda\Gamma\mu}{(\lambda+u)}e^{-(\lambda+u)q}(\mu+v)e^{-as}$$

$$\times \textstyle\int_{r=0}^{s}e^{-(\mu+v-a)r}I_0\big(2\sqrt{\lambda\Gamma\mu qr}\big)dr. \qquad (4.2.11)$$

Note that,

$$\frac{\lambda\Gamma\mu}{(\lambda+u)}\frac{1}{a} = \frac{\lambda\mu\gamma}{(\lambda+u)(\mu+v)(\gamma+\theta)-\lambda\mu\gamma}. \tag{4.2.12}$$

Using (4.2.12) in (4.2.11) and applying (4.2.1) in (4.2.10) we arrive at (4.2.6-4.2.9) which completes the proof. □

### 4.2.2 The Marginal Transform of $\tau_\rho$.

Letting $u = v = 0$ in Theorem 15 we arrive at

$$Ee^{-\theta\tau_\rho} = G_1 e^{-as} + H_1 e^{-\lambda q}\int_{r=0}^{s} e^{-\mu r} I_0\big(2\sqrt{\lambda G_1 \mu q r}\big)\,dr$$
$$+ (K_1 - H_1)e^{-\lambda q} e^{-a_1 s}\int_{r=0}^{s} e^{-b_1 r} I_0\big(2\sqrt{\lambda G_1 \mu q r}\big)\,dr$$

where

$$G_1 = \frac{\gamma}{\gamma+\theta}, \qquad H_1 = \frac{\mu\gamma}{\gamma+\theta}, \qquad K_1 = \frac{\mu^2\gamma\theta}{(\gamma+\theta)^2},$$

$$a_1 = \frac{\theta}{\gamma+\theta}, \qquad b_1 = \frac{\mu\lambda\gamma}{\lambda(\gamma+\theta)}, \quad \Gamma_1 = \frac{\gamma}{\gamma+\theta} = G_1.$$ □

## Chapter 5

# Discrete Operational Calculus in Non Monotone Games.

Consider a more general game, in which the values of cumulative casualties to each player, instead of monotone increasing, oscillate, that is non monotone. This takes place when successive increments of casualties are real-valued. Most typical examples of such situations are portfolios of financial instruments. Suppose an investor holds an energy stock, and at the same time he or she watches over the fluctuations of the oil index. So the investor decides to sell the stock before the stock exhibits a second drop in price value or so does the oil index, whichever of the two comes first.

Since our analysis of such events cannot handle components of marked point processes other than non negative, we introduce so-called auxiliary components attached to non monotone components and being strictly non negative. For this reason we need to explicitly distinguish those components referred to as *active* from the others referred to as *passive*. Suppose

$$(\mathcal{A},\mathcal{B},\mathcal{P},\mathcal{Q},\mathcal{T}) = \sum_{n=1}^{\infty}(X_n, Y_n, Z_n, W_n)\varepsilon_{t_n}$$

is a multi-dimensional marked point process, with mutually dependent components, and let

$$A_n = \sum_{i=1}^{n} X_i, B_n = \sum_{i=1}^{n} Y_i, \mathcal{P}_n = \sum_{i=1}^{n} Z_i, \mathcal{Q}_n = \sum_{i=1}^{n} W_i.$$

Of the four named components (not counting the time), we identify $\mathcal{A}$ and $\mathcal{B}$ as active and the rest as passive components. The increments $X$'s and $Y$'s of the two active components are non negative and with either of them we associate thresholds $M$ and $N$, respectively. The increments $Z$'s and $W$'s of the passive components $\mathcal{P}$ and $\mathcal{Q}$ are real-valued and for this reason, $\mathcal{P}$ and $\mathcal{Q}$ are non monotone. If the process $(\mathcal{A}, \mathcal{B}, \mathcal{P}, \mathcal{Q}, \mathcal{T})$ models a game, it ends when one of both active components cross their respective thresholds. The passive components $\mathcal{P}$ and $\mathcal{Q}$, including the time $\mathcal{T}$, will assume their values upon those crossings.

In the context of a "game" between an energy stock and the oil index, their price values $\mathcal{P}$ and $\mathcal{Q}$ will represent passive components, whereas active components $\mathcal{A}$ and $\mathcal{B}$ will "watch over" $\mathcal{P}$ and $\mathcal{Q}$ and assume zero values upon times $\mathcal{T} = \{\tau_1, \tau_2, \dots\}$ whenever $Z_1, Z_2, \dots$ and $W_1, W_2, \dots$ do not depreciate more than, say $1 each. Otherwise, they assume values 1 when the incremental prices $Z$'s and $W$'s drop below $1. The investor will sell the stock when either $A_n$ or $B_n$ or both cross $M$ or $N$ upon some time $\tau_n$. We thus assume that the investor watches over these instruments upon random times $\mathcal{T}$. The objective is to predict the crossing or crossings in terms of the time and the price values.

In another situation with a patient undergoing a treatment, as before, we consider the effects of the treatment against its side effects, now up and down varying in their behaviors. Such real-world scenarios are numerous. Unfortunately, it is virtually impossible to obtain functionals of such recurrent processes in tractable forms if their non monotone components are active. Lajos Takács [46] studied such recurrent processes, but his seminal results are far from tractable. A break came in Dshalalow [19], followed by Dshalalow and Liew [26] and Dshalalow and Robinson [27], with the introduction of auxiliary active components that opened a gate to various situations predicting repeated drops or increases of non monotone components and then applied exclusively to finance. Their results look tame, but they were not sufficiently explored and partially applied to single, non monotone components or with no game context.

### 5.1 The Formalism of a Non Monotone Model

Consider the following game of two players A and B. As before, the mutual attacks are observed upon the times $\mathcal{T} = \{\tau_1, \tau_2, \dots\}$. At time $\tau_n$ players A and B undergo damages in quantities $Z_n$ and $W_n$, respectively. We now assume that they are real-valued. That being said, they can be negative. A negative damage at time $\tau_n$ can occur if between $\tau_{n-1}$ and $\tau_n$ there were some maintenance or reparations that offset casualties taking place during the same time interval. Similarly, we define

$$P_n = Z_1 + \dots + Z_n \text{ and } Q_n = W_1 + \dots + W_n$$

which will give us the status of each player upon $\tau_n$. Now because the components $\mathcal{P} = \{P_n\}$ and $\mathcal{Q} = \{Q_n\}$ are non monotone, we cannot apply to them the same technique as for monotone processes. We therefore introduce auxiliary components as follows. Let

$$X_n = \begin{cases} 0, & W_n \geq 0 \\ 1, & W_n < 0 \end{cases} \tag{5.1.1}$$

$$Y_n = \begin{cases} 0, & Z_n \geq 0 \\ 1, & Z_n < 0 \end{cases} \tag{5.1.2}$$

that watch over $\mathcal{P}$ and $\mathcal{Q}$ recording every single drop or precisely, a change of monotonicity. If $M$ and $N$ are two positive integers, by reaching $M$ and $N$ by the components

$$A_n = X_1 + \ldots + X_n \text{ and } B_n = Y_1 + \ldots + Y_n \tag{5.1.3}$$

respectively, we ensure that $\mathcal{P}$ and $\mathcal{Q}$ dropped $M$ and $N$ times upon $\tau_n$. As we see it, the components $\mathcal{A} = \{A_n\}$ and $\mathcal{B} = \{B_n\}$ are identified as *active components*, because they are the ones that fluctuate around critical thresholds. The other two components $\mathcal{P}$ and $\mathcal{Q}$, that $\mathcal{A}$ and $\mathcal{B}$ "watch," are *passive* and so is the time component $\mathcal{T}$.

In the context of a game, we will make $\mathcal{P}$ and $\mathcal{Q}$ compete with each other and find the nearest observation epoch $\tau_n$ when one of the two mentioned events take place. Correspondingly, let

$$\nu_1 = \min\{n \in \mathbb{N} : A_n = M\} \text{ and } \nu_2 = \min\{n \in \mathbb{N} : B_n = N\} \tag{5.1.4}$$

and let

$$\rho = \nu_1 \wedge \nu_2. \tag{5.1.5}$$

Thus, $\tau_\rho$ is the first passage time when the game ends, that is when $\mathcal{P}$ or $\mathcal{Q}$ drop $M$ or $N$ times, while increasing. The associated functional of interest will be defined as

$$\Phi_\rho(u, v, \alpha, \beta, \theta) = Eu^{A_\rho} v^{B_\rho} e^{i\alpha P_\rho + i\beta Q_\rho} e^{-\theta\tau_\rho}, \|u\| \leq 1, \|v\| \leq 1, \mathrm{Re}\theta \geq 0,$$

and $\alpha$,$\beta \in \mathbb{C}$, (5.1.6)

that is a joint integer and continuous transforms including Laplace-Stieltjes and Fourier-Stieltjes transforms. Suppose we can obtain the functional

$$\gamma(u, v, \alpha, \beta, \theta) = Eu^{X_1} v^{Y_1} e^{i\alpha W_1 + i\beta Z_1} e^{-\theta\tau_\rho} \tag{5.1.7}$$

of the transform of the joint distribution of all underlying components. Then a modification of Theorem 1 will read

**Theorem 16**. *Under assumptions* (5.1.1- 5.1.7), *the functional* $\Phi_\rho$ *satisfies the following formula*:

$$\Phi_\rho(u, v, \alpha, \beta, \theta) = Eu^{A_\rho} v^{B_\rho} e^{i\alpha P_\rho + i\beta Q_\rho} e^{-\theta\tau_\rho}$$

$$= 1 - [1 - \gamma(u, v, \alpha, \beta, \theta)]\mathcal{D}_y^{N-1}\left(\mathcal{D}_x^{M-1}\left\{\frac{1}{1-\gamma(ux, vy, \alpha, \beta, \theta)}\right\}\right) \tag{5.1.8}$$

*where the operator* $\mathcal{D}$ *applied to a function* $\varphi : \mathbb{C}^3 \to \mathbb{C}$ *analytic at* $x = y = 0$ *is defined as*

$$\mathcal{D}_{x,y}^{k,m}\varphi(x, y, z)$$

$$= \begin{cases} \lim_{x\to 0, y\to 0} \frac{1}{k!m!} \frac{\partial^{k+m}}{\partial x^k \partial y^m}\left[\frac{1}{(1-x)(1-y)}\varphi(x, y, z)\right], & k, m \geq 0 \\ 0, & k < 0 \text{ or } m < 0. \end{cases} \quad \square$$

Consider a special case where the casualties $(X_n, W_n)$ and $(Y_n, Z_n)$ are independent. We assume that the inter-observation times $\tau_1, \tau_2 - \tau_1, \ldots$ are independent and identically distributed according to the common exponential distribution with parameter $\gamma$.

Suppose further that the increments $W_n$ and $Z_n$ are Laplace-distributed r.v.'s with parameters $\lambda$ and $\mu$, that is, their probability density functions are:

$$f_{W_1}(x) = \tfrac{1}{2\lambda} e^{-\lambda|x|}, x \in \mathbb{R} \tag{5.1.9}$$

and

$$f_{Z_1}(x) = \tfrac{1}{2\mu} e^{-\mu|x|}, x \in \mathbb{R}. \tag{5.1.10}$$

Under the above assumptions,

$$\gamma(u, v, \alpha, \beta, \theta) = \delta_1(u, \alpha)\delta_2(v, \beta)\tfrac{\gamma}{\gamma+\theta} \tag{5.1.11}$$

where,

$$\delta_1(u, \alpha) = Eu^{X_1} e^{i\alpha W_1}$$

and

$$\delta_2(v, \beta) = Ev^{Y_1} e^{i\alpha Z_1}.$$

**Theorem 17.** *Under the assumptions of (5.1.1) - (5.1.11), the joint functional $\Phi_\rho$ of the non monotone game satisfies the following formula*:

$$\Phi_\rho = \left(\frac{(a+b)df}{4-(a+b)cf}\right)^N + \left(\frac{bcf}{4-acf}\right)^M \sum_{k=0}^{M} \binom{M}{k} \left(\frac{d}{c}\right)^k \sum_{j=0}^{N-k-1} \binom{M+j-1}{j}$$

$$\times \left[\left(\frac{adf}{4-acf}\right)^j - \left(\frac{(a+b)df}{4-(a+b)cf}\right)^{N-k} \left(\frac{a}{(4-acf)} \frac{4-(a+b)cf}{(a+b)}\right)^j\right], \quad (5.1.12)$$

for $N-k-1 \geq 0$, where

$$a = \frac{\lambda}{\lambda - i\alpha}, \qquad b = \frac{\lambda u}{\lambda + i\alpha}, \qquad c = \frac{\mu}{\mu - i\beta}. \quad (5.1.13)$$

$$d = \frac{\mu}{\mu + i\beta}, \qquad f = \frac{\gamma}{\gamma + \theta}. \quad (5.1.14)$$

**Proof.**

From Theorem 16 (5.1.8), we need to find $\mathcal{D}_x^{M-1}\left\{\frac{1}{1-\gamma(ux,vy,\alpha,\beta,\theta)}\right\}$.

From (5.1.11),

$$\delta_1(u,\alpha) = Eu^{X_1}e^{i\alpha W_1} = Eu^{X_1}e^{i\alpha W_1}(\mathbf{1}_{X_1=0} + \mathbf{1}_{X_1=1}),$$

where,

$$Eu^{X_1}e^{i\alpha W_1}\mathbf{1}_{X_1=0} = \delta_1(0,\alpha)$$

$$= E[\mathbf{1}_{X_1=0}e^{i\alpha W_1}] = E\left[e^{i\alpha W_1}\mathbf{1}_{\{W_1 \geq 0\}}\right]$$

$$= \int_{x=0}^{\infty} e^{i\alpha x} \frac{1}{2\lambda} e^{-\lambda x} d(x)$$

$$= \frac{1}{2}\frac{\lambda}{\lambda - i\alpha}, \quad (5.1.15)$$

and

$$Eu^{X_1}e^{i\alpha W_1}\mathbf{1}_{X_1=1} = uEe^{i\alpha W_1}\mathbf{1}_{X_1=1}$$

$$= uE\big[e^{i\alpha W_1}\mathbf{1}_{\{W_1<0\}}\big]$$

$$= u\int_{x=-\infty}^{0} e^{i\alpha x}\tfrac{1}{2\lambda}e^{-\lambda(-x)}d(-x)(-1)$$

$$= u(-1)\int_{y=\infty}^{0} e^{-i\alpha y}\tfrac{1}{2\lambda}e^{-\lambda y}dy$$

$$= \tfrac{u}{2}\int_{0}^{\infty}\tfrac{1}{\lambda}e^{-y(\lambda+i\alpha)}dy$$

$$= \tfrac{u}{2}\tfrac{\lambda}{\lambda+i\alpha}. \tag{5.1.16}$$

Therefore,

$$\delta_1(u,\alpha) = \tfrac{1}{2}\tfrac{\lambda}{\lambda-i\alpha} + \tfrac{u}{2}\tfrac{\lambda}{\lambda+i\alpha}. \tag{5.1.17}$$

Similarly,

$$\delta_2(v,\beta) = \tfrac{1}{2}\tfrac{\mu}{\mu-i\beta} + \tfrac{v}{2}\tfrac{\mu}{\mu+i\beta}. \tag{5.1.18}$$

Thus using (5.1.17) and (5.1.18) in (5.1.11) we have that

$$\gamma(u,v,\alpha,\beta,\theta) = \delta_1(u,\alpha)\delta_2(v,\beta)\tfrac{\gamma}{\gamma+\theta}$$

$$= \left(\tfrac{1}{2}\tfrac{\lambda}{\lambda-i\alpha} + \tfrac{u}{2}\tfrac{\lambda}{\lambda+i\alpha}\right)\left(\tfrac{1}{2}\tfrac{\mu}{\mu-i\beta} + \tfrac{v}{2}\tfrac{\mu}{\mu+i\beta}\right)\tfrac{\gamma}{\gamma+\theta}. \tag{5.1.19}$$

From (5.1.19) we can write,

$$\gamma(ux, vy, \alpha, \beta, \theta) = \tfrac{1}{4}\left(\tfrac{\lambda}{\lambda - i\alpha} + \tfrac{\lambda u}{\lambda + i\alpha}x\right)\left(\tfrac{\mu}{\mu - i\beta} + \tfrac{\mu v}{\mu + i\beta}y\right)\tfrac{\gamma}{\gamma + \theta}$$

$$= \tfrac{1}{4}(a + bx)(c + dy)f, \tag{5.1.20}$$

where,

$$a = \tfrac{\lambda}{\lambda - i\alpha}, \qquad b = \tfrac{\lambda u}{\lambda + i\alpha}, \qquad c = \tfrac{\mu}{\mu - i\beta},$$

$$d = \tfrac{\mu v}{\mu + i\beta}, \qquad f = \tfrac{\gamma}{\gamma + \theta}.$$

Using (5.1.20),

$$1 - \gamma(ux, vy, \alpha, \beta, \theta) = 1 - \tfrac{1}{4}(a + bx)(c + dy)f$$

$$= \tfrac{4 - acf - adfy - (bcf + bdfy)x}{4}.$$

$$\tfrac{1}{1 - \gamma(ux, vy, \alpha, \beta, \theta)} = \tfrac{4}{4 - acf - adfy - (bcf + bdfy)x}. \tag{5.1.21}$$

Therefore using (5.1.21) in (5.1.8) we have,

$$\mathcal{D}_x^{M-1}\left\{\tfrac{1}{1 - \gamma(ux, vy, \alpha, \beta, \theta)}\right\} = \mathcal{D}_x^{M-1}\left\{\tfrac{4}{4 - acf - adfy - (bcf + bdfy)x}\right\}$$

$$= \tfrac{4}{4 - acf - adfy}\mathcal{D}_x^{M-1}\left\{\tfrac{1}{\left(1 - \frac{bcf + bdfy}{4 - acf - adfy}x\right)}\right\}.$$

Such that for

$$bcf + bdfy \neq 4 - acf - adfy,$$

and applying Theorem 3 $(iv)$,

$$\mathcal{D}_x^{M-1}\left\{\frac{1}{1-\gamma(ux,vy,\alpha,\beta,\theta)}\right\} = \frac{4}{4-acf-adfy}\left[\frac{1-\left(\frac{bcf+bdfy}{4-acf-adfy}\right)^M}{1-\left(\frac{bcf+bdfy}{4-acf-adfy}\right)}\right]$$

$$= \frac{4}{4-acf-bcf-(adf+bdf)y}\left[1-\left(\frac{bcf}{4-acf}\cdot\frac{1+\frac{d}{c}y}{1-\frac{adf}{4-acf}y}\right)^M\right].$$

For brevity we let,

$$a_1 = 4-acf-bcf, \qquad b_1 = adf+bdf, \qquad g = \frac{bcf}{4-acf}.$$

$$= 4\left[\frac{1}{a_1-b_1y} - \frac{1}{a_1-b_1y}g^M\left(\frac{1+\frac{d}{c}y}{1-\frac{adf}{4-acf}y}\right)^M\right]$$

$$= 4\left[\frac{1}{a_1-b_1y} - g^M\frac{1}{a_1-b_1y}\left(1+\frac{d}{c}y\right)^M\left(\frac{1}{1-\frac{adf}{4-acf}y}\right)^M\right].$$

Therefore,

$$\mathcal{D}_y^{N-1}\circ\mathcal{D}_x^{M-1}\left\{\frac{1}{1-\gamma(ux,vy,\alpha,\beta,\theta)}\right\}$$

$$= 4\mathcal{D}_y^{N-1}\left\{\frac{1}{a_1-b_1y} - g^M\frac{1}{a_1-b_1y}\left(1+\frac{d}{c}y\right)^M\left(\frac{1}{1-\frac{adf}{4-acf}y}\right)^M\right\}$$

$$= \frac{4}{a_1}\mathcal{D}_y^{N-1}\left\{\frac{1}{a_1-b_1y}\right\} - \frac{4}{a_1}g^M$$

$$\times\,\mathcal{D}_y^{N-1}\left\{\sum_{k=0}^{M}\binom{M}{k}\left(\frac{d}{c}\right)^k y^k\frac{1}{1-\frac{b_1}{a_1}y}\left(\frac{1}{1-\frac{adf}{4-acf}y}\right)^M\right\}.$$

By Theorem 3 $(iv)$ and $(vi)$ this gives,

$$\mathcal{D}_y^{N-1} \circ \mathcal{D}_x^{M-1}\left\{\frac{1}{1-\gamma(ux,vy,\alpha,\beta,\theta)}\right\}$$

$$= \frac{4}{a_1}\mathcal{D}_y^{N-1}\left\{\frac{1}{a_1-b_1 y}\right\} - \frac{4}{a_1} g^M \sum_{k=0}^{M} \binom{M}{k} \left(\frac{d}{c}\right)^k$$

$$\times \mathcal{D}_y^{N-k-1}\left\{\frac{1}{1-\frac{b_1}{a_1}y}\left(\frac{1}{1-\frac{adf}{4-acf}y}\right)^M\right\},$$

for $N-k-1 \geq 0$ and $a_1 \neq b_1$,

$$= \frac{4}{a_1}\left\{\frac{1-\left(\frac{b_1}{a_1}\right)^N}{1-\frac{b_1}{a_1}}\right\} - \frac{4}{a_1} g^M \sum_{k=0}^{M} \binom{M}{k} \left(\frac{d}{c}\right)^k \frac{1}{1-\frac{b_1}{a_1}}$$

$$\times \sum_{j=0}^{N-k-1} \binom{M+j-1}{j} \left[\left(\frac{adf}{4-acf}\right)^j - \left(\frac{b_1}{a_1}\right)^{N-k} \left(\frac{adfa_1}{(4-acf)b_1}\right)^j\right]$$

$$= \frac{4}{a_1-b_1}\left\{\left[1-\left(\frac{b_1}{a_1}\right)^N\right] - g^M \sum_{k=0}^{M} \binom{M}{k} \left(\frac{d}{c}\right)^k\right.$$

$$\left.\times \sum_{j=0}^{N-k-1} \binom{M+j-1}{j} \left[\left(\frac{adf}{4-acf}\right)^j - \left(\frac{b_1}{a_1}\right)^{N-k} \left(\frac{adfa_1}{(4-acf)b_1}\right)^j\right]\right\}. \quad (5.1.22)$$

Note that,

$$a_1 - b_1 = 4 - acf - bcf - adf - bdf, \quad (5.1.23)$$

$$\gamma(u,v,\alpha,\beta,\theta) = \tfrac{1}{4}(a+b)(c+d), \quad (5.1.24)$$

and

$$1-\gamma(u,v,\alpha,\beta,\theta) = \frac{4-acf-adf-bcf-bdf}{4}. \quad (5.1.25)$$

Substituting (5.1.23) in (5.1.22) and then multiplying the resulting expression by $1-\gamma(u,v,\alpha,\beta,\theta)$ of (5.1.25) we obtain

$$1-\gamma(u,v,\alpha,\beta,\theta)]\mathcal{D}_{yx}^{N-1,M-1}\left\{\frac{1}{1-\gamma(ux,vy,\alpha,\beta,\theta)}\right\}$$

$$=1-\left(\frac{b_1}{a_1}\right)^N-g^M\sum_{k=0}^{M}\binom{M}{k}\left(\frac{d}{c}\right)^k$$

$$\times\sum_{j=0}^{N-k-1}\binom{M+j-1}{j}\left[\left(\frac{adf}{4-acf}\right)^j-\left(\frac{b_1}{a_1}\right)^{N-k}\left(\frac{adfa_1}{(4-acf)b_1}\right)^j\right]. \quad (5.1.26)$$

Using (5.1.26) in formula (5.1.8) of Theorem 16 we arrive at (5.1.12) - (5.1.14) and herewith complete the proof. □

## 5.2 Marginal Transform of $\tau_\rho$, $Ee^{-\theta\tau_\rho}$

If we let $u=v=1,\alpha=\beta=0,$ in (5.1.12) - (5.1.14) and after some algebra, we arrive at

$$Ee^{-\theta\tau_\rho}=\left(\frac{\gamma}{\gamma+2\theta}\right)^N+\left(\frac{\gamma}{3\gamma+4\theta}\right)^M\sum_{k=0}^{M}\binom{M}{k}\sum_{j=0}^{N-k-1}\binom{M+j-1}{j}$$

$$\times\left[\frac{1}{(3\gamma+4\theta)^j}\left(\gamma^j-\left(\frac{\gamma}{\gamma+2\theta}\right)^{N-k}(\gamma+2\theta)^j\right)\right], \quad (5.2.1)$$

for $N-k-1\geq 0\,(k\leq N-1)$.

## 5.3 Special Case of $Ee^{-\theta\tau_\rho}$.

Revisiting subsection 5.2 we will target the special case of the marginal transform of $\tau_\rho$ with $M = N = 1$. From Theorem in the equation (5.2.1),

$$Ee^{-\theta\tau_\rho} = \left(\frac{\gamma}{\gamma+2\theta}\right)^1 + \left(\frac{\gamma}{3\gamma+4\theta}\right)^1 \sum_{k=0}^{1} \binom{1}{k} \sum_{j=0}^{1-k-1} \binom{1+j-1}{j}$$

$$\times \left[\frac{1}{(3\gamma+4\theta)^j}\left(\gamma^j - \left(\frac{\gamma}{\gamma+2\theta}\right)^{1-k} (\gamma+2\theta)^j\right)\right], \quad (5.3.1)$$

for $k \leq N-1$.

The last inequality above implies that for the special case of $M = N = 1$ which we are considering, $k \leq 0$. Hence, (5.3.1) becomes,

$$\begin{aligned}
Ee^{-\theta\tau_\rho} &= \left(\frac{\gamma}{\gamma+2\theta}\right) + \left(\frac{\gamma}{3\gamma+4\theta}\right)\binom{1}{0}\binom{0}{0}\left[\frac{1}{(3\gamma+4\theta)^0}\left(\gamma^0 - \left(\frac{\gamma}{\gamma+2\theta}\right)^1 (\gamma+2\theta)^0\right)\right] \\
&= \left(\frac{\gamma}{\gamma+2\theta}\right) + \left(\frac{\gamma}{3\gamma+4\theta}\right)\left[1 - \frac{\gamma}{\gamma+2\theta}\right] \\
&= \left(\frac{\gamma}{\gamma+2\theta}\right) + \left(\frac{\gamma}{3\gamma+4\theta}\right)\frac{2\theta}{\gamma+2\theta} \\
&= \frac{\gamma}{\gamma+2\theta}\left[1 + \frac{2\theta}{3\gamma+4\theta}\right] \\
&= \frac{\gamma}{\gamma+2\theta}\left[\frac{3\gamma+6\theta}{3\gamma+4\theta}\right] \\
&= \frac{3\gamma}{\gamma+2\theta}\left[\frac{\gamma+2\theta}{3\gamma+4\theta}\right] \\
&= \frac{3\gamma}{3\gamma+4\theta} = \frac{\frac{3}{4}\gamma}{\frac{3}{4}\gamma+\theta}.
\end{aligned}$$

Therefore, the first passage time of the game end is exponential with parameter $\frac{3}{4}\gamma$, that is, the $4/3$ longer than the mean time of $\tau_1, \tau_2 - \tau_1, \dots$ .

# Conclusion

We have studied purely antagonistic games (in Delayed-time and Real-time) of two players modeled by marked stochastic processes with marks representing simultaneous independent attacks (for the Real-time game) of random magnitudes. We focused primarily on the real time game which has two versions. The games differed in the cardinals of marks, being discrete- and continuous-valued, respectively. We chose them to be geometrically and exponentially distributed, respectively. Otherwise, the games were congruent. Because geometric and exponential distributions are the only two of the kind memoryless representatives of either class, we believe that an analytic comparison between the two games was warranted.

We came to the point that for the games $\mathcal{G}_d$ and $\mathcal{G}_c$ (as we called them so in Chapter 4) we needed to apply a LST-inverse to obtain the probability density function of the first passage time, that is the epoch when the game ends. We could do it explicitly only for game $\mathcal{G}_d$. For the other game, $\mathcal{G}_c$ our only option was a numerical inversion, which we did not do, although it is readily accomplishable.

In the comparison of the two games, it turned out that the discrete game $\mathcal{G}_d$ that underwent a discrete operational calculus was superior to the continuous game $\mathcal{G}_c$ that underwent the more traditional Laplace inversion.

In Chapter 5, we started exploring a five component antagonistic game where two of them were declared non-monotonic and active and the other three were passive. Interestingly, we found out that for the special case where the threshold for the two active component was 1, we obtained an explicit result for the marginal of the first passage time to be exponential with parameter $\frac{3}{4}\gamma$.

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