5-2020

Optimal Control of the Second Order Elliptic Equations with Biomedical Applications

Saleheh Seif

Optimal Control of the Second Order Elliptic Equations with Biomedical Applications

by

Saleheh Seif

Bachelor of Science
Pure Mathematics
Damghan University
2004

Masters of Science
Applied Mathematics
Tabriz University
2008

A dissertation
submitted to Department of Mathematical Sciences of Florida Institute of Technology
in partial fulfillment of the requirements
for the degree of

Doctorate of Philosophy
in
Operations Research

Melbourne, Florida
May, 2020

We the undersigned committee
hereby approve the attached dissertation

Optimal Control of the Second Order Elliptic Equations with Biomedical Applications

by
Saleheh Seif

Ugur G. Abdulla, Ph.D., Dr.Sci., Dr.rer.nat.habil.
Professor
Department of Mathematical Sciences
Committee Chair

Ted Conway, Ph.D.
Professor
Department of Biomedical and Chemical Engineering and Sciences
Outside Committee Member

Jian Du, Ph.D.
Associate Professor
Department of Mathematical Sciences
Committee Member

Vladislav Bukshtynov, Ph.D.
Assistant Professor
Department of Mathematical Sciences
Committee Member

Munevver Mine Subasi, Ph.D.
Associate Professor and Department Head
Department of Mathematical Sciences

ABSTRACT

Title:

Optimal Control of the Second Order Elliptic Equations with Biomedical Applications

Author:

Saleheh Seif

Major Advisor:

Ugur G. Abdulla, Ph.D., Dr.Sci., Dr.rer.nat.habil.

Dissertation analyzes optimal control of systems with distributed parameters described by the general boundary value problems in a bounded Lipschitz domain for the linear second order uniformly elliptic partial differential equations (PDE) with bounded measurable coefficients. Broad class of elliptic optimal control problems under Dirichlet or Neumann boundary conditions are considered, where the control parameter is the density of sources, and the cost functional is the L_2-norm difference of the weak solution of the elliptic problem from measurement along the boundary or subdomain. The optimal control problems are fully discretized using the method of finite differences. Two types of discretization of the elliptic boundary value problem depending on Dirichlet or Neumann type boundary condition are introduced. Convergence of the sequence of finite-dimensional discrete optimal control problems both with respect to the cost functional and the control is proved. The methods of the proof are based on energy estimates in discrete Sobolev spaces, Lax-Milgram theory, weak compactness and convergence of interpolations of solutions of discrete elliptic problems, and delicate estimation of the cost functional along the sequence of interpolations of the minimizers for the discrete optimal control problems. Dissertation pursues application of the optimal control theory of elliptic systems with distributed parameters to biomedical problem on the identification of cancerous tumor. The Inverse Electrical Impedance Tomography (EIT) problem on recovering electrical conductivity tensor and potential in the body based on the measurement of the boundary voltages on the m electrodes for a given electrode current is analyzed. A PDE constrained optimal control framework in Besov space is developed, where the electrical conductivity tensor and boundary voltages are control parameters, and the cost functional is the norm difference of the boundary electrode current from the given current pattern and boundary electrode voltages from the measurements. The state vector is a solution of the second order elliptic PDE in divergence form with bounded measurable coefficients under mixed Neumann/Robin type boundary condition. The novelty of the control theoretic model is its adaptation to clinical situation when additional "voltage-to-current" measurements can increase the size of the input data from m up to $m!$ while keeping the size of the unknown parameters fixed. Existence of the optimal control is established. Fréchet differentiability in the Banach-Besov spaces framework is proved and the formula for the Frechet gradient expressed in terms of the adjoined state vector is derived. Optimality condition is formulated, and gradient type iterative algorithm in Hilbert-Besov spaces setting is developed. EIT optimal control problem is fully discretized using the method of finite differences. New Sobolev-Hilbert space is introduced, and the convergence of the sequence of finite-dimensional optimal control problems to EIT coefficient optimal control problem is proved both with respect to functional and control in 2- and 3-dimensional domains.

Table of Contents

List of Notations

In this section, assume Q is a domain in $\mathbb{R}^n$. $B_r(x) \subset \mathbb{R}^n$ - ball of radius r and center x; $m_d(\cdot)$ - d-dimensional Lebesgue measure;

- For $1 \leq p < \infty$, $L_p(Q)$ is a Banach space of measurable functions on Q with finite norm

  $$\|u\|_{L_p(Q)} := \Big(\int_Q |u(x)|^p dx \Big)^{\frac{1}{p}}$$

  In particular if $p = 2$, $L_2(Q)$ is a Hilbert space with inner product

  $$(f,g)_{L_2(Q)} = \int_Q f(x)g(x)dx$$

- $L_\infty(Q)$ is a Banach space of measurable functions on Q with finite norm

  $$\|u\|_{L_\infty(Q)} := \operatorname*{ess\,sup}_{x\in Q} |u(x)|$$

- For $s \in \mathbb{Z}_+$, $W_p^s(Q)$ is the Banach space of measurable functions on Q with finite norm

  $$\|u\|_{W_p^s(Q)} := \Big(\int_Q \sum_{|\alpha|\leq s} |D^\alpha u(x)|^p dx \Big)^{\frac{1}{p}},$$

  where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_j are nonnegative integers, $|\alpha| = \alpha_1 + ... + \alpha_n$, $D_k =$

$\frac{\partial}{\partial x_k}$, $D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n}$. In particular if $p=2$, $H^s(Q) := W_2^s(Q)$ is a Hilbert space with inner product

$$(f,g)_{H^s(Q)} = \sum_{|\alpha|\leq s} (D^\alpha f(x), D^\alpha g(x))_{L_2(Q)}$$

- $H_0^1(Q)$ - linear subspace of elements u of $H^1(Q)$ which satisfy

$$u\Big|_S = 0,$$

  in the sense of traces.

- For $s \notin \mathbb{Z}_+$, $B_p^s(Q)$ is the Banach space of measurable functions on Q with finite norm

$$\|u\|_{B_p^s(Q)} := \|u\|_{W_p^{[s]}(Q)} + [u]_{B_p^s(Q)}$$

  where

$$[u]_{B_p^s(Q)} := \int_Q \int_Q \frac{\left|\frac{\partial^{[s]}u(x)}{\partial x^{[s]}} - \frac{\partial^{[s]}u(y)}{\partial x^{[s]}}\right|^p}{|x-y|^{1+p(s-[s])}} dxdy\Big)^{\frac{1}{p}}$$

  $H^\varepsilon(Q) := B_2^\varepsilon(Q)$ is a Hilbert space.

- $\tilde{H}_1(Q),\ n=2,3$ is a linear subspace of $H^1(Q)$ which is defined as follows:

$$\tilde{H}_1(Q) = \{u \in H^1(Q) | u_{x_1x_2} \in L_2(Q)\}, \quad \text{if } Q \in \mathbb{R}^2 \tag{1}$$

$$\tilde{H}_1(Q) = \{u \in H^1(Q) | u_{x_1x_2}, u_{x_1x_3}, u_{x_2x_3}, u_{x_1x_2x_3} \in L_2(Q)\}, \quad \text{if } Q \in \mathbb{R}^3 \tag{2}$$

and their respective norms are defined accordingly:

$$\|u\|^2_{\tilde{H}_1(Q)} = \|u\|^2_{H^1(Q)} + \|u_{x_1x_2}\|^2_{L_2(Q)}, \quad \text{if } Q \in \mathbb{R}^2 \tag{3}$$

$$\|u\|^2_{\tilde{H}_1(Q)} = \|u\|^2_{H^1(Q)} + \sum_{\substack{i,j=1 \\ i<j}}^{3} \|u_{x_ix_j}\|^2_{L_2(Q)} + \|u_{x_1x_2x_3}\|^2_{L_2(Q)}, \quad \text{if } Q \in \mathbb{R}^3 \tag{4}$$

- $\mathbf{ba}(Q) = \big(L_\infty(Q)\big)'$ is the Banach space of bounded and finitely additive signed measures on Q and the dual space of $L_\infty(Q)$ with finite norm

$$\|\phi\|_{\mathbf{ba}(Q)} = |\phi|(Q),$$

  $|\phi|(Q)$ is total variation of ϕ and defined as $|\phi|(Q) = \sup\sum_i \phi(E_i)$, where the supremum is taken over all partitions $\cup E_i$ of E into measurable subsets E_i.

- $\mathbb{M}^{m\times n}$ is a space of real $m\times n$ matrices.

- $\mathscr{L} := L_\infty(Q;\mathbb{M}^{n\times n})$ is the Banach space of $n\times n$ matrices of $L_\infty(Q)$ functions.

- $\mathscr{L}' := \mathbf{ba}(Q;\mathbb{M}^{n\times n}) = \big(L_\infty(Q;\mathbb{M}^{n\times n})\big)'$ is the Banach space of $n\times n$ matrices of $\mathbf{ba}(Q)$ measures.

Acknowledgements

I like to start by sending my greatest gratitude to my PhD advisor Prof. Ugur G. Abdulla. He was more than an advisor for me and ever since I started my program, he offered his great support to enable me to professionally grow and pursue my research. Whenever I had a question, he was always there for me and I am sure this achievement would not be possible if I have not been blessed with his tremendous support and advice.

Special gratitude goes to my committee members, Drs. Vladislav Bukshtynov, Jian Du, and Ted Conway for evaluating my work.

I need to acknowledge my previous and current lab mates Elise Aspray, Adam Prinkey, Amna Abuweden, Lamees Alzaki, Jonathan Goldfarb, Ali Hagverdyev, Naveed Iqbal and Evan Cosgrove, for their constructive feedbacks on my research and their supportive friendship.

I like to thank my best friend and my husband, Alireza, but honestly, I do not know how. During these years, he was always there for me and had my back. Whenever he found me tired of work, he sat by my side, patiently listened to my complaints, and tried to understand my problem and helped me to solve it. Every time I fell down throughout this journey, he was the one who held my hands, picked me up, dusted me off, and pushed me to challenge myself again and again.

Dedication

This work is dedicated to

my Amazing Parents

my Beautiful Sisters and Brother

and my Lovely Husband

Whose love and support for me has always been beyond imagination.

Chapter 1

Introduction

The understanding of optimal control problems for systems with distributed parameters is one of the most important and challenging problems in modern industrial, economical, and biomedical applications, to name a few. Solving optimal control problems is a crucial step in the transition from mathematical model based simulations to the actual design and control of complex systems. The development of effective methods of optimization of complex systems modeled by partial differential equations (or PDE-constrained optimization), combined with powerful software development, also plays a central role in many areas of operations research and management science. Yet another important application of the mathematical theory of optimal control in infinite-dimensional settings arises in the early stages of mathematical modeling of complex systems that provide the most effective tools to solve inverse problems for the identification of parameters based on experimental data.

Optimal control of systems with distributed parameters described by second order linear elliptic PDEs has a well established theory. The book [81] was the first with systematic outline of the mathematical theory of optimal control of systems with distributed parameters. Literature in this field within the last half-century is enormous. Without any ambition to pursue a complete survey, we refer to monographs [81, 95, 43, 94, 52] for the outline of the mathematical theory of the optimal control of elliptic PDEs, including the questions of existence and uniqueness of the optimal control in Banach spaces setting, necessary and sufficient optimality conditions via adjoined approach, Fréchet and Gateaux differentiability, Pontryagin's Maximum Principle, theory of constrained optimal control including both control and state constrains. Theoretical advance along with development of the powerful computational tools opened the way for the development and implementation of numerical methods for solving PDE constrained optimal control problems [93, 95, 28, 82, 47, 94, 52, 33]. The most effective idea for the development of effective numerical methods for optimal control problems in infinite-dimensional setting is based on approximation with the sequence of finite-dimensional optimal control problems via discretization by methods of finite elements or finite differences. Necessary and sufficient condition for the convergence of the sequence of discrete finite-dimensional optimal control problems to the infinite-dimensional optimal control problem both with respect to cost functional and control was formulated and proved in abstract setting in [95]. State-to-the-art introduction and survey of the results on discrete concepts, and numerical algorithms for the Elliptic PDE constrained optimization we refer to [52, 53, 33], the latter having more focus on computational aspects. In general, to solve optimal control numerically there are two approaches: *first Discretize, then Optimize* (DO) vs. *first Optimize, then Discretize* (OD) [52]. DO approach first discretizes optimal control problem via method of finite elements, followed by derivation of the necessary optimality condition for the finite-dimensional problem. In particular, this requires an introduction of the discrete adjoined vector determined by the ansatz of the discrete state vector. OD approach first derives first order optimality condition in infinite-dimensional setting, followed by the discretization of all the variables, including the adjoined variable. The latter includes some freedom, and in fact choice of the ansatz space of the adjoined variable forms the difference between two approaches. In [52] both DO and OD approaches are analyzed and the convergence of the finite element approximation, along with error estimates, is established for the uniquely solvable PDE optimality system in canonical convex optimal control problem for the Poisson equation with zero Dirichlet boundary values. There is a broad literature on convergence and error estimates for finite element approximation of the PDE optimality systems in linear elliptic control problems in space dimension two or three [26, 25, 39, 38, 36, 84, 37, 35, 22]. In contrast, convergence of the method of finite differences in optimal control problems for elliptic PDEs is not as widely investigated. In [33, 32, 31] convergence of the finite difference multigrid solution to PDE optimality system for the same control problem in two dimensional rectangular domain such that boundaries coincide with the grid lines.

Despite its importance, the result on the convergence of the finite differences method for optimal control problem for the general elliptic PDEs in arbitrary domains is not available in the literature. **One of the main goals of the dissertation is to prove such convergence result for a broad class of elliptic optimal control problems under Dirichlet and Neumann boundary conditions.** It should be pointed out that we are not analyzing PDE optimality system, but aiming to prove that the necessary and sufficient condition for the convergence of the sequence of discrete optimal control problems is satisfied. This is essential both for OD and DO approaches, and in particular it provides legitimacy for the solution of the finite-dimensional discrete optimal control problems instead of infinite-dimensional optimal control problem.

In Chapter 2 we analyze optimal control problem for the general linear elliptic PDE *with bounded measurable coefficients, where control parameter is the density of sources and the cost functional is the L_2-norm difference of the weak solution of the elliptic Dirichlet or Neumann problem from measurement along the boundary or subdomain. The optimal control problems are fully discretized using the method of finite differences. Two types of discretization of the elliptic boundary value problem depending on Dirichlet or Neumann type boundary condition are introduced. The main result of the Chapter 2 is the following:*

- *Convergence of the sequence of finite-dimensional discrete optimal control problems both with respect to the cost functional and the control is proved. The methods of the proof are based on energy estimates in discrete Sobolev spaces, Lax-Milgram theory, weak compactness and convergence of interpolations of solutions of discrete elliptic problems, and delicate estimation of the cost functional along the sequence of interpolations of the minimizers for the discrete optimal control problems.*

Another major goal of the dissertation is to apply elliptic PDE constrained optimal control theory to solve an inverse Electrical Impedance Tomography (EIT) problem of estimating an unknown electrical conductivity tensor inside the body based on voltage measurements on the surface of the body when electric currents are applied through a set of contact electrodes. Inverse EIT problem has many important applications in medicine, industry, gepphysics and material science [54]. We are especially motivated with medical applications on the detection of cancerous tumors from breast tissue or other parts of the body. Relevance of the inverse EIT problem for cancer detection is based on the fact that the conductivity of the cancerous tumor is higher than the condactivity of normal tissues [77]. Inverse EIT Problem is an ill-posed problem and belongs to the class of so-called Calderon type inverse problems, due to celebrated work by [34] where well-posedness of the inverse problem for the identification of the conductivity coefficient $\sigma : \Omega \to \mathbb{R}$ of the second order elliptic PDE

$$div(\sigma(x)\nabla u) = 0 \tag{1.1}$$

through Dirichlet-to-Neumann or Neumann-to-Dirichlet boundary maps is presented. Significant development in Calderon's inverse problem in the class of smooth conductivity function with spatial dimension $n \geq 3$, concerning questions on uniqueness, stability, reconstruction procedure, reconstruction with partial data was achieved in [92, 85, 15, 24, 61, 62]. Global uniqueness in spatial dimension $n = 2$ and reconstruction procedure through scattering transform and employment of the, so-called D-bar method was presented in a key paper [86]. Further essential development of the D-bar method for the reconstruction of discontinuous parameters, regularization due to inaccuracy of measurements, joint recovery of the shape of domain and conductivity are pursued in [64, 65, 66, 69]. Inverse EIT problem with unknown anisotropic conductivity tensor as in (3.3) is highly ill-posed, and even with perfect Dirichlet to Neumann map there is a non-uniqueness [91]. This is the structural non-uniqueness, and one can talk about the identification of the conductivity tensor up to diffeomorphisms which keep the boundary fixed [79, 91, 86, 75, 76, 27, 23]. Alternative approach is based on imposing apriori structural constraints on the class of anisotropies [67, 68, 80, 17, 44, 45, 16].

Mathematical model for the EIT Problem, referred as complete electrode model, was suggested in [90] in the case of given isotropic electrical conductivity tensor. The model suggests replacement of the complete potential measurements along the boundary with measurements of constant potential along the electrodes with contact impedances. In [90] it was demonstrated that the complete electrode model is physically more relevant, and it is capable of predicting the experimentally measured voltages to within 0.1 percent. Existence and uniqueness of the solution to the EIT problem was proved in [90]. **Inverse EIT Problem** is more difficult than the Calderon's problem due to the fact that the infinite-dimensional conductivity function σ (or tensor A) and finite-dimensional voltage vector U must be identified based on the finitely many boundary electrode voltage measurements. Indeed, there are only finitely many electrodes are available where input current pattern can be injected for the successful measurement of the voltage. Hence the input data is finite-dimensional current vector, while in Calderon's problem input data is given through infinite-dimensional boundary operator "Dirichlet-to-Neuman" or "Neuman-to-Dirichlet". Therefore, inverse EIT problem is highly ill-posed and powerful regularization methods are required for its solution. It is essential to note that the size of the input current vector is limited to the number of electrodes, and there is no flexibility to increase its size. It would be natural to suggest that multiple data sets - input currents can be implemented for the identification of the same conductivity function. However, note that besides unknown conductivity function, there is unknown boundary voltage vector with size directly proportional to the size of the input current vector. Accordingly, multiple experiments with "current-to-voltage" measurements is not reducing underdeterminacy of the inverse problem. One can prove uniqueness and stability results by restricting isotropic conductivity to the finite-dimensional subset of piecewise analytic functions provided that the number of electrodes is large enough [78, 50]. Within last three decades many methods developed for numerical solution of the ill-posed inverse EIT problem both in isotropic and anisotropic conductivities. Without any ambition to present a full review we refer to some significant developments such as recovery of small inclusions from boundary measurements [21, 70]; hybrid conductivity imaging methods [20, 89, 96]; multi-frequency EIT imaging methods

[19, 88]; finite element and adaptive finite element method [57, 83]; imaging algorithms based on the sparsity reconstruction [19, 56]; globally convergent method for shape reconstruction in EIT [51]; D-bar method, diction reconstruction method, recovering boundary shape and imaging the anisotropic electrical conductivity [18, 40, 48, 49, 55]; globally convergent regularization method using Carleman weight function [63]. Inverse EIT problem was widely studied in the framework of Bayesian statistics [60]. In [58] inverse EIT problem is formulated as a Bayesian problem of statistical inference and Markov Chain Monte Carlo method with various prior distributions is implemented for calculation of the posterior distributions of the unknown parameters conditioned on measurement data. In [59] Bayesian model of the regularized version of the inverse EIT problem is analyzed. In [73] the Bayesian method with Whittle-Matérn priors is applied to inveres EIT problem. In general the strategy of the Bayesian approach to inverse EIT problem in infinite-dimensional setting is twofold. First approach is based on discretization followed by the application of finite-dimensional Bayesian methods. All the described papers are following this approach, which is nicely outlined in [60]. Alternative approach is based on direct application of the Bayesian methods in functional spaces before discretization [74, 41].

Dissertation introduces new variational formulation of the inverse EIT problem as a PDE constrained optimal control problem in a Besov space. The methods of Chapter 2 are developed and applied to biomedical problem on the detection of the cancerous tumor. In Chapters 3-5 we analyze the inverse EIT problem in a PDE constrained optimal control framework in Besov space, where the electrical conductivity tensor and boundary voltages are control parameters, and the cost functional is the norm difference of the boundary electrode current from the given current pattern and boundary electrode voltages from the measurements. The state vector is a solution of the sec-

ond order elliptic PDE in divergence form with bounded measurable coefficients under mixed Neumann/Robin type boundary condition. The following are the main results of Chapters 3-5:

- *The novelty of the control theoretic model is its adaptation to clinical situation when additional "voltage-to-current" measurements can increase the size of the input data from the number of boundary electrodes* m *up to* $m!$ *while keeping the size of the unknown parameters fixed.*

- *Existence of the optimal control and Fréchet differentiability in the Besov space setting is proved. The formula for the Fréchet gradient and optimality condition is derived. Numerical method based on the projective gradient method in Hilbert-Besov spaces is developed.*

- *EIT optimal control problem is fully discretized using the method of finite differences. New Sobolev-Hilbert space is introduced, and the convergence of the sequence of finite-dimensional optimal control problems to EIT coefficient optimal control problem is proved both with respect to functional and control in 2- and 3-dimensional domains.*

The organization of the dissertation is as follows. Chapter 2 pursues discretization and convergence for the method of finite differences for the optimal control problems for the second order elliptic PDEs. In Section 2.1 we introduce optimal control problems, outline the well-posedness facts of the formulated optimal control problems, pursue discretization by the method of finite differences, formulate discrete optimal control problems and describe the main results on the convergence of the sequence of discrete optimal control problems both with respect to functional and control. Some preliminary results are formulated in Section 2.2. In particular, in Section 2.2.1 we prove approximation lemma on the convergence of the interpolations of the discrete state vectors to weak solutions of the respective elliptic PDE problems. In Section 2.3 we prove the main results.

Chapter 3 analyzes inverse EIT problem in framework of optimal control of elliptic PDEs. In Section 3.1 we describe inverse EIT problem. Section 3.2 introduces variational formulation of the inverse EIT problem in a optimal control framework. Main results of Chapter 3 are formulated in Section 3.3. Finally, in Section 3.4 we prove the main results.

Chapter 4 analyzes discretization and convergence of the EIT optimal control problem in 2D domains. In Section 4.1 we describe 2D inverse EIT problem with isotropic conductivity map, its formulation as an optimal control problem and discretization with method of finite differences. Main convergence result is formulated in Section 4.2. In Section 4.3 we prove energy estimates and some essential interpolation theorems. Approximation lemma is established in Section 4.4. Finally, the proof of the main convergence theorem is completed in Section 4.5.

In Chapter 5 we analyze discretization and convergence of the EIT optimal control problem in 3D domains. Section 5.1 introduces EIT optimal control in 3D domains and its finite difference discretization. Main result is formulated in Section 5.2. We prove the main result in Section 5.3.

Finally, in Chapter 6 we describe main conclusions.

Chapter 2

Discretization and Convergence of Optimal Control Problems for Second Order Elliptic PDEs

2.1 Introduction and Main Results

2.1.1 Optimal Control Problems

Let $Q \subset \mathbb{R}^n$ is bounded domain with Lipschitz boundary $S = \partial Q$. Let $D \subseteq Q$ be an open subset of Q. Consider the optimal control problem on the minimization of the cost functional

$$\mathscr{I}(f) = \int_D |u(x;f) - g(x)|^2 dx + \beta \int_Q |f - \bar{f}|^2 dx \tag{2.1}$$

on a control set

$$\mathscr{F}^R = \left\{ f \in L_2(Q) \middle| \|f\|_2 \leq R \right\} \tag{2.2}$$

where $g \in L^2(D), \bar{f} \in L^2(Q)$ are given, $\beta \geq 0$, and $u = u(\cdot; f) \in H_0^1(Q)$ is a solution of the following Dirichlet problem for the second order linear elliptic PDE

$$\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + \sum_i b_i(x)u_{x_i} + a(x)u = f(x) \quad x \in Q \tag{2.3}$$

$$u(x) = 0 \qquad x \in S \tag{2.4}$$

with bounded measurable coefficients a_{ij}, b_i, a which satisfy the structural condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j - \sum_{i=1}^{n} b_i(x)\xi_i\xi_0 - a(x)\xi_0^2 \geq \nu \sum_{i=1}^{n} \xi_i^2 + \lambda \xi_0^2 \tag{2.5}$$

for arbitrary $\xi_i \in \mathbb{R},\ i = 0, 1, ..., n$ with positive constants ν and λ. This optimal control problem will be called Problem $\mathscr{D}$.

Next, we formulate optimal control problem for the elliptic PDE (2.3) under Neumann boundary condition. Let $\Gamma \subset S$ is a subset with positive $N-1$-dimensional Hausdorff measure and $z \in L^2(\Gamma)$ is a given function. Consider minimization of the cost functional

$$\mathscr{J}(f) = \beta_1 \int_D |u(x;f) - g(x)|^2 dx + \beta_2 \int_\Gamma |u(x;f) - z(x)|^2 ds + \beta \int_Q |f - \bar{f}|^2 dx \tag{2.6}$$

on the control set (2.2), where $\beta, \beta_i \geq 0$, $i = 1, 2$ and $\beta_1^2 + \beta_2^2 > 0$ and $u = u(\cdot; f) \in H^1(Q)$ is a solution of the Neumann problem for the elliptic PDE (2.3) under the boundary condition

$$\frac{\partial u(x)}{\partial \mathscr{N}} + k(x)u(x) = 0 \qquad\qquad x \in S \tag{2.7}$$

where $\frac{\partial u(x)}{\partial \mathscr{N}} = \sum_{i,j} a_{ij}(x) u_{x_j} \nu^i$ is a conormal derivative, $k \in L_\infty(S)$. If $k \geq 0$ then we assume that the ν and λ are arbitrary positive numbers, while if k is non-positive, then the constant λ in (2.5) should be sufficiently large. The formulated optimal control problem will be called Problem $\mathscr{N}$. We refer to Chapter 1 for literature review on elliptic optimal control theory. The goal of this chapter is to discretize both optimal control problems using the method of finite differences and prove the convergence of the sequence of discrete finite-dimensional optimal control problems to original problem both with respect to cost functional and control.

2.1.2 Well-posedness of the Optimal Control Problems

Let bilinear form $B : H(Q) \times H(Q) \to \mathbb{R}$ be defined as follows

$$B[u, \eta] = \int_Q \Big(\sum_{i,j=1}^{n} a_{ij} u_{x_j} \eta_{x_i} - \sum_i b_i u_{x_i} \eta - a u \eta \Big) dx$$

where H stands for $H_0^1(Q)$ in Dirichlet problem, and for $H^1(Q)$ in Neumann problem. The following are definitions of the weak solutions of the Dirichlet and Neumann problems (2.3),(2.4) and (2.3),(2.7) respectively.

Definition 2.1.1. $u \in H_0^1(Q)$ is called the weak solution to the problem (2.3),(2.4) if

$$B[u, \eta] = -(f, \eta)_{L_2}, \quad \forall \eta \in H_0^1(Q) \tag{2.8}$$

Definition 2.1.2. $u \in H^1(Q)$ is called the weak solution to the problem (2.3)-(2.7) if

$$B[u,\eta] + \int_S ku\eta ds = -(f,\eta)_{L_2}, \quad \forall \eta \in H^1(Q) \tag{2.9}$$

From Lax-Milgram theory ([46, 72, 71]) it follows that for a given $f \in \mathscr{F}^R$ there exist unique weak solutions to the problems (2.3),(2.4) and (2.3),(2.7) respectively, and the following energy estimate is valid

$$\|u\|_H \leq C\|f\|_{L_2(Q)}, \tag{2.10}$$

Due to bounded embedding $H^1(Q) \hookrightarrow L_2(Q)$, cost functionals $\mathscr{I}$ and $\mathscr{J}$ in both problems are well defined. Cost functionals are weakly lower semicontinuous and convex in a bounded, closed and convex control set $\mathscr{F}^R$. Therefore, there exists an optimal control in both problems [81, 95, 43]. If $\beta > 0$, then functionals $\mathscr{I}$ and $\mathscr{J}$ are strictly convex, and therefore there is a unique optimal control in both problems [81, 95, 43].

2.1.3 Discrete Optimal Control Problems

To discretize optimal control problems $\mathscr{D}$ and $\mathscr{N}$ we pursue finite difference method following the framework introduced in [14]. Let $h > 0$ and cut $\mathbb{R}^n$ by the planes

$$x_i = k_i h, \qquad i = 1, \dots, n, \qquad \forall k_i \in \mathbb{Z}.$$

into a collection of elementary cells with length h in each x_i-direction. We denote the discretization with step size h by Δ. We introduce an ordering in the class of discretizations by setting $\Delta(h_1) \leq \Delta(h_2)$ if $h_1 \leq h_2$. For every discretization Δ and multi-index

$\alpha = (k_1, k_2, \ldots, k_n)$ we define a cell C_Δ^α as

$$C_\Delta^\alpha = \{x \in \mathbb{R}^n |\ k_i h \leq x_i \leq (k_i + 1)h,\ i = 1, \ldots, n\}, \tag{2.11}$$

and consider the collection of these cells

$$\mathscr{C}_\Delta = \{C_\Delta^\alpha \mid \alpha \in \mathbb{Z}^n\}. \tag{2.12}$$

Denote the subcollection of cells which lie in $\overline{Q}$ as $\mathscr{C}_\Delta^Q$, and the subcollection of cells which have non-empty intersection with Q as $\mathscr{C}_\Delta^{Q^*}$:

$$\mathscr{C}_\Delta^Q = \{C_\Delta^\alpha \in \mathscr{C}_\Delta \mid C_\Delta^\alpha \subset \bar{Q}\},\ \mathscr{C}_\Delta^{Q^*} = \{C_\Delta^\alpha \in \mathscr{C}_\Delta \mid C_\Delta^\alpha \cap Q \neq \emptyset\} \tag{2.13}$$

We now introduce interior and exterior approximations of $\overline{Q}$ as follows:

$$Q_\Delta = \bigcup_{C_\Delta^\alpha \in \mathscr{C}_\Delta^Q} C_\Delta^\alpha, \qquad Q_\Delta^* = \bigcup_{C_\Delta^\alpha \in \mathscr{C}_\Delta^{Q^*}} C_\Delta^\alpha \tag{2.14}$$

Obviously, we have $Q_\Delta \subset \overline{Q} \subset Q_\Delta^*$. Let $S_\Delta = \partial Q_\Delta$ and $S_\Delta^* = \partial Q_\Delta^*$.

The vertex of the prism C_Δ^α whose coordinates are smallest relative to the other vertices, is called its *natural corner*. We are going to identify each prism (cell) by its natural corner.

Now define the lattice of points

$$\mathscr{L} = \Big\{x \in \mathbb{R}^d \mid \exists \alpha \in \mathbb{Z}^d \text{ s.t. } x_i = k_i h,\ i = 1, \ldots, n\Big\}.$$

We will write $x_\alpha = (k_1h, k_2h, \ldots, k_nh)$. Note the obvious bijections $\alpha \mapsto x_\alpha$; bijections of this form will henceforth be referred as natural. Given a set X which is in natural bijection with a subset of the set of multi-indexes α, we write $\mathscr{A}(X)$ as the indexing set. Moreover, if $X \subset \mathbb{R}^d$, then $\mathscr{L}(X) := \mathscr{L} \cap X$. When $X = \mathscr{L}(Y) \subset \mathbb{R}^d$, we'll agree to write $\mathscr{A}(Y)$ instead of $\mathscr{A}(\mathscr{L}(Y))$. These indexes are also in natural bijection with the natural corners of these prisms. In particular, some of the corresponding lattice points may fall on the boundary S_Δ. We contrast this set to the set $\mathscr{A}(Q'_\Delta)$ of indexes in natural bijection to the lattice points that lie strictly in the interior of Q_Δ, and to the set $\mathscr{A}(Q_\Delta)$, of all indexes which are in natural bijection with the lattice points that lie in Q_Δ. We will write

$$\sum_{\mathscr{A}(X)} \quad \text{instead of} \quad \sum_{\alpha \in \mathscr{A}(X)},$$

and likewise for other expressions requiring subscripts. We adopt the notation

$$\alpha \pm e_i := (k_1, ..., k_i \pm 1, ..., k_n).$$

To discretize optimal control problem $\mathscr{N}$, we need to introduce some refined subsets of grid pints of Q^*_Δ. Let

$$Q^{*+}_\Delta = \{x_\alpha \in Q^*_\Delta : C^\alpha_\Delta \cap Q \neq \emptyset\}$$

be a subset of natural corners of the cells in Q^*_Δ. We denote as

$$Q^{*(i)}_\Delta = \{x_\alpha \in Q^*_\Delta : x_{\alpha+e_i} \in Q^*_\Delta\}$$

the subset of all grid points x_α in Q^*_Δ such that the edge $[x_\alpha, x_{\alpha+e_i}]$ is in Q^*_Δ too. The sets Q^+_Δ and $Q^{(i)}_\Delta$ are defined similarly. Subset of natural corners x_α of cells in Q^*_Δ which intersect the boundary S is denoted as

$$\hat{S}^*_\Delta = \{x_\alpha \in Q^*_\Delta : C^\alpha_\Delta \cap S \neq \emptyset\}$$

Similar to Q_Δ, Q^*_Δ and $\hat{S}^*_\Delta$ we define

$$D_\Delta = \{\ x_\alpha \in \mathscr{C}^Q_\Delta \mid C^\alpha_\Delta \subset \overline{D}\ \}$$

$$D^*_\Delta = \{\ x_\alpha \in \mathscr{C}^Q_\Delta \mid C^\alpha_\Delta \cap D \neq \emptyset\ \}$$

$$\Gamma_\Delta = \{\ x_\alpha \in \hat{S}^*_\Delta \mid C^\alpha_\Delta \cap \Gamma \neq \emptyset\}$$

Note that we don't use superscript $+$ in preceding definition. We are going to assume that the coeffcients $a_{ij}, b_i, a \in L^\infty(Q)$ are extended to a larger set $Q + B_1(0)$ as bounded measurable functions with preservation of the structural condition (2.5). Any control vector $f \in \mathscr{F}^R$ and given function $\bar{f} \in L_2(Q)$ are continued as zero to $Q + B_1(0)$. We introduce discrete grid functions by discretizing a_{ij}, b_i, a, f and $\bar{f}$ through Steklov averages:

$$\phi_\alpha = \frac{1}{h^n} \int\limits_{x_1}^{x_1+h} \int\limits_{x_2}^{x_2+h} \cdots \int\limits_{x_n}^{x_n+h} \phi(x)dx,\ \alpha \in \mathscr{A}(Q^*_\Delta),\ \text{where } x_i \text{ is the i-th coordinate of } x_\alpha, \tag{2.15}$$

and ϕ stands for any of the functions a_{ij}, b_i, a, f and $\bar{f}$. Similar grid function is introduced for $g \in L_2(D)$ after zero continuation to $D + B_1(0)$. For $k \in L_\infty(S)$ and $z \in L_2(\Gamma)$ we define

$$k_\alpha := \int_{S_\alpha} k(x)ds, \quad \alpha \in \mathscr{A}(\hat{S}^*_\Delta),\ S_\alpha := S \cap C^\alpha_\Delta \tag{2.16}$$

$$z^\Gamma_\alpha = \frac{1}{|\Gamma_\alpha|} \int_{\Gamma_\alpha} z(x)ds, \quad \alpha \in \mathscr{A}(\Gamma_\Delta),\ \Gamma_\alpha := C^\alpha_\Delta \cap \Gamma. \tag{2.17}$$

For a given discretization Δ, we employ the notation $[f]_\Delta = \{f_\alpha\}$, $\alpha \in \mathscr{A}(Q^*_\Delta)$ and define the discrete ℓ_2 norm of $[f]_\Delta$ as

$$\|[f]_\Delta\|_{\ell_2} := \Big(\sum_{\mathscr{A}(Q^*_\Delta)} h^n f_\alpha^2\Big)^{\frac{1}{2}}.$$

We use standard notation for finite differences of grid function $u_\alpha, \alpha \in \mathscr{A}(Q^*_\Delta)$:

$$u_{\alpha x_i} = \frac{u_{\alpha+e_i} - u_\alpha}{h},\ u_{\alpha \bar{x}_i} = \frac{u_\alpha - u_{\alpha-e_i}}{h}.$$

For fixed $R > 0$, define the discrete control sets as

$$\mathscr{F}^R_\Delta := \big\{[f]_\Delta : \|[f]_\Delta\|_{\ell_2} \leq R\big\} \tag{2.18}$$

and the interpolating map $\mathscr{P}_\Delta$ as

$$\mathscr{P}_\Delta : \bigcup_R \mathscr{F}^R_\Delta \to \bigcup_R \mathscr{F}^R, \qquad \mathscr{P}_\Delta([f]_\Delta) = f^\Delta$$

where

$$f^\Delta\Big|_{C^\alpha_\Delta} = f_\alpha, \qquad \alpha \in \mathscr{A}(Q^*_\Delta).$$

Also, we define the discretizing map $\mathscr{Q}_\Delta$ as

$$\mathscr{Q}_\Delta : \bigcup_R \mathscr{F}^R \to \bigcup_R \mathscr{F}^R_\Delta, \qquad \mathscr{Q}_\Delta(f) = [f]_\Delta$$

where $[f]_\Delta = \{f_\alpha\}$ and f_α is given by (2.15) for each $\alpha \in \mathscr{A}(Q^*_\Delta)$.

Next, we define a solution of the discrete Dirichlet problem.

Definition 2.1.3. Given $[f]_\Delta \in \mathscr{F}_\Delta^R$, the discrete valued function

$$[u([f]_\Delta)]_\Delta = \{u_\alpha \in \mathbb{R} : \alpha \in \mathscr{A}(Q_\Delta)\}$$

is called a discrete state vector of problem $\mathscr{D}$, or solution of the discrete Dirichlet problem if $u_\alpha = 0$ for $\alpha \in \mathscr{A}(S_\Delta)$ and it satisfies

$$\sum_{\mathscr{A}(Q_\Delta^+)} h^n \left[\sum_{i,j=1}^{n} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} + (-\sum_i b_{i\alpha} u_{\alpha x_i} - a_\alpha u_\alpha + f_\alpha^\Delta)\eta_\alpha \right] = 0 \tag{2.19}$$

for arbitrary collection of values $\{\eta_\alpha\},\ \alpha \in \mathscr{A}(Q_\Delta)$ such that $\eta_\alpha\big|_{S_\Delta} = 0$.

Note that the collection $\{f_\alpha^\Delta\}$ in the (2.19) is the function $\mathscr{Q}_\Delta(\mathscr{P}_\Delta([f]_\alpha))$.
In Section 2.2 it will be proved that for a given $[f]_\alpha \in \mathscr{F}_\Delta^R$, there exists a unique discrete state vector $[u([f]_\Delta)]_\Delta$ of problem $\mathscr{D}$. Consider minimization of the discrete cost functional

$$\mathscr{I}_\Delta([f]_\Delta) = \sum_{\mathscr{A}(D_\Delta^+)} h^n |u_\alpha - g_\alpha|^2 + \beta \sum_{\mathscr{A}(Q_\Delta^+)} h^n |f_\alpha - \bar{f}_\alpha|^2 \tag{2.20}$$

on a control set $\mathscr{F}_\Delta^R$, where u_α's are components of the discrete state vector $[u([f]_\Delta)]_\Delta$, g_α and $\bar{f}_\alpha$ are Steklov averages of g and $\bar{f}$ based on formula (2.15). The formulated discrete optimal control problem will be called Problem $\mathscr{D}_\Delta$.
Next, we define a solution of the discrete Neumann problem.

Definition 2.1.4. Given $[f]_\Delta \in \mathscr{F}_\Delta^R$, the discrete valued function

$$[u([f]_\Delta)]_\Delta = \{u_\alpha \in \mathbb{R} : \alpha \in \mathscr{A}(Q_\Delta^*)\}$$

is called a discrete state vector of problem $\mathscr{N}$, or solution of the discrete Neumann problem if it satisfies

$$\sum_{\mathscr{A}(Q_\Delta^{*+})} h^n \left[\sum_{i,j=1}^{n} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} - \sum_i b_{i\alpha} u_{\alpha x_i} \eta_\alpha - a_\alpha u_\alpha \eta_\alpha + f_\alpha^\Delta \eta_\alpha \right]$$

$$+ J_\alpha(u_\alpha, \eta_\alpha) + \sum_{\mathscr{A}(\hat{S}_\Delta^*)} k_\alpha u_\alpha \eta_\alpha = 0 \qquad (2.21)$$

for arbitrary collection of values $\{\eta_\alpha\},\ \alpha \in \mathscr{A}(Q_\Delta^*)$,where

$$J_\alpha(u_\alpha, \eta_\alpha) = h^n \sum_{\mathscr{A}(S_\Delta^*)} \left[\theta_\alpha u_\alpha \eta_\alpha + \sum_{i=1}^{n} \theta_\alpha^i u_{\alpha x_i} \eta_{\alpha x_i} \right], \qquad (2.22)$$

$$\theta_\alpha = \begin{cases} 1 & \text{if } \alpha \in \mathscr{A}(S_\Delta^* \setminus \hat{S}_\Delta^*) \\ 0 & \text{otherwise} \end{cases}, \quad \theta_\alpha^i = \begin{cases} 1 & \text{if } \alpha \in \mathscr{A}(Q_\Delta^{*(i)}) \setminus \mathscr{A}(Q_\Delta^{*+}) \\ 0 & \text{otherwise} \end{cases}.$$

The necessity of adding J_α to (2.21) is that the term $h^n \sum_{\mathscr{A}(Q_\Delta^{*+})} \eta_\alpha u_\alpha$ of (2.21) does not extend to whole grid points in Q_Δ^*. The missing terms will be added through the first term of J_α. Moreover, some $u_{\alpha x_i}$ and $\eta_{\alpha x_i}$ values on S_Δ^* are not present in the term $h^n \sum_{\mathscr{A}(Q_\Delta^{*+})} \eta_{\alpha x_i} u_{\alpha x_i}$ of (2.21). These values are added to (2.21) through the second term of J_α. For stability of our discrete scheme, it is essential to add these two terms to the discrete integral identity (2.21).

In Section 2.2 it will be proved that for a given $[f]_\alpha \in \mathscr{F}_\Delta^R$, there exists a unique discrete state vector $[u([f]_\Delta)]_\Delta$ of problem $\mathscr{N}$. Consider minimization of the discrete cost functional

$$\mathscr{J}_\Delta([f]_\Delta) = \sum_{\mathscr{A}(D_\Delta^{*+})} h^n |u_\alpha - g_\alpha|^2 + \sum_{\mathscr{A}(\Gamma_\Delta)} |\Gamma_\alpha| |u_\alpha - z_\alpha^\Gamma|^2 + \beta \sum_{\mathscr{A}(Q_\Delta^{*+})} h^n |f_\alpha - \bar{f}_\alpha|^2 \quad (2.23)$$

on a control set $\mathscr{F}_\Delta^R$, where u_α's are components of the discrete state vector $[u([f]_\Delta)]_\Delta$

of the Problem $\mathscr{N}$, g_α, $\bar{f}_\alpha$ and z_α^Γ are Steklov averages of g, $\bar{f}$ and z based on (2.15), (2.17). The formulated discrete optimal control problem will be called Problem $\mathscr{N}_\Delta$.

Definition 2.1.5. The discrete $\mathscr{H}^1(Q_\Delta^*)$ norm for $[u([f]_\Delta)]_\Delta = \{u_\alpha\}$ is defined as

$$\|[u([f]_\Delta)]_\Delta\|_{\mathscr{H}^1(Q_\Delta^*)} := \Big(\sum_{\mathscr{A}(Q_\Delta^*)} h^n u_\alpha^2 + \sum_{i=1}^n \sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^n u_{\alpha x_i}^2 \Big)^{\frac{1}{2}}.$$

The discrete norm $\mathscr{H}^1(Q_\Delta)$ is defined similarly.

2.1.4 Main Results

Theorem 2.1.6. *The sequence of discrete optimal control problems $\mathscr{D}_\Delta$ approximates the optimal control problem $\mathscr{D}$ with respect to functional, i.e.*

$$\lim_{\Delta \to 0} \mathscr{I}_{\Delta_*} = \mathscr{I}_*, \tag{2.24}$$

where

$$\mathscr{I}_{\Delta_*} = \inf_{\mathscr{F}_\Delta^R} \mathscr{I}_\Delta([f]_\Delta), \; \mathscr{I}_* = \inf_{\mathscr{F}^R} \mathscr{I}(f). \tag{2.25}$$

Furthermore, let $\{\varepsilon_\Delta\}$ be a sequence of positive real numbers with $\lim_{\Delta\to 0} \varepsilon_\Delta = 0$. If the sequence $[f]_{\Delta,\varepsilon} \in \mathscr{F}_\Delta^R$ is chosen so that

$$\mathscr{I}_{\Delta_*} \leq \mathscr{I}_\Delta([f]_{\Delta,\varepsilon}) \leq \mathscr{I}_{\Delta_*} + \varepsilon_\Delta, \tag{2.26}$$

then we have

$$\lim_{\Delta \to 0} \mathscr{I}(\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})) = \mathscr{I}_* \tag{2.27}$$

Also, the sequence $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ *is uniformly bounded in* $L_2(Q)$ *and all of its* $L_2(Q)$*-weak limit points lie in optimal control set*

$$\mathscr{F}_* := \{f \in \mathscr{F}^R | \mathscr{I}(f) = \mathscr{I}_*\}$$

In particular, if $\beta > 0$*, then the sequence* $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ *converges weakly in* $L_2(Q)$ *to unique optimal control* f_**. Moreover, if* f_* *is such a weak limit point, then there is a subsequence* Δ' *such that the multilinear interpolations of the discrete state vectors* $[u([f]_{\Delta',\varepsilon})]_{\Delta'}$ *converge to weak solution* $u = u(x; f_*)$ *of the Dirichlet problem* (2.3)–(2.4) *weakly in* $H_0^1(Q)$*, strongly in* $L_2(Q)$*, and almost everywhere on* Q*.*

Theorem 2.1.7. *The sequence of discrete optimal control problems* $\mathscr{N}_\Delta$ *approximates the optimal control problem* $\mathscr{N}$ *with respect to functional, i.e.*

$$\lim_{\Delta \to 0} \mathscr{J}_{\Delta_*} = \mathscr{J}_*, \tag{2.28}$$

where

$$\mathscr{J}_{\Delta_*} = \inf_{\mathscr{F}_\Delta^R} \mathscr{J}_\Delta([f]_\Delta), \ \mathscr{J}_* = \inf_{\mathscr{F}^R} \mathscr{J}(f). \tag{2.29}$$

Furthermore, let $\{\varepsilon_\Delta\}$ *be a sequence of positive real numbers with* $\lim_{\Delta \to 0} \varepsilon_\Delta = 0$*. If the sequence* $[f]_{\Delta,\varepsilon} \in \mathscr{F}_\Delta^R$ *is chosen so that*

$$\mathscr{J}_{\Delta_*} \le \mathscr{J}_\Delta([f]_{\Delta,\varepsilon}) \le \mathscr{J}_{\Delta_*} + \varepsilon_\Delta, \tag{2.30}$$

then we have

$$\lim_{\Delta \to 0} \mathscr{J}(\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})) = \mathscr{J}_* \tag{2.31}$$

Also, the sequence $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ *is uniformly bounded in* $L_2(Q)$ *and all of its* $L_2(Q)$*-*

weak limit points lie in optimal control set

$$\mathscr{F}_* := \{f \in \mathscr{F}^R | \mathscr{J}(f) = \mathscr{J}_*\}$$

In particular, if $\beta > 0$*, then the sequence* $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ *converges weakly in* $L_2(Q)$ *to unique optimal control* f_**. Moreover, the multilinear interpolations of the discrete state vectors* $[u([f]_{\Delta',\varepsilon})]_{\Delta'}$ *converge to weak solution* $u = u(x; f_*)$ *of the Neumann problem* (2.3)*,* (2.7) *weakly in* $H^1(Q)$*, strongly in* $L_2(Q)$*, and almost everywhere on* Q*.*

2.2 Preliminary Results

In this section we recall known results about the unique solvability of the discrete Dirichlet and Neumann problems for the second order elliptic PDEs [71]. The next proposition formulates discrete Dirichlet problem as a system of linear algebraic equations for the unknown grid components of the discrete state vector of the problem $\mathscr{D}$.

Proposition 2.2.1. *For a given discretization* Δ *and control* $[f]_\Delta \in \mathscr{F}^R_\Delta$*, a vector* $[u([f]_\Delta)]_\Delta$ *is a solution of the discrete Dirichlet problem if and only if it satisfies the conditions*

$$\sum_{i,j=1}^{n} \left(a_{ij\alpha}(x) u_{\alpha x_j}\right)_{\bar{x}_i} + \sum_i b_{i\alpha}(x) u_{\alpha x_i} + a_\alpha u_\alpha - f^\Delta_\alpha = 0, \ \forall \alpha \in \mathscr{A}(Q'_\Delta) \tag{2.32}$$

$$u_\alpha = 0, \qquad \forall \alpha \in \mathscr{A}(S_\Delta) \tag{2.33}$$

Proof. Suppose $[u([f]_\Delta)]_\Delta$ satisfies (2.32). Take an arbitrary collection of $\{\eta_\alpha\}$ for $\alpha \in \mathscr{A}(Q_\Delta)$ which satisfies $\eta_\alpha = 0$ for $\alpha \in \mathscr{A}(S_\Delta)$. Multiplying (2.32) by $h^n \eta_\alpha$ and adding them for all $\alpha \in \mathscr{A}(Q'_\Delta)$ we have

$$\sum_{\mathscr{A}(Q'_\Delta)} h^n \eta_\alpha \Big[\sum_{i,j=1}^{n} \big(a_{ij\alpha}(x) u_{\alpha x_j}\big)_{\bar{x}_i} + \sum_i b_{i\alpha}(x) u_{\alpha x_i} + a_\alpha u_\alpha - f_\alpha^\Delta \Big] = 0 \tag{2.34}$$

Observe that for any $i, j = 1, ..., n$ we have

$$\begin{aligned}
\sum_{\mathscr{A}(Q'_\Delta)} \big(a_{ij\alpha} u_{\alpha x_j}\big)_{\bar{x}_i} \eta_\alpha &= \sum_{\mathscr{A}(Q'_\Delta)} \frac{a_{ij\alpha} u_{\alpha x_j}}{h} \eta_\alpha - \sum_{\mathscr{A}(Q'_\Delta)} \frac{a_{ij(\alpha-e_i)} u_{(\alpha-e_i)x_j}}{h} \eta_\alpha \\
&= \sum_{\mathscr{A}(Q'_\Delta)} \frac{a_{ij\alpha} u_{\alpha x_j}}{h} \eta_\alpha - \sum_{\mathscr{A}(Q'_\Delta - e_i)} \frac{a_{ij\alpha} u_{\alpha x_j}}{h} \eta_{\alpha+e_i} = \\
&- \sum_{\mathscr{A}(Q'_\Delta) \cap \mathscr{A}(Q'_\Delta - e_i)} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} + \sum_{\mathscr{A}(Q'_\Delta) \setminus \mathscr{A}(Q'_\Delta - e_i))} \frac{a_{ij\alpha} u_{\alpha x_j}}{h} \eta_\alpha \\
&- \sum_{\mathscr{A}(Q'_\Delta - e_i) \setminus \mathscr{A}(Q'_\Delta)} \frac{a_{ij\alpha} u_{\alpha x_j}}{h} \eta_{\alpha+e_i} \\
&= - \sum_{\mathscr{A}(Q'_\Delta) \cap \mathscr{A}(Q'_\Delta - e_i)} - \sum_{\mathscr{A}(Q'_\Delta) \setminus \mathscr{A}(Q'_\Delta - e_i)} - \sum_{\mathscr{A}(Q'_\Delta - e_i) \setminus \mathscr{A}(Q'_\Delta)} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} \\
&= - \sum_{\mathscr{A}(Q'_\Delta) \cup \mathscr{A}(Q'_\Delta - e_i)} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} = - \sum_{\mathscr{A}(Q^+_\Delta)} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i}.
\end{aligned} \tag{2.35}$$

Plugging this calculation into (2.34) and using the fact that $\eta_\alpha = 0$ for each $\alpha \in \mathscr{A}(S_\Delta)$, we derive (2.19). Conversely, from (2.19) and (2.35), (2.34) easily follows. Since $\eta_\alpha, \alpha \in \mathscr{A}(Q'_\Delta)$ are arbitrary, from (2.34), (2.32) follows. □

The next lemma presents energy estimate for the discrete Dirichlet problem.

Lemma 2.2.2 (Energy Estimate for Discrete Dirichlet Problem [71])**.** *For any $R > 0$, Δ and $[f]_\Delta \in \mathscr{F}_\Delta^R$ the discrete state vector $[u([f]_\Delta)]_\Delta$ of the problem $\mathscr{D}$ satisfies the following energy estimate:*

$$\| [u([f]_\Delta)]_\Delta \|_{\mathscr{H}^1(Q_\Delta)} \le M \|f\|_{L^2(Q)} \tag{2.36}$$

with M independent of Δ, R.

In particular, energy estimate implies the existence and uniqueness of the discrete state vector of the problem $\mathscr{D}$.

Corollary 2.2.3. *For a fixed* Δ *and any* $R > 0$*, there exists a unique discrete state vector* $[u([f]_\Delta)]_\Delta$ *in a problem* $\mathscr{D}$ *for each* $[f]_\Delta \in \mathscr{F}_\Delta^R$.

Indeed, the number of unknowns in the system (2.32)-(2.33) are the same as the number of equations. From the energy estimate (2.36) it follows that the corresponding homogeneous system has only trivial solution. Well known linear algebra fact implies the claim of the corollary.

Next lemma formulates the energy estimate for the discrete Neumann problem.

Lemma 2.2.4 (Energy Estimate for Discrete Neumann Problem [71])**.** *For any* $R > 0$*,* Δ *and* $[f]_\Delta \in \mathscr{F}_\Delta^R$ *the discrete state vector* $[u([f]_\Delta)]_\Delta$ *of the problem* $\mathscr{N}$ *satisfies the following energy estimate:*

$$\| [u([f]_\Delta)]_\Delta \|_{\mathscr{H}^1(Q_\Delta^*)} \leq M \|f\|_{L^2(Q_\Delta)} \tag{2.37}$$

As in Corollary 2.2.3, the energy estimate implies the existence and uniqueness of the discrete state vector in a problem $\mathscr{N}$.

Corollary 2.2.5. *For a fixed* Δ *and any* $R > 0$*, there exists a unique discrete state vector* $[u([f]_\Delta)]_\Delta$ *in a problem* $\mathscr{N}$ *for each* $[f]_\Delta \in \mathscr{F}_\Delta^R$.

Indeed, by explicitly writing $u_{\alpha x_i}$ and $\eta_{\alpha x_i}$ in terms of u_α and η_α from (2.21) it follows that

$$\sum_{\mathscr{A}(Q_\Delta^*)} \mathscr{L}_\Delta(u_\alpha).\eta_\alpha = \sum_{\mathscr{A}(Q_\Delta^*)} \mathscr{F}_\Delta(f_\alpha).\eta_\alpha \tag{2.38}$$

where $\mathscr{L}_\Delta$ and $\mathscr{F}_\Delta$ are linear mappings. Due to arbitrariness of values of η_α at $\alpha \in \mathscr{A}(Q^*_\Delta)$ we have

$$\mathscr{L}_\Delta(u_\alpha) = \mathscr{F}_\Delta(f_\alpha), \quad \forall \alpha \in \mathscr{A}(Q^*_\Delta) \tag{2.39}$$

which presents a system of linear algebraic equations for the unknown values $\{u_\alpha\}$ of the discrete state vector in a problem $\mathscr{N}$ at the grid points of Q^*_Δ. System (2.39) has the same number of equations as unknowns. From the energy estimate it follows that the corresponding homogeneous system has only trivial solution. Therefore, the system (2.39) has a unique solution.

Next, we recall the well-known necessary and sufficient condition for the convergence of the discrete optimal control problems to the continuous optimal control problem, formulated in the context of the optimal control problem $\mathscr{N}$:

Theorem 2.2.6. *[95] The sequence of discrete optimal control problems $\mathscr{N}_\Delta$ approximates the continuous optimal control problem $\mathscr{N}$ with respect to the functional if and only if the following conditions are satisfied:*

1. *For any $f \in \mathscr{F}^R$, we have $\mathscr{Q}_\Delta(f) \in \mathscr{F}^R_\Delta$, and the following inequality is satisfied*

$$\limsup_{\Delta\to 0}\left(\mathscr{J}_\Delta(\mathscr{Q}_\Delta(f)) - \mathscr{J}(f)\right) \le 0.$$

2. *For each $[f]_\Delta \in \mathscr{F}^R_\Delta$, we have $\mathscr{P}_\Delta([f]_\Delta) \in \mathscr{F}^R$, and the following inequality is satisfied*

$$\limsup_{\Delta\to 0}\left(\mathscr{J}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{J}_\Delta([f]_\Delta)\right) \le 0.$$

Similar necessary and sufficient condition can be formulated for the optimal control problem $\mathscr{B}$. Next proposition proves that mappings $\mathscr{P}_\Delta$ and $\mathscr{Q}_\Delta$ satisfy the conditions of the Theorem 2.2.6.

Proposition 2.2.7. *The maps $\mathscr{P}_\Delta$and $\mathscr{Q}_\Delta$ satisfy the conditions of Theorem 2.2.6*

Proof. Fix $\varepsilon > 0$ and Δ arbitrary. First let $f \in \mathscr{F}^R$. Then we note

$$\|\mathscr{Q}_\Delta(f)\|_{\ell_2}^2 = h^n \sum_{\mathscr{A}(Q_\Delta^*)} f_\alpha^2 = h^n \sum_{\mathscr{A}(Q_\Delta^{*+})} f_\alpha^2 = h^n \sum_{\mathscr{A}(Q_\Delta^{*+})} \left(\frac{1}{h^n}\int_{C_\Delta^\alpha} f\,dx\right)^2 \tag{2.40}$$

$$\leq \sum_{\mathscr{A}(Q_\Delta^{*+})} \int_{C_\Delta^\alpha} f^2\,dx = \int_{Q_\Delta^*} f^2\,dx = \int_Q f^2\,dx \leq R^2 \tag{2.41}$$

Now let $[f]_\Delta \in \mathscr{F}_\Delta^R$which implies $\|[f]_\Delta\|_{l_2} = \Big(\sum_{\mathscr{A}(Q_\Delta^*)} h^n f_\alpha^2\Big)^{\frac{1}{2}} \leq R$. We have

$$\|\mathscr{P}_\Delta([f]_\Delta)\|_{L_2(Q)}^2 = \int_Q (f^\Delta)^2\,dx \leq \int_{Q_\Delta^*} (f^\Delta)^2\,dx = \sum_{\mathscr{A}(Q_\Delta^{*+})} \int_{C_\Delta^\alpha} (f^\Delta)^2\,dx$$

$$= \sum_{\mathscr{A}(Q_\Delta^{*+})} h^n f_\alpha^2 \leq \sum_{\mathscr{A}(Q_\Delta^*)} h^n f_\alpha^2 = \|[f]_\Delta\|_{\ell_2}^2 \leq R^2$$

which proves the claim of the proposition. □

Following the frame of the recent paper [14] we define three types of interpolations of the discrete state vector in problem $\mathscr{D}$.

The first interpolation is denoted by $\tilde{U}_\Delta$ which is a piece-wise constant function $\tilde{U}_\Delta : Q \to R$, which assigns to the interior of each cell in Q_Δ the value of u_α at its natural corner and it is defined as following

$$\tilde{U}_\Delta\big|_{C_\Delta^\alpha} = u_\alpha, \qquad \forall \alpha \in \mathscr{A}(Q_\Delta^+) \tag{2.42}$$

and we let $\tilde{U}_\Delta$ be 0 elsewhere in Q that it is not already defined.

Now for each $i = 1, 2, .., n$, we define the second piece-wise constant interpolating function $\tilde{U}_\Delta^i : Q \to R$ which assign to each cell in Q_Δ the value of the forward spatial difference at the natural corner and it is defined as following

$$\tilde{U}_\Delta^i\big|_{C_\Delta^\alpha} = u_{\alpha x_i}, \qquad \forall \alpha \in \mathscr{A}(Q_\Delta^+) \tag{2.43}$$

and 0 elsewhere in Q that it is not already defined.

Next, we define $U'_\Delta : Q \to R$ which assigns the value u_α to each grid point in $\mathscr{L}(Q_\Delta)$, and it is a peicewise linear with respect to each variable x_i when the rest of variables are fixed. It is also extended as 0 on $Q - Q_\Delta$.

$U'_\Delta(x)$ can be represented as the weighted average of u_{α^*} values for each $\alpha^* \in \mathscr{A}(C_\Delta^\alpha)$

$$U'_\Delta(x) = \sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} w_{\alpha^*}(x) u_{\alpha^*}, \qquad x \in C_\Delta^\alpha \tag{2.44}$$

where coefficient functions, $w_{\alpha^*} : C_\Delta^\alpha \to [0,1]$, are continuous in C_Δ^α and satisfy

$$\sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} w_{\alpha^*}(x) = 1, \qquad x \in C_\Delta^\alpha \tag{2.45}$$

Using this property we obtain the following estimation:

$$\int_Q |U'_\Delta|^2 dx = \int_{Q_\Delta} |U'_\Delta|^2 dx \le \sum_{\mathscr{A}(Q_\Delta)} h^n \max_{\mathscr{A}(Q'_\Delta)} |u_{\alpha^*}|^2 \le 2^n \sum_{\mathscr{A}(Q_\Delta)} h^n |u_\alpha|^2. \tag{2.46}$$

Now, we claim that $\frac{\partial}{\partial x_i} U'_\Delta(x)$ can be represented as weighted average of forward differences in a fixed cell. To this end, we fix a x_i direction and denote the forward differences defined on one dimensional lines parallel to the x_i direction which join the vertices of C_Δ^α to each other with $u_{\alpha^* x_i}$ where $\alpha^* \in \mathscr{A}(C_\Delta^\alpha, i) := \mathscr{A}(C_\Delta^\alpha) \cap \{\alpha_i^* = \alpha_i\}$. Note that there are 2^{n-1} forward spatial differences of this type.

Then, for each $x \in C_\Delta^\alpha$, the value $\frac{\partial}{\partial x_i} U'_\Delta(x)$ will be represented as weighted average of $u_{\alpha^* x_i}$ where $\alpha^* \in \mathscr{A}(C_\Delta^\alpha, i)$.

$$\frac{\partial}{\partial x_i} U'_\Delta(x) = \sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha, i)} W_{\alpha^*}(x) u_{\alpha^* x_i}, \qquad x \in C_\Delta^\alpha \tag{2.47}$$

where the weight functions $W_{\alpha^*} : C_\Delta^\alpha \to [0,1]$ are continuous and satisfy

$$\sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha, i)} W_{\alpha^*}(x) = 1, \qquad x \in C_\Delta^\alpha \tag{2.48}$$

Using this property we obtain the following estimation:

$$\int_Q \left| \frac{\partial}{\partial x_i} U'_\Delta \right|^2 dx \le 2^{n-1} \sum_{\alpha \in \mathscr{A}(Q_\Delta)} h^n |u_{\alpha x_i}|^2. \tag{2.49}$$

Lemma 2.2.8 (Interpolation of a Discrete Dirichlet Problem [14])**.** *For each Δ, let $\{[f]_\Delta\} \in \mathscr{F}_\Delta^R$ be a sequence of discrete control vectors for some $R > 0$. The following statements hold for interpolations of the discrete state variable of the problem $\mathscr{D}$:*
(a) The sequences $\{U'_\Delta\}$ and $\{\tilde{U}_\Delta\}$ are uniformly bounded in $L_2(Q)$.
(b) For each $i \in \{1,2,\dots,n\}$, the sequences $\{\tilde{U}_\Delta^i\}, \{\frac{\partial U'_\Delta}{\partial x_i}\}$ are uniformly bounded in $L_2(Q)$.
(c) the sequence $\{\tilde{U}_\Delta - U'_\Delta\}$ converges strongly to 0 in $L_2(Q)$ as $h \to 0$.
(d) For each $i \in \{1,2,\dots,n\}$, the sequences $\{\frac{\partial U'_\Delta}{\partial x_i} - \tilde{U}_\Delta^i\}$ converges weakly to zero in $L_2(Q)$ as $h \to 0$.

We define interpolations of the discrete state vector of the problem $\mathscr{N}$ in a similar manner.

- $\tilde{V}_\Delta : Q_\Delta^* \to R$, $\tilde{V}_\Delta\big|_{C_\Delta^\alpha} = u_\alpha, \qquad \forall \alpha \in \mathscr{A}(Q_\Delta^{*+})$.

- $\tilde{V}^i_\Delta : Q^*_\Delta \to R$, $\tilde{V}^i_\Delta\big|_{C^\alpha_\Delta} = u_{\alpha x_i}, \qquad \forall \alpha \in \mathscr{A}(Q^{*+}_\Delta)$

- $V'_\Delta : Q^*_\Delta \to R$ which assigns the value u_α to each grid point in $\mathscr{L}(Q^*_\Delta)$, and it is a peicewise linear with respect to each variable x_i when the rest of variables are fixed.

By the same techniques that we used before we obtain these two evaluations for $V'_\Delta(x)$ and $\frac{\partial}{\partial x_i} V'_\Delta(x)$

$$\int_Q |V'_\Delta|^2 dx = \int_{Q_\Delta} |V'_\Delta|^2 dx \leq \sum_{\mathscr{A}(Q^*_\Delta)} h^n \max_{\mathscr{A}(Q'_\Delta)} |u_{\alpha^*}|^2 \leq 2^n \sum_{\mathscr{A}(Q^*_\Delta)} h^n |u_\alpha|^2. \tag{2.50}$$

$$\int_Q \Big| \frac{\partial}{\partial x_i} V'_\Delta \Big|^2 dx \leq 2^{n-1} \sum_{\mathscr{A}(Q^*_\Delta)} h^n |u_{\alpha x_i}|^2. \tag{2.51}$$

Lemma 2.2.9 (Interpolation of a Discrete Neumann Problem [14])**.** *For each Δ, let $\{[f]_\Delta\} \in \mathscr{F}^R_\Delta$ be a sequence of discrete control vectors for some $R > 0$. The following statements hold:*

(a) The sequences $\{V'_\Delta\}$ and $\{\tilde{V}_\Delta\}$ are uniformly bounded in $L_2(Q^*_\Delta)$.

(b) For each $i \in \{1,2,\dots,n\}$, the sequences $\{\tilde{V}^i_\Delta\}, \{\frac{\partial V'_\Delta}{\partial x_i}\}$ are uniformly bounded in $L_2(Q^*_\Delta)$.

(c) the sequence $\{\tilde{V}_\Delta - V'_\Delta\}$ converges strongly to 0 in $L_2(Q)$ as $h \to 0$.

(d) For each $i \in \{1,2,\dots,n\}$, the sequences $\{\frac{\partial V'_\Delta}{\partial x_i} - \tilde{V}^i_\Delta\}$ converges weakly to zero in $L_2(Q)$ as $h \to 0$.

(e) the sequence $\{\tilde{V}_\Delta - V'_\Delta\}$ converges strongly to 0 in $L_2(S)$ as $h \to 0$.

The claims (a)-(d) are proved in Theorem 14 of [14]. We present the proof of the claim (e), which is similar to the proof of the claim (d) in Theorem 14 of [14]. We observe that for each $\alpha \in \mathscr{A}(Q_\Delta^{*+})$

$$|\tilde{V}_\Delta(x) - V'_\Delta(x)|^2 = |u_\alpha - \sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} w_{\alpha^*}(x) u_{\alpha^*}|^2 \tag{2.52}$$

$$= |\sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} w_{\alpha^*}(x)(u_\alpha - u_{\alpha^*})|^2 \leq \sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} |u_\alpha - u_{\alpha^*}|^2, \quad \text{a.e. } x \in C_\Delta^\alpha \tag{2.53}$$

We note that if $\alpha = (k_1, k_2, \ldots, k_n)$, then each $\alpha^* \in C_\Delta^\alpha$ satisfies that $\alpha_i^* = \{k_i, k_i + 1\}$. Therefore, for each fixed $\alpha^* \in \mathscr{A}(C_\Delta^\alpha)$, there is a (not necessarily unique) path along the edges of the cell C_Δ^α which starts at x_α, ends at x_{α^*} , and is made up of gluing together at most n one-dimensional edges of the cell. Call such a path $P_{\alpha\to\alpha^*}$, and $T_P(x)$ the tangent vector to the path at point x. It is easy to see then that we can write

$$u_{\alpha^*} - u_\alpha = \int_{P_{\alpha\to\alpha^*}} D_x V_\Delta . dP = \sum_{P_{\alpha\to\alpha^*}} h u_{\alpha' x_j} \tag{2.54}$$

where the sum on the right-hand side of (2.54) is taken over the α' that correspond to vertices of C_Δ^α which lie on the path $P_{\alpha\to\alpha^*}$ (except for the end-point x_α^*)), and j corresponds to the spatial direction that the path $P_{\alpha\to\alpha^*}$ takes in moving from $x_{\alpha'}$ to the next vertex that lies on the path. With this observation in hand and using the Cauchy-Schwartz inequality, the following estimate is true, uniformly over the path chosen, and uniformly over α^*

$$|u_{\alpha^*} - u_\alpha|^2 \leq n \sum_{\text{edges of } P_{\alpha\to\alpha^*}} h^2 |u_{\alpha' x_j}|^2 \leq n \sum_{\text{edges of } C_\Delta^\alpha} h^2 |u_{\alpha' x_j}|^2 \tag{2.55}$$

where the sum on the right-hand side of (2.55) is taken over all α' and j such that $\alpha' \in \mathscr{A}(C_\Delta^\alpha)$ and $\alpha' + e_j \in \mathscr{A}(C_\Delta^\alpha)$ (intuitively, recall that the spatial differences $u_{\alpha' x_j}$ are in natural bijection with the edges of the lattice. So effectively, the sum is over all edges of the cell C_Δ^α). Therefore, using (2.53) and (2.55), we have

$$
\begin{aligned}
|\tilde{V}_\Delta(x) - V'_\Delta(x)|^2 &\le \sum_{\alpha^* \in \mathscr{A}(C_\Delta^\alpha)} n \sum_{\text{edges of } C_\Delta^\alpha} h^2 |u_{\alpha' x_j}|^2 \\
&\le (2^n - 1) n \sum_{\text{edges of } C_\Delta^\alpha} h^2 |u_{\alpha' x_j}|^2, \quad \text{a.e. } x \in C_\Delta^\alpha
\end{aligned}
\tag{2.56}
$$

since there are $2^n - 1$ vertexes x_{α^*} other than x_α in C_Δ^α. Now, using (2.56) we evaluate $\|\tilde{V}_\Delta - V'_\Delta\|^2_{L_2(S)}$

$$
\int_S |\tilde{V}_\Delta(x) - V'_\Delta(x)|^2 = \sum_{\mathscr{A}(\hat{S}^*_\Delta)} \int_{\Gamma_\alpha} |\tilde{V}_\Delta(x) - V'_\Delta(x)|^2 dx
$$

$$
\le \sum_{\mathscr{A}(\hat{S}^*_\Delta)} \int_{\Gamma_\alpha} (2^n - 1) n \sum_{\text{edges of } C_\Delta^\alpha} h^2 |u_{\alpha' x_j}|^2 ds \le \sum_{\mathscr{A}(\hat{S}^*_\Delta)} (2^n - 1) n 2^{n-1} \sum_{i=1}^{n} h^2 |u_{\alpha x_j}|^2 \int_{\Gamma_\alpha} ds
$$

as Γ_α is part of the smooth boundary S, for a fixed x_0 on Γ_α there exists $r > 0$ and $\gamma \in C^1 : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$
B_r(x_0) \cap \Gamma_\alpha = \{x \in B_r(x_0) | x_n = \gamma(x_1, x_2, \ldots, x_{n-1})\} \tag{2.57}
$$

then for each $x \in \Gamma_\alpha \cap B_r(x_0)$ we use the following coordinate change:

$$
\begin{aligned}
y_i &= x_i =: \phi^i(x), \quad i = \overline{1 : n-1} \\
y_n &= x_n - \gamma(x_1, \ldots, x_{n-1}) =: \phi^n(x)
\end{aligned}
$$

where $|D\boldsymbol{\phi}| = 1$, then we have

$$
\int_{\Gamma_\alpha \cap B_r(x_0)} ds = \int_{y_n = 0} \sqrt{1 + (\frac{\partial \gamma}{\partial y_1})^2 + \cdots + (\frac{\partial \gamma}{\partial y_{n-1}})^2}\, dy_1 \ldots dy_{n-1} \le C h^{n-1}
$$

since Γ_α is compact, so

$$\int_{\Gamma_\alpha} ds \le C_1 h^{n-1}$$

so

$$\int_S |\tilde{V}_\Delta(x) - V'_\Delta(x)|^2 \le C \sum_{\mathscr{A}(\hat{S}^*_\Delta)} (2^n - 1) n 2^{n-1} \sum_{i=1}^{n} h^2 |u_{\alpha x_j}|^2 h^{n-1}$$
$$\le Ch(2^n - 1) n 2^{n-1} \sum_{\mathscr{A}} h^n \sum_{i=1}^{n} |u_{\alpha x_j}|^2$$

By energy estimate (2.37) proof of this part is complete.

2.2.1 Approximation Lemmas

In this section we prove convergence of the interpolations of the discrete state vectors to weak solutions of the respective elliptic PDE problems. In Lemma 2.2.10 we prove the convergence of the multilinear interpolations of the discrete state vector of the problem $\mathscr{D}$ to the weak solution of the Dirichlet problem. In the following Lemma 2.2.11 we prove similar approximation result for the problem $\mathscr{N}$. The proofs are similar to the proofs given in [71].

Lemma 2.2.10. *Let* $\{[f]_\Delta\}$ *be a sequence of discrete control vectors such that there exists* $R > 0$ *for which* $[f]_\Delta \in \mathscr{F}^R_\Delta$ *for each* Δ*, and such that the sequence of interpolations* $\{\mathscr{P}_\Delta([f]_\Delta)\}$ *converges weakly to* f *in* $L_2(Q)$*. Then the sequence of interpolations* $\{U'_\Delta\}$ *of associated discrete state vectors converges weakly in* $H^1(Q)$ *to* $u = u(x; f) \in H^1_0(Q)$*, with* u *the unique weak solution to the* (2.3)–(2.4) *in the sense of Definition 2.1.1.*

Proof. From (a) and (b) of Lemma 2.2.8, it follows that $\{U'_\Delta\}$ is uniformly bounded in $H^1(Q)$. Consequently, $\{U'_\Delta\}$ has a weak limit point in $H^1(Q)$. Let $u \in H^1(Q)$ be any weak limit point of $\{U'_\Delta\}$ in $H^1(Q)$. By the Rellich-Kondrachev Theorem [42], it is known that a subsequence of $\{U'_\Delta\}$ converges strongly to u in $L_2(Q)$. Moreover, by construction, $U'_\Delta = 0$ on S for each Δ. Due to u being a weak limit point of $\{U'_\Delta\}$ in $H^1(Q)$, it follows that

$$\lim_{\Delta' \to 0} \|u|_S - U'_{\Delta'}|_S\|_{L_2(S)} = \|u|_S\|_{L_2(S)} = 0$$

from which we conclude $u|_S = 0$. Thus $u \in H^1_0(Q)$. Now, we proceed to show that u satisfies the integral identity (2.8). For simplicity of notation we write the subsequence of $\{U'_\Delta\}$ that converges weakly to u in $H^1(Q)$ as the whole sequence Δ. Let $\eta \in \overset{\bullet}{\mathscr{C}}{}^1(Q)$, where $\overset{\bullet}{\mathscr{C}}{}^1(Q)$ be a space of all continuously differentiable functions on $\bar{Q}$ whose support is a positive distance away from S. Since $Q_{\Delta'} \nearrow Q$, it follows that there exists Δ^* small enough so that $\overline{\operatorname{supp}\eta} \subset Q_\Delta$ for all $\Delta \le \Delta^*$. The collection of values $\{\eta_\alpha\},\ \alpha \in \mathscr{A}(Q_\Delta)$ is an admissible test collection for the summation identity (2.19). Let us remind the equation (2.19)

$$\sum_{\mathscr{A}(Q_\Delta)} h^n \left[\sum_{i,j=1}^{n} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} + (-\sum_i b_{i\alpha} u_{\alpha x_i} - a_\alpha u_\alpha + f^\Delta_\alpha) \eta_\alpha \right] = 0 \tag{2.58}$$

Then we define the piecewise constant interpolations $\bar{a}_{ij_\Delta}, \bar{b}_{i_\Delta},\ \bar{a}_\Delta$ of discrete valued functions $a_{ij\alpha}, b_{i\alpha}, a_\alpha$ as following

$$\bar{\beta}_\Delta \Big|_{C^\alpha_\Delta} = \beta_\alpha, \qquad \bar{\beta}_\Delta \equiv 0 \text{ elsewhere on } Q, \quad \forall \alpha \in \mathscr{A}(Q_\Delta)$$

where $\bar{\beta}_\Delta$ represents $\bar{a}_{ij_\Delta}, \bar{b}_{i_\Delta},\ \bar{a}_\Delta$.

in addition, we define the interpolations for η_α and $\eta_{\alpha x_i}$ for each $\alpha \in \mathscr{A}(Q_\Delta)$ as following

$$\bar{\eta}_\Delta\Big|_{C^\alpha_\Delta} = \eta_\alpha, \qquad \bar{\eta}_\Delta \equiv 0 \text{ elsewhere on } Q,$$

$$\bar{\eta}^i_\Delta\Big|_{C^\alpha_\Delta} = \eta_{\alpha x_i}, \qquad \bar{\eta}^i_\Delta \equiv 0 \text{ elsewhere on } Q,$$

With these functions and with the interpolations described for discrete state vector, identity (2.58) becomes

$$\sum_{\mathscr{A}(Q_\Delta)} \int_{C^\alpha_\Delta} \left[\sum_{i,j=1}^{n} \bar{a}_{ij_\Delta} \tilde{U}^i_\Delta \bar{\eta}^i_\Delta + \Big(-\sum_i \bar{b}_{i_\Delta} \tilde{U}^i_\Delta - \bar{a}_\Delta \tilde{U}_\Delta + f^\Delta\Big) \bar{\eta}_\Delta \right] = 0 \tag{2.59}$$

It can be easily proved that interpolations $\bar{\eta}_\Delta$ and $\bar{\eta}^i_\Delta$ converge uniformly on $\bar{Q}$ to the functions η and η_{x_i} as $\Delta \to 0$. Consequently, the above identity can be written as

$$\sum_{\mathscr{A}(Q_\Delta)} \int_{C^\alpha_\Delta} \left[\sum_{i,j=1}^{n} \bar{a}_{ij_\Delta} \tilde{U}^i_\Delta \eta_{x_i} + \Big(-\sum_i \bar{b}_{i_\Delta} \tilde{U}^i_\Delta - \bar{a}_\Delta \tilde{U}_\Delta + f^\Delta\Big) \eta \right] + J = 0 \tag{2.60}$$

where

$$J = \sum_{\mathscr{A}(Q_\Delta)} \int_{C^\alpha_\Delta} \left[\sum_{i,j=1}^{n} (\bar{a}_{ij_\Delta} \tilde{U}^i_\Delta)(\bar{\eta}^i_\Delta - \eta_{x_i}) + \Big(-\sum_i \bar{b}_{i_\Delta} \tilde{U}^i_\Delta - \bar{a}_\Delta \tilde{U}_\Delta + f^\Delta\Big)(\bar{\eta}_\Delta - \eta) \right]$$

We claim $|J| \to 0$ as $\Delta \to 0$. $\{\bar{a}_{ij_\Delta}\}$, $\{\bar{b}_{i_\Delta}\}$, $\{\bar{a}_\Delta\}$ are uniformly bounded in $L^\infty(Q)$, and $\{\tilde{U}^i_\Delta\}$, $\{\tilde{U}_\Delta\}$, $\{f^\Delta\}$ are uniformly bounded in $L_2(Q)$ and $\bar{\eta}_\Delta$ and $\bar{\eta}^i_\Delta$ converge uniformly on $\bar{Q}$ to the functions η and η_{x_i} as $\Delta \to 0$, they imply $|J| \to 0$ as $\Delta \to 0$. In the following, we show the convergence to zero for just one term.

$$\left| \sum_{\mathscr{A}(Q_\Delta)} \int_{C^\alpha_\Delta} \sum_{i,j=1}^{n} \bar{a}_{ij_\Delta} \tilde{U}^i_\Delta (\bar{\eta}^i_\Delta - \eta_{x_i}) \right| \le \|\bar{a}_{ij_\Delta}\|_{L^\infty(Q_\Delta)} \|\tilde{U}^i_\Delta\|_{L_2(Q_\Delta)} \Big(\int_{Q_\Delta} (\bar{\eta}^i_\Delta - \eta_{x_i})^2 \Big)^{\frac{1}{2}} \to 0$$

It can be easily proved that $\bar{a}_{ij_\Delta}, \bar{b}_{i_\Delta}, \bar{a}_\Delta, f^\Delta$ converge to the functions a_{ij}, b_i, a, f strongly in $L_2(Q)$ norm, hence in the following relation

$$\sum_{\mathscr{A}(Q_\Delta)} \int_{C_\Delta^\alpha} \left[\sum_{i,j=1}^{n} a_{ij} \tilde{U}_\Delta^i \eta_{x_i} + (-\sum_i b_i \tilde{U}_\Delta^i - a\tilde{U}_\Delta + f)\eta \right] + I = 0 \tag{2.61}$$

where

$$I = \sum_{\mathscr{A}(Q_\Delta)} \int_{C_\Delta^\alpha} \left[\sum_{i,j=1}^{n} (\bar{a}_{ij_\Delta} - a_{ij}) \tilde{U}_\Delta^i \eta_{x_i} \right.$$
$$\left. - \sum_i \Big((\bar{b}_{i_\Delta} - b_i)\tilde{U}_\Delta^i - (\bar{a}_\Delta - a)\tilde{U}_\Delta + (f^\Delta - f) \Big) \eta \right]$$

$|I| \to 0$ as $\Delta \to 0$. In the following, we show the convergence to zero for just one term.

$$\left| \sum_{\mathscr{A}(Q_\Delta)} \int_{C_\Delta^\alpha} \sum_{i,j=1}^{n} (\bar{a}_{ij_\Delta} - a_{ij}) \tilde{U}_\Delta^i \eta_{x_i} \right| \le c\, \|\tilde{U}_\Delta^i\|_{L_2(Q_\Delta)} \Big(\int_{C_\Delta^\alpha} (\bar{a}_{ij_\Delta} - a_{ij})^2 \Big)^{\frac{1}{2}} \to 0$$

Finally, from Lemma 2.2.8 (c) and (d), we know that sequence $\{\tilde{U}_\Delta^i\}$ converges weakly to u_{x_i} in $L_2(Q)$ and sequence $\{\tilde{U}_\Delta\}$ converges strongly to u in $L_2(Q)$. It follows that taking $\Delta \to 0$ on (2.61) gives the identity

$$\int_Q \Big[\sum_{i,j=1}^{n} a_{ij} u_{x_i} \eta_{x_i} - \sum_i b_i u_{x_i} \eta - au\eta \Big] dx = \int_Q f\eta\, dx, \quad \forall \eta \in \overset{\bullet}{\mathscr{C}}{}^1(Q) \tag{2.62}$$

which is (2.8). Since $\overset{\bullet}{\mathscr{C}}{}^1(Q)$ is dense in set of admissible test functions for integral identity (2.8) we have that u is a weak solution to the Problem (2.3)–(2.4) in the sense of Definition 2.1.1. Therefore, we have proved that if u is a weak limit point of $\{U_\Delta'\}$ then it must be a weak solution to the Problem (2.3)–(2.4). Due to uniqueness of the weak solution it follows that $\{U_\Delta'\}$ has one and only one weak limit point, which shows that the whole sequence $\{U_\Delta'\}$ converges weakly to u in $H^1(Q)$. Lemma is proved. □

Lemma 2.2.11. *Let $\{[f]_\Delta\}$ be a sequence of discrete control vectors such that there exists $R > 0$ for which $[f]_\Delta \in \mathscr{F}_\Delta^R$ for each Δ, and such that the sequence of interpolations $\{\mathscr{P}_\Delta([f]_\Delta)\}$ converges weakly to f in $L_2(Q)$. Then the sequence of interpolations $\{V'_\Delta\}$ of associated discrete state vectors converges weakly in $H^1(Q)$ to $u = u(x; f) \in H^1(Q)$, with u the unique weak solution to the* (2.3), (2.7) *in the sense of Definition 2.1.2.*

Proof. From (a) and (b) of Lemma 2.2.9, it follows that $\{V'_\Delta\}$ is uniformly bounded in $H^1(Q)$. Consequently, $\{V'_\Delta\}$ has a weak limit point in $H^1(Q)$. Let $u \in H^1(Q)$ be any weak limit point of $\{V'_\Delta\}$ in $H^1(Q)$. By the Rellich-Kondrachev Theorem, it is known that a subsequence of $\{V'_\Delta\}$ converges strongly to u in $L_2(Q)$. In addition, $\{V'_\Delta\}$ converges to u on the boundary S in $L_2(S)$ norm. Now, we proceed to show that u satisfies the integral identity (2.9). For simplicity of notation we write the subsequence of $\{V'_\Delta\}$ that converges weakly to u in $H^1(Q)$ as the whole sequence Δ. Let $\eta \in \mathscr{C}^1(\tilde{Q})$, where $\bar{Q} \subset \tilde{Q}$ and $\mathscr{C}^1(Q)$ be a space of all continuously differentiable functions on $\bar{Q}$. We also assume that $h > 0$ is small enough that $Q^*_\Delta \subset \tilde{Q}$. Then the collection of values $\{\eta_\alpha\}$, $\alpha \in \mathscr{A}$ is an admissible test collection for the summation identity (2.21). We claim that the limit function, u, satisfies the integral identity (2.9) . Let call the discrete integral identity (2.21) as I_Δ and the continuous integral identity (2.9) as I.

$$I_\Delta := \sum_{\mathscr{A}(Q_\Delta^{*+})} h^n \left[\sum_{i,j=1}^{n} a_{ij\alpha} u_{\alpha x_j} \eta_{\alpha x_i} - \sum_i b_{i\alpha} u_{\alpha x_i} \eta_\alpha - a_\alpha u_\alpha \eta_\alpha + f_\alpha^\Delta \eta_\alpha \right] + J_\alpha(u_\alpha, \eta_\alpha) + \sum_{\mathscr{A}(\hat{S}_\Delta^*)} k_\alpha u_\alpha \eta_\alpha \quad (2.63)$$

$$I := \int_Q \Big(\sum_{i,j=1}^{n} a_{ij} u_{x_j} \eta_{x_i} - \sum_i b_i u_{x_i} \eta - au\eta \Big) dx + \int_S ku\eta\, ds + \int_Q u\eta\, dx$$

Then we define the piecewise constant interpolations $\bar{a}_{ij_\Delta}$, $\bar{b}_{i_\Delta}$, $\bar{a}_\Delta$ of discrete valued functions $a_{ij\alpha}, b_{i\alpha}, a_\alpha$ as following

$$\bar{\beta}_\Delta\Big|_{C^\alpha_\Delta} = \beta_\alpha, \qquad \bar{\beta}_\Delta \equiv 0 \text{ elsewhere on } Q, \quad \forall \alpha \in \mathscr{A}(Q^*_\Delta)$$

where $\bar{\beta}_\Delta$ represents $\bar{a}_{ij_\Delta}, \bar{b}_{i_\Delta}, \bar{a}_\Delta$. In addition, we define the interpolations for η_α and $\eta_{\alpha x_i}$ for each $\alpha \in \mathscr{A}(Q^*_\Delta)$ as following

$$\bar{\eta}_\Delta\Big|_{C^\alpha_\Delta} = \eta_\alpha, \quad \forall \alpha \in \mathscr{A}(Q^*_\Delta)$$

$$\bar{\eta}^i_\Delta\Big|_{C^\alpha_\Delta} = \eta_{\alpha x_i}, \quad \forall \alpha \in \mathscr{A}(Q^*_\Delta)$$

Note that f^Δ is the interpolation of f^Δ_α. Using these interpolation functions I_Δ is transformed as follows:

$$\begin{aligned}
I_\Delta := & \sum_{\mathscr{A}(Q^{*+}_\Delta)} \int_{C^\alpha_\Delta} \left[\sum_{i,j=1}^n \bar{a}_{ij\Delta} \tilde{V}^j_\Delta \bar{\eta}^i_\Delta - \sum_i \bar{b}_{i\Delta} \tilde{V}^i_\Delta \bar{\eta}_\Delta - \bar{a}_\Delta \tilde{V}_\Delta \bar{\eta}_\Delta + f^\Delta \bar{\eta}_\Delta \right] \\
& + J_\alpha(u_\alpha, \eta_\alpha) + \sum_{\mathscr{A}(\hat{S}^*_\Delta)} \int_{S_\alpha} k(x) \tilde{V}_\Delta \bar{\eta}_\Delta ds \\
= & \int_{Q^*_\Delta} \left[\sum_{i,j=1}^n \bar{a}_{ij\Delta} \tilde{V}^j_\Delta \bar{\eta}^i_\Delta - \sum_i \bar{b}_{i\Delta} \tilde{V}^i_\Delta \bar{\eta}_\Delta - \bar{a}_\Delta \tilde{V}_\Delta \bar{\eta}_\Delta + f^\Delta \bar{\eta}_\Delta \right] \\
& + J_\alpha(u_\alpha, \eta_\alpha) + \int_S k(x) \tilde{V}_\Delta \bar{\eta}_\Delta ds = 0
\end{aligned}$$

Adding and subtracting some terms to I_Δ, we obtain the following identity:

$$I_\Delta = I + \sum_{i=1}^5 R_i,$$

where

$$R_1 = \int_{Q_\Delta^* \setminus Q} \left[\sum_{i,j=1}^{n} \bar{a}_{ij\Delta} \tilde{V}_\Delta^j \bar{\eta}_\Delta^i - \sum_i \bar{b}_{i\Delta} \tilde{V}_\Delta^i \bar{\eta}_\Delta - \bar{a}_\Delta \tilde{V}_\Delta \bar{\eta}_\Delta + f^\Delta \bar{\eta}_\Delta \right],$$

$$R_2 = J_\alpha(u_\alpha, \eta_\alpha) = h^n \sum_{\mathscr{A}(S_\Delta^*)} \left[\theta_\alpha u_\alpha \eta_\alpha + \sum_{i=1}^{n} \theta_\alpha^i u_{\alpha x_i} \eta_{\alpha x_i} \right],$$

$$R_3 = \int_Q \left[\sum_{i,j=1}^{n} \bar{a}_{ij\Delta} \tilde{V}_\Delta^j (\bar{\eta}_\Delta^i - \eta_{x_i}) - \sum_i \bar{b}_{i\Delta} \tilde{V}_\Delta^i (\bar{\eta}_\Delta - \eta) - \bar{a}_\Delta \tilde{V}_\Delta (\bar{\eta}_\Delta - \eta) + f^\Delta (\bar{\eta}_\Delta - \eta) \right] + \int_S k(x) \tilde{V}_\Delta (\bar{\eta}_\Delta - \eta) ds,$$

$$R_4 = \int_Q \left[\sum_{i,j=1}^{n} (\bar{a}_{ij\Delta} - a_{ij}) \tilde{V}_\Delta^j \eta_{x_i} - \sum_i (\bar{b}_{i\Delta} - b_i) \tilde{V}_\Delta^i \eta - (\bar{a}_\Delta - a) \tilde{V}_\Delta \eta + (f_\alpha^\Delta - f) \eta \right],$$

$$R_5 = \int_Q \left[\sum_{i,j=1}^{n} a_{ij} (\tilde{V}_\Delta^j - u_{x_j}) \eta - \sum_i b_i (\tilde{V}_\Delta^i - u_{x_i}) \eta + a (\tilde{V}_\Delta - u) \eta \right] + \int_S k(x) (\tilde{V}_\Delta - u) \eta ds.$$

We claim that by passing to the limit when $\Delta \to 0$, $I_\Delta \to I$ and $R_i \to 0$ for $i = 1, \ldots, 5$. Using Cauchy Schwartz inequality and extending the region of integration from $Q_\Delta^* \setminus Q$ to Q_Δ^* for functions $\tilde{V}_\Delta^i$ and $\tilde{V}_\Delta$ we obtain the following estimate for R_1:

$$|R_1| \le C_1 \sum_{i,j=1}^{n} \|\tilde{V}_\Delta^j\|_{L_2(Q_\Delta^*)} \|\bar{\eta}_\Delta^i\|_{L_2(Q_\Delta^* \setminus Q)} + C_2 \sum_{i=1}^{n} \|\tilde{V}_\Delta^i\|_{L_2(Q_\Delta^*)} \|\bar{\eta}_\Delta\|_{L_2(Q_\Delta^* \setminus Q)} + C_3 \|\tilde{V}_\Delta\|_{L_2(Q_\Delta^*)} \|\bar{\eta}_\Delta\|_{L_2(Q_\Delta^* \setminus Q)} + C_4 \|f^\Delta\|_{L_2(Q_\Delta^*)} \|\bar{\eta}_\Delta\|_{L_2(Q_\Delta^* \setminus Q)}$$

Lemma 2.2.9 (a) and (b), Proposition 2.2.7, and the fact that all a_{ij}, b_i, a are bounded functions imply that

$$|R_1| \leq C_5 \sum_{i=1}^{n} \|\bar{\eta}_{\Delta}^{i}\|_{L_2(Q_{\Delta}^{*}\setminus Q)} + C_6 \|\bar{\eta}_{\Delta}\|_{L_2(Q_{\Delta}^{*}\setminus Q)}$$

interpolations $\bar{\eta}_{\Delta}$ and $\bar{\eta}_{\Delta}^{i}$ converge uniformly on $\bar{Q}$ to the functions η and η_{x_i} as $\Delta \to 0$ and since $\eta \in \mathscr{C}^1(\tilde{Q})$ and $|Q_{\Delta}^{*} \setminus Q| \to 0$ and we have

$$|R_1| \to 0, \quad \text{as } \Delta \to 0$$

Now we try to show that R_2 is small.

$$R_2 = h^n \sum_{\mathscr{A}(S_{\Delta}^{*})} \left[\theta_{\alpha} u_{\alpha} \eta_{\alpha} + \sum_{i=1}^{n} \theta_{\alpha}^{i} u_{\alpha x_i} \eta_{\alpha x_i}\right] = R_{21} + R_{22}$$

$$\begin{aligned} |R_{21}| = |h^n \sum_{\mathscr{A}(S_{\Delta}^{*})} \theta_{\alpha} u_{\alpha} \eta_{\alpha}| &\leq C 2^{n-1} n h \|[u([f]_{\Delta})]_{\Delta}\|_{\mathscr{H}^1(Q_{\Delta}^{*})}^{2} + \sum_{\mathscr{A}(\hat{S}_{\Delta}^{*})} h^n u_{\alpha} \eta_{\alpha} \\ &\leq C 2^{n-1} n h \|[u([f]_{\Delta})]_{\Delta}\|_{\mathscr{H}^1(Q_{\Delta}^{*})}^{2} + \|\tilde{V}_{\Delta}\|_{L_2(Q_{\Delta}^{*})} \|\bar{\eta}_{\Delta}\|_{L_2(Q_{\Delta}^{*}\setminus Q)} \\ &\leq Mh + N \|\bar{\eta}_{\Delta}\|_{L_2(Q_{\Delta}^{*}\setminus Q)} \to 0 \end{aligned}$$

where 2^{n-1} is the number of vertices in C_{Δ}^{α} other than the natural corner, and n is the maximum number of edges that connect x_α to the natural corner in C_Δ^α.

$$
\begin{aligned}
|R_{22}| &= |h^n \sum_{\mathscr{A}(S_\Delta^*)} \sum_{i=1}^{n} \theta_\alpha^i u_{\alpha x_i} \eta_{\alpha x_i}| \\
&\leq \Big(\sum_{\mathscr{A}(S_\Delta^*)} \sum_{i=1}^{n} h^n \theta_\alpha^i u_{\alpha x_i}^2 \Big)^{\frac{1}{2}} \Big(\sum_{\mathscr{A}(S_\Delta^*)} \sum_{i=1}^{n} h^n \theta_\alpha^i \eta_{\alpha x_i}^2 \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{i=1}^{n} \sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^n u_{\alpha x_i}^2 \Big)^{\frac{1}{2}} \|\eta\|_{C^1} \Big(\sum_{i=1}^{n} \sum_{\mathscr{A}(S_\Delta^*)} h^n \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{i=1}^{n} \sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^n u_{\alpha x_i}^2 \Big)^{\frac{1}{2}} \|\eta\|_{C^1} \sqrt{n} \Big(\sum_{\mathscr{A}(S_\Delta^*)} h^n \Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{i=1}^{n} \sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^2 u_{\alpha x_i}^2 \Big)^{\frac{1}{2}} \|\eta\|_{C^1} \sqrt{nh}\, (2|S|)^{\frac{1}{2}} \to 0
\end{aligned}
$$

Sum with respect to all grid points of S_Δ^* is bounded by the Lebesgue measure of S_Δ^*. Since S is Lipschitz, the latter converges to Lebesgue measure of S as $h \to 0$; This imply that for sufficiently small h, it will be bounded by $2|S|$. The same argument that we used for R_1 implies that $R_{21}, R_{22} \to 0$ as $\Delta \to 0$.

Using Cauchy Schwartz inequality and Lemma 2.2.9 (a) and (b) we get the following estimation for R_3:

$$
\begin{aligned}
|R_3| = \Bigg| \int_Q \Big[\sum_{i,j=1}^{n} \bar{a}_{ij\Delta} \tilde{V}_\Delta^j (\bar{\eta}_\Delta^i - \eta_{x_i}) - \sum_i \bar{b}_{i\Delta} \tilde{V}_\Delta^i (\bar{\eta}_\Delta - \eta) - \bar{a}_\Delta \tilde{V}_\Delta (\bar{\eta}_\Delta - \eta) + f^\Delta (\bar{\eta}_\Delta - \eta) \Big] \\
+ \int_S k(x) \tilde{V}_\Delta (\bar{\eta}_\Delta - \eta) ds \Bigg| \leq N_1 \sum_{i=1}^{n} \|\bar{\eta}_\Delta^i - \eta_{x_i}\|_{L_2(Q)} + N_2 \|\bar{\eta}_\Delta - \eta\|_{L_2(Q)} + N_3 \|\bar{\eta}_\Delta - \eta\|_{L_2(S)}
\end{aligned}
$$

It can be easily proved that interpolations $\bar{\eta}_\Delta$ and $\bar{\eta}_\Delta^i$ converge uniformly on $\bar{Q}$ to the functions η and η_{x_i} as $\Delta \to 0$, so $R_3 \to 0$.

Using Cauchy Schwartz inequality and Lemma 2.2.9 (a) and (b) and the fact that $\eta \in$

$\mathscr{C}^1(\tilde{Q})$ we get the following estimation for R_4:

$$|R_4| = \left| \int_Q \Big[\sum_{i,j=1}^{n} (\bar{a}_{ij\Delta} - a_{ij}) \tilde{V}_\Delta^j \eta_{x_i} - \sum_i (\bar{b}_{i\Delta} - b_i) \tilde{V}_\Delta^i \eta - (\bar{a}_\Delta - a) \tilde{V}_\Delta \eta + (f^\Delta - f)\eta \Big] \right|$$
$$\leq \sum_{i,j=1}^{n} H_1 \|\bar{a}_{ij\Delta} - a_{ij}\|_{L_2(Q)} + H_2 \sum_{i=1}^{n} \|\bar{b}_{i\Delta} - b_i\|_{L_2(Q)} + H_3 \|\bar{a}_\Delta - a\|_{L_2(Q)} + H_4 \|f^\Delta - f\|_{L_2(Q)} \tag{2.64}$$

By convergence of the Steklov averages to the original function in L_2, it implies that $\bar{a}_{ij_\Delta}, \bar{b}_{i_\Delta}, \bar{a}_\Delta, f^\Delta$ converge to a_{ij}, b_i, a, f strongly in $L_2(Q)$ norm; Thus, it follows that $R_4 \to 0$

By adding and subtracting V'_Δ and $\frac{\partial V'_\Delta}{\partial x_j}$ and using the fact that a_{ij}, b_i, a are bounded we calculate the following estimate:

$$|R_5| = \left| \int_Q \Big[\sum_{i,j=1}^{n} a_{ij} (\tilde{V}_\Delta^j - u_{x_j}) \eta - \sum_i b_i (\tilde{V}_\Delta^i - u_{x_i}) \eta + a(\tilde{V}_\Delta - u)\eta \Big] \right.$$
$$\left. + \int_S k(x)(\tilde{V}_\Delta - u)\eta ds \right| \leq K_1 \sum_{j=1}^{n} \|\tilde{V}_\Delta^j - \frac{\partial V'_\Delta}{\partial x_j}\|_{L_2(Q)} + K_1 \sum_{j=1}^{n} \|\frac{\partial V'_\Delta}{\partial x_j} - u_{x_j}\|_{L_2(Q)}$$
$$+ K_2 \sum_{i=1}^{n} \|\tilde{V}_\Delta^i - \frac{\partial V'_\Delta}{\partial x_i}\|_{L_2(Q)} + K_2 \sum_{i=1}^{n} \|\frac{\partial V'_\Delta}{\partial x_i} - u_{x_i}\|_{L_2(Q)} + K_3 \|\tilde{V}_\Delta - V'_\Delta\|_{L_2(Q)}$$
$$+ K_3 \|V'_\Delta - u\|_{L_2(Q)} + K_4 \|\tilde{V}_\Delta - V'_\Delta\|_{L_2(S)} + K_4 \|V'_\Delta - u\|_{L_2(S)}$$

Lemma 2.2.9 (c),(d), (e) and Approximation Lemma implies that $R_5 \to 0$.

Finally, since $\mathscr{C}^1(Q)$ is dense in set of admissible test functions for integral identity (2.9) we have that u is a weak solution to the Problem (2.3)–(2.7) in the sense of Definition 2.1.2. Therefore, we have proved that if u is a weak limit point of $\{V'_\Delta\}$ then it must be a weak solution to the Problem (2.3)–(2.7). Due to uniqueness of the weak solution it follows that $\{V'_\Delta\}$ has one and only one weak limit point, which shows that the whole sequence $\{V'_\Delta\}$ converges weakly to u in $H^1(Q)$. Lemma is proved. $\square$

2.3 Proofs of Main Results

2.3.1 Proof of Theorem 2.1.6

Proof of Theorem 2.1.6. To prove 2.24 and 2.27, it is enough to show that conditions (i) and (ii) of Lemma 2.2.6 for problem $\mathscr{D}$ are satisfied.

Step 1. In this step we show that for any $f \in \mathscr{F}^R$,

$$\lim_{\Delta\to 0} |\mathscr{I}_\Delta(\mathscr{Q}_\Delta(f)) - \mathscr{I}(f)| = 0 \tag{2.65}$$

In Preposition 2.2.7 it is shown that $\mathscr{Q}_\Delta(f) = [f]_\Delta \in \mathscr{F}^R_\Delta$, and the sequence $\{\mathscr{P}_\Delta(\mathscr{Q}_\Delta(f))\}$ converges strongly to f

$$\mathscr{P}_\Delta([f]_\Delta) \to f \text{ strongly in } L_2(Q) \text{ as } \Delta \to 0 \tag{2.66}$$

this shows that the requirement of the Theorem 2.2.10 is satisfied and it follows that the interpolations $\{U'_\Delta\}$ of the discrete state vectors $[u([f]_\Delta)]_\Delta$ converge weakly in $H^1(Q)$ to the unique weak solution $u = u(x : f)$ of the PDE problem with control f.

Let define $\tilde{G}_\Delta, \tilde{F}_\Delta$ and $\tilde{\bar{F}}_\Delta$ as piece wise constant interpolation of collection $\{g_\alpha\}, \{f_\alpha\}$ and $\{\bar{f}_\alpha\}$which is defined by the formula (2.15)

$$.\tilde{G}_\Delta\Big|_{C^\alpha_\Delta} = g_\alpha, \quad \forall\alpha \in \mathscr{A}(D_\Delta), \qquad \tilde{G}_\Delta = 0 \quad \text{elsewhere on } D$$

$$\tilde{F}_\Delta\Big|_{C^\alpha_\Delta} = f_\alpha, \quad \forall\alpha \in \mathscr{A}(Q_\Delta), \qquad \tilde{F}_\Delta = 0 \quad \text{elsewhere on } Q$$

$$\tilde{\bar{F}}_\Delta\Big|_{C^\alpha_\Delta} = \bar{f}_\alpha, \quad \forall\alpha \in \mathscr{A}(Q_\Delta), \qquad \tilde{\bar{F}}_\Delta = 0 \quad \text{elsewhere on } Q$$

then we note that

$$\begin{aligned}
\mathscr{I}_\Delta(\mathscr{Q}_\Delta(f)) &= \sum_{\mathscr{A}(D_\Delta)} h^n|u_\alpha - g_\alpha|^2 + \beta \sum_{\mathscr{A}(Q^+_\Delta)} h^n|f_\alpha - \bar{f}_\alpha|^2 \\
&= \sum_{\mathscr{A}(D_\Delta)} \int_{C^\alpha_\Delta} |\tilde{U}_\Delta - \tilde{G}_\Delta|^2 + \beta \sum_{\mathscr{A}(Q^+_\Delta)} h^n|\tilde{F}_\Delta - \tilde{\bar{F}}_\Delta|^2 \\
&= \|\tilde{U}_\Delta - \tilde{G}_\Delta\|_{L_2(D_\Delta)} + \|\tilde{F}_\Delta - \tilde{\bar{F}}_\Delta\|_{L_2(Q_\Delta)} \\
&= \|\tilde{U}_\Delta \pm u \pm g - \tilde{G}_\Delta\|_{L_2(D)} + \|\tilde{F}_\Delta \pm f \pm \bar{f} - \tilde{G}_\Delta\|_{L_2(Q)} \\
&\le \|\tilde{U}_\Delta - u\|_{L_2(D_\Delta)} + \|\tilde{G}_\Delta - g\|_{L_2(D_\Delta)} + \|u - g\|_{L_2(D_\Delta)} \\
&+ \|\tilde{F}_\Delta - f\|_{L_2(Q_\Delta)} + \|\tilde{\bar{F}}_\Delta - \bar{f}\|_{L_2(Q_\Delta)} + \|f - \bar{f}\|_{L_2(Q_\Delta)} \\
&= \|\tilde{U}_\Delta - u\|_{L_2(D_\Delta)} + \|\tilde{G}_\Delta - g\|_{L_2(D_\Delta)} + \|\tilde{F}_\Delta - f\|_{L_2(Q_\Delta)} + \|\tilde{\bar{F}}_\Delta - \bar{f}\|_{L_2(Q_\Delta)} + \mathscr{I}(f)
\end{aligned}$$

Lemma 2.2.8(c) and Theorem 2.2.10 imply that when $\Delta \to 0$

$$\|\tilde{U}_\Delta - u\|_{L_2} \to 0$$

strong convergence of Steklov average to the original function as $\Delta \to 0$ implies that

$$\|\tilde{G}_\Delta - g\|_{L_2} \to 0, \quad \|\tilde{F}_\Delta - f\|_{L_2(Q_\Delta)} \to 0, \quad \|\tilde{\bar{F}}_\Delta - \bar{f}\|_{L_2(Q_\Delta)} \to 0$$

so

$$\lim_{\Delta\to 0} \mathscr{I}(\mathscr{Q}_\Delta(f)) - \mathscr{I}(f) = 0$$

Hence, (2.65) is proved.

Step 2. In this step we show that for any collection of numbers $\{[f]_\Delta\}$ such that $[f]_\Delta \in \mathscr{F}_\Delta^R$,

$$\lim_{\Delta\to 0} |\mathscr{I}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{I}_\Delta([f]_\Delta))| = 0 \tag{2.67}$$

Preposition 2.2.7 implies that the sequence $\mathscr{P}_\Delta([f]_\Delta)$ is uniformly bounded in $L_2(Q)$ norm. Hence, there exists a subsequence of that converging weakly in $L_2(Q)$ to some $\bar{f} \in \mathscr{F}^R$. We know that there is a unique state vector $\bar{u} := u(x;\bar{f}) \in H_0^1(Q)$ which solves the problem (2.3)–(2.4). By Theorem 2.2.10, we also know that the sequence of interpolations $\{U'_\Delta\}$ of $[u([f]_\Delta)]_\Delta$, discrete state vectors associated to $[f]_\Delta$, converges weakly in $H^1(Q)$ to $\bar{u}$. For simplicity, we use the whole sequence $\mathscr{P}_\Delta([f]_\Delta)$ instead of the subsequence.

To prove (2.67), we add and subtract $\mathscr{I}(\bar{f})$ to (2.67) and we get the following inequality

$$|\mathscr{I}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{I}_\Delta([f]_\Delta))| \leq I_1 + I_2$$

where

$$I_1 = |\mathscr{I}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{I}(\bar{f})|$$

and

$$I_2 = |\mathscr{I}(\bar{f}) - \mathscr{I}_\Delta([f]_\Delta))|$$

Then weak continuity of $\mathscr{I}$ implies that $I_1 \to 0$ as $\Delta \to 0$.

To show that $I_2 \to 0$ as $\Delta \to 0$, let consider the following

$$
\begin{aligned}
|\mathscr{I}_\Delta([f]_\Delta)) - \mathscr{I}(\bar{f})| &= \left| \sum_{\mathscr{A}} h^n |u_\alpha - g_\alpha|^2 - \int_Q |\bar{u} - g(x)|^2 dx \right| \\
&= \left| \int_Q |\tilde{U}_\Delta - \tilde{G}_\Delta \pm \bar{u}|^2 - \int_Q |\bar{u} - g(x) \pm \tilde{G}_\Delta|^2 dx \right| \\
&\le \int_Q |\tilde{U}_\Delta - \bar{u}|^2 + \int_Q |\tilde{G}_\Delta - g|^2 + 2 \int_Q |\tilde{U}_\Delta - \bar{u}||\bar{u} - \tilde{G}_\Delta| \\
&+ 2 \int_Q |\tilde{G}_\Delta - g||\bar{u} - \tilde{G}_\Delta| \le A_1 + A_2 + 2A_3 + 2A_4
\end{aligned}
$$

By Lemma 2.2.10 and Theorem 2.2.8, it follows that $A_1 \to 0$. By convergence of the interpolation of Steklov average to the original function, $A_2 \to 0$. Theorem 2.2.10, Lemma 2.2.8, and the fact that $g \in L_2(Q)$ and $\bar{u} \in H_0^1(Q)$ imply that $A_3, A_4 \to 0$. Therefore, it is proved that $I_2 \to 0$.

In *step 1* and *step 2*, we have proved conditions of the Lemma 2.2.6. Thus, 2.24 and 2.27 are satisfied. In order to prove the rest of Theorem 2.1.6, we consider the sequence $\{[f]_{\Delta,\varepsilon}\} \in \mathscr{F}_\Delta^R$. It is followed by lemma 2.2.7 that $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ is uniformly bounded in $L_2(Q)$. Assume $f_* \in L_2(Q)$ is a weak limit point of this sequence. Weak continuity of $\mathscr{I}$ and 2.27 implies that $\mathscr{I}(f_*) = \mathscr{I}_*$ and $f_* \in \mathscr{V}_*$. In addition, referring to Theorem 2.2.10 there exists a unique discrete state vector $[u([f]_{\Delta,\varepsilon})]_\Delta$ corresponding to $[f]_{\Delta,\varepsilon}$ whose interpolations, $\{U'_\Delta\}$, converge weakly in $W_2^1(Q)$ to $u_* = u(x; f_*)$, a weak solution to the (2.3)–(2.4) in the sense of (2.1.1).□

2.3.2 Proof of Theorem 2.1.7

Proof of Theorem 2.1.7. To prove (2.28) and (2.31), it is enough to show that conditions (1) and (2) of Lemma 2.2.6 are satisfied.

Step 1. In this step we show that for any $f \in \mathscr{F}^R$,

$$\lim_{\Delta \to 0} |\mathscr{J}_\Delta(\mathscr{Q}_\Delta(f)) - \mathscr{J}(f)| = 0 \tag{2.68}$$

In Preposition 2.2.7 it is shown that $\mathscr{Q}_\Delta(f) = [f]_\Delta \in \mathscr{F}^R_\Delta$, and by convergence of the Steklov averages to the original function in L_2, it follows that the sequence $\{\mathscr{P}_\Delta(\mathscr{Q}_\Delta(f))\}$ converges to f

$$\mathscr{P}_\Delta([f]_\Delta) \to f \text{ strongly in } L_2(Q) \text{ as } \Delta \to 0 \tag{2.69}$$

Therefore, the conditions of the Lemma 2.2.11 is satisfied and it follows that the interpolations $\{V'_\Delta\}$ of the discrete state vectors $[u([f]_\Delta)]_\Delta$ converge weakly in $H^1(Q)$ to the unique weak solution $u = u(x : f)$ of the PDE problem with control f.

Let define $\tilde{G}_\Delta$ and $\tilde{Z}_\Delta$ as piece wise constant interpolation of collection $\{g_\alpha\}$ and $\{z^\Gamma_\alpha\}$ which are defined by the formula (2.15).

$$\tilde{G}_\Delta\Big|_{C^\alpha_\Delta} = g_\alpha, \quad \forall \alpha \in \mathscr{A}(D^{*+}_\Delta), \qquad \tilde{Z}_\Delta\Big|_{\Gamma_\alpha} = z^\Gamma_\alpha, \quad \forall \alpha \in \mathscr{A}(\Gamma_\Delta)$$

where

$$\|\tilde{Z}_\Delta\|^2_{L_2(\Gamma)} = \int_\Gamma |\tilde{Z}_\Delta|^2 \, ds = \sum_{\mathscr{A}(\Gamma_\Delta)} \int_{\Gamma_\alpha} |\tilde{Z}_\Delta|^2 \, ds = \sum_{\mathscr{A}(\Gamma_\Delta)} |\Gamma_\alpha| \Big(\frac{1}{|\Gamma_\alpha|} \int_{\Gamma_\alpha} z ds\Big)^2$$
$$\leq \sum_{\mathscr{A}(\Gamma_\Delta)} \frac{1}{|\Gamma_\alpha|} |\Gamma_\alpha| \int_{\Gamma_\alpha} z^2 ds = \int_\Gamma z^2 ds$$

In the proof, we use the following identity for elements a, b, c of the Hilbert space H

$$\|a-b\|^2_H - \|c-b\|^2_H = \langle a-c, a-c\rangle - 2\langle a-c, b-c\rangle \tag{2.70}$$

In this section, we skip the third term of $\mathscr{J}_\Delta(\mathscr{Q}_\Delta(f))$ and $\mathscr{J}(f)$ due to strong convergence of Steklov average to the original function.

$$\begin{aligned}
\mathscr{J}_\Delta(\mathscr{Q}_\Delta(f)) - \mathscr{J}(f) = \\
&\sum_{\mathscr{A}(D_\Delta^{*+})} h^n |u_\alpha - g_\alpha|^2 + \sum_{\mathscr{A}(\Gamma_\Delta)} |\Gamma_\alpha||u_\alpha - z_\alpha^\Gamma|^2 \\
&- \int_D |u(x;f) - g(x)|^2 dx - \int_\Gamma |u(x;f) - z(x)|^2 ds \\
&= J_1 + J_2
\end{aligned}$$

where

$$J_1 := \sum_{\mathscr{A}(D_\Delta^{*+})} h^n |u_\alpha - g_\alpha|^2 - \int_D |u(x;f) - g(x)|^2 dx$$

$$J_2 := \sum_{\mathscr{A}(\Gamma_\Delta)} |\Gamma_\alpha||u_\alpha - z_\alpha^\Gamma|^2 - \int_\Gamma |u(x;f) - z(x)|^2 ds$$

We claim that $J_1 \to 0$ as $\Delta \to 0$

$$\begin{aligned}
J_1 &:= \sum_{\mathscr{A}(D_\Delta^{*+})} \int_{C_\Delta^\alpha} |\tilde{V}_\Delta - \tilde{G}_\Delta|^2 - \int_D |u(x;f) - g(x)|^2 dx \\
&= \|\tilde{V}_\Delta - \tilde{G}_\Delta\|^2_{L_2(\bar{D}_\Delta^*)} - \|u - g\|^2_{L_2(D)} \\
&= \|\tilde{V}_\Delta - \tilde{G}_\Delta\|^2_{L_2(\bar{D}_\Delta^*)} - \|u - g\|^2_{L_2(D)} - \|u - g\|^2_{L_2(\bar{D}_\Delta^*)} + \|u - g\|^2_{L_2(\bar{D}_\Delta^*)} \\
&= \|\tilde{V}_\Delta - \tilde{G}_\Delta\|^2_{L_2(\bar{D}_\Delta^*)} - \|u - g\|^2_{L_2(\bar{D}_\Delta^*)} + \|u - g\|^2_{L_2(\bar{D}_\Delta^* \setminus D)} \\
&= J_{11} + J_{12} + J_{13}
\end{aligned}$$

Where we extended g to a bigger set $\tilde{D}$ which covers $\bar{D}^*_\Delta$ when $h > 0$ is small enough, then due to absolute continuity of integral we have

$$J_{13} = \|u - g\|^2_{L_2(\bar{D}^*_\Delta \setminus D)} \to 0, \text{ as } \Delta \to 0$$

Using (2.70) and adding and subtracting some additional terms to J_1 we get

$$\begin{aligned} J_{11} + J_{12} &= \|\tilde{V}_\Delta - \tilde{G}_\Delta\|^2_{L_2(\bar{D}^*_\Delta)} \pm \|u - \tilde{G}_\Delta\|^2_{L_2(\bar{D}^*_\Delta)} - \|u - g\|^2_{L_2(\bar{D}^*_\Delta)} \\ &= \|\tilde{V}_\Delta - u\|^2_{L_2(\bar{D}^*_\Delta)} - 2\langle \tilde{V}_\Delta - u, \tilde{G}_\Delta - u\rangle \\ &+ \|\tilde{G}_\Delta - g\|^2_{L_2(\bar{D}^*_\Delta)} - 2\langle \tilde{G}_\Delta - g, u - g\rangle \end{aligned}$$

By convergence of the Steklov averages to the original function in L_2, it follows

$$\|\tilde{G}_\Delta - g\|^2_{L_2(\bar{D}^*_\Delta)} \to 0, \quad \text{as} \quad \Delta \to 0 \tag{2.71}$$

By Lemma (c) and Approximation Lemma, we also have

$$\|\tilde{V}_\Delta - u\|^2_{L_2(\bar{D}^*_\Delta)} \le \|\tilde{V}_\Delta - V'_\Delta\|^2_{L_2(\bar{D}^*_\Delta)} + \|V'_\Delta - u\|^2_{L_2(\bar{D}^*_\Delta)} \tag{2.72}$$

$$\le \|\tilde{V}_\Delta - V'_\Delta\|^2_{L_2(Q^*_\Delta)} + \|V'_\Delta - u\|^2_{L_2(Q^*_\Delta)} \to 0, \quad \text{as} \quad \Delta \to 0 \tag{2.73}$$

Cauchy Schwartz inequality implies

$$|J_{11} + J_{12}| \le \|\tilde{V}_\Delta - u\|^2_{L_2(\bar{D}^*_\Delta)} \|\tilde{G}_\Delta - u\|^2_{L_2(\bar{D}^*_\Delta)} + \|\tilde{G}_\Delta - g\|^2_{L_2(\bar{D}^*_\Delta)} \|u - g\|^2_{L_2(\bar{D}^*_\Delta)} \tag{2.74}$$

(2.71), (2.73) and the fact that $\tilde{G}_\Delta$ is bounded in $L_2(\bar{D}^*_\Delta)$ proves that

$$J_1 \to 0, \quad \text{as} \quad \Delta \to 0 \tag{2.75}$$

In addition, we claim that $J_2 \to 0$ as $\Delta \to 0$

$$\begin{aligned} J_2 &:= \sum_{\mathscr{A}(\Gamma_\Delta)} |\Gamma_\alpha||u_\alpha - z^\Gamma_\alpha|^2 - \int_\Gamma |u(x;f) - z(x)|^2 ds \\ &= \sum_{\mathscr{A}(\Gamma_\Delta)} \int_{\Gamma_\alpha} |\tilde{V}_\Delta - \tilde{Z}_\Delta|^2 - \int_\Gamma |u(x;f) - z(x)|^2 dx \\ &= \|\tilde{V}_\Delta - \tilde{Z}_\Delta\|^2_{L_2(\Gamma)} - \|u - z\|^2_{L_2(\Gamma)} \end{aligned}$$

Using (2.70), J_2 becomes

$$\begin{aligned} J_2 &= \|\tilde{V}_\Delta - \tilde{Z}_\Delta\|^2_{L_2(\Gamma)} \pm \|u - \tilde{Z}_\Delta\|^2_{L_2(\Gamma)} - \|u - z\|^2_{L_2(\Gamma)} \\ &= \|\tilde{V}_\Delta - u\|^2_{L_2(\Gamma)} - 2\langle \tilde{V}_\Delta - u, \tilde{Z}_\Delta - u\rangle \\ &+ \|\tilde{Z}_\Delta - z\|^2_{L_2(\Gamma)} - 2\langle \tilde{Z}_\Delta - z, u - z\rangle \end{aligned}$$

By convergence of the Steklov averages to the original function in L_2, it follows

$$\|\tilde{Z}_\Delta - z\|^2_{L_2(\Gamma)} \to 0, \quad \text{as} \quad \Delta \to 0 \tag{2.76}$$

By Lemma (e) and Approximation Lemma, we also have

$$\begin{aligned} &\|\tilde{V}_\Delta - u\|^2_{L_2(\Gamma)} \le \|\tilde{V}_\Delta - V'_\Delta\|^2_{L_2(\Gamma)} + \|V'_\Delta - u\|^2_{L_2(\Gamma)} \\ &\le \|\tilde{V}_\Delta - V'_\Delta\|^2_{L_2(S)} + \|V'_\Delta - u\|^2_{L_2(S)} \to 0, \quad \text{as} \quad \Delta \to 0 \end{aligned} \tag{2.77}$$

Cauchy Schwartz inequality implies

$$|J_2| \leq \|\tilde{V}_\Delta - u\|^2_{L_2(\Gamma)} \|\tilde{Z}_\Delta - u\|^2_{L_2(\Gamma)} + \|\tilde{Z}_\Delta - z\|^2_{L_2(\Gamma)} \|u - z\|^2_{L_2(\Gamma)} \tag{2.78}$$

(2.76), (2.77) and the fact that $\tilde{Z}_\Delta$ is bounded in $L_2(\Gamma)$ proves that

$$J_2 \to 0, \quad \text{as} \quad \Delta \to 0 \tag{2.79}$$

so

$$\lim_{\Delta\to 0} \mathscr{J}(\mathscr{Q}_\Delta(f)) - \mathscr{J}(f) = 0$$

Hence, (2.68) is proved.

Step 2. In this step we show that for any sequence $\{[f]_\Delta\}$ such that $[f]_\Delta \in \mathscr{F}^R_\Delta$, we have

$$\lim_{\Delta\to 0} |\mathscr{J}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{J}_\Delta([f]_\Delta))| = 0 \tag{2.80}$$

Preposition 2.2.7 implies that the sequence $\mathscr{P}_\Delta([f]_\Delta)$ is uniformly bounded in $L_2(Q)$ norm. Hence, there exists a subsequence of that converging weakly in $L_2(Q)$ to some $\bar{f} \in \mathscr{F}^R$. For simplicity, we use the whole sequence $\mathscr{P}_\Delta([f]_\Delta)$ instead of the subsequence.

$$\mathscr{P}_\Delta([f]_\Delta) \to \bar{f} \text{ weakly in } L_2(Q) \text{ as } \Delta \to 0 \tag{2.81}$$

this shows that the requirement of the Theorem 2.2.11 is satisfied, so the same argument that proved (2.68) can lead us to the following assertion

$$\lim_{\Delta\to 0} |\mathscr{J}_\Delta([f]_\Delta) - \mathscr{J}(\bar{f})| = 0 \tag{2.82}$$

To prove (2.80), we add and subtract $\mathscr{J}(\bar{f})$ to (2.80) and we get the following inequality

$$|\mathscr{J}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{J}_\Delta([f]_\Delta))| \leq |\mathscr{J}(\mathscr{P}_\Delta([f]_\Delta)) - \mathscr{J}(\bar{f})| + |\mathscr{J}(\bar{f}) - \mathscr{J}_\Delta([f]_\Delta))|$$

Then weak continuity of $\mathscr{J}$ implies that $I_1 \to 0$ as $\Delta \to 0$.

In *step 1* and *step 2*, we have proved conditions of the Lemma 2.2.6. Thus, 2.28 and 2.31 are satisfied. In order to prove the rest of Theorem 2.1.7, we consider the sequence $\{[f]_{\Delta,\varepsilon}\} \in \mathscr{F}_\Delta^R$. It is followed by lemma 2.2.7 that $\{\mathscr{P}_\Delta([f]_{\Delta,\varepsilon})\}$ is uniformly bounded in $L_2(Q)$. Assume $f_* \in L_2(Q)$ is a weak limit point of this sequence. Weak continuity of $\mathscr{J}$ and 2.31 implies that $\mathscr{J}(f_*) = \mathscr{J}_*$ and $f_* \in \mathscr{F}_*$. In addition, referring to Theorem 2.2.11 there exists a unique discrete state vector $[u([f]_{\Delta,\varepsilon})]_\Delta$ corresponding to $[f]_{\Delta,\varepsilon}$ whose interpolations, $\{V'_\Delta\}$, converge weakly in $W_2^1(Q)$ to $u_* = u(x; f_*)$, a weak solution to the Neumann problem (2.3), (2.7) in the sense of (2.1.2).□

Remark 2.3.1. It is an important open problem to extend the methods of this Chapter to analyze the optimal control problem for elliptic and parabolic PDEs in domains with non-compact boundaries by employing well-posedness and regularity theory of elliptic and parabolic PDEs in general unbounded open sets [1, 2, 3, 4, 7].

Chapter 3

Cancer Detection through Electrical Impedance Tomography and Optimal Control of Elliptic PDEs

3.1 Introduction and Problem Description

This chapter of the dissertation analyzes inverse EIT problem of estimating an unknown conductivity inside the body based on voltage measurements on the surface of the body when electric currents are applied through a set of contact electrodes. Let $Q \in \mathbb{R}^n$ be an open and bounded set representing body, and assume $A(x) = \big(a_{ij}(x)\big)_{ij=1}^n$ be a matrix representing the electrical conductivity tensor at the point $x \in Q$. Electrodes, $(E_l)_{l=1}^m$, with contact impedances vector $Z := (Z_l)_{l=1}^m \in \mathbb{R}_+^m$ are attached to the periphery of the body, ∂Q. Electric current vector $I := (I_l)_{l=1}^m \in \mathbb{R}^m$ is applied to the electrodes. Vector

I is called *current pattern* if it satisfies conservation of charge condition

$$\sum_{l=1}^{m} I_l = 0 \tag{3.1}$$

The induced constant voltage on electrodes is denoted by $U := (U_l)_{l=1}^m \in \mathbb{R}^m$. By specifying ground or zero potential it is assumed that

$$\sum_{l=1}^{m} U_l = 0 \tag{3.2}$$

EIT problem is to find the electrostatic potential $u : Q \to \mathbb{R}$ and boundary voltages U on $(E_l)_{l=1}^m$. The mathematical model of the EIT problem is described through the following boundary value problem for the second order elliptic partial differential equation:

$$-\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right)_{x_i} = 0, \qquad x \in Q \tag{3.3}$$

$$\frac{\partial u(x)}{\partial \mathcal{N}} = 0, \qquad x \in \partial Q - \bigcup_{l=1}^{m} E_l \tag{3.4}$$

$$u(x) + Z_l \frac{\partial u(x)}{\partial \mathcal{N}} = U_l, \qquad x \in E_l,\ l = \overline{1,m} \tag{3.5}$$

$$\int_{E_l} \frac{\partial u(x)}{\partial \mathcal{N}} ds = I_l, \qquad l = \overline{1,m} \tag{3.6}$$

where

$$\frac{\partial u(x)}{\partial \mathcal{N}} = \sum_{i,j} a_{ij}(x) u_{x_j} \nu^i$$

be a co-normal derivative at x, and $\nu = (\nu^1, ..., \nu^n)$ is the outward normal at a point x to ∂Q. Electrical conductivity matrix $A = (a_{ij})$ is positive definite with

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2,\ \forall \xi \in \mathbb{R}^n; \quad \mu > 0. \tag{3.7}$$

The following is the **EIT Problem:** *Given electrical conductivity tensor A, electrode contact impedance vector Z, and electrode current pattern I it is required to find electrostatic potential u and electrode voltages U satisfying* (3.2)–(3.6):

$$(A,Z,I) \longrightarrow (u,U)$$

The goal of this chapter is to analyze inverse EIT problem of determining conductivity tensor A from the measurements of the boundary voltages U^*. **Inverse EIT Problem:** *Given electrode contact impedance vector Z, electrode current pattern I and boundary electrode measurement U^*, it is required to find electrostatic potential u and electrical conductivity tensor A satisfying* (3.2)–(3.6) with $U = U^*$.

We refer to Chapter 1 for literature review on inverse EIT problem.

3.2 Optimal Control Problem

We formulate Inverse EIT Problem as the following PDE constrained optimal control problem. Consider the minimization of the cost functional

$$\mathscr{J}(v) = \sum_{l=1}^{m} \Big| \int_{E_l} \frac{U_l - u(x)}{Z_l} ds - I_l \Big|^2 + \beta |U - U^*|^2 \tag{3.8}$$

on the control set

$$V_R = \Big\{ v = (A,U) \in \Big(L_\infty(Q;\mathbb{M}^{n\times n}) \bigcap H^\varepsilon(Q;\mathbb{M}^{n\times n}) \Big) \times \mathbb{R}^m \Big| \sum_{l=1}^{m} U_l = 0,$$
$$\|A\|_{L_\infty} + \|A\|_{H^\varepsilon} + |U| \leq R,\ \xi^T A \xi \geq \mu |\xi|^2,\ \forall \xi \in \mathbb{R}^n,\ \mu > 0 \Big\}$$

where $\beta > 0$, and $u = u(\cdot; v) \in H^1(Q)$ is a solution of the elliptic problem (3.3)–(3.5). This optimal control problem will be called Problem $\mathscr{J}$. The first term in the cost functional $\mathscr{J}(v)$ characterizes the mismatch of the condition (3.6) in light of the Robin condition (3.5).

Note that the variational formulation of the EIT Problem is a particular case of the Problem $\mathscr{J}$, when the conductivity tensor A is known, and therefore is removed from the control set by setting $R = +\infty$ and $\beta = 0$:

$$\mathscr{I}(U) = \sum_{l=1}^{m} \Big| \int_{E_l} \frac{U_l - u(x)}{Z_l} ds - I_l \Big|^2 \to \inf \tag{3.9}$$

in a control set

$$W = \Big\{ U \in \mathbb{R}^m \Big| \sum_{l=1}^{m} U_l = 0 \Big\} \tag{3.10}$$

where $u = u(\cdot; v) \in H^1(Q)$ is a solution of the elliptic problem (3.3)–(3.5). This optimal control problem will be called Problem $\mathscr{I}$. It is a convex PDE constrained optimal control problem (Remark 3.4.3, Section 2.3).

Inverse EIT problem on the identification of the electrical conductivity tensor A with m input data $(I_l)_{l=1}^m$ is highly ill-posed. Next, we formulate an optimal control problem which is adapted to the situation when the size of input data can be increased through additional measurements while keeping the size of the unknown parameters fixed. Let $U^1 = U, I^1 = I$ and consider $m-1$ new permutations of boundary voltages

$$U^j = (U_j, ..., U_m, U_1, ..., U_{j-1}), \ j = 2, ..., m \tag{3.11}$$

applied to electrodes $E_1, E_2, \ldots, E_m$ respectively. Assume that the "voltage–to–current" measurement allows us to measure associated currents $I^j = (I_1^j, \cdots, I_m^j)$. By setting

$U^1 = U^*$ and having a new set of m^2 input data $(I^j)_{j=1}^m$, we now consider optimal control problem on the minimization of the new cost functional

$$\mathscr{K}(v) = \sum_{j=1}^m \sum_{l=1}^m \Big| \int_{E_l} \frac{U_l^j - u^j(x)}{Z_l} ds - I_l^j \Big|^2 + \beta |U - U^*|^2 \tag{3.12}$$

on a control set V_R, where each function $u^j(\cdot; A, U^j)$, $j = 1, ..., m$, solves elliptic PDE problem (3.3)–(3.5) with U replaced by U^j. This optimal control problem will be called Problem $\mathscr{K}$. Note that the number of input currents in the Problem $\mathscr{K}$ has increased from m to m^2. However, the size of unknown control vector is unchanged, and in particular there are only m unknown voltages $U_1, \cdots, U_m$. The price we pay for this gain is the increase of the number of PDE constrains, which has increased from 1 to m. It should be noted that similar approach can be pursued to increase the size of input data up to $m!$ by adding possible permutations of U in (3.11).

We effectively use Problem $\mathscr{I}$ to generate model examples of the inverse EIT problem which adequately represents the diagnosis of the breast cancer in reality. Computational analysis based on the Fréchet differentiability result and gradient method in Besov spaces for the Problems $\mathscr{J}$ and $\mathscr{K}$ is pursued in realistic model examples.

3.3 Main Results

Let bilinear form $B : H^1(Q) \times H^1(Q) \to \mathbb{R}$ be defined as

$$B[u, \eta] = \int_Q \sum_{i,j=1}^n a_{ij} u_{x_j} \eta_{x_i} dx + \sum_{l=1}^m \frac{1}{Z_l} \int_{E_l} u \eta \, ds, \tag{3.13}$$

Definition 3.3.1. For a given $v \in V_R$, $u = u(\cdot; v) \in H^1(Q)$ is called a solution of the problem (3.3)–(3.5) if

$$B[u,\eta] = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l ds, \quad \forall \eta \in H^1(Q). \tag{3.14}$$

For a given control vector $v \in V_R$ and corresponding $u(\cdot;v) \in H_1(Q)$, consider the adjoined problem:

$$\sum_{ij} (a_{ij} \psi_{x_i})_{x_j} = 0, \qquad x \in Q \tag{3.15}$$

$$\frac{\partial \psi}{\partial \mathcal{N}} = 0, \qquad x \in \partial Q - \bigcup_{l=1}^{m} E_l \tag{3.16}$$

$$\psi + Z_l \frac{\partial \psi}{\partial \mathcal{N}} = 2 \int_{E_l} \frac{u - U_l}{Z_l} ds + 2I_l, \quad x \in E_l,\ l = \overline{1,m} \tag{3.17}$$

Definition 3.3.2. $\psi \in H^1(Q)$ is called a solution of the adjoined problem (3.15)–(3.17) if

$$B[\psi,\eta] = \sum_{l} \int_{E_l} \frac{\eta}{Z_l} \Big[2 \int_{E_l} \frac{u - U_l}{Z_l} ds + 2I_l \Big] ds,\ \forall \eta \in H^1(Q). \tag{3.18}$$

In Lemma 3.4.1, Section 2.3 it is demonstrated that for a given $v \in V_R$, both elliptic problems are uniquely solvable.

Definition 3.3.3. Let V be a convex and closed subset of the Banach space H. We say that the functional $\mathcal{J} : V \to \mathbb{R}$ is differentiable in the sense of Fréchet at the point $v \in V$ if there exists an element $\mathcal{J}'(v) \in H'$ of the dual space such that

$$\mathcal{J}(v+h) - \mathcal{J}(v) = \left\langle \mathcal{J}'(v), h \right\rangle_H + o(h,v), \tag{3.19}$$

where $v + h \in V \cap \{u : \|u\| < \gamma\}$ for some $\gamma > 0$; $\langle \cdot, \cdot \rangle_H$ is a pairing between H and its dual H', and

$$\frac{o(h,v)}{\|h\|} \to 0, \quad \text{as } \|h\| \to 0.$$

The expression $d\mathscr{J}(v) = \langle \mathscr{J}'(v), \cdot \rangle_H$ is called a Fréchet differential of $\mathscr{J}$ at $v \in V$, and the element $\mathscr{J}'(v) \in H'$ is called Fréchet derivative or gradient of $\mathscr{J}$ at $v \in V$.

Note that if Fréchet gradient $\mathscr{J}'(v)$ exists at $v \in V$, then the Fréchet differential $d\mathscr{J}(v)$ is uniquely defined on a convex cone ([5, 6, 10, 12, 13, 11, 9])

$$\mathscr{H}_v = \{w \in H : w = \lambda(u-v), \lambda \in [0,+\infty), u \in V\}.$$

The following are the main theoretical results of this chapter:

Theorem 3.3.4. *(Existence of an Optimal Control). Problem $\mathscr{J}$ has a solution, i.e.*

$$V_* = \{v = (A,U) \in V_R; \mathscr{J}(v) = \mathscr{J}_* = \inf_{v \in V_R} \mathscr{J}(v)\} \neq \emptyset \tag{3.20}$$

Theorem 3.3.5. *(Fréchet Differentiability): The functional $\mathscr{J}(v)$ is differentiable on V_R in the sense of Fréchet; the Fréchet differential $d\mathscr{J}(v)$ and the gradient $\mathscr{J}'(A,U) \in \mathscr{L}' \times \mathbb{R}^m$ are*

$$\left\langle \mathscr{J}'(v), \delta v \right\rangle_H = -\int_Q \sum_{i,j=1}^n u_{x_j} \psi_{x_i} \delta a_{ij} dx$$
$$+\sum_{k=1}^m \Big(\sum_{l=1}^m 2\Big[\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l\Big] \int_{E_l} \frac{1}{Z_l}(\delta_{lk} - w^k(s))ds + 2\beta(U_k - U_k^*)\Big)\delta U_k \tag{3.21}$$

$$\mathscr{J}'(A,U) = \Big(\mathscr{J}'_A(A,U), \mathscr{J}'_U(A,U)\Big)$$
$$= \left(-\big(\psi_{x_i} u_{x_j}\big)_{i,j=1}^n, \Big(\sum_{l=1}^m 2\Big[\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l\Big] \int_{E_l} \frac{1}{Z_l}(\delta_{lk} - w^k(s))ds + 2\beta(U_k - U_k^*)\Big)_{k=1}^m \right) \tag{3.22}$$

where $u = u(\cdot;v), \psi = \psi(\cdot;v)$*;* $w^k = \frac{\partial u}{\partial U_k} = u(\cdot;A,e_k),\ k = 1,2,..,m$ *is a solution of* (3.3)–(3.5) *with* $v = (A,e_k)$*,* $e_k \in \mathbb{R}$ *is a unit ort vector in* x_k*-direction;* δ_{lk} *is a Kronecker delta;* $\delta v = (\delta A, \delta U) = ((\delta a_{ij})_{i,j=1}^n, (\delta U_k)_{k=1}^m)$ *is a variation of the control vector* $v \in V_R$ *such that* $v + \delta v \in V_R$*.*

Corollary 3.3.6. *(Optimality Condition) If* $v \in V_R$ *is an optimal control in Problem* $\mathscr{J}$*, then the following variational inequality is satisfied:*

$$\left\langle \mathscr{J}'(\boldsymbol{v}), v - \boldsymbol{v} \right\rangle_H \geq 0,\ \forall v \in V_R. \tag{3.23}$$

Corollary 3.3.7. *(Fréchet Differentiability): The functional* $\mathscr{K}(v)$ *is differentiable on* V_R *in the sense of Fréchet and the Fréchet gradient* $\mathscr{K}'(\sigma, U) \in \mathscr{L}' \times \mathbb{R}^m$ *is*

$$\begin{gathered}\mathscr{K}'(v) = \Big(\mathscr{K}_A'(A,U), \mathscr{K}_U'(A,U)\Big) = \\ \Bigg(-\Big(\sum_{j=1}^m \psi_{x_p}^j u_{x_q}^j\Big)_{p,q=1}^n, \Big(\sum_{j=1}^m \sum_{l=1}^m 2\Big[\int_{E_l} \frac{U_l^j - u_j}{Z_l} ds - I_l^j\Big] \int_{E_l} \frac{\delta_{l,\theta_{kj}} - w^{\theta_{kj}}(s)}{Z_l} ds \\ +2\beta(U_k - U_k^*)\Big)_{k=1}^m \Bigg)\end{gathered} \tag{3.24}$$

where $\psi^j(\cdot), j = 1, ..., m$*, be a solution of the adjoined PDE problem* (3.15)–(3.17) *with* $u(\cdot), U$ *and* I *replaced with* $u^j(\cdot), U^j$*,* I^j *respectively, and*

$$\theta_{kj} = \begin{cases} k - j + 1, & \text{if } j \leq k, \\ m + k - j + 1, & \text{if } j > k. \end{cases}$$

3.3.1 Gradient Method in Banach Space

Fréchet differentiability result of Theorem 3.3.5 and the formula (3.22) for the Fréchet derivative suggest the following algorithm based on the projective gradient method in Banach space H for the Problem $\mathscr{J}$.

Step 1. Set $N = 0$ and choose initial vector function $(A^0, U^0) \in V_R$ where

$$A^0 = (a^0_{ij})^n_{ij=1},\ U^0 = (U^0_1, ..., U^0_m),\ \sum_{l=0}^{m} U^0_l = 0$$

Step 2. Solve the PDE problem (3.3)–(3.5) to find $u^N = u(\cdot; A^N, U^N)$ and $\mathscr{J}(A^N, U^N)$.

Step 3. If $N = 0$, move to Step 4. Otherwise, check the following criteria:

$$\left| \frac{\mathscr{J}(A^N, U^N) - \mathscr{J}(A^{N-1}, U^{N-1})}{\mathscr{J}(A^{N-1}, U^{N-1})} \right| < \varepsilon, \quad \frac{\|A^N - A^{N-1}\|}{\|A^{N-1}\|} < \varepsilon,\ \frac{|U^N - U^{N-1}|}{|U^{N-1}|} < \varepsilon \tag{3.25}$$

where ε is the required accuracy. If the criteria are satisfied, then terminate the iteration. Otherwise, move to Step 4.

Step 4. Solve the PDE problem (3.3)–(3.5) to find $w^N_k = u(\cdot; A^N, e_k), k = 1, ..., m$,

Step 5. Solve the adjoined PDE problem (3.15)–(3.17) to find $\psi_N = \psi(\cdot; A^N, U^N, u^N)$.

Step 6. Choose stepsize parameter $\alpha_N > 0$ and compute a new control vector components $\tilde{A}^{N+1} = (\tilde{a}^{N+1}_{ij}(x))^n_{i,j=1}, \tilde{U}^{N+1} \in \mathbb{R}^m$ as follows:

$$\tilde{a}^{N+1}_{ij}(x) = a^N_{ij}(x) + \alpha_N \psi^N_{x_i} u^N_{x_j},\ i, j = 1, ..., n, \tag{3.26}$$

$$\tilde{U}^{N+1}_k = U^N_k - \alpha_N \Big[\sum_{l=1}^{m} 2 \Big(\int_{E_l} \frac{U^N_l - u^N(s)}{Z_l} ds - I_l \Big) \int_{E_l} \frac{1}{Z_l} (\delta_{lk} - w^N_k(s)) ds + 2\beta(U^N_k - U^*_k) \Big], k = 1, ..., m. \tag{3.27}$$

Step 7. Replace $(\tilde{A}^{N+1}, \tilde{U}^{N+1})$ with $(A^{N+1}, U^{N+1}) \in V_R$ as follows

$$a_{ij}^{N+1}(x) = \begin{cases} \mu, & if\ \tilde{a}_{ij}^{N+1}(x) \leq \mu, \\ \tilde{a}_{ij}^{N+1}(x), & if\ \mu \leq \tilde{a}_{ij}^{N+1}(x) \leq R, \\ R, & if\ \tilde{a}_{ij}^{N+1}(x) > R. \end{cases} \tag{3.28}$$

$$U_k^{N+1} = \tilde{U}_k^{N+1} - \frac{1}{m}\sum_{k=1}^{m} \tilde{U}_k^{N+1},\ k = 1, ..., m \tag{3.29}$$

Then replace N with $N+1$ and move to Step 2.

Based on formula (3.24) similar algorithm is implemented for solving Problem $\mathscr{K}$.

Remark 3.3.8. Differentiability result and optimality condition similar to Theorem 3.3.5 and Corollary 3.3.6 are true for the Problem $\mathscr{I}$ and the gradient $\mathscr{I}'_U$ coincides with $\mathscr{J}'_U$ from (3.22). Similar algorithm for the gradient method in $\mathbb{R}^m$ applies to the Problem $\mathscr{I}$ in which case only iteration of the parameter U is pursued.

3.4 Proofs of the Main Results

Well-posedness of the elliptic problems (3.3)–(3.5) and (3.15)–(3.17) follow from the Lax-Milgram theorem ([42]).

Lemma 3.4.1. *For* $\forall v \in V_R$ *there exists a unique solution* $u = u(\cdot, v) \in H^1(Q)$ *to the problem* (3.3)–(3.5) *which satisfy the energy estimate*

$$\|u\|^2_{H^1(Q)} \leq C \sum_{l=1}^{m} Z_l^{-2} U_l^2 \tag{3.30}$$

Proof: *Step 1. Introduction of the equivalent norm in $H^1(Q)$.* Let

$$|||u|||_{H^1(Q)} := \left[\int_Q |\nabla u|^2 dx + \sum_{l=1}^{m} \int_{E_l} u^2 ds \right]^{\frac{1}{2}}, \tag{3.31}$$

and prove that this is equivalent to the standard norm of $H^1(Q)$, i.e. there is $c > 1$ such that $\forall u \in H^1(Q)$

$$c^{-1}\|u\|_{H^1(Q)} \leq |||u|||_{H^1(Q)} \leq c\|u\|_{H^1(Q)} \tag{3.32}$$

The second inequality immediately follows due to bounded embedding $H^1(Q) \hookrightarrow L^2(\partial Q)$ ([42]). To prove the first inequality assume on the contrary that

$$\forall k > 0, \quad \exists u_k \in H^1(Q) \quad \text{such that } \|u_k\|_{H^1(Q)} > k|||u_k|||_{H^1(Q)}.$$

Without loss of generality we can assume that $\|u_k\| = 1$, and therefore

$$\|\nabla u_k\|_{L_2(Q)} \longrightarrow 0, \ \|u_k\|_{L_2(E_l)} \longrightarrow 0, \quad \text{as } k \to \infty, \quad l = 1, 2, ...m. \tag{3.33}$$

Since $\{u_k\}$ is a bounded sequence in $H^1(Q)$, it is weakly precompact in $H^1(Q)$ and strongly precompact in both $L_2(Q)$ and $L_2(\partial Q)$ ([87, 29, 30]). Therefore, there exists a subsequence $\{u_{k_j}\}$ and $u \in H^1(Q)$ such that u_{k_j} converges to u weakly in $H^1(Q)$ and strongly in $L_2(Q)$ and $L_2(\partial Q)$. Without loss of generality we can assume that the whole sequence $\{u_k\}$ converges to u. From the first relation of (3.33) it follows that ∇u_k converges to zero strongly, and therefore also weakly in $L^2(Q)$. Due to uniqueness of the limit $\nabla u = 0$, and therefore $u = const$ a.e. in Q, and on the ∂Q in the sense of traces. According to the second relation in (3.33), and since $|E_l| > 0$, it follows that $const = 0$. This fact contradicts with $\|u_k\| = 1$, and therefore the second inequality is proved.

Step 2. Application of the Lax-Milgram theorem. Since $v \in V_R$, by using Cauchy-Bunyakowski-Schwartz (CBS) inequality, bounded trace embedding $H^1(Q) \hookrightarrow L^2(\partial Q)$ and (3.32) we have the following estimations for the bilinear form B:

$$|B[u,\eta]| \leq \alpha \|u\|_{H^1(Q)} \|\eta\|_{H^1(Q)}, \; B[u,u] \geq \beta \|u\|^2_{H^1(Q)} \tag{3.34}$$

where $\alpha, \beta > 0$ are independent of u, η. Note that the component U of the control vector v defines a bounded linear functional $\hat{U} : H^1(Q) \to \mathbb{R}$ according to the right-hand side of (3.14):

$$\hat{U}(\eta) := \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} \eta ds. \tag{3.35}$$

Indeed, by using CBS inequality and bounded trace embedding $H^1(Q) \hookrightarrow L^2(\partial Q)$ we have

$$|\hat{U}(\eta)| \leq |Q|^{\frac{1}{2}} (\sum_{l=1}^{m} Z_l^{-2} U_l^2)^{\frac{1}{2}} \|\eta\|_{L_2(\partial Q)} \leq C \|\eta\|_{H^1(Q)} \tag{3.36}$$

From (3.34),(3.36) and Lax-Milgram theorem ([42]) it follows that there exists a unique solution of the problem (3.3)–(3.5) in the sense of Definition 3.14.

Step 3. Energy estimate. By choosing η as a weak solution u in (3.14), using (3.7) and Cauchy's inequality with ε we derive

$$\mu \|\nabla u\|^2_{L_2(Q)} + z_0 \sum_{l=1}^{m} \|u\|^2_{L_2(E_l)} \leq \frac{c}{\varepsilon} \sum_{l=1}^{m} Z_l^{-2} U_l^2 + \varepsilon |\partial Q| \sum_{l=1}^{m} \left(\int_{E_l} |u|^2 ds \right) \tag{3.37}$$

where $z_0 = \min_{1 \leq l \leq m} Z_l^{-1}$. By choosing $\varepsilon = (2|\partial Q|)^{-1} z_0$ from (3.37) it follows that

$$|||u|||_{H^1(Q)} \leq C \sum_{l=1}^{m} Z_l^{-2} U_l^2. \tag{3.38}$$

From (3.32) and (3.38), energy estimate (3.30) follows. Lemma is proved.■

Corollary 3.4.2. *For* $\forall v \in V_R$ *there exists a unique solution* $\psi = \psi(\cdot, v) \in H^1(Q)$ *of the adjoined problem* (3.15)–(3.17) *which satisfy the energy estimate*

$$\|\psi\|^2_{H^1(Q)} \leq C \sum_{l=1}^{m} Z_l^{-2} \Big[\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l \Big]^2 \tag{3.39}$$

where $u = u(\cdot; v) \in H^1(Q)$ *is a solution of the problem* (3.3)–(3.5) .

Proof of Theorem 3.3.4. Let $\{v_k\} = \{(A^k, U^k)\} \subset V_R$ be a minimizing sequence

$$\lim_{k \to \infty} \mathscr{J}(v_k) = \mathscr{J}_*$$

Since $\{A^k\}$ is a bounded sequence in $H^\varepsilon(Q; \mathbb{M}^{n \times n})$, it is weakly precompact in $H^\varepsilon(Q; \mathbb{M}^{n \times n})$ and strongly precompact in $L_2(Q; \mathbb{M}^{n \times n})$ ([87, 29, 30]). Therefore, there exists a subsequence $\{A^{k_p}\}$ which converges weakly in $H^\varepsilon(Q; \mathbb{M}^{n \times n})$ and strongly in $L_2(Q; \mathbb{M}^{n \times n})$ to some element $A \in H^\varepsilon(Q; \mathbb{M}^{n \times n})$. Since any strong convergent sequence in $L_2(Q; \mathbb{M}^{n \times n})$ has a subsequence which converges a.e. in Q, without loss of generality one can assume that the subsequence A^{k_p} converges to A a.e. in Q, which implies that $A \in L_\infty(Q; \mathbb{M}^{n \times n}) \cap H^\varepsilon(Q; \mathbb{M}^{n \times n}) \cap V_R$. Since U^k is a bounded sequence in $\mathbb{R}^m$ it has a subsequence which converges to some $U \in \mathbb{R}^m, |U| \leq R$. Without loss of generality we cam assume that the whole minimizing sequence $v_k = (A_k, U^k)$ converges $v = (A, U) \in V_R$ in the indicated way.

Let $u_k = u(x; v_k)$, $u = u(x; v) \in H^1(Q)$ are weak solutions of (3.3)–(3.5) corresponding to v_k and v respectively. By Lemma 3.4.1 u_k satisfy the energy estimate (3.30) with U^k on the right hand side, and therefore it is uniformly bounded in $H^1(Q)$. By the Rellich-Kondrachov compact embedding theorem there exists a subsequence $\{u_{k_p}\}$ which converges weakly in $H^1(Q)$ and strongly in both $L_2(Q)$ and $L_2(\partial Q)$ to some function $\tilde{u} \in H^1(Q)$([87, 29, 30]). Without loss of generality assume that the whole sequence u_k converges to $\tilde{u}$ weakly in $H^1(Q)$ and strongly both in $L_2(Q)$ and $L_2(\partial Q)$. For any fixed $\eta \in C^1(Q)$ weak solution u_k satisfies the following integral identity

$$\int_Q \sum_{i,j=1}^{n} a_{ij}^k u_{k_{x_j}} \eta_{x_i} dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} u_k \eta ds = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l^k ds. \tag{3.40}$$

Due to weak convergence of ∇u_k to $\nabla \tilde{u}$ in $L_2(Q;\mathbb{R}^n)$, strong convergence of u_k to $\tilde{u}$ in $L_2(\partial Q)$, strong convergence of a_{ij}^k to a_{ij} in $L_2(Q)$ and convergence of U^k to U, passing to the limit as $k \to \infty$, from (3.40) it follows

$$\int_Q \sum_{i,j=1}^{n} a_{ij} \tilde{u}_{x_j} \eta_{x_i} dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \tilde{u} \eta ds = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l ds. \tag{3.41}$$

Due to density of $C^1(Q)$ in $H^1(Q)$ ([87, 29, 30]) the integral identity (3.41) is true for arbitrary $\eta \in H^1(Q)$. Hence, $\tilde{u}$ is a weak solution of the problem (3.3)–(3.5) corresponding to the control vector $v = (A,U) \in V_R$. Due to uniqueness of the weak solution it follows that $\tilde{u} = u$, and the sequence u_k converges to the weak solution $u = u(x;v)$ weakly in $H^1(Q)$, and strongly both in $L_2(Q)$ and $L_2(\partial Q)$. The latter easily implies that

$$\mathscr{J}(v) = \lim_{n\to\infty} \mathscr{J}(v_n) = \mathscr{J}_*$$

Therefore, $v \in V_*$ is an optimal control and (3.20) is proved. ■

Proof of Theorem 3.3.5. Let $v = (A,U) \in V_R$ is fixed and $\delta v = (\delta A, \delta U)$ is an increment such that $\bar{v} = v + \delta v \in V_R$ and $u = u(\cdot;v), \bar{u} = u(\cdot;v+\delta v) \in H^1(Q)$ are respective weak solutions of the problem (3.3)–(3.5). Since $u(\cdot;A,U)$ is a linear function of U it easily follows that

$$w^k = \frac{\partial u}{\partial U_k} = u(\cdot;A,e_k) \in H^1(Q),\ k = 1,2,..,m$$

is a solution of (3.3)–(3.5) with $v = (A,e_k)$, $e_k \in \mathbb{R}^m$ is a unit ort vector in x_k-direction. Straightforward calculation imply that

$$\frac{\partial \mathscr{J}}{\partial U_k} = \sum_{l=1}^{m} 2\Big[\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l\Big] \int_{E_l} \frac{1}{Z_l}(\delta_{lk} - w^k) ds + 2\beta(U_k - U_k^*),\ k = 1,...,m.$$

where δ_{lk} is a Kronecker delta.

In order to prove the Fréchet differentiability with respect to A, assume that $\delta U = 0$ and transform the increment of $\mathscr{J}$ as follows

$$\delta \mathscr{J} := \mathscr{J}(v+\delta v) - \mathscr{J}(v) = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} 2\left(\int_{E_l} \frac{u-U_l}{Z_l} ds + I_l\right) \delta u ds + R_1, \quad (3.42)$$

$$R_1 = \sum_{l=1}^{m} Z_l^{-2} \left(\int_{E_l} \delta u ds\right)^2 \leq \sum_{l=1}^{m} |E_l| Z_l^{-2} |||\delta u|||^2_{H^1(Q)}, \quad (3.43)$$

where $\delta u = \bar{u} - u$. By subtracting integral identities (3.14) for $\bar{u}$ and u, and by choosing test function $\eta = \psi(\cdot;v)$ as a solution of the adjoined problem (3.15)–(3.17) we have

$$\int_Q \sum_{ij} \left(\delta a_{ij} u_{x_j} + a_{ij}(\delta u)_{x_j} + \delta a_{ij}(\delta u)_{x_j}\right) \psi_{x_i} dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \psi \delta u ds = 0. \quad (3.44)$$

By choosing $\eta = \delta u$ in the integral identity (3.18) for the weak solution ψ of the adjoined problem we have

$$-\int_Q \sum_{ij} a_{ij} \psi_{x_i} \delta u_{x_j} dx + \sum_{l} \int_{E_l} \frac{\delta u}{Z_l} \big[2 \int_{E_l} \frac{u - U_l}{Z_l} ds + 2I_l - \psi\big] ds = 0 \quad (3.45)$$

Adding (3.44) and (3.45) we derive

$$\sum_{l=1}^{m}\frac{1}{Z_l}\int_{E_l}2\left(\int_{E_l}\frac{u-U_l}{Z_l}dS(x)+I_l\right)\delta u ds=\int_Q(-\sum_{ij}\delta a_{ij}u_{x_j}\psi_{x_i}-\sum_{ij}\delta a_{ij}(\delta u)_{x_j}\psi_{x_i})dx. \tag{3.46}$$

From (3.42) and (3.46) it follows that

$$\delta\mathscr{J}=-\int_Q\sum_{ij}u_{x_j}\psi_{x_i}\delta a_{ij}dx+R_1+R_2 \tag{3.47}$$

where

$$R_2=-\int_Q\sum_{ij}\delta a_{ij}(\delta u)_{x_j}\psi_{x_i}dx. \tag{3.48}$$

To complete the proof it remains to prove that

$$R_1+R_2=o(\|\delta A\|_{L_\infty(Q;M^{n\times n})})\quad \text{as } \|\delta A\|_{L_\infty(Q;M^{n\times n})}\to 0. \tag{3.49}$$

By subtracting integral identities (3.14) for $\bar{u}$ and u again, and by choosing test function $\eta=\delta u$ we have

$$\int_Q\sum_{ij}\bar{a}_{ij}(\delta u)_{x_j}(\delta u)_{x_i}dx+\sum_{l=1}^{m}\frac{1}{Z_l}\int_{E_l}(\delta u)^2ds=-\int_Q\sum_{ij}\delta a_{ij}u_{x_j}(\delta u)_{x_i}dx. \tag{3.50}$$

By using positive definiteness of $\bar{A}\in V_R$ and by applying Cauchy inequality with $\varepsilon>0$ to the right hand side, from (3.50) it follows that

$$\mu\int_Q|\nabla\delta u|^2dx+\sum_{l=1}^{m}\frac{1}{Z_l}\int_{E_l}(\delta u)^2ds\le\varepsilon\int_Q|\nabla\delta u|^2dx+\frac{c}{\varepsilon}\int_Q|\sum_{ij}\delta a_{ij}|^2|\nabla u|^2. \tag{3.51}$$

By choosing $\varepsilon=\mu/2$ and by applying the energy estimate (3.30) from (3.51) we derive

$$|||\delta u|||^2_{H^1(Q)} \le C\|\delta A\|^2_{L_\infty(Q;\mathbb{M}^{n\times n})}. \tag{3.52}$$

From (3.48) it follows that

$$|R_2| \le C\|\delta A\|_{L_\infty(Q;\mathbb{M}^{n\times n})}\|\nabla\delta u\|_{L_2(Q)}\|\nabla\psi\|_{L_2(Q)}. \tag{3.53}$$

From (3.30), (3.32), (3.39), (3.43), (3.52) and (3.53), desired estimation (3.49) follows. Theorem is proved.■

Remark 3.4.3. Functional (3.9) in the optimal control Problem $\mathscr{I}$ is convex due to the following formula

$$\mathscr{I}(\alpha U^1+(1-\alpha)U^2) = \alpha\mathscr{I}(U^1)+(1-\alpha)\mathscr{I}(U^2)-\alpha(1-\alpha)\sum_{l=1}^{m} Z_l^{-2}\left|\int_{E_l}(U_l^1-U_l^2-u^1+u^2)ds\right|^2$$

where $U^1,U^2 \in W, \alpha \in [0,1]; u^i = u(\cdot;U^i), i=1,2$ is a solution of (3.3)–(3.5) with $U = U^i$. Therefore, unique solution of the EIT problem would be a unique global minimizer of the Problem $\mathscr{I}$.

Results of this Chapter are contained in a recent preprint [8].

Chapter 4

Discretization and Convergence of the EIT Optimal Control Problem in 2D Domains

4.1 Introduction and Problem Description

In this chapter, we consider the following EIT problem for $Q \in \mathbb{R}^2$:

$$\operatorname{div}\big(\sigma(x)\nabla u\big) = 0, \qquad x \in Q \tag{4.1}$$

$$\frac{\partial u(x)}{\partial n} = 0, \qquad x \in S - \bigcup_{l=1}^{m} E_l \tag{4.2}$$

$$u(x) + Z_l \sigma(x) \frac{\partial u(x)}{\partial n} = U_l, \qquad x \in E_l,\ l = \overline{1,m} \tag{4.3}$$

$$\int_{E_l} \sigma(x) \frac{\partial u(x)}{\partial n} ds = I_l, \qquad l = \overline{1,m} \tag{4.4}$$

where

$$\frac{\partial u(x)}{\partial n} = \sum_{i=1}^{2} u_{x_i} \nu^i$$

and $\nu = (\nu^1, \nu^2)$ is the outward normal at a point x to S, electrical conductivity σ is a positive function. The difference of this problem with one in Chapter 3 is that we remove the assumption on anisotropy for electrical conductivity tensor $A(x)$, i.e. $A(x) = \sigma(x)I$, where I is a 2×2 unit matrix.

4.1.1 Optimal Control Problem

Consider the optimal control problem on the minimization of the cost functional

$$\mathscr{J}(v) = \sum_{l=1}^{m} \Big| \int_{E_l} \frac{U_l - u(x)}{Z_l} ds - I_l \Big|^2 + \beta |U - U^*|^2 \tag{4.5}$$

on the control set

$$\mathscr{F}^R = \Big\{ v = (\sigma, U) \in \big(L_\infty(Q) \bigcap \tilde{H}^1(Q)\big) \times \mathbb{R}^m \Big| \sum_{l=1}^{m} U_l = 0, \, \|\sigma\|^2_{\tilde{H}^1} + |U|^2 \leq R^2 \tag{4.6}$$
$$0 < \sigma_0 \leq \sigma(x) \leq R, \forall x \in Q \Big\}$$

where $\beta > 0$, and $u = u(\cdot; v) \in H^1(Q)$ is a solution of the elliptic problem (4.1)–(4.3). The following is the definition of the weak solution of problem (4.1)–(4.3):

Definition 4.1.1. For a given $v \in \mathscr{F}^R$, $u = u(\cdot; v) \in H^1(Q)$ is called the weak solution of the problem (4.1)–(4.3) if

$$\int_Q \sigma \nabla u \cdot \nabla \eta dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} u \eta ds = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l ds, \quad \forall \eta \in H^1(Q). \tag{4.7}$$

This optimal control problem will be called Problem $\mathscr{E}$. The first term in the cost functional $\mathscr{J}(v)$ characterizes the mismatch of the condition (4.4) in light of the Robin condition (4.3).

4.1.2 Discrete Optimal Control Problem

To discretize optimal control problems $\mathscr{E}$ we pursue finite difference method which is outlined in Chapter 2. In addition to discrete sets, $Q_\Delta^*, Q_\Delta^{*+}, Q_\Delta^{*(i)}$, and S_Δ^* that we defined in Chapter 2, we introduce the following notation as well:

$$\hat{E}_{l\Delta} = \{\, x_\alpha \in Q_\Delta^* : \ C_\Delta^\alpha \cap E_l \neq \emptyset \,\}, \quad l = 1, \cdots, m$$

which is a collection of grid pints which are natural corners of C_Δ^α containing boundary curve E_l, and

$$E_{l\alpha} = C_\Delta^\alpha \cap E_l, \quad l = 1, \cdots, m$$

is a portion of the boundary curve which is contained in C_Δ^α. $\Gamma_{l\alpha} = |E_{l\alpha}|$, $l = 1, \cdots, m$ is $(n-1)$ dimensional Lebesgue measure of $E_{l\alpha}$. We are going to assume that any control vector σ is extended to a larger set $Q + B_1(0)$ as bounded measurable functions with preservation of conditions in the control set (4.6). We introduce discrete grid function by discretizing σ through Steklov average (2.15). For a given discretization Δ, we employ the notation $[\sigma]_\Delta = \{\sigma_\alpha\}$, $\alpha \in \mathscr{A}(Q_\Delta^*)$ where $\sigma_\alpha \in \mathbb{R}$. Then We define the discrete $\tilde{\mathscr{H}}^1(Q_\Delta^*)$ norm as

$$\|[\sigma]_\Delta\|^2_{\tilde{\mathscr{H}}^1(Q_\Delta^*)} = \sum_{\mathscr{A}(Q_\Delta^*)} h^2 \sigma_\alpha^2 + \sum_{i=1}^{2} \sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^2 \sigma_{\alpha x_i}^2 + \sum_{\mathscr{A}(Q_\Delta^{*+})} h^2 \sigma_{\alpha x_1 x_2}^2$$

$$\|[\sigma]_\Delta\|_{L_\infty(Q_\Delta^*)} = \max_{\alpha \in \mathscr{A}(Q_\Delta^*)} |\sigma_\alpha|$$

We use standard notation for finite differences of grid function u_α, σ_α:

$$u_{\alpha x_i} = \frac{u_{\alpha+e_i} - u_\alpha}{h},\ u_{\alpha \bar{x}_i} = \frac{u_\alpha - u_{\alpha-e_i}}{h}, \quad i = 1,2$$

$$\sigma_{\alpha x_1} = \frac{\sigma_{\alpha+e_1} - \sigma_\alpha}{h}, \sigma_{\alpha x_2} = \frac{\sigma_{\alpha+e_2} - \sigma_\alpha}{h}$$

and

$$\sigma_{\alpha x_1 x_2} = \frac{\sigma_{(\alpha+e_2)x_1} - \sigma_{\alpha x_1}}{h} = \frac{\sigma_{\alpha+e_2+e_1} - \sigma_{\alpha+e_2} - \sigma_{\alpha+e_1} + \sigma_\alpha}{h^2}$$

For fixed $R > 0$, define the discrete control sets $\mathscr{F}^R_\Delta$ as

$$\mathscr{F}^R_\Delta := \Bigg\{ [v]_\Delta = ([\sigma]_\Delta, U) \Big| \sum_{l=1}^m U_l = 0,\ \|[\sigma]_\Delta\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)} + |U|^2_{\mathbb{R}^m} \le R^2$$
$$0 < \sigma_0 \le \sigma_\alpha \le R, \forall \alpha \in \mathscr{A}(Q^*_\Delta) \Bigg\} \tag{4.8}$$

and the interpolating map $\mathscr{P}_\Delta$ as

$$\mathscr{P}_\Delta : \bigcup_R \mathscr{F}^R_\Delta \to \bigcup_R \mathscr{F}^R, \qquad \mathscr{P}_\Delta([v]_\Delta) = (\mathscr{P}_\Delta([\sigma]_\Delta), U) = (\sigma^\Delta, U)$$

where σ^Δ in each cell C^α_Δ is a multilinear interpolation which assigns the value σ_α to each grid point of C^α_Δ, and it is a peicewise linear with respect to each variable x_i when the other variable is fixed.

$$\sigma^\Delta(x) = \sigma_\alpha + \sigma_{\alpha x_1}(x_1 - k_1 h) + \sigma_{\alpha x_2}(x_2 - k_2 h) + \sigma_{\alpha x_1 x_2}(x_1 - k_1 h)(x_2 - k_2 h), \forall x \in C^\alpha_\Delta \tag{4.9}$$

Also, we define the discretizing map $\mathscr{Q}_\Delta$ as

$$\mathscr{Q}_\Delta : \bigcup_R \mathscr{F}^R \to \bigcup_R \mathscr{F}^R_\Delta, \qquad \mathscr{Q}_\Delta(v) = (\mathscr{Q}_\Delta(\sigma), U) = ([\sigma]_\Delta, U)$$

where $[\sigma]_\Delta = \{\sigma_\alpha\}$ where σ_α is given by (2.15) for each $\alpha \in \mathscr{A}(Q^*_\Delta)$.

Using the newly introduced notations we can define a solution of the discrete elliptic problem (4.1)–(4.3)

Definition 4.1.2. Given $[v]_\Delta$, the discrete valued function

$$[u([v]_\Delta)]_\Delta = \{u_\alpha \in \mathbb{R} : \alpha \in \mathscr{A}(Q^*_\Delta)\}$$

is called a discrete state vector of problem $\mathscr{E}$ if it satisfies

$$h^2 \sum_{\mathscr{A}(Q^{*+}_\Delta)} \sigma^\Delta_\alpha \sum_{i=1}^{2} u_{\alpha x_i} \eta_{\alpha x_i} + \sum_{l=1}^{m} \frac{1}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u_\alpha \eta_\alpha + J_\alpha(u_\alpha, \eta_\alpha) = \sum_{l=1}^{m} \frac{U_l}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} \eta_\alpha \tag{4.10}$$

for arbitrary collection of values $\{\eta_\alpha\},\ \alpha \in \mathscr{A}(Q^*_\Delta)$, where

$$J_\alpha(u_\alpha, \eta_\alpha) = h^2 \sum_{\mathscr{A}(S^*_\Delta)} \sum_{i=1}^{2} \theta^i_\alpha u_{\alpha x_i} \eta_{\alpha x_i}, \tag{4.11}$$

$$\theta^i_\alpha = \begin{cases} 1 & \text{if } \alpha \in \mathscr{A}(Q^{*(i)}_\Delta \setminus Q^{*+}_\Delta) \\ 0 & \text{otherwise} \end{cases}$$

The necessity of adding J_α to (4.10) is that some $u_{\alpha x_i}$ and $\eta_{\alpha x_i}$ values on S^*_Δ are not present in the term $h^2 \sum_{\mathscr{A}(Q^{*+}_\Delta)} \eta_{\alpha x_i} u_{\alpha x_i}$ of (4.10). These values are added to (4.10) through $h^2 \sum_{\mathscr{A}(S^*_\Delta)} \sum_{i=1}^{2} \theta^i_\alpha u_{\alpha x_i} \eta_{\alpha x_i}$ of J_α. For stability of our discrete scheme, it is essential to add this term to the discrete integral identity (4.10).

In Section 4.3, it will be proved that for a given $[\sigma]_\alpha \in \mathscr{F}_\Delta^R$ there exists a unique discrete state vector of problem E. Consider minimization of the discrete cost functional

$$\mathscr{I}_\Delta([v]_\Delta) = \sum_l \Big(\sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} \frac{U_l - u_\alpha}{Z_l} - I_l \Big)^2 + \beta |U - U^*|^2 \to inf \tag{4.12}$$

on a control set $\mathscr{F}_\Delta^R$, where u_α's are components of the discrete state vector $[u([v]_\Delta)]_\Delta$ of the Problem $\mathscr{E}$. The formulated discrete optimal control problem will be called Problem $\mathscr{E}_\Delta$.

4.2 Main Result

The following is the main result on the convergence of the sequence of finite-dimensional discrete optimal control problems to EIT optimal control problem both with respect to functional and control.

Theorem 4.2.1. *The sequence of discrete optimal control problems $\mathscr{E}_\Delta$ approximates the optimal control problem $\mathscr{E}$ with respect to functional, i.e.*

$$\lim_{\Delta \to 0} \mathscr{I}_{\Delta_*} = \mathscr{I}_*, \tag{4.13}$$

where

$$\mathscr{I}_{\Delta_*} = \inf_{\mathscr{F}_\Delta^R} \mathscr{I}_\Delta([v]_\Delta), \tag{4.14}$$

Furthermore, let $\{\varepsilon_\Delta\}$ be a sequence of positive real numbers with $\lim_{\Delta \to 0} \varepsilon_\Delta = 0$. If the *sequence* $[v]_{\Delta,\varepsilon} \in \mathscr{F}_{\Delta}^{R}$ *is chosen so that*

$$\mathscr{J}_{\Delta_*} \leq \mathscr{J}_{\Delta}([v]_{\Delta,\varepsilon}) \leq \mathscr{J}_{\Delta_*} + \varepsilon_{\Delta}, \tag{4.15}$$

then we have

$$\lim_{\Delta \to 0} \mathscr{J}\left(\mathscr{P}_{\Delta}([v]_{\Delta,\varepsilon})\right) = \mathscr{J}_* \tag{4.16}$$

Also, the sequence $\{\mathscr{P}_{\Delta}([\sigma]_{\Delta,\varepsilon})\}$ *is uniformly bounded in* $\tilde{H}^1(Q)$ *and all of its* $\tilde{H}^1(Q)$*-weak limits points lie in* $\mathscr{F}_*$*. Moreover, the multilinear interpolations of the discrete state vectors* $[u([v]_{\Delta',\varepsilon})]_{\Delta'}$ *converge weakly in* $H^1(Q)$ *to* $u = u(x; v_*)$*, a weak solution to the* (4.1)-(4.3).

4.3 Preliminary Results

Following the frame of the interpolations in Chapter 2, we have three interpolations $\tilde{U}_{\Delta}, \tilde{U}_{\Delta}^{i}$ and U'_{Δ} which are defined in Chapter 2. As in Chapter 2, we have the following estimations for U'_{Δ} and $\frac{\partial}{\partial x_i} U'_{\Delta}$:

$$\int_{Q_{\Delta}^{*}} |U'_{\Delta}|^2 dx \leq \sum_{\mathscr{A}(Q_{\Delta}^{*})} h^2 \max_{\mathscr{A}(Q_{\Delta}^{*})} |u_{\alpha^*}|^2 \leq 2^2 \sum_{\mathscr{A}(Q_{\Delta}^{*})} h^2 |u_{\alpha}|^2. \tag{4.17}$$

$$\int_{Q_{\Delta}^{*}} \left| \frac{\partial}{\partial x_i} U'_{\Delta} \right|^2 dx \leq 2 \sum_{\mathscr{A}(Q_{\Delta}^{*(i)})} h^2 u_{\alpha x_i}^2 \tag{4.18}$$

The following lemma is a discrete analogy of the norm equivalency result in space $H^1(Q)$ proved in Step 1 of Lemma 3.4.1.

Lemma 4.3.1. *For any $[u]_\Delta = \{u_\alpha : \ \alpha \in \mathscr{A}(Q^*_\Delta)\}$, we have an estimation*

$$\| [u]_\Delta \|_{\mathscr{H}^1(Q^*_\Delta)} \leq c||| [u]_\Delta |||_{\mathscr{H}^1(Q^*_\Delta)} \tag{4.19}$$

where C is independent of $[u]_\Delta$ and

$$|||[u]_\Delta|||^2_{\mathscr{H}^1(Q^*_\Delta)} := \sum_{i=1}^{2} h^2 \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} u^2_{\alpha x_i} + \sum_{l=1}^{m} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u^2_\alpha \tag{4.20}$$

Proof: To prove this inequality assume on the contrary that

$$\forall k > 0, \quad \exists [u]^k_\Delta \in \mathscr{H}^1(Q^*_\Delta);\ k||| [u]^k_\Delta |||_{\mathscr{H}^1(Q^*_\Delta)} < \| [u]^k_\Delta \|_{\mathscr{H}^1(Q^*_\Delta)}$$

where $[u]^k_\Delta = \{u^k_\alpha\}$. Without loss of generality we can assume that

$$\|[u]^k_\Delta\|_{\mathscr{H}^1(Q^*_\Delta)} = 1$$

and therefore

$$||| [u]^k_\Delta |||_{\mathscr{H}^1(Q^*_\Delta)} < \frac{1}{k}, \quad \forall k > 0 \tag{4.21}$$

which means

$$\sum_{i=1}^{2} h^2 \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} |u^k_{\alpha x_i}|^2 < \frac{1}{k}, \qquad \text{and} \qquad \sum_{l=1}^{m} \sum_{\mathscr{A}(\hat{E}_{l\alpha})} \Gamma_{l\alpha} |u^k_\alpha|^2 < \frac{1}{k}, \quad \forall k > 0 \tag{4.22}$$

and consequently

$$\sum_{i=1}^{2} \int_{Q^*_\Delta} |\tilde{U}^{ik}_\Delta|^2 < \frac{1}{k}, \qquad \text{and} \qquad \sum_{l=1}^{m} \int_{E_l} |\tilde{U}^k_\Delta|^2 < \frac{1}{k}, \quad \forall k > 0 \tag{4.23}$$

where $\tilde{U}^{ik}_{\Delta}$ and $\tilde{U}^{k}_{\Delta}$ are piece-wise constant interpolations of $u^k_{\alpha x_i}$ and u^k_{α}. These two inequalities imply that

$$\|\tilde{U}^{ik}_{\Delta}\|_{L_2(Q^*_{\Delta})} \to 0, \quad i=1,2, \quad \|\tilde{U}^{k}_{\Delta}\|_{L_2(E_l)} \to 0, \quad l=1,2,...m, \quad \text{as } k\to\infty \tag{4.24}$$

On the other hand, $[u]^k_{\Delta}$ is a bounded sequence in $\mathscr{H}^1(Q^*_{\Delta})$, by relations (4.17) and (4.18), it follows that corresponding multilinear interpolations $\{U^{k'}_{\Delta}\}$ is weakly precompact in $H^1(Q)$ and strongly precompact in both $L_2(Q)$ and $L_2(S)$ [87, 29, 30]. Therefore, there exists a subsequence of $\{U^{k'}_{\Delta}\}$ and $u\in H^1(Q)$ such that the subsequence converges weakly to u in $H^1(Q)$ and strongly in $L_2(Q)$ and $L_2(S)$. Without loss of generality we can take the whole sequence $\{U^{k'}_{\Delta}\}$ instead of the subsequence and summarize the useful results of the last paragraph into the following:

$$U^{k'}_{\Delta} \to u, \text{ in } L^2(Q), \tag{4.25}$$

$$\frac{\partial U^{k'}_{\Delta}}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}, \text{ weakly in } L^2(Q),\ i=1,2, \tag{4.26}$$

$$U^{k'}_{\Delta} \to u, \text{ in } L^2(S), \tag{4.27}$$

From the first claim of (4.24), it follows that

$$\tilde{U}^{ik}_{\Delta} \rightharpoonup 0, \text{ weakly on } L^2(Q),\ i=1,2, \tag{4.28}$$

In [14] (Theorem 14, parts (e) and (f)), it is proved that the sequences $\tilde{U}^{ik}_{\Delta}$ and $\frac{\partial U^{k'}_{\Delta}}{\partial x_i}$ are equivalent in a weak topology of $L^2(Q)$. Therefore, from (4.28) and (4.26) it follows that

$$\frac{\partial u}{\partial x_i} = 0$$

and hence $u = c$ a.e. in Q. By the second relation (4.24), and the fact that $|E_l| > 0$, it follows that $u = 0$ almost everywhere. This fact contradicts with $\|[u]^k_\Delta\|_{\mathscr{H}^1(Q^*_\Delta)} = 1$, and therefore the inequality is proved.

Lemma 4.3.2 (Discrete Energy Estimate)**.** *Let* $[u([v]_\Delta)]_\Delta$ *be the discrete state vector, then it satisfies the following energy estimate:*

$$\|[u([v]_\Delta)]_\Delta\|_{\mathscr{H}^1(Q^*_\Delta)} \leq M\Big(\sum_{l=1}^{m}\Big(\frac{U_l}{Z_l}\Big)^2\Big)^{\frac{1}{2}} \tag{4.29}$$

Proof: The proof follows the method developed in [71]. We set $\eta_\alpha = u_\alpha$ in (4.10) which implies

$$h^2 \sum_{\mathscr{A}(Q^{*+}_\Delta)} \sigma^\Delta_\alpha \sum_{i=1}^{2} u^2_{\alpha x_i} + \sum_{l=1}^{m} \frac{1}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u^2_\alpha + J_\alpha(u_\alpha, u_\alpha) = \sum_{l=1}^{m} \frac{U_l}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u_\alpha \tag{4.30}$$

recalling the definition of J_α and the norm $|||[u([v]_\Delta)]_\Delta|||_{\mathscr{H}^1(Q^*_\Delta)}$ and the fact that $0 < \sigma_0 \leq \sigma_\alpha$ we have

$$\mu|||[u([v]_\Delta)]_\Delta|||^2_{\mathscr{H}^1(Q^*_\Delta)} \leq \sum_{l=1}^{m} \frac{U_l}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u_\alpha \tag{4.31}$$

where $\mu = \min\{1, \sigma_0, \min_l \big(\frac{1}{Z_l}\big)\}$. Using Cauchy–Schwarz inequality we can estimate the right hand side as following

$$
\begin{aligned}
\sum_{l=1}^{m}\frac{U_l}{Z_l}\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha &\le \left(\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m}(\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha)^2\right)^{\frac{1}{2}} \\
&\le \left(\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m}(\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha})(\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha^2)\right)^{\frac{1}{2}} \\
&\le \left(\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}\left(|\partial Q|\sum_{l=1}^{m}\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha^2\right)^{\frac{1}{2}} \\
&\le \left(|\partial Q|\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m}\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha^2\right)^{\frac{1}{2}} \\
&\le \left(|\partial Q|\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m}\sum_{\mathscr{A}(\hat{E}_{l\Delta})}\Gamma_{l\alpha}u_\alpha^2+\sum_{i=1}^{2}h^2\sum_{\mathscr{A}(Q_\Delta^{*(i)})}u_{\alpha x_i}^2\right)^{\frac{1}{2}} \\
&= \left(|\partial Q|\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}}|||[u([v]_\Delta)]_\Delta|||_{\mathscr{H}^1(Q_\Delta^*)}
\end{aligned}
$$

using this estimate, the inequality (4.31) turns into

$$
\mu|||[u([v]_\Delta)]_\Delta|||_{\mathscr{H}^1(Q_\Delta^*)} \le \left(|\partial Q|\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}} \tag{4.32}
$$

By Lemma 4.3.1, it follows that

$$
\|[u([v]_\Delta)]_\Delta\|_{\mathscr{H}^1(Q_\Delta^*)} \le \mu^{-1}|\partial Q|^{1/2}\left(\sum_{l=1}^{m}(\frac{U_l}{Z_l})^2\right)^{\frac{1}{2}} \quad \square \tag{4.33}
$$

In particular, energy estimate implies the existence and uniqueness of the discrete state vector of the problem $\mathscr{E}$.

Corollary 4.3.3. *For a fixed Δ and any $R>0$, there exists a unique discrete state vector $[u([v]_\Delta)]_\Delta$ in a problem $\mathscr{E}$ for each $[v]_\Delta \in \mathscr{F}_\Delta^R$.*

This corollary can be proved as the Corollary 2.2.5

Lemma 4.3.4. *For each Δ, let $\{[v]_\Delta\} \in \mathscr{F}^R_\Delta$ be a sequence of discrete control vectors for some $R > 0$, and $[u([v]_\Delta)]_\Delta$ be the corresponding state variable. Then the following statements hold:*

(a) The sequences $\{U'_\Delta\}$ and $\{\tilde{U}_\Delta\}$ are uniformly bounded in $L_2(Q^*_\Delta)$.

(b) For each $i \in \{1,2\}$, the sequences $\{\tilde{U}^i_\Delta\}$, $\{\frac{\partial U'_\Delta}{\partial x_i}\}$ are uniformly bounded in $L_2(Q^*_\Delta)$.

(c) the sequence $\{\tilde{U}_\Delta - U'_\Delta\}$ converges strongly to 0 in $L_2(Q)$ as $h \to 0$.

(d) For each $i \in \{1,2\}$, the sequences $\{\frac{\partial U'_\Delta}{\partial x_i} - \tilde{U}^i_\Delta\}$ converges weakly to zero in $L_2(Q)$ as $h \to 0$.

(e) the sequence $\{\tilde{U}_\Delta - U'_\Delta\}$ converges strongly to 0 in $L_2(S)$ as $h \to 0$.

The proof of this theorem is similar to the proof in Theorem 14 of [14] by using (4.29).

Next, we recall the suitable version of the necessary and sufficient condition for the convergence of the discrete optimal control problems to the continuous optimal control problem formulated in the context of the optimal control problem $\mathscr{E}$.

Lemma 4.3.5. *[95] The sequence of discrete optimal control problems $\mathscr{E}_\Delta$ approximates the continuous optimal control problem $\mathscr{E}$ with respect to the functional if and only if the following conditions are satisfied:*

1. *For arbitrary sufficiently small $\varepsilon > 0$ there exists $\Delta_1 = \Delta_1(\varepsilon)$ such that $\mathscr{Q}_\Delta(v) \in \mathscr{F}^R_\Delta$ for all $v \in \mathscr{F}^{(R-\varepsilon)}$ and $\Delta \le \Delta_1$; Moreover, for any fixed $\varepsilon > 0$ and for all $v \in \mathscr{F}^{(R-\varepsilon)}$ the following inequality is satisfied:*

$$\limsup_{\Delta \to 0} (\mathscr{J}_\Delta(\mathscr{Q}_\Delta(v)) - \mathscr{J}(v)) \le 0.$$

2. *For arbitrary sufficiently small $\varepsilon > 0$ there exists $\Delta_2 = \Delta_2(\varepsilon)$ such that $\mathscr{P}_\Delta([v]_\Delta) \in \mathscr{F}^{(R+\varepsilon)}$ for all $[v]_\Delta \in \mathscr{F}^R_\Delta$ and $\Delta \le \Delta_2$; moreover, for all $[v]_\Delta \in \mathscr{F}^R_\Delta$, the following inequality is satisfied:*

$$\limsup_{\Delta \to 0} (\mathscr{J}(\mathscr{P}_\Delta([v]_\Delta)) - \mathscr{J}_\Delta([v]_\Delta)) \le 0.$$

3. *For arbitrary sufficiently small $\varepsilon > 0$, the following inequalities are satisfied:*

$$\limsup_{\varepsilon \to 0} \mathscr{J}_*(\varepsilon) \ge \mathscr{J}_*, \ \liminf_{\varepsilon \to 0} \mathscr{J}_*(-\varepsilon) \le \mathscr{J}_*,$$

where $\mathscr{J}_*(\pm\varepsilon) = \inf_{\mathscr{F}^{R\pm\varepsilon}} \mathscr{J}(v)$.

Now, our goal is to show $\mathscr{P}_\Delta$ and $\mathscr{Q}_\Delta$ satisfy the conditions of Lemma 4.3.5. The following lemma plays a key role to prove this claim. The proof is similar to the proof of Proposition in [14].

Lemma 4.3.6. *For $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{\mathscr{A}(Q_\Delta^{*+})} h^2 |\sigma_{\alpha x_1 x_2}|^2 \le (1+\varepsilon) \left\| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|^2_{L_2(Q_\Delta^*)}$$

whenever $h < \delta$.

Proof: For each $h > 0$, define the function $\tilde{\sigma}_h^{12}$ as

$$\tilde{\sigma}_h^{12}\Big|_{C_\Delta^\alpha} = \sigma_{\alpha x_1 x_2}, \quad \forall \alpha \in \mathscr{A}(Q_\Delta^{*+}) \tag{4.34}$$

where

$$\sigma_{\alpha x_1 x_2} = \frac{\sigma_{\alpha+e_2+e_1} - \sigma_{\alpha+e_2} - \sigma_{\alpha+e_1} + \sigma_\alpha}{h^2}$$

In the following we will prove that

$$\tilde{\sigma}_h^{12} \to \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \quad \text{strongly in } L_2(Q) \text{ as } h \to 0 \tag{4.35}$$

As an element of $\tilde{H}^1(Q)$, almost all restrictions of σ to lines parallel to the x_1 and x_2 direction are absolutely continuous, moreover, restrictions of σ_{x_1} to lines parallel to the x_2 direction are absolutely continuous and restrictions of σ_{x_2} to lines parallel to the x_1 direction are absolutely continuous. Hence if we let $z = (z_1, z_2)$ and $x = (x_1, x_2)$, then for almost every $z \in Q$ we have

$$\begin{aligned}
&\int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t)\, dy\, dt \\
&= \sigma(z_1+h, z_2+h) - \sigma(z_1+h, z_2) - \sigma(z_1, z_2+h) + \sigma(z_1, z_2) \\
&= \sigma(z+he_2+he_1) - \sigma(z+he_2) - \sigma(z+he_1) + \sigma(z)
\end{aligned} \tag{4.36}$$

In this lemma, for simplicity, instead of $\sum\limits_{\alpha \in \mathscr{A}(Q_\Delta^{*+})}$ we use $\sum\limits_{\mathscr{A}}$. Using the definition of

Steklov average (2.15) and Cauchy-Schwartz inequality, we get

$$\left\|\tilde{\sigma}_h^{12}-\frac{\partial^2\sigma}{\partial x_1\partial x_2}\right\|_{L_2(Q_\Delta^*)}^2=\int\limits_{Q_\Delta^*}\left|\tilde{\sigma}_h^{12}-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\right|^2dx=\sum_{\mathscr{A}}\int\limits_{C_\Delta^\alpha}\left|\sigma_{\alpha x_1x_2}-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\right|^2dx$$

$$=\sum_{\mathscr{A}}\int\limits_{C_\Delta^\alpha}\Big|\frac{1}{h^4}\Big[\int\limits_{C_\Delta^{\alpha+e_1+e_2}}dz-\int\limits_{C_\Delta^{\alpha+e_1}}dz-\int\limits_{C_\Delta^{\alpha+e_2}}dz+\int\limits_{C_\Delta^\alpha}dz\sigma(z)\Big]-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dx$$

$$=\sum_{\mathscr{A}}\int\limits_{C_\Delta^\alpha}\Big|\frac{1}{h^4}\int\limits_{C_\Delta^\alpha}[\sigma(z+he_1+he_2)-\sigma(z+he_2)-\sigma(z+he_1)+\sigma(z)]dz-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dx$$

$$=\sum_{\mathscr{A}}\int\limits_{C_\Delta^\alpha}\Big|\frac{1}{h^4}\int\limits_{C_\Delta^\alpha}[\int_{z_2}^{z_2+h}\int_{z_1}^{z_1+h}\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)\,dy\,dt]dz-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\frac{h^4}{h^4}\Big|^2dx$$

$$=\sum_{\mathscr{A}}\frac{1}{h^8}\int\limits_{C_\Delta^\alpha}\Big|\int\limits_{C_\Delta^\alpha}[\int_{z_2}^{z_2+h}\int_{z_1}^{z_1+h}\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)\,dy\,dt-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)h^2]dz\Big|^2dx$$

$$=\sum_{\mathscr{A}}\frac{1}{h^8}\int\limits_{C_\Delta^\alpha}\Big|\int\limits_{C_\Delta^\alpha}[\int_{z_2}^{z_2+h}\int_{z_1}^{z_1+h}\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\,dy\,dt]dz\Big|^2dx$$

$$\leq\sum_{\mathscr{A}}\frac{1}{h^6}\int\limits_{C_\Delta^\alpha}\int\limits_{C_\Delta^\alpha}\Big|\int_{z_2}^{z_2+h}\int_{z_1}^{z_1+h}\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\,dy\,dt\Big|^2dzdx$$

$$\leq\sum_{\mathscr{A}}\frac{1}{h^4}\int\limits_{C_\Delta^\alpha}\int\limits_{C_\Delta^\alpha}\Big[\int_{z_2}^{z_2+h}\int_{z_1}^{z_1+h}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dydt\Big]dzdx \tag{4.37}$$

Assume $m_\alpha=(m_1,m_2)$ be the natural corner of C_Δ^α. Now, we employ Fubini theorem to switch the order of integration with respect to y and z_1. Hence we observe

$$\sum_{\mathscr{A}} \frac{1}{h^4} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1}^{m_1+h} \Big[\int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dy dt \Big] dz_1 dz_2 \right) dx$$

$$= \sum_{\mathscr{A}} \frac{1}{h^4} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1}^{m_1+h} \Big[\int_{z_2}^{z_2+h} \int_{m_1}^{y} \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dz_1 dt \Big] dy dz_2 \right) dx$$

$$+ \sum_{\mathscr{A}} \frac{1}{h^4} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1+h}^{m_1+2h} \Big[\int_{z_2}^{z_2+h} \int_{y-h}^{m_1+h} \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dz_1 dt \Big] dy dz_2 \right) dx$$

$$= \sum_{\mathscr{A}} \frac{1}{h^4} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1}^{m_1+h} \Big[\int_{z_2}^{z_2+h} (y - m_1) \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dt \Big] dy dz_2 \right) dx +$$

$$\sum_{\mathscr{A}} \frac{1}{h^4} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1+h}^{m_1+2h} \Big[\int_{z_2}^{z_2+h} (m_1 + h - y + h) \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dt \Big] dy dz_2 \right) dx$$

$$\leq \sum_{\mathscr{A}} \frac{1}{h^3} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1}^{m_1+h} \Big[\int_{z_2}^{z_2+h} \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dt \Big] dy dz_2 \right) dx$$

$$+ \sum_{\mathscr{A}} \frac{1}{h^3} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1+h}^{m_1+2h} \Big[\int_{z_2}^{z_2+h} \Big| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \Big|^2 dt \Big] dy dz_2 \right) dx \quad (4.38)$$

We utilize Fubini theorem again to switch the order of integration with respect to t and

z_2. Hence we observe

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1}^{m_1+h}\Big[\int_{z_2}^{z_2+h}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dt\Big]dydz_2\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1+h}^{m_1+2h}\Big[\int_{z_2}^{z_2+h}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dt\Big]dydz_2\right)dx$$

$$=\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1}^{m_1+h}\Big[\int_{m_2}^{t}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2\Big]dydt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2+h}^{m_2+2h}\int_{m_1}^{m_1+h}\Big[\int_{t-h}^{m_2+h}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2\Big]dydt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1+h}^{m_1+2h}\Big[\int_{m_2}^{t}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2\Big]dydt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2+h}^{m_2+2h}\int_{m_1+h}^{m_1+2h}\Big[\int_{t-h}^{m_2+h}\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2\Big]dydt\right)dx$$

$$=\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1}^{m_1+h}(t-m_2)\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dydt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2+h}^{m_2+2h}\int_{m_1}^{m_1+h}(m_2+h-t+h)\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2dt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2}^{m_2+h}\int_{m_1+h}^{m_1+2h}(t-m_2)\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dydt\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C_\Delta^\alpha}\left(\int_{m_2+h}^{m_2+2h}\int_{m_1+h}^{m_1+2h}(m_2+h-t+h)\Big|\frac{\partial^2\sigma}{\partial x_1\partial x_2}(y,t)-\frac{\partial^2\sigma}{\partial x_1\partial x_2}(x)\Big|^2dz_2dt\right)dx$$

$$\leq \sum_{\mathscr{A}} \frac{1}{h^2} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1}^{m_1+h} \left| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \right|^2 dy dt \right) dx$$

$$+ \sum_{\mathscr{A}} \frac{1}{h^2} \int_{C_\Delta^\alpha} \left(\int_{m_2+h}^{m_2+2h} \int_{m_1}^{m_1+h} \left| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \right|^2 dy dt \right) dx$$

$$+ \sum_{\mathscr{A}} \frac{1}{h^2} \int_{C_\Delta^\alpha} \left(\int_{m_2}^{m_2+h} \int_{m_1+h}^{m_1+2h} \left| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \right|^2 dy dt \right) dx$$

$$+ \sum_{\mathscr{A}} \frac{1}{h^2} \int_{C_\Delta^\alpha} \left(\int_{m_2+h}^{m_2+2h} \int_{m_1+h}^{m_1+2h} \left| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(y,t) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \right|^2 dy dt \right) dx \tag{4.39}$$

without loss of generality, we replace (y,t) with (z_1,z_2) and we get

$$\left\| \tilde{\sigma}_h^{12} - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L_2(Q_\Delta^*)}^2 \leq \sum_{\mathscr{A}} \frac{1}{h^2} \int_{C_\Delta^\alpha} \left(\int_{C_\Delta^{\alpha+e_1+e_2}} dz \right.$$

$$\left. + \int_{C_\Delta^{\alpha+e_1}} dz + \int_{C_\Delta^{\alpha+e_2}} dz + \int_{C_\Delta^\alpha} dz \left| \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(z_1,z_2) - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2}(x) \right|^2 \right) dx \tag{4.40}$$

For fixed $\varepsilon > 0$, we pick a $g \in C^2(Q + B_1(0))$ such that

$$\|\sigma - g\|_{\tilde{H}^1(Q+B_1(0))}^2 \leq c(\varepsilon) \tag{4.41}$$

Now, we add and subtract $\frac{\partial^2 g(z)}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 g(x)}{\partial x_1 \partial x_2}$ to the integrands of (4.40)

$$\left\| \tilde{\sigma}_h^{12} - \frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right\|_{L_2(Q_\Delta^*)}^2 \leq I_1 + I_2 + I_3 \tag{4.42}$$

where

$$I_1 = \sum_{\mathscr{A}} \frac{3}{h^2} \int_{C_\Delta^\alpha} \left(\int_{C_\Delta^{\alpha+he_1+he_2}} dz + \int_{C_\Delta^{\alpha+he_1}} dz + \int_{C_\Delta^{\alpha+he_2}} dz + \int_{C_\Delta^\alpha} dz \left| \frac{\partial^2 \sigma(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(z)}{\partial x_1 \partial x_2} \right|^2 \right) dx$$

$$I_2 = \sum_{\mathscr{A}} \frac{3}{h^2} \int_{C_\Delta^\alpha} \left(\int_{C_\Delta^{\alpha+he_1+he_2}} dz + \int_{C_\Delta^{\alpha+he_1}} dz + \int_{C_\Delta^{\alpha+he_2}} dz + \int_{C_\Delta^\alpha} dz \left| \frac{\partial^2 g(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(x)}{\partial x_1 \partial x_2} \right|^2 \right) dx$$

$$I_3 = \sum_{\mathscr{A}} \frac{3}{h^2} \int_{C_\Delta^\alpha} \left(\int_{C_\Delta^{\alpha+he_1+he_2}} dz + \int_{C_\Delta^{\alpha+he_1}} dz + \int_{C_\Delta^{\alpha+he_2}} dz + \int_{C_\Delta^\alpha} dz \left| \frac{\partial^2 g(x)}{\partial x_1 \partial x_2} - \frac{\partial^2 \sigma(x)}{\partial x_1 \partial x_2} \right|^2 \right) dx$$

Since $g \in C^2(Q + B_1(0))$, it follows that $\frac{\partial^2 g}{\partial x_1 \partial x_2}$ is uniformly continuous on $Q + B_1(0)$. Therefore, there exists $\delta = \delta(g, \varepsilon) > 0$ such that

$$\left|\frac{\partial^2 g(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(x)}{\partial x_1 \partial x_2}\right|^2 \leq c(\varepsilon) \tag{4.43}$$

whenever $|z - x| < \delta$. Let $h_\varepsilon > 0$ satisfy

$$\sqrt{8}\, h_\varepsilon < \delta \tag{4.44}$$

Then it follows that for each $h < h_\varepsilon$, any $\alpha \in \mathscr{A}$, and any $x, z \in C_\Delta^{\alpha+he_1+he_2} \cup C_\Delta^{\alpha+he_1} \cup C_\Delta^{\alpha+he_2} \cup C_\Delta^\alpha$,

$$\left|\frac{\partial^2 g(z)}{\partial x_1 \partial x_2} - \frac{\partial^2 g(x)}{\partial x_1 \partial x_2}\right|^2 \leq c(\varepsilon). \tag{4.45}$$

Therefore,

$$I_1 = \frac{3}{h^2}\sum_{\mathscr{A}} h^2\Bigg(\int\limits_{C_\Delta^{\alpha+he_1+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_1}} dz + \int\limits_{C_\Delta^{\alpha+he_2}} dz + \int\limits_{C_\Delta^{\alpha}} dz\Big|\frac{\partial^2\sigma(z)}{\partial x_1\partial x_2} - \frac{\partial^2 g(z)}{\partial x_1\partial x_2}\Big|^2\Bigg)$$

$$\leq 12 \int\limits_{Q+B_1(0)} \Big|\frac{\partial^2\sigma(z)}{\partial x_1\partial x_2} - \frac{\partial^2 g(z)}{\partial x_1\partial x_2}\Big|^2 dz \leq 12\|\sigma - g\|^2_{\tilde{H}^1(Q+B_1(0))}$$

$$I_2 \leq \sum_{\mathscr{A}} \frac{3}{h^2}\int\limits_{C_\Delta^{\alpha}} \Bigg(\int\limits_{C_\Delta^{\alpha+he_1+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_1}} dz + \int\limits_{C_\Delta^{\alpha+he_2}} dz + \int\limits_{C_\Delta^{\alpha}} dz\, c(\varepsilon)\Bigg)dx \leq 12c(\varepsilon)m(Q_\Delta^*)$$

$$I_3 = 12\int_{Q_\Delta^*}\Big|\frac{\partial^2 g(x)}{\partial x_1\partial x_2} - \frac{\partial^2\sigma(x)}{\partial x_1\partial x_2}\Big|^2 dx \leq 12\|\sigma - g\|^2_{\tilde{H}^1(Q_\Delta^*)}$$

If we take $c(\varepsilon) = \frac{\varepsilon}{24+12m(Q_\Delta^*)}$, these calculations imply that

$$I_1 + I_2 + I_3 < \varepsilon, \quad \forall h \leq h_\varepsilon \tag{4.46}$$

This proves the strong convergence of $\tilde{\sigma}_h^{12}$ to $\frac{\partial^2\sigma}{\partial x_1\partial x_2}$ in $L_2(Q_\Delta^*)$, and strong convergence implies the claim of the lemma. Lemma is proved. □

Proposition 4.3.7. *For arbitrary sufficiently small $\varepsilon > 0$ there exists h_ε such that*

$$\mathscr{Q}_\Delta(v) \in \mathscr{F}_\Delta^R \quad \textit{for all} \quad v \in \mathscr{F}^{(R-\varepsilon)} \quad \textit{and} \quad h \leq h_\varepsilon \tag{4.47}$$

$$\mathscr{P}_\Delta([v]_\Delta) \in \mathscr{F}^{(R+\varepsilon)} \quad \textit{for all} \quad [v]_\Delta \in \mathscr{F}_\Delta^R \quad \textit{and} \quad h \leq h_\varepsilon \tag{4.48}$$

Proof. Let $0 < \varepsilon << R$ and Δ arbitrary. First let $\sigma \in \mathscr{F}^{(R-\varepsilon)}$. Then we note

$$\|\mathscr{Q}_\Delta(\sigma)\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)} = \sum_{\mathscr{A}(Q^*_\Delta)} h^2\sigma^2_\alpha + \sum_{i=1}^{2} \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2\sigma^2_{\alpha x_i} + \sum_{\mathscr{A}(Q^{*+}_\Delta)} h^2\sigma^2_{\alpha x_1 x_2} = Q_1 + Q_2 + Q_3$$

where

$$Q_1 = h^2 \sum_{\mathscr{A}(Q^*_\Delta)} \sigma^2_\alpha = h^2 \sum_{\mathscr{A}(Q^*_\Delta)} \left(\frac{1}{h^2}\int_{C^\alpha_\Delta} \sigma\, dx\right)^2 \leq \sum_{\mathscr{A}(Q^*_\Delta)} \int_{C^\alpha_\Delta} \sigma^2\, dx \leq \int_{Q+B_1(0)} \sigma^2\, dx$$

for Q_2 and Q_3, referring to the Proposition 11 in [14] and Lemma 4.3.6 respectively, we deduce that for any $0 < \varepsilon_1 < \left(\frac{R}{R-\varepsilon}\right)^2 - 1$ there exists a positive δ such that for $h \leq \delta$ we have

$$Q_1 = h^2 \sum_{\mathscr{A}(Q^*_\Delta)} \sigma^2_\alpha \leq (1+\varepsilon_1)||\sigma||^2_{L_2(Q^*_\Delta)}. \tag{4.49}$$

$$Q_2 = \sum_{i=1}^{2} \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2\sigma^2_{\alpha x_i} \leq (1+\varepsilon_1)\|D\sigma\|^2_{L_2(Q^*_\Delta)}$$

$$Q_3 = \sum_{\mathscr{A}(Q^{*+}_\Delta)} h^2\sigma^2_{\alpha x_1 x_2} \leq (1+\varepsilon_1)\left\|\frac{\partial^2\sigma}{\partial x_1 \partial x_2}\right\|^2_{L_2(Q^*_\Delta)}$$

Then, for small enough $\varepsilon_1 > 0$ we have

$$\begin{aligned}\|\mathscr{Q}_\Delta(\sigma)\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)} &\leq (1+\varepsilon_1)\|\sigma\|^2_{L_2(Q)} + (1+\varepsilon_1)\|D\sigma\|^2_{L_2(Q)} + (1+\varepsilon_1)\|\sigma_{x_1x_2}\|^2_{L_2(Q)} \\ &\leq (1+\varepsilon_1)(R-\varepsilon)^2 \leq R^2\end{aligned}$$

In addition, $\max_{\alpha \in \mathscr{A}(Q^*_\Delta)} |\sigma_\alpha| \leq R$ is automatically correct since $\sigma \in \mathscr{F}^{(R-\varepsilon)}$.

Now let $[\sigma]_\Delta \in \mathscr{F}_\Delta^R$ which implies

$$\max\Big\{\max_{\alpha\in\mathscr{A}(Q_\Delta^*)}|\sigma_\alpha| + \sum_{\mathscr{A}(Q_\Delta^*)} h^2\sigma_\alpha^2 + \sum_{i=1}^{2}\sum_{\mathscr{A}(Q_\Delta^{*(i)})} h^2\sigma_{\alpha x_i}^2 + \sum_{\mathscr{A}(Q_\Delta^{*+})} h^2\sigma_{\alpha x_1 x_2}^2\Big\} \le R \tag{4.50}$$

we claim that $\sigma^\Delta := \mathscr{P}_\Delta([\sigma]_\Delta) \in \mathscr{F}^R$. where

$$\sigma^\Delta(x) = \sigma_\alpha + \sigma_{\alpha x_1}(x_1 - k_1 h) + \sigma_{\alpha x_2}(x_2 - k_2 h) + \sigma_{\alpha x_1 x_2}(x_1 - k_1 h)(x_2 - k_2 h),\ \forall x \in C_\Delta^\alpha \tag{4.51}$$

In order to prove this claim we first prove

$$\|\sigma^\Delta\|_{L_\infty(Q_\Delta^*)} \le R + \varepsilon \tag{4.52}$$

and then we need to prove

$$\begin{gathered}\|\sigma^\Delta\|^2_{L_2(Q_\Delta^*)} + \|D\sigma^\Delta\|^2_{L_2(Q_\Delta^*)} + \|\sigma^\Delta_{x_1x_2}\|^2_{L_2(Q_\Delta^*)} \\ = \sum_{\mathscr{A}(Q_\Delta^*)} \int_{C_\Delta^\alpha} \big[(\sigma^\Delta)^2 + (\sigma^\Delta(x))^2_{x_1} + (\sigma^\Delta(x))^2_{x_2} + (\sigma^\Delta(x))^2_{x_1x_2}\big] dx_1 dx_2 \le (R+\varepsilon)^2\end{gathered} \tag{4.53}$$

The proof of (4.52) is obvious, since the interpolation σ^Δ is multilinear and it takes its maximum on one of the corners of the cell; therefore, considering the fact that $[\sigma]_\Delta \in \mathscr{F}_\Delta^R$, claim (4.52) is proved .
We prove (4.53) directly by evaluating the L_2 norm of σ^Δ, $D\sigma^\Delta$ and $\sigma^\Delta_{x_1x_2}$ over a fixed cell C_Δ^α, $\alpha \in \mathscr{A}(Q_\Delta^*)$.

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h} (\sigma^\Delta(x))^2_{x_1x_2} dx_1 dx_2 = \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h} |\sigma_{\alpha x_1 x_2}|^2 dx_2 dx_1 = h^2\sigma^2_{\alpha x_1 x_2} \tag{4.54}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}(\sigma^{\Delta}(x))^2_{x_1}dx_2dx_1 = \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}|\sigma_{\alpha x_1}+\sigma_{\alpha x_1x_2}(x_2-k_2h)|^2dx_2dx_1$$

$$= \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}|\sigma_{\alpha x_1}|^2+|\sigma_{\alpha x_1x_2}(x_2-k_2h)|^2+|2\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2}(x_2-k_2h)|dx_2dx_1$$

$$= h^2\sigma^2_{\alpha x_1}+\frac{h^4}{3}\sigma^2_{\alpha x_1x_2}+h^3\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2} \tag{4.55}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}(\sigma^{\Delta}(x))^2_{x_2}dx_2dx_1 = h^2\sigma^2_{\alpha x_2}+\frac{h^4}{3}\sigma^2_{\alpha x_1x_2}+h^3\sigma_{\alpha x_2}\sigma_{\alpha x_1x_2} \tag{4.56}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}(\sigma^{\Delta}(x))^2dx_2dx_1 = h^2\sigma^2_{\alpha}+\frac{h^4}{3}\sigma^2_{\alpha x_1}+\frac{h^4}{3}\sigma^2_{\alpha x_2}+\frac{h^6}{9}\sigma^2_{\alpha x_1x_2}+h^3\sigma_{\alpha}\sigma_{\alpha x_1}$$
$$+h^3\sigma_{\alpha}\sigma_{\alpha x_2}+\frac{h^4}{2}\sigma_{\alpha}\sigma_{\alpha x_1x_2}+\frac{h^4}{2}\sigma_{\alpha x_1}\sigma_{\alpha x_2}+\frac{h^5}{3}\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2}+\frac{h^5}{3}\sigma_{\alpha x_2}\sigma_{\alpha x_1x_2} \tag{4.57}$$

except the first terms of each evaluation, the rest of the terms are higher order terms, since $[\sigma]_{\Delta}\in\mathscr{F}^R_{\Delta}$. we show this fact for all the terms in the last integral:

$$|h^3\sigma_{\alpha}\sigma_{\alpha x_1}|\le\frac{h}{2}(h^2\sigma^2_{\alpha}+h^2\sigma^2_{\alpha x_1})$$

$$|\frac{h^4}{3}\sigma^2_{\alpha x_1}|=\frac{h^2}{3}(h^2\sigma^2_{\alpha x_1})\le\frac{h^2}{3}R^2$$

$$|\frac{h^6}{9}\sigma^2_{\alpha x_1 x_2}| = \frac{h^4}{9}(h^2\sigma^2_{\alpha x_1 x_2}) \leq \frac{h^4}{9}R^2$$

$$|\frac{h^4}{2}\sigma_\alpha \sigma_{\alpha x_1 x_2}| \leq \frac{h^2}{4}(h^2\sigma^2_\alpha + h^2\sigma^2_{\alpha x_1 x_2}) \leq \frac{h^2}{2}R^2$$

$$|\frac{h^5}{3}\sigma_{\alpha x_2}\sigma_{\alpha x_1 x_2}| \leq \frac{h^3}{6}(h^2\sigma^2_{\alpha x_2} + h^2\sigma^2_{\alpha x_1 x_2}) \leq \frac{h^2}{2}R^2$$

So, we have

$$\|\mathscr{P}_\Delta([\sigma]_\Delta)\|^2_{\tilde{H}^1(Q^*_\Delta)} = \|\sigma^\Delta\|^2_{\tilde{H}^1(Q^*_\Delta)} \leq \|[\sigma]_\Delta\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)} + O(h)$$

which proves that $\mathscr{P}_\Delta([v]_\Delta) \in \mathscr{F}^{(R+\varepsilon)}$.□

4.4 Approximation Theorem

Theorem 4.4.1. *Let $\{[v]_\Delta\} = \{([\sigma]_\Delta, U)\}$ be a sequence of discrete control vectors such that there exists $R > 0$ for which $[v]_\Delta \in \mathscr{F}^R_\Delta$ for each Δ, and such that the sequence of interpolations $\{\mathscr{P}_\Delta([\sigma]_\Delta)\}$ converges weakly to some σ in $\tilde{H}^1(Q)$ (strongly in $L_2(Q)$ and $L_2(S)$). Then the sequence of interpolations $\{U'_\Delta\}$ of associated discrete state vectors converges weakly in $H^1(Q)$ to $u = u(x;v) \in H^1(Q)$, with u the unique weak solution to the* (4.1)–(4.3).

Proof. Proof follows the method of the similar result proved in [14]. From (a) and (b) of Lemma 4.3.4, it follows that $\{U'_\Delta\}$ is uniformly bounded in $H^1(Q)$. Consequently, $\{U'_\Delta\}$ has a weak limit point in $H^1(Q)$. Let $u \in H^1(Q)$ be any weak limit point of $\{U'_\Delta\}$ in $H^1(Q)$. By the Rellich-Kondrachev Theorem, it is known that a subsequence of $\{U'_\Delta\}$

converges strongly to u in $L_2(Q)$. In addition, $\{U'_\Delta\}$ converges to u on the boundary S in $L_2(S)$ norm. Now, we proceed to show that u satisfies the integral identity (4.1.1). For simplicity of notation we write the subsequence of $\{U'_\Delta\}$ that converges weakly to u in $H^1(Q)$ as the whole sequence Δ. Let $\eta \in \mathscr{C}^1(\tilde{Q})$, where $\bar{Q} \subset \tilde{Q}$ and $\mathscr{C}^1(\tilde{Q})$ be a space of all continuously differentiable functions on $\tilde{Q}$. We also assume that $h > 0$ is small enough that $Q^*_\Delta \subset \tilde{Q}$. Then the collection of values $\{\eta_\alpha\}$, $\alpha \in \mathscr{A}$ is an admissible test collection for the summation identity (4.10). We claim that the limit function ,u, satisfies the integral identity (4.7). Let call the discrete integral identity (4.10) as I_Δ and the continuous integral identity (4.7) as I.

$$I_\Delta = h^2 \sum_{\mathscr{A}(Q^{*+}_\Delta)} \sigma^\Delta_\alpha \sum_{i=1}^{2} u_{\alpha x_i}\eta_{\alpha x_i} + \sum_{l=1}^{m} \frac{1}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u_\alpha \eta_\alpha + J_\alpha(u_\alpha, \eta_\alpha) - \sum_{l=1}^{m} \frac{U_l}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha}\eta_\alpha \tag{4.58}$$

$$I := \int_Q \sigma \nabla u \cdot \nabla \eta \, dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} u\eta \, ds - \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l \, ds$$

We define the interpolations for η_α and $\eta_{\alpha x_i}$ for each $\alpha \in \mathscr{A}(Q^{*+}_\Delta)$ as following

$$\bar{\eta}_\Delta \Big|_{C^\alpha_\Delta} = \eta_\alpha, \quad \forall \alpha \in \mathscr{A}(Q^{*+}_\Delta)$$

$$\bar{\eta}^i_\Delta \Big|_{C^\alpha_\Delta} = \eta_{\alpha x_i}, \quad \forall \alpha \in \mathscr{A}(Q^{*+}_\Delta)$$

Using these interpolations and the ones described in the interpolation Section 4.3 and definition steklov average in 2.15.

$$I_\Delta := \sum_{\mathscr{A}(Q_\Delta^{*+})} \int_{C_\Delta^\alpha} \left[\sigma^\Delta \sum_{i=1}^{2} \tilde{U}_\Delta^i \bar{\eta}_\Delta^i \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \tilde{U}_\Delta \bar{\eta}_\Delta ds + J_\alpha(u_\alpha, \eta_\alpha) - \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} \bar{\eta}_\Delta ds$$
$$= \int_{Q_\Delta^*} \left[\sigma^\Delta \sum_{i=1}^{2} \tilde{U}_\Delta^i \bar{\eta}_\Delta^i \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \tilde{U}_\Delta \bar{\eta}_\Delta ds + J_\alpha(u_\alpha, \eta_\alpha) - \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} \bar{\eta}_\Delta ds$$

Adding and subtracting some terms to I_Δ, we obtain the following identity :

$$I_\Delta = I + \sum_{i=1}^{5} R_i$$

where

$$R_1 = \int_{Q_\Delta^* \setminus Q} \left[\sigma^\Delta \sum_{i=1}^{2} \tilde{U}_\Delta^i \bar{\eta}_\Delta^i \right] dx \tag{4.59}$$

$$R_2 = J_\alpha(u_\alpha, \eta_\alpha) = h^2 \sum_{\mathscr{A}(S_\Delta^*)} \sum_{i=1}^{2} \theta_\alpha^i u_{\alpha x_i} \eta_{\alpha x_i} \tag{4.60}$$

$$R_3 = \int_Q \left[\sigma^\Delta \sum_{i=1}^{2} \tilde{U}_\Delta^i (\bar{\eta}_\Delta^i - \eta_{x_i}) \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \tilde{U}_\Delta (\bar{\eta}_\Delta - \eta) ds - \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} (\bar{\eta}_\Delta - \eta) ds \tag{4.61}$$

$$R_4 = \int_Q (\sigma^\Delta - \sigma) \sum_{i=1}^{2} \tilde{U}_\Delta^i \eta_{x_i} dx \tag{4.62}$$

$$R_5 = \int_Q \left[\sigma \sum_{i=1}^{2} (\tilde{U}_\Delta^i - u_{x_i}) \eta_{x_i} \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} (\tilde{U}_\Delta - u) \eta ds \tag{4.63}$$

We claim that by passing to the limit when $\Delta \to 0$, $I_\Delta \to I$ and $R_i \to 0$ for $i = 1, \ldots, 5$. Using Cauchy Schwartz inequality and extending the region of integration for function

$\tilde{U}^i_\Delta$ from $(Q^*_\Delta \setminus Q)$ to Q^*_Δ we obtain the following estimate for R_1:

$$|R_1| \leq C_1 \sum_{i=1}^{2} \|\tilde{U}^i_\Delta\|_{L_2(Q^*_\Delta)} \|\bar{\eta}^i_\Delta\|_{L_2(Q^*_\Delta \setminus Q)}$$

Lemma 4.3.4 (b), and Proposition 4.3.7 implies that

$$|R_1| \leq C_2 \sum_{i=1}^{2} \|\bar{\eta}^i_\Delta\|_{L_2(Q^*_\Delta \setminus Q)}$$

interpolation $\bar{\eta}^i_\Delta$ converges uniformly on $\bar{Q}$ to the function η_{x_i} as $\Delta \to 0$ and since $\eta \in \mathscr{C}^1(\tilde{Q})$ and $|Q^*_\Delta \setminus Q| \to 0$ and we have

$$|R_1| \to 0, \quad \text{as } \Delta \to 0$$

Now we try to show that R_2 is small.

$$\begin{aligned}
|R_2| &= |h^2 \sum_{\mathscr{A}(S^*_\Delta)} \sum_{i=1}^{2} \theta^i_\alpha u_{\alpha x_i} \eta_{\alpha x_i}| \\
&\leq (\sum_{\mathscr{A}(S^*_\Delta)} \sum_{i=1}^{2} h^2 \theta^i_\alpha u^2_{\alpha x_i})^{\frac{1}{2}} (\sum_{\mathscr{A}(S^*_\Delta)} \sum_{i=1}^{2} h^2 \theta^i_\alpha \eta^2_{\alpha x_i})^{\frac{1}{2}} \\
&\leq (\sum_{i=1}^{2} \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2 u^2_{\alpha x_i})^{\frac{1}{2}} \|\eta\|_{C^1} (\sum_{i=1}^{2} \sum_{\mathscr{A}(S^*_\Delta)} h^2)^{\frac{1}{2}} \\
&\leq (\sum_{i=1}^{2} \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2 u^2_{\alpha x_i})^{\frac{1}{2}} \|\eta\|_{C^1} \sqrt{2} (\sum_{\mathscr{A}(S^*_\Delta)} h^2)^{\frac{1}{2}} \\
&\leq (\sum_{i=1}^{2} \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2 u^2_{\alpha x_i})^{\frac{1}{2}} \|\eta\|_{C^1} \sqrt{2h} (2|S|)^{\frac{1}{2}} \to 0
\end{aligned}$$

Sum with respect to all grid points of S^*_Δ is bounded by the Lebesgue measure of S^*_Δ. Since S is Lipschitz, the latter converges to Lebesgue measure of S as $h \to 0$; This imply that for sufficiently small h, it will be bounded by $2|S|$. The same argument that we used for R_1 implies that $R_2 \to 0$ as $\Delta \to 0$.

Using Cauchy Schwartz inequality and Lemma (4.3.4) (a) and (b) we get the following estimation for R_3:

$$
\begin{aligned}
|R_3| = &\left| \int_Q \left[\sigma_\Delta \sum_{i=1}^{2} \tilde{U}_\Delta^i (\bar{\eta}_\Delta^i - \eta_{x_i}) \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \tilde{U}_\Delta (\bar{\eta}_\Delta - \eta) ds - \sum_{l=1}^{m} \frac{U_l}{Z_l} \int_{E_l} (\bar{\eta}_\Delta - \eta) ds \right| \\
&\leq N_1 \, \|\sigma_\Delta\|_{L^\infty(Q)} \sum_{i=1}^{2} \|\tilde{U}_\Delta^i\|_{L_2(Q_\Delta^*)} \, \|\bar{\eta}_\Delta^i - \eta_{x_i}\|_{L_2(Q)} \\
&+ N_2 \max_{1 \leq l \leq m} \left| \frac{1}{Z_l} \right| \, \|\bar{\eta}_\Delta - \eta\|_{L_2(S)} \left(\|\tilde{U}_\Delta - U_\Delta'\|_{L_2(S)} + \|U_\Delta' - u\|_{L_2(S)} \right) \\
&+ N_3 \max_{1 \leq l \leq m} \left| \frac{U_l}{Z_l} \right| \, \|\bar{\eta}_\Delta - \eta\|_{L_2(S)}
\end{aligned}
$$

It can be easily proved that interpolations $\bar{\eta}_\Delta$ and $\bar{\eta}_\Delta^i$ converge uniformly on $\bar{Q}$ to the functions η and η_{x_i} on Q and S as $\Delta \to 0$, so $R_3 \to 0$. Using Cauchy Schwartz inequality and Lemma (4.3.4) (a) and (b) and the fact that $\eta \in \mathscr{C}^1(\tilde{Q})$ we get the following estimation for R_4:

$$
|R_4| = \left| \int_Q (\sigma^\Delta - \sigma) \sum_{i=1}^{2} \tilde{U}_\Delta^i \eta_{x_i} dx \right| \leq H \sum_{i=1}^{2} \|\tilde{U}_\Delta^j\|_{L_2(Q_\Delta^*)} \|\sigma^\Delta - \sigma\|_{L_2(Q)} \leq H_1 \|\sigma^\Delta - \sigma\|_{L_2(Q)}
$$

R_4 goes to zero because of the theorem assumption.

By adding and subtracting U_Δ' and $\frac{\partial U_\Delta'}{\partial x_j}$, we calculate the following estimate:

$$
R_5 = \int_Q \left[\sigma \sum_{i=1}^{2} (\tilde{U}_\Delta^i - u_{x_i}) \eta_{x_i} \right] dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} (\tilde{U}_\Delta - u) \eta \, ds = R_{51} + R_{52}
$$

$$R_{51} = \int_Q \Big[\sigma \sum_{i=1}^{2} (\tilde{U}^i_\Delta - \frac{\partial U'_\Delta}{\partial x_i})\eta_{x_i} + (\frac{\partial U'_\Delta}{\partial x_i} - u_{x_i})\eta_{x_i}\Big] dx$$

R_{51} converges to zero since $\{\frac{\partial U'_\Delta}{\partial x_i} - \tilde{U}^i_\Delta\}$ converges weakly to zero (Lemma (4.3.4)(d)) and $\{U'_\Delta\}$ converges weakly to u in $H^1(Q)$.

$$|R_{52}| = \Big|\sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \big[(\tilde{U}_\Delta - U'_\Delta)\eta + (U'_\Delta - u)\eta\big] ds\Big|$$
$$\leq K_1 \|\tilde{U}_\Delta - U'_\Delta\|_{L_2(S)} + K_2 \|U'_\Delta - u\|_{L_2(S)}$$

Lemma (4.3.4) (e) implies that $R_5 \to 0$.

Finally, since $\mathscr{C}^1(Q)$ is dense in set of admissible test functions for integral identity (4.7) we have that u is a weak solution to the Problem (4.1)–(4.3) in the sense of Definition 4.1.1. Therefore, we have proved that if u is a weak limit point of $\{U'_\Delta\}$ then it must be a weak solution to the Problem (4.1)–(4.3). Due to uniqueness of the weak solution it follows that $\{U'_\Delta\}$ has one and only one weak limit point, which shows that the whole sequence $\{U'_\Delta\}$ converges weakly to u in $H^1(Q)$. This ends the proof of the theorem.□

4.5 Convergence of the Discrete Optimal Control Problem

Existence of the optimal control in Problem $\mathscr{E}$ is proved in Theorem 3.3.4. In particular, from the proof of Theorem 3.3.4 it follows that the functional $\mathscr{J}$ is weakly continuous.

Proof of Theorem 4.2.1. To prove 4.13 and 4.16, it is enough to show that conditions (1) and (2) of Lemma 4.3.5 are satisfied.

Step 1. In this step we show that for any $v \in \mathscr{F}^{(R-\varepsilon)}$,

$$\lim_{\Delta \to 0} |\mathscr{J}_\Delta(\mathscr{Q}_\Delta(v)) - \mathscr{J}(v)| = 0 \tag{4.64}$$

From the Proposition 4.3.7 it follows that $\mathscr{Q}_\Delta(\sigma) = [\sigma]_\Delta \in \mathscr{F}_\Delta^R$. Applying Proposition 4.3.7 again, we deduce that $\mathscr{P}_\Delta([\sigma]_\Delta)$ belong to $\mathscr{F}^{(R+\varepsilon)}$, and hence,

$$\|\mathscr{P}_\Delta([\sigma]_\Delta)\|_{\tilde{H}^1(Q_\Delta^*)} \leq R + \varepsilon$$

Therefore, there exists a $\sigma_0 \in \tilde{H}^1(Q)$ and a subsequence of $\mathscr{P}_\Delta([\sigma]_\Delta)$ converging weakly to σ_0 in $\tilde{H}^1(Q)$. Without loss of generality we can assume that the whole sequence $\mathscr{P}_\Delta([\sigma]_\Delta)$ is weakly convergent to σ_0 in $\tilde{H}^1(Q)$. By using compact embedding theorems, we therefore have

$$\begin{aligned}
&\mathscr{P}_\Delta([\sigma]_\Delta) \rightharpoonup \sigma_0, \text{ weakly in } \tilde{H}^1(Q) \\
&\mathscr{P}_\Delta([\sigma]_\Delta) \to \sigma_0, \text{ strongly in } L^2(Q) \\
&\mathscr{P}_\Delta([\sigma]_\Delta) \to \sigma_0, \text{ strongly in } L^2(S)
\end{aligned} \tag{4.65}$$

On the other side, we know that the piecewise constant interpolation of $[\sigma]_\Delta$ converges strongly to σ. From Lemma 4.3.4 part (c), it follows that $\sigma = \sigma_0$ almost everywhere on Q. Therefore, we have

$$\mathscr{P}_\Delta([\sigma]_\Delta) \rightharpoonup \sigma, \text{ in } \tilde{H}^1(Q)$$

By applying approximation Theorem 4.4.1 it follows that the interpolations $\{U'_\Delta\}$ of the discrete state vectors $[u([v]_\Delta)]_\Delta$ converge weakly in $H^1(Q)$, and strongly in $L_2(Q)$ and $L_2(S)$ to the unique weak solution $u = u(x;v)$ of the PDE problem with control v.

We transform the first term in the discrete cost functional (4.12) as folllows:

$$\begin{aligned}
\sum_l \Big(\sum_{\alpha \in \mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} \frac{U_l - u_\alpha}{Z_l} - I_l \Big)^2 &= \sum_l \Big(\int_{E_l} \frac{U_l - \tilde{U}_\Delta}{Z_l} ds - I_l \Big)^2 \\
&= \sum_l \Big(\int_{E_l} \frac{U_l - \tilde{U}_\Delta + U'_\Delta - U'_\Delta + u - u}{Z_l} ds - I_l \Big)^2 \\
&= \sum_l \Big(\int_{E_l} \frac{U'_\Delta - \tilde{U}_\Delta}{Z_l} ds + \frac{u - U'_\Delta}{Z_l} ds + \frac{U_l - u}{Z_l} ds - I_l \Big)^2 \\
&= \sum_l \Big(\int_{E_l} \frac{U'_\Delta - \tilde{U}_\Delta}{Z_l} ds \Big)^2 + \Big(\int_{E_l} \frac{u - U'_\Delta}{Z_l} ds \Big)^2 + \Big(\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l \Big)^2 \\
&\quad + 2 \Big(\int_{E_l} \frac{U'_\Delta - \tilde{U}_\Delta}{Z_l} ds \Big) \Big(\int_{E_l} \frac{u - U'_\Delta}{Z_l} ds \Big) \\
&\quad + 2 \Big(\int_{E_l} \frac{U'_\Delta - \tilde{U}_\Delta}{Z_l} ds \Big) \Big(\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l \Big) \\
&\quad + 2 \Big(\int_{E_l} \frac{u - U'_\Delta}{Z_l} ds \Big) \Big(\int_{E_l} \frac{U_l - u}{Z_l} ds - I_l \Big)
\end{aligned} \tag{4.66}$$

Since $\{U'_\Delta\}$ converge strongly in $L^2(S)$ to $u = u(x:v)$, we have

$$\int_{E_l} \frac{u - U'_\Delta}{Z_l} ds \leq |S| \bigg(\int_S (\frac{u - U'_\Delta}{Z_l})^2 ds \bigg)^{\frac{1}{2}} \to 0, \quad \text{as } \Delta \to 0 \tag{4.67}$$

By part (e) of Lemma 4.3.4, it follows that

$$\int_{E_l} \frac{U'_\Delta - \tilde{U}_\Delta}{Z_l} ds \leq |S| \bigg(\int_S (\frac{U'_\Delta - \tilde{U}_\Delta}{Z_l})^2 ds \bigg)^{\frac{1}{2}} \to 0, \quad \text{as } \Delta \to 0 \tag{4.68}$$

By (4.67) and (4.68), it follows that

$$\lim_{\Delta \to 0} (\mathscr{J}_\Delta(\mathscr{Q}_\Delta(v)) - \mathscr{J}(v)) = 0.$$

which completes the proof of the ***Step 1***.

Step 2. In this step we show that for any sequence $\{[v]_\Delta\}$ such that $[v]_\Delta \in \mathscr{F}^R_\Delta$, we have

$$\lim_{\Delta\to 0} |\mathscr{J}(\mathscr{P}_\Delta([v]_\Delta)) - \mathscr{J}_\Delta([v]_\Delta))| = 0 \tag{4.69}$$

Proposition 4.3.7 implies that the sequence $\mathscr{P}_\Delta([v]_\Delta)$ is uniformly bounded in $\tilde{H}^1(Q)$. Hence, it has a a subsequence converging weakly in $\tilde{H}^1(Q)$ and strongly in $L_2(Q)$ and $L_2(S)$ to some $\bar{v} = (\bar{\sigma}, U) \in \mathscr{F}^R$. Without loss of generality we can assume that the whole sequence

$$\mathscr{P}_\Delta([v]_\Delta) \rightharpoonup \bar{v} \text{ weakly in } \tilde{H}^1(Q) \text{ as } \Delta \to 0. \tag{4.70}$$

By applying Theorem 4.4.1 as in the proof of (4.64) in Step 1, it follows that

$$\lim_{\Delta\to 0} |\mathscr{J}_\Delta([v]_\Delta) - \mathscr{J}(\bar{v})| = 0 \tag{4.71}$$

To prove (4.69), we add and subtract $\mathscr{J}(\bar{v})$ to (4.69) and we get the following inequality

$$\begin{aligned} |\mathscr{J}(\mathscr{P}_\Delta([v]_\Delta)) - \mathscr{J}_\Delta([v]_\Delta))| &\le |\mathscr{J}(\mathscr{P}_\Delta([v]_\Delta)) - \mathscr{J}(\bar{v})| + |\mathscr{J}(\bar{v}) - \mathscr{J}_\Delta([v]_\Delta))| \\ &= I_1 + I_2 \end{aligned}$$

Weak continuity of $\mathscr{J}$ implies that $I_1 \to 0$ as $\Delta \to 0$. We proved $I_2 \to 0$ in (4.71), and hence (4.69) follows.

Thus ***Step 1*** and ***Step 2*** of the proof implies that the conditions of the Lemma 4.3.5 are satisfied. Therefore, assertiona (4.13) and (4.16) of Theorem 4.2.1 are proved. In order to prove the rest of Theorem 4.2.1, we consider the sequence $\{[v]_{\Delta,\varepsilon}\} \in \mathscr{F}^R_\Delta$. From the Proposition 4.3.7 it follows that $\{\mathscr{P}_\Delta([v]_{\Delta,\varepsilon})\}$ is uniformly bounded in $\tilde{H}^1(Q)$. Assume

$v_* \in \tilde{H}^1(Q)$ is a weak limit point of this sequence. Weak continuity of $\mathscr{J}$ and (4.16) implies that

$$\lim_{\Delta \to 0} \mathscr{J}(\mathscr{P}_\Delta([v]_{\Delta,\varepsilon})) = \mathscr{J}(v_*) = \mathscr{J}_*$$

and $v_* \in \mathscr{F}_*$. In addition, referring to Theorem 4.4.1 there exists a unique discrete state vector $[u([v]_{\Delta,\varepsilon})]_\Delta$ corresponding to $[v]_{\Delta,\varepsilon}$ whose interpolations, $\{U'_\Delta\}$, converge weakly in $H^1(Q)$ to $u_* = u(x;v_*)$, a weak solution to the (4.1)-(4.3). To complete the proof, it remains to demonstrate that

$$\lim_{\varepsilon \to 0} \mathscr{J}_*(\varepsilon) = \mathscr{J}_* = \lim_{\varepsilon \to 0} \mathscr{J}_*(-\varepsilon) \tag{4.72}$$

where $\mathscr{J}_*(\pm\varepsilon) = \inf_{\mathscr{F}^{R\pm\varepsilon}} \mathscr{J}(v)$. The proof of (4.72) coincides with the proof of similar fact in [6]. Theorem is proved. ■

Chapter 5

Discretization and Convergence of the EIT Optimal Control Problem in 3D Domains

5.1 EIT Optimal Control in 3D Domains

In this chapter, we consider the EIT problem in $Q \subset \mathbb{R}^3$ with electrical conductivity tensor $A(x) = \sigma(x)I$, where I is a 3×3 unit matrix:

$$\operatorname{div}\big(\sigma(x)\nabla u\big) = 0, \qquad x \in Q \tag{5.1}$$

$$\frac{\partial u(x)}{\partial n} = 0, \qquad x \in S - \bigcup_{l=1}^{m} E_l \tag{5.2}$$

$$u(x) + Z_l \sigma(x) \frac{\partial u(x)}{\partial n} = U_l, \qquad x \in E_l,\ l = \overline{1,m} \tag{5.3}$$

$$\int_{E_l} \sigma(x) \frac{\partial u(x)}{\partial n} ds = I_l, \qquad l = \overline{1,m} \tag{5.4}$$

where

$$\frac{\partial u(x)}{\partial n} = \sum_{i=1}^{3} u_{x_i} \nu^i$$

and $\nu = (\nu^1, \nu^2, \nu^3)$ is the outward normal at a point x to S, electrical conductivity σ is a positive function. Consider the optimal control problem on the minimization of the cost functional

$$\mathscr{J}(v) = \sum_{l=1}^{m} \Big| \int_{E_l} \frac{U_l - u(x)}{Z_l} ds - I_l \Big|^2 + \beta |U - U^*|^2 \tag{5.5}$$

on the control set

$$\mathscr{F}^R = \Big\{ v = (\sigma, U) \in \big(L_\infty(Q) \bigcap \tilde{H}^1(Q) \big) \times \mathbb{R}^m \Big| \sum_{l=1}^{m} U_l = 0, \, \|\sigma\|^2_{\tilde{H}^1} + |U|^2 \leq R^2 \tag{5.6}$$
$$0 < \sigma_0 \leq \sigma(x) \leq R, \forall x \in Q \Big\}$$

where $\beta > 0$, and $u = u(\cdot; v) \in H^1(Q)$ is a solution of the elliptic problem (5.1)–(5.3). The following is the definition of the weak solution of problem (5.1)–(5.3):

Definition 5.1.1. For a given $v \in \mathscr{F}^R$, $u = u(\cdot; v) \in H^1(Q)$ is called the weak solution of the problem (5.1)–(5.3) if

$$\int_Q \sigma \nabla u \cdot \nabla \eta dx + \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} u \eta ds = \sum_{l=1}^{m} \frac{1}{Z_l} \int_{E_l} \eta U_l ds, \quad \forall \eta \in H^1(Q). \tag{5.7}$$

This optimal control problem will be called Problem $\mathscr{E}'$. The first term in the cost functional $\mathscr{J}(v)$ characterizes the mismatch of the condition (5.4) in light of the Robin condition (5.3).

5.1.1 Discrete EIT Optimal Control Problem

To discretize optimal control problems $\mathscr{E}'$ we pursue finite difference method which is explained in Chapter 2. In addition to discrete sets, $Q^*_\Delta, Q^{*+}_\Delta, Q^{*(i)}_\Delta, S^*_\Delta$, and $\hat{E}_{l\Delta}$ that we defined in Section 4.1.2 we introduce the following set as well:

$$Q^{*(i,j)}_\Delta = \{x_\alpha \in Q^*_\Delta : x_\alpha + e_i + e_j \in Q^*_\Delta\}, \quad i,j = 1,2,3 \tag{5.8}$$

For a given discretization Δ, we employ the notation $[\sigma]_\Delta = \{\sigma_\alpha\}$, $\alpha \in \mathscr{A}(Q^*_\Delta)$ where $\sigma_\alpha \in \mathbb{R}$. Then We define the discrete $\tilde{\mathscr{H}}^1(Q^*_\Delta)$ norm as

$$\|[\sigma]_\Delta\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)} = \sum_{\mathscr{A}(Q^*_\Delta)} h^3\sigma^2_\alpha + \sum_{i=1}^{3}\sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^3\sigma^2_{\alpha x_i} + \sum_{\substack{i,j=1\\ i<j}}^{3}\sum_{\mathscr{A}(Q^{*(i,j)}_\Delta)} h^3\sigma^2_{\alpha x_i x_j}$$
$$+ \sum_{\mathscr{A}(Q^{*+}_\Delta)} h^3\sigma^2_{\alpha x_1 x_2 x_3} \tag{5.9}$$

We use standard notation for finite differences of grid function u_α, σ_α and

$$\sigma_{\alpha x_1 x_2 x_3} = \frac{\sigma_{(\alpha+e_3)x_1x_2} - \sigma_{\alpha x_1 x_2}}{h} = \frac{\sigma_{(\alpha+e_3+e_2)x_1} - \sigma_{(\alpha+e_3)x_1} - (\sigma_{(\alpha+e_2)x_1} - \sigma_{\alpha x_1})}{h^2}$$
$$= (\sigma_{(\alpha+e_3+e_2+e_1)} - \sigma_{(\alpha+e_3+e_2)} - \sigma_{(\alpha+e_3+e_1)} - \sigma_{(\alpha+e_2+e_1)} + \sigma_{(\alpha+e_3)}$$
$$+\sigma_{(\alpha+e_2)} + \sigma_{(\alpha+e_1)} - \sigma_\alpha)/h^3$$

We define discrete control set $\mathscr{F}^R_\Delta$ as in (4.1.2) with the discrete $\tilde{\mathscr{H}}^1(Q^*_\Delta)$ - norm defined as in (5.9). Discretizing map $\mathscr{Q}_\Delta$ and interpolating map $\mathscr{P}_\Delta$ are defined as in Chapter 4. Interpolating map $\mathscr{P}_\Delta$ assigns multilinear interpolation to discrete control vector, which is defined as follows in 3D case:

$$\sigma^{\Delta}(x) = \sigma_\alpha + \sum_{i=1}^{3} \sigma_{\alpha x_i}(x_i - k_i h) + \sum_{\substack{i,j=1 \\ i<j}}^{3} \sigma_{\alpha x_i x_j}(x_i - k_i h)(x_j - k_j h)$$

$$+\sigma_{\alpha x_1 x_2 x_3} \prod_{1\le i\le 3} (x_i - k_i h), \quad \forall x \in C_\Delta^\alpha \tag{5.10}$$

Definition 5.1.2. Given $[v]_\Delta$, the discrete valued function

$$[u([v]_\Delta)]_\Delta = \{u_\alpha \in \mathbb{R} : \alpha \in \mathscr{A}(Q_\Delta^*)\}$$

is called a discrete state vector of problem $\mathscr{E}'$ if it satisfies

$$h^3 \sum_{\mathscr{A}(Q_\Delta^{*+})} \sigma_\alpha^\Delta \sum_{i=1}^{3} u_{\alpha x_i} \eta_{\alpha x_i} + \sum_{l=1}^{m} \frac{1}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u_\alpha \eta_\alpha + J_\alpha(u_\alpha, \eta_\alpha) = \sum_{l=1}^{m} \frac{U_l}{Z_l} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} \eta_\alpha \tag{5.11}$$

for arbitrary collection of values $\{\eta_\alpha\},\ \alpha \in \mathscr{A}(Q_\Delta^*)$, where

$$J_\alpha(u_\alpha, \eta_\alpha) = h^3 \sum_{\mathscr{A}(S_\Delta^*)} \sum_{i=1}^{3} \theta_\alpha^i u_{\alpha x_i} \eta_{\alpha x_i}, \tag{5.12}$$

$$\theta_\alpha^i = \begin{cases} 1 & \text{if } \alpha \in \mathscr{A}(Q_\Delta^{*(i)} \setminus Q_\Delta^{*+}) \\ 0 & \text{otherwise} \end{cases}$$

Discrete optimal control problem on the minimization of the cost functional $\mathscr{J}_\Delta([v]_\Delta)$ (defined as in (4.12)) on a control set $\mathscr{F}_\Delta^R$, with discrete state vector $[u([v]_\Delta)]_\Delta$ being defined according to Definition 5.1.2, will be called Problem $\mathscr{E}_\Delta'$.

5.2 Main Result

Theorem 5.2.1. *The sequence of discrete optimal control problems $\mathscr{E}'_\Delta$ approximates the optimal control problem $\mathscr{E}'$ with respect to functional, i.e.*

$$\lim_{\Delta\to 0} \mathscr{J}_{\Delta_*} = \mathscr{J}_*, \tag{5.13}$$

where

$$\mathscr{J}_{\Delta_*} = \inf_{\mathscr{F}^R_\Delta} \mathscr{J}_\Delta([v]_\Delta), \tag{5.14}$$

Furthermore, let $\{\varepsilon_\Delta\}$ be a sequence of positive real numbers with $\lim_{\Delta\to 0} \varepsilon_\Delta = 0$. If the sequence $[v]_{\Delta,\varepsilon} \in \mathscr{F}^R_\Delta$ is chosen so that

$$\mathscr{J}_{\Delta_*} \leq \mathscr{J}_\Delta([v]_{\Delta,\varepsilon}) \leq \mathscr{J}_{\Delta_*} + \varepsilon_\Delta, \tag{5.15}$$

then we have

$$\lim_{\Delta\to 0} \mathscr{J}(\mathscr{P}_\Delta([v]_{\Delta,\varepsilon})) = \mathscr{J}_* \tag{5.16}$$

Also, the sequence $\{\mathscr{P}_\Delta([\sigma]_{\Delta,\varepsilon})\}$ is uniformly bounded in $\tilde{H}^1(Q)$ and all of its $\tilde{H}^1(Q)$-weak limits points lie in $\mathscr{F}_*$. Moreover, the multilinear interpolations of the discrete state vectors $[u([v]_{\Delta',\varepsilon})]_{\Delta'}$ converge weakly in $H^1(Q)$ to $u = u(x; v_*)$, a weak solution to the (5.1)-(5.3).

5.3 Proof of the Main Result

We pursue three interpolations $\tilde{U}_\Delta, \tilde{U}^i_\Delta$ and U'_Δ as in in Chapter 2. The following estimations for U'_Δ and $\frac{\partial}{\partial x_i}U'_\Delta$ are proved as in Chapter 2:

$$\int_{Q^*_\Delta} |U'_\Delta|^2 dx \leq \sum_{\mathscr{A}(Q^*_\Delta)} h^2 \max_{\mathscr{A}(Q^*_\Delta)} |u_{\alpha^*}|^2 \leq 2^3 \sum_{\mathscr{A}(Q^*_\Delta)} h^2 |u_\alpha|^2, \tag{5.17}$$

$$\int_{Q^*_\Delta} \left|\frac{\partial}{\partial x_i} U'_\Delta\right|^2 dx \leq 2^2 \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} h^2 u^2_{\alpha x_i} \tag{5.18}$$

Lemma 4.3.1, Lemma 4.3.2 and Corollary 4.3.3 apply to 3D case without any change. The new discrete norm $|||\cdot|||_{\mathscr{H}^1(Q^*_\Delta)}$ introduced in Lemma 4.3.1 modified as follows:

$$|||[u([v]_\Delta)]_\Delta|||^2_{\mathscr{H}^1(Q^*_\Delta)} := \sum_{i=1}^{3} h^3 \sum_{\mathscr{A}(Q^{*(i)}_\Delta)} u^2_{\alpha x_i} + \sum_{l=1}^{m} \sum_{\mathscr{A}(\hat{E}_{l\Delta})} \Gamma_{l\alpha} u^2_\alpha \tag{5.19}$$

Index set is updated to $i \in \{1,2,3\}$ for the part (b) and (d) of Lemma 4.3.4.

Lemma 5.3.1. *For $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{\mathscr{A}(Q^{*+}_\Delta)} h^3 |\sigma_{\alpha x_1 x_2 x_3}|^2 \leq (1+\varepsilon) \left\| \frac{\partial^3 \sigma}{\partial x_1 \partial x_2 \partial x_3} \right\|^2_{L_2(Q^*_\Delta)}$$

whenever $h < \delta$.

Proof: For each $h > 0$, define the function $\tilde{\sigma}^{123}_h$ as

$$\tilde{\sigma}^{123}_h \Big|_{C^\alpha_\Delta} = \sigma_{\alpha x_1 x_2 x_3}, \quad \forall \alpha \in \mathscr{A}(Q^{*+}_\Delta) \tag{5.20}$$

where

$$\sigma_{\alpha x_1 x_2 x_3} =$$
$$(\sigma_{(\alpha+e_3+e_2+e_1)} - \sigma_{(\alpha+e_3+e_2)} - \sigma_{(\alpha+e_3+e_1)} - \sigma_{(\alpha+e_2+e_1)} + \sigma_{(\alpha+e_3)}$$
$$+\sigma_{(\alpha+e_2)} + \sigma_{(\alpha+e_1)} - \sigma_{\alpha})/h^3$$

In the following we will prove that

$$\tilde{\sigma}_h^{123} \to \frac{\partial^3 \sigma}{\partial x_1 \partial x_2 x_3} \quad \text{strongly in } L_2(Q) \text{ as } h \to 0 \tag{5.21}$$

As an element of $\tilde{H}^1(Q)$, almost all restrictions of σ to lines parallel to the x_1, x_2 and x_3 direction are absolutely continuous; Moreover, restrictions of $\sigma_{x_1 x_2}$ to lines parallel to the x_3 direction, $\sigma_{x_1 x_3}$ to lines parallel to the x_2 direction, $\sigma_{x_2 x_3}$ to lines parallel to the x_1 direction are absolutely continuous; Also, restrictions of σ_{x_1} to lines parallel to the $x_2$and x_3, σ_{x_2} to lines parallel to the $x_1$and x_3 and σ_{x_3} to lines parallel to the $x_2$and x_1 directions are absolutely continuous. Hence if we let $z = (z_1, z_2, z_3)$ and $x = (x_1, x_2, x_3)$, then for almost every $z \in Q$ we have

$$\int_{z_3}^{z_3+h} \int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \frac{\partial^3 \sigma(y,t,w)}{\partial x_1 \partial x_2 \partial x_3} \, dy\, dt\, dw = \sigma(z+he_3+he_2+he_1) - \sigma(z+he_3+he_2)$$
$$-\sigma(z+he_3+he_1) - \sigma(z+he_2+he_1) + \sigma(z+he_3) + \sigma(z+he_2) + \sigma(z+he_1) - \sigma(z) \tag{5.22}$$

In this lemma, for simplicity, instead of $\sum_{\alpha \in \mathscr{A}(Q_\Delta^{*+})}$ we write $\sum_{\mathscr{A}}$. Using the definition of

Steklov average (2.15) and Cauchy-Schwartz inequality, we get

$$\left\|\tilde{\sigma}_h^{123} - \frac{\partial^3 \sigma}{\partial x_1 \partial x_2 \partial x_3}\right\|^2_{L_2(Q^*_\Delta)} = \int\limits_{Q^*_\Delta} \left|\tilde{\sigma}_h^{123} - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3}\right|^2 dx$$

$$= \sum_{\mathscr{A}} \int\limits_{C^\alpha_\Delta} \left|\sigma_{\alpha x_1 x_2 x_3} - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3}\right|^2 dx$$

$$= \sum_{\mathscr{A}} \int\limits_{C^\alpha_\Delta} \Big|\frac{1}{h^6}\Big[\int\limits_{C_\Delta^{\alpha+he_1+he_2+he_3}} dz - \int\limits_{C_\Delta^{\alpha+he_1+he_2}} dz - \int\limits_{C_\Delta^{\alpha+he_3+he_2}} dz - \int\limits_{C_\Delta^{\alpha+he_1+he_3}} dz$$

$$+ \int\limits_{C_\Delta^{\alpha+he_1}} dz + \int\limits_{C_\Delta^{\alpha+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_3}} dz - \int\limits_{C_\Delta^{\alpha}} dz \sigma(z)\Big] - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3}\Big|^2 dx$$

$$= \sum_{\mathscr{A}} \int\limits_{C^\alpha_\Delta} \Big|\frac{1}{h^6} \int\limits_{C^\alpha_\Delta} [\sigma(z+he_3+he_2+he_1) - \sigma(z+he_3+he_2) - \sigma(z+he_3+he_1)$$

$$-\sigma(z+he_2+he_1) + \sigma(z+he_3) + \sigma(z+he_2) + \sigma(z+he_1) - \sigma(z)]dz - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3}\Big|^2 dx$$

$$= \sum_{\mathscr{A}} \int\limits_{C^\alpha_\Delta} \Big|\frac{1}{h^6} \int\limits_{C^\alpha_\Delta} [\int_{z_3}^{z_3+h} \int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \frac{\partial^3 \sigma(y,t,w)}{\partial x_1 \partial x_2 \partial x_3} dy\, dt\, dw]dz - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3} \frac{h^6}{h^6}\Big|^2 dx$$

$$= \sum_{\mathscr{A}} \frac{1}{h^{12}} \int\limits_{C^\alpha_\Delta} \Big| \int\limits_{C^\alpha_\Delta} [\int_{z_3}^{z_3+h} \int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \frac{\partial^3 \sigma(y,t,w)}{\partial x_1 \partial x_2 \partial x_3} dy\, dt\, dw - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3} h^3]dz\Big|^2 dx$$

$$= \sum_{\mathscr{A}} \frac{1}{h^{12}} \int\limits_{C^\alpha_\Delta} \Big| \int\limits_{C^\alpha_\Delta} [\int_{z_3}^{z_3+h} \int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \frac{\partial^3 \sigma(y,t,w)}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3} dydtdw]dz\Big|^2 dx$$

$$\leq \sum_{\mathscr{A}} \frac{1}{h^6} \int\limits_{C^\alpha_\Delta} \int\limits_{C^\alpha_\Delta} [\int_{z_3}^{z_3+h} \int_{z_2}^{z_2+h} \int_{z_1}^{z_1+h} \Big|\frac{\partial^3 \sigma(y,t,w)}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3}\Big|^2 dydtdw]dzdx$$

Assume $m_\alpha = (m_1, m_2, m_3)$ be the natural corner of C^α_Δ. Now, we employ Fubini theorem three times to first switch the order of integration with respect to y and z_1, then t

and z_2, and finally w and z_3 . Hence we observe the

$$\left\|\tilde{\sigma}_h^{123}-\frac{\partial^3\sigma}{\partial x_1\partial x_2\partial x_3}\right\|^2_{L_2(Q^*_\Delta)} \leq$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3}^{m_3+h}\int_{m_2}^{m_2+h}\int_{m_1}^{m_1+h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx$$

$$+\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3}^{m_3+h}\int_{m_2}^{m_2+h}\int_{m_1+h}^{m_1+2h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3}^{m_3+h}\int_{m_2+h}^{m_2+2h}\int_{m_1}^{m_1+h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3+h}^{m_3+2h}\int_{m_2}^{m_2+h}\int_{m_1}^{m_1+h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3}^{m_3+h}\int_{m_2+h}^{m_2+2h}\int_{m_1+h}^{m_1+2h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3+h}^{m_3+2h}\int_{m_2+h}^{m_2+2h}\int_{m_1}^{m_1+h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3+h}^{m_3+2h}\int_{m_2}^{m_2+h}\int_{m_1+h}^{m_1+2h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx+$$

$$\sum_{\mathscr{A}}\frac{1}{h^3}\int_{C^\alpha_\Delta}\left(\int_{m_3+h}^{m_3+2h}\int_{m_2+h}^{m_2+2h}\int_{m_1+h}^{m_1+2h}\left|\frac{\partial^3\sigma(y,t,w)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2 dydtdw\right)dx$$

without loss of generality, we replace (y,t,w) with (z_1,z_2,z_3) and we get

$$\left\|\tilde{\sigma}_h^{123}-\frac{\partial^2\sigma}{\partial x_1\partial x_2\partial x_3}\right\|^2_{L_2(Q^*_\Delta)} \leq \sum_{\mathscr{A}}\frac{1}{h^3}\int\limits_{C^\alpha_\Delta}\Bigg(\int\limits_{C^{\alpha+he_1+he_2+he_3}_\Delta} dz$$

$$+\int\limits_{C^{\alpha+he_1+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_3+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_1+he_3}_\Delta} dz+\int\limits_{C^{\alpha+he_1}_\Delta} dz+\int\limits_{C^{\alpha+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_3}_\Delta} dz$$

$$+\int\limits_{C^\alpha_\Delta} dz\left|\frac{\partial^3\sigma(z_1,z_2,z_3)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\right|^2\Bigg)dx \tag{5.23}$$

For fixed $\varepsilon>0$, we pick a $g\in C^3(Q+B_1(0))$ such that

$$\|\sigma-g\|^2_{\tilde{H}^1(Q+B_1(0))}\leq c(\varepsilon) \tag{5.24}$$

Now, we add and subtract $\frac{\partial^3 g(z)}{\partial x_1\partial x_2\partial x_3}$ and $\frac{\partial^3 g(x)}{\partial x_1\partial x_2\partial x_3}$ to the integrands of (5.23)

$$\left\|\tilde{\sigma}_h^{123}-\frac{\partial^3\sigma}{\partial x_1\partial x_2\partial x_3}\right\|^2_{L_2(Q^*_\Delta)}\leq I_1+I_2+I_3 \tag{5.25}$$

where

$$I_1=\sum_{\mathscr{A}}\frac{3}{h^3}\int\limits_{C^\alpha_\Delta}\Bigg(\int\limits_{C^{\alpha+he_1+he_2+he_3}_\Delta} dz+\int\limits_{C^{\alpha+he_1+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_3+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_1+he_3}_\Delta} dz$$

$$+\int\limits_{C^{\alpha+he_1}_\Delta} dz+\int\limits_{C^{\alpha+he_2}_\Delta} dz+\int\limits_{C^{\alpha+he_3}_\Delta} dz+\int\limits_{C^\alpha_\Delta} dz\left|\frac{\partial^3\sigma(z)}{\partial x_1\partial x_2\partial x_3}-\frac{\partial^3 g(z)}{\partial x_1\partial x_2\partial x_3}\right|^2\Bigg)dx$$

$$I_2 = \sum_{\mathscr{A}} \frac{3}{h^3} \int_{C_\Delta^\alpha} \Bigg(\int_{C_\Delta^{\alpha+he_1+he_2+he_3}} dz + \int_{C_\Delta^{\alpha+he_1+he_2}} dz + \int_{C_\Delta^{\alpha+he_3+he_2}} dz + \int_{C_\Delta^{\alpha+he_1+he_3}} dz$$

$$+ \int_{C_\Delta^{\alpha+he_1}} dz + \int_{C_\Delta^{\alpha+he_2}} dz + \int_{C_\Delta^{\alpha+he_3}} dz + \int_{C_\Delta^\alpha} dz \Big| \frac{\partial^3 g(z)}{\partial x_1 \partial x_2 \partial x_3} \frac{\partial^3 g(x)}{\partial x_1 \partial x_2 \partial x_3} \Big|^2 \Bigg) dx$$

$$I_3 = \sum_{\mathscr{A}} \frac{3}{h^3} \int_{C_\Delta^\alpha} \Bigg(\int_{C_\Delta^{\alpha+he_1+he_2+he_3}} dz + \int_{C_\Delta^{\alpha+he_1+he_2}} dz + \int_{C_\Delta^{\alpha+he_3+he_2}} dz + \int_{C_\Delta^{\alpha+he_1+he_3}} dz$$

$$+ \int_{C_\Delta^{\alpha+he_1}} dz + \int_{C_\Delta^{\alpha+he_2}} dz + \int_{C_\Delta^{\alpha+he_3}} dz + \int_{C_\Delta^\alpha} dz \Big| \frac{\partial^3 g(x)}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 \sigma(x)}{\partial x_1 \partial x_2 \partial x_3} \Big|^2 \Bigg) dx$$

Since $g \in C^3(Q + B_1(0))$, it follows that $\frac{\partial^3 g}{\partial x_1 \partial x_2 \partial x_3}$ is uniformly continuous on $Q + B_1(0)$. Therefore, there exists $\delta = \delta(g, \varepsilon) > 0$ such that

$$\Big| \frac{\partial^3 g(z)}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 g(x)}{\partial x_1 \partial x_2 \partial x_3} \Big|^2 \leq c(\varepsilon) \tag{5.26}$$

whenever $|z - x| < \delta$. Let $h_\varepsilon > 0$ satisfy

$$\sqrt{12}\, h_\varepsilon < \delta \tag{5.27}$$

Then it follows that for each $h < h_\varepsilon$, any $\alpha \in \mathscr{A}$, and any $x, z \in C_\Delta^{\alpha+he_1+he_2+he_3} \cup C_\Delta^{\alpha+he_1+he_2} \cup C_\Delta^{\alpha+he_3+he_2} \cup C_\Delta^{\alpha+he_1+he_3} \cup C_\Delta^{\alpha+he_3} \cup C_\Delta^{\alpha+he_1} \cup C_\Delta^{\alpha+he_2} \cup C_\Delta^\alpha$,

$$\Big| \frac{\partial^3 g(z)}{\partial x_1 \partial x_2 \partial x_3} - \frac{\partial^3 g(x)}{\partial x_1 \partial x_2 \partial x_3} \Big|^2 \leq c(\varepsilon). \tag{5.28}$$

Therefore,

$$I_1 = \frac{3}{h^3}\sum_{\mathscr{A}} h^3\Bigg(\int\limits_{C_\Delta^{\alpha+he_1+he_2+he_3}} dz + \int\limits_{C_\Delta^{\alpha+he_1+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_3+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_1+he_3}} dz$$

$$+ \int\limits_{C_\Delta^{\alpha+he_1}} dz + \int\limits_{C_\Delta^{\alpha+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_3}} dz + \int\limits_{C_\Delta^{\alpha}} dz \Big|\frac{\partial^3\sigma(z)}{\partial x_1\partial x_2\partial x_3} - \frac{\partial^3 g(z)}{\partial x_1\partial x_2\partial x_3}\Big|^2\Bigg)$$

$$\leq 24 \int\limits_{Q+B_1(0)} \Big|\frac{\partial^2\sigma(z)}{\partial x_1\partial x_2} - \frac{\partial^2 g(z)}{\partial x_1\partial x_2}\Big|^2 dz \leq 24\|\sigma - g\|^2_{\tilde{H}^1(Q+B_1(0))}$$

$$I_2 \leq \sum_{\mathscr{A}} \frac{3}{h^3}\int\limits_{C_\Delta^{\alpha}}\Bigg(\int\limits_{C_\Delta^{\alpha+he_1+he_2+he_3}} dz + \int\limits_{C_\Delta^{\alpha+he_1+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_3+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_1+he_3}} dz$$

$$+ \int\limits_{C_\Delta^{\alpha+he_1}} dz + \int\limits_{C_\Delta^{\alpha+he_2}} dz + \int\limits_{C_\Delta^{\alpha+he_3}} dz + \int\limits_{C_\Delta^{\alpha}} dz\, c(\varepsilon)\Bigg) dx \leq 24c(\varepsilon)m(Q_\Delta^*)$$

$$I_3 = 24\int_{Q_\Delta^*}\Big|\frac{\partial^3 g(x)}{\partial x_1\partial x_2\partial x_3} - \frac{\partial^3\sigma(x)}{\partial x_1\partial x_2\partial x_3}\Big|^2 dx \leq 24\|\sigma - g\|^2_{\tilde{H}^1(Q_\Delta^*)}$$

If we take $c(\varepsilon) = \frac{\varepsilon}{48+24m(Q_\Delta^*)}$, these calculations imply that

$$I_1 + I_2 + I_3 < \varepsilon, \quad \forall h \leq h_\varepsilon \tag{5.29}$$

Hence (5.21) is proved. Assertion of the lemma follows from (5.21). Lemma is proved.

Proposition 5.3.2. *For arbitrary sufficiently small* $\varepsilon > 0$ *there exists* h_ε *such that*

$$\mathscr{Q}_\Delta(v) \in \mathscr{F}_\Delta^R \quad \textit{for all} \quad v \in \mathscr{F}^{(R-\varepsilon)} \quad \textit{and} \quad h \le h_\varepsilon \tag{5.30}$$

$$\mathscr{P}_\Delta([v]_\Delta) \in \mathscr{F}^{(R+\varepsilon)} \quad \textit{for all} \quad [v]_\Delta \in \mathscr{F}_\Delta^R \quad \textit{and} \quad h \le h_\varepsilon \tag{5.31}$$

Proof. First side of proposition can be proved similar to Proposition 4.3.7 by using Lemma 5.3.1. Now let $[\sigma]_\Delta \in \mathscr{F}_\Delta^R$. we claim that $\sigma^\Delta := \mathscr{P}_\Delta([\sigma]_\Delta) \in \mathscr{F}^{(R+\varepsilon)}$. In order to prove our claim, we need to show

$$\|\sigma\|^2_{\tilde{H}_1(Q)} = \sum_{\mathscr{A}(Q^*_\Delta)} \int_{C^\alpha_\Delta} \big[(\sigma^\Delta)^2 + (\sigma^\Delta(x))^2_{x_1} + (\sigma^\Delta(x))^2_{x_2} + (\sigma^\Delta(x))^2_{x_1x_2} + (\sigma^\Delta(x))^2_{x_1x_3}$$
$$(\sigma^\Delta(x))^2_{x_3x_2} + (\sigma^\Delta(x))^2_{x_1x_2x_3}\big] dx_1 dx_2 \le (R+\varepsilon)^2 \tag{5.32}$$

We prove (5.32) directly by evaluating the L_2 norm of every term in it over a fixed cell C^α_Δ, $\alpha \in \mathscr{A}(Q^*_\Delta)$.

$$\begin{aligned}
&\int_{k_1h}^{(k_1+1)h} \int_{k_2h}^{(k_2+1)h} \int_{k_3h}^{(k_3+1)h} (\sigma^\Delta(x))^2_{x_1x_2x_3} dx_1 dx_2 dx_3 \\
&= \int_{k_1h}^{(k_1+1)h} \int_{k_2h}^{(k_2+1)h} \int_{k_3h}^{(k_3+1)h} |\sigma_{\alpha x_1x_2x_3}|^2 dx_2 dx_1 dx_3 \\
&= h^3 \sigma^2_{\alpha x_1x_2x_3},
\end{aligned} \tag{5.33}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} (\sigma^{\Delta}(x))^2_{x_1x_2} dx_1dx_2dx_3$$

$$= \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} |\sigma_{\alpha x_1x_2} + \sigma_{\alpha x_1x_2x_3}(x_3 - k_3h)|^2 dx_1dx_2dx_3$$

$$= h^3\sigma^2_{\alpha x_1x_2} + \sigma^2_{\alpha x_1x_2x_3}\frac{h^5}{3} + \sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_2x_3}\frac{h^4}{2}$$

where

$$|\frac{h^4}{2}\sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_2x_3}| \leq \frac{h}{4}(h^3\sigma^2_{\alpha x_1x_2} + h^3\sigma^2_{\alpha x_1x_2x_3}), \quad \text{and}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} (\sigma^{\Delta}(x))^2_{x_1x_3} dx_1dx_2dx_3$$

$$= \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} |\sigma_{\alpha x_1x_3} + \sigma_{\alpha x_1x_2x_3}(x_2 - k_2h)|^2 dx_1dx_2dx_3$$

$$= h^3\sigma^2_{\alpha x_1x_3} + \sigma^2_{\alpha x_1x_2x_3}\frac{h^5}{3} + \sigma_{\alpha x_1x_3}\sigma_{\alpha x_1x_2x_3}\frac{h^4}{2}$$

where

$$|\frac{h^4}{2}\sigma_{\alpha x_1x_3}\sigma_{\alpha x_1x_2x_3}| \leq \frac{h}{4}(h^3\sigma^2_{\alpha x_1x_3} + h^3\sigma^2_{\alpha x_1x_2x_3}), \quad \text{and}$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} (\sigma^\Delta(x))^2_{x_2x_3} dx_1dx_2dx_3$$
$$= \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} |\sigma_{\alpha x_2x_3} + \sigma_{\alpha x_1x_2x_3}(x_1 - k_1h)|^2 dx_1dx_2dx_3$$
$$= h^3\sigma^2_{\alpha x_2x_3} + \sigma^2_{\alpha x_1x_2x_3}\frac{h^5}{3} + \sigma_{\alpha x_2x_3}\sigma_{\alpha x_1x_2x_3}\frac{h^4}{2}$$

where

$$|\frac{h^4}{2}\sigma_{\alpha x_2x_3}\sigma_{\alpha x_1x_2x_3}| \leq \frac{h}{4}(h^3\sigma^2_{\alpha x_2x_3} + h^3\sigma^2_{\alpha x_1x_2x_3}).$$

We demonstrate the calculation for the term $\left(\sigma^\Delta(x)\right)_{x_1}$, and omit similar calculations for the terms $\left(\sigma^\Delta(x)\right)_{x_2}, \left(\sigma^\Delta(x)\right)_{x_3}$:

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} \left(\sigma^\Delta(x)\right)^2_{x_1} dx_2dx_1dx_3 = \int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h} \big(\sigma_{\alpha x_1} + \sigma_{\alpha x_1x_2}(x_2 - k_2h)$$
$$+\sigma_{\alpha x_1x_3}(x_3 - k_3h) + \sigma_{\alpha x_1x_2x_3}(x_2 - k_2h)(x_3 - k_3h)\big)^2 dx_2dx_1dx_3$$
$$= \sigma^2_{\alpha x_1}h^3 + \sigma^2_{\alpha x_1x_3}\frac{h^5}{3} + \sigma^2_{\alpha x_1x_2}\frac{h^5}{3} + \sigma^2_{\alpha x_1x_2x_3}\frac{h^7}{9}$$
$$+2\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2}\frac{h^4}{2} + 2\sigma_{\alpha x_1}\sigma_{\alpha x_1x_3}\frac{h^4}{2} + 2\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2x_3}\frac{h^5}{4}$$
$$+2\sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_3}\frac{h^5}{4} + 2\sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_2x_3}\frac{h^6}{6}$$
$$+2\sigma_{\alpha x_1x_3}\sigma_{\alpha x_1x_2x_3}\frac{h^6}{6}$$

where

$$|h^4\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2}| \leq \frac{h}{2}(h^3\sigma^2_{\alpha x_1} + h^3\sigma^2_{\alpha x_1x_2})$$

$$|h^4\sigma_{\alpha x_1}\sigma_{\alpha x_1 x_3}| \leq \frac{h}{2}(h^3\sigma^2_{\alpha x_1} + h^3\sigma^2_{\alpha x_1 x_3})$$

$$|\frac{h^5}{2}\sigma_{\alpha x_1}\sigma_{\alpha x_1 x_2 x_3}| \leq \frac{h^2}{4}(h^3\sigma^2_{\alpha x_1} + h^3\sigma^2_{\alpha x_1 x_2 x_3})$$

$$|\frac{h^5}{2}\sigma_{\alpha x_1 x_2}\sigma_{\alpha x_1 x_3}| \leq \frac{h^2}{4}(h^3\sigma^2_{\alpha x_1 x_2} + h^3\sigma^2_{\alpha x_1 x_3})$$

$$|\frac{h^6}{3}\sigma_{\alpha x_1 x_2}\sigma_{\alpha x_1 x_2 x_3}| \leq \frac{h^3}{6}(h^3\sigma^2_{\alpha x_1 x_2} + h^3\sigma^2_{\alpha x_1 x_2 x_3})$$

$$|\frac{h^6}{3}\sigma_{\alpha x_1 x_3}\sigma_{\alpha x_1 x_2 x_3}| \leq \frac{h^3}{6}(h^3\sigma^2_{\alpha x_1 x_3} + h^3\sigma^2_{\alpha x_1 x_2 x_3})$$

$$\int_{k_1h}^{(k_1+1)h}\int_{k_2h}^{(k_2+1)h}\int_{k_3h}^{(k_3+1)h}(\sigma^{\Delta}(x))^2dx_dx_2dx_3$$

$$= h^3\sigma_\alpha^2+\frac{h^5}{3}\sigma_{\alpha x_1}^2+\frac{h^5}{3}\sigma_{\alpha x_2}^2+\frac{h^5}{3}\sigma_{\alpha x_3}^2+\frac{h^7}{9}\sigma_{\alpha x_1x_2}^2+\frac{h^7}{9}\sigma_{\alpha x_1x_3}^2$$

$$+\frac{h^7}{9}\sigma_{\alpha x_2x_3}^2+\frac{h^9}{27}\sigma_{\alpha x_1x_2x_3}^2+h^4\sigma_\alpha\sigma_{\alpha x_1}+h^4\sigma_\alpha\sigma_{\alpha x_2}+h^4\sigma_\alpha\sigma_{\alpha x_3}$$

$$+\frac{h^5}{2}\sigma_\alpha\sigma_{\alpha x_1x_2}+\frac{h^5}{2}\sigma_\alpha\sigma_{\alpha x_1x_3}+\frac{h^5}{2}\sigma_\alpha\sigma_{\alpha x_2x_3}+\frac{h^6}{4}\sigma_\alpha\sigma_{\alpha x_1x_2x_3}$$

$$+\frac{h^5}{2}\sigma_{\alpha x_1}\sigma_{\alpha x_2}+\frac{h^5}{2}\sigma_{\alpha x_1}\sigma_{\alpha x_3}+\frac{h^6}{3}\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2}+\frac{h^6}{3}\sigma_{\alpha x_1}\sigma_{\alpha x_1x_3}$$

$$+\frac{h^6}{4}\sigma_{\alpha x_1}\sigma_{\alpha x_2x_3}+\frac{h^7}{6}\sigma_{\alpha x_1}\sigma_{\alpha x_1x_2x_3}$$

$$+\frac{h^5}{2}\sigma_{\alpha x_2}\sigma_{\alpha x_3}+\frac{h^6}{3}\sigma_{\alpha x_2}\sigma_{\alpha x_1x_2}+\frac{h^6}{4}\sigma_{\alpha x_2}\sigma_{\alpha x_1x_3}+\frac{h^6}{3}\sigma_{\alpha x_2}\sigma_{\alpha x_2x_3}+\frac{h^7}{6}\sigma_{\alpha x_2}\sigma_{\alpha x_1x_2x_3}$$

$$+\frac{h^6}{4}\sigma_{\alpha x_3}\sigma_{\alpha x_1x_2}+\frac{h^6}{3}\sigma_{\alpha x_3}\sigma_{\alpha x_1x_3}+\frac{h^6}{3}\sigma_{\alpha x_3}\sigma_{\alpha x_2x_3}+\frac{h^7}{6}\sigma_{\alpha x_3}\sigma_{\alpha x_1x_2x_3}$$

$$+\frac{h^7}{6}\sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_3}+\frac{h^7}{6}\sigma_{\alpha x_1x_2}\sigma_{\alpha x_2x_3}+\frac{h^8}{9}\sigma_{\alpha x_1x_2}\sigma_{\alpha x_1x_2x_3}$$

$$+\frac{h^7}{6}\sigma_{\alpha x_1x_3}\sigma_{\alpha x_2x_3}+\frac{h^8}{9}\sigma_{\alpha x_1x_3}\sigma_{\alpha x_1x_2x_3}+\frac{h^8}{9}\sigma_{\alpha x_2x_3}\sigma_{\alpha x_1x_2x_3}$$

After summation and using all these inequalities we deduce that

$$\|\mathscr{P}_\Delta([\sigma]_\Delta)\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)}=\|\sigma^\Delta\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)}\le\|[\sigma]_\Delta\|^2_{\tilde{\mathscr{H}}^1(Q^*_\Delta)}+O(h)$$

which easily implies that for some $h_\varepsilon>0$, we have $\mathscr{P}_\Delta([v]_\Delta)\in\mathscr{F}^{(R+\varepsilon)}$ for all $h<h_\varepsilon$. Lemma is proved. □

Having Lemma 5.3.1 and Proposition 5.3.2, the rest of the proof of Theorem 5.2.1 coincides with the proof of Theorem 4.2.1 given in Chapter 4.

Chapter 6

Conclusions

Dissertation research is on the analysis of optimal control problems for the systems with distributed parameters described by general boundary value problems for the second order linear elliptic PDEs with bounded measurable coefficients in Lipschitz domains. Chapter 2 analyzes elliptic optimal control problem where control parameter is the density of sources and the cost functional is the L_2-norm difference of the weak solution of the elliptic Dirichlet or Neumann problem from measurement along the boundary or subdomain. The optimal control problems are fully discretized using the method of finite differences. Two types of discretization of the elliptic boundary value problem depending on Dirichlet or Neumann type boundary condition are introduced. The main result of the Chapter 2 is the following:

- Convergence of the sequence of finite-dimensional discrete optimal control problems both with respect to the cost functional and the control is proved. The methods of the proof are based on energy estimates in discrete Sobolev spaces, Lax-Milgram theory, weak compactness and convergence of interpolations of solutions of discrete elliptic problems, and delicate estimation of the cost functional along the sequence of interpolations of the minimizers for the discrete optimal control problems.

The methods of Chapter 2 are developed and applied to biomedical problem on the detection of the cancerous tumor. Chapters 3-5 analyze the inverse EIT problem in a PDE constrained optimal control framework in Besov space, where the electrical conductivity tensor and boundary voltages are control parameters, and the cost functional is the norm difference of the boundary electrode current from the given current pattern and boundary electrode voltages from the measurements. The state vector is a solution of the second order elliptic PDE in divergence form with bounded measurable coefficients under mixed Neumann/Robin type boundary condition. The following are the main results of Chapters 3-5:

- The novelty of the control theoretic model is its adaptation to clinical situation when additional "voltage-to-current" measurements can increase the size of the input data from the number of boundary electrodes m up to $m!$ while keeping the size of the unknown parameters fixed.

- Existence of the optimal control and Fréchet differentiability in the Besov space setting is proved. The formula for the Fréchet gradient and optimality condition is derived. Numerical method based on the projective gradient method in Hilbert-Besov spaces is developed.

- EIT optimal control problem is fully discretized using the method of finite differences. New Sobolev-Hilbert space is introduced, and the convergence of the sequence of finite-dimensional optimal control problems to EIT coefficient optimal control problem is proved both with respect to functional and control in 2-

and 3-dimensional domains.

Bibliography

[1] UG Abdulla. Wiener's criterion for the unique solvability of the Dirichlet problem in arbitrary open sets with non-compact boundaries. *Nonlinear Analysis*, 67(2):563–578, 2007.

[2] UG Abdulla. Wiener's criterion at ∞ for the heat equation. *Advances in Differential Equation*, 13(5-6):457–488, 2008.

[3] UG Abdulla. Wiener's criterion at ∞ for the heat equation and its measure-theoretical counterpart. *Electron. Res. Announc. Math. Sci.*, 15:44–51, 2008.

[4] UG Abdulla. Regularity of ∞ for elliptic equations with measurable coefficients and its consequences. *Discrete and Continuous Dynamical Systems - Series A (DCDS-A)*, 32(10):3379–3397, 2012.

[5] UG Abdulla. On the optimal control of the free boundary problems for the second order parabolic equations. I. Well-posedness and convergence of the method of lines. *Inverse Problems and Imaging*, 7(2):307–340, 2013.

[6] UG Abdulla. On the optimal control of the free boundary problems for the second order parabolic equations. II. Convergence of the method of finite differences. *Inverse Problems and Imaging*, 10(4):869–898, 2016.

[7] UG Abdulla. Removability of the logarithmic singularity for the elliptic PDEs with measurable coefficients and its consequences. *Calculus of Variations and Partial Differential Equations*, 57:157(6), 2018.

[8] UG Abdulla, V Bukshtynov, and S Seif. Cancer detection through electrical impedance tomography and optimal control theory: Theoretical and computational analysis. *arXiv:1809.05936*, 2018.

[9] UG Abdulla and E Cosgrove. Optimal control of the multiphase free boundary problems for nonlinear parabolic equations. *Applied Mathematics & Optimization*, 2020.

[10] UG Abdulla, E Cosgrove, and JM Goldfarb. On the Fréchet differentiability in optimal control of coefficients in parabolic free boundary problems. *Evolution Equations and Control Theory*, 6(3):319–344, 2017.

[11] UG Abdulla, J Goldfarb, and A Hagverdiyev. Optimal control of coefficients in parabolic free boundary problems modeling laser ablation. *Journal of Computational and Applied Mathematics*, 372, 2020.

[12] UG Abdulla and JM Goldfarb. Fréchet differentiability in Besov spaces in the optimal control of parabolic free boundary problems. *Journal of Inverse and Ill-posed Problems*, 26(2):211–227, 2018.

[13] UG Abdulla and B Poggi. Optimal control of the multiphase Stefan problem. *Applied Mathematics & Optimization*, 80(2):479–513, 2019.

[14] UG Abdulla and B Poggi. Optimal Stefan problem. *Calculus of Variations and Partial Differential Equations, to appear; arXiv:1901.04663*, 2020.

[15] G Alessandrini. Stable determination of conductivity by boundary measurements. *Applicable Analysis*, 27(1-3):153–172, 1988.

[16] G Alessandrini, MV de Hoop, and R Gaburro. Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities. *Inverse Problems*, 33(12), 2017.

[17] G Alessandrini and R Gaburro. The local calderon problem and the determination at the boundary of the conductivity. *Comm. PDEs*, 34:918–936, 2009.

[18] M Alsaker and JL Mueller. A d-bar algorithm with a priori information for 2-dimensional electrical impedance tomography. *SIAM J. Imaging Science*, 9:1619–1654, 2016.

[19] H Ammari, GS Alberti, B Jin, JK Seo, and W Zhang. The linearized inverse problem in multifrequency electrical impedance tomography. *SIAM J. Imaging Science*, 9:1525–1551, 2016.

[20] H Ammari, L Qiu, F Santosa, and W Zhang. Determining anisotropic conductivity using diffusion tensor imaging data in magneto-acoustic tomography with magnetic induction. *Inverse Problems*, 34:201–224, 2017.

[21] M Ammari and H Kang. *Reconstruction of Small Inhomogeneities from Boundary Measurements*. Springer, 2004.

[22] T Apel, M Mateos, J Pfefferer, and A Rösch. On the regularity of the solutions of dirichlet optimal control problems in polygonal domains. *SIAM J Control Optim.*, 53(6):3620–3641, 2015.

[23] K Astala, M Lassas, and L Palvarinta. Calderon's inverse problem for anisotropic conductivity in the plane. *Comm. PDEs*, 30:207–224, 2005.

[24] K Astala and L Palvarinta. Calderon's inverse conductivity problem in the plane. *Annals of Mathematics*, 163(2):265–299, 2006.

[25] W Bangerth and R Rannacher. *Adaptive finite element methods for differential equations*. Lectures inMathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003.

[26] R Becker, H Kapp, and R Rannacher. Adaptive finite element methods for optimal control of partial differential equations: Basic concept. *SIAM Journal on Control and Optimization*, 39(1):113–132, 2000.

[27] MI Belishev. The calderon problem for two-dimensional manifolds by the bc-method. *SIAM J Math. Anal.*, 35(1):172–182, 2003.

[28] JD Benamou. Décomposition de domaine pour le contrôle optimal de systemes gouvernés par des équations aux dérivées partielles elliptiques. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 317(2):205–209, 1993.

[29] OV Besov, VP Il'in, and SM Nikol'skii. *Integral Representations of Functions and Imbedding Theorems*, volume Vol. 1. John Wiley & Sons, 1979.

[30] OV Besov, VP Il'in, and SM Nikol'skii. *Integral Representations of Functions and Imbedding Theorems*, volume Vol. 2. John Wiley & Sons, 1979.

[31] A Borzì and K Kunisch. A multigrid scheme for elliptic constrained optimal control problems. *Computational Optimization and Applications*, 31(3):309–333, 2005.

[32] A Borzì, K Kunisch, and DY Kwak. Accuracy and convergence properties of the finite difference multigrid solution of an optimal control optimality system. *SIAM Journal on Control and Optimization*, 41(5):1477–1497, 2002.

[33] A Borzì and V Schulz. *Computational optimization of systems governed by partial differential equations*, volume 8. SIAM, 2011.

[34] AP Calderon. On an inverse boundary value problem. In *Seminar on Numerical Analysis and Its Applications to Continuum Physics*, pages 65–73. Soc. Brasileira de Mathematica, Rio de Janeiro, 1980.

[35] E Casas, A Günther, and M Mateos. A paradox in the approximation of dirichlet control problems in curved domains. *SIAM J Control Optim.*, 49(5):1998–2007, 2011.

[36] E Casas and JP Raymond. Error estimates for the numerical approximation of dirich- let boundary control for semilinear elliptic equations. *SIAM J Control Optim.*, 45(5):1586–1611, 2006.

[37] E Casas and J Sokolowski. Approximation of boundary control problems on curved domains. *SIAM J Control Optim.*, 48(6):3746–3780, 2010.

[38] K Deckelnick, A Günther, and M Hinze. Finite element approximation of dirichlet boundary control for elliptic pdes on two-and three-dimensional curved domains. *SIAM Journal on Control and Optimization*, 48(4):2798–2819, 2009.

[39] K Deckelnick and M Hinze. Convergence of a finite element approximation to a state-constrained elliptic control problem. *SIAM Journal on Numerical Analysis*, 45(5):1937–1953, 2007.

[40] M Dodd and J Mueller. A real-time *d*-bar algorithm for 2d electrical impedance tomography data. *Inverse Problems and Imaging*, 8:1013–1031, 2014.

[41] MM Dunlop and AM Stuart. The Bayesian formulation of EIT: analysis and algorithms. *Inverse Problems and Imaging*, 10(4):1007–1036, 2016.

[42] LC Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 1998.

[43] AV Fursikov. *Optimal control of distributed systems. Theory and applications*. American Mathematical Soc., 1999.

[44] R Gaburro and WRB Lionheart. Recovering riemannian metrics in monotone families from boundary data. *Inverse Problems*, 25(4), 2009.

[45] R Gaburro and E Sinicich. Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities. *Inverse Problems*, 31, 2015.

[46] D Gilbarg and NS Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[47] R Glowinski, JL Lions, and J He. *Exact and approximate controllability for distributed parameter systems: a numerical approach (Encyclopedia of Mathematics and its Applications)*. Cambridge University Press, 2008.

[48] SJ Hamilton, M Lassas, and S Siltanen. A direct reconstruction method for anisotropic electrical impedance tomography. *Inverse Problems*, 30, 2014.

[49] SJ Hamilton, M Lassas, and S Siltanen. A hybrid segmentation and *d*-bar method for electrical impedance tomography. *SIAM J. Imaging Science*, 9:770–793, 2016.

[50] B Harrach. Uniqueness and lipschitz stability in electrical impedance tomography with finitely many electrodes. *Inverse problems*, 35(2):19, 2019.

[51] B Harrah and MN Minh. Enhancing residual-based techniques with shape reconstruction features in electrical impedance tomography. *Inverse Problems*, 32, 2016.

[52] M Hinze, R Pinnau, M Ulbrich, and S Ulbrich. *Optimization with PDE constraints*, volume 23 of *Mathematical Modelling: Theory and Applications*. Springer Science & Business Media, 2009.

[53] M Hinze and A Rösch. Discretization of optimal control problems. In *Constrained optimization and optimal control for partial differential equations*, volume 160 of *Internat. Ser. Numer. Math.*, pages 391–430. Birkhäuser/Springer Basel AG, Basel, 2012.

[54] DS Holder. *Electrical impedance tomography: methods, history and applications*. CRC Press, 2004.

[55] N Hyvönen, L Pälvärinta, and JP Tamminen. Enhancing d-bar reconstructions for electrical impedance tomography with conformal maps. *Inverse Problems and Imaging*, 12:373–400, 2018.

[56] B Jin, T Khan, and P Maass. A reconstruction algorithm for electrical impedance tomography based on sparsity regularization. *Int. J. Numer. Methods*, 89:337–353, 2012.

[57] B Jin, Y Xu, and Zou J. A convergent adaptive finite element method for electrical impedance tomography. *IMA J. Numer. Anal.*, 37:1520–1550, 2017.

[58] JP Kaipio, V Kolehmainen, E Somersalo, and M Vauhkonen. Statistical inversion and Monte Carlo sampling methods in electrical impedance tomography. *Inverse problems*, 16(5):1487, 2000.

[59] JP Kaipio, V Kolehmainen, M Vauhkonen, and E Somersalo. Inverse problems with structural prior information. *Inverse problems*, 15(3):713, 1999.

[60] JP Kaipio and E Somersalo. *Statistical and Computational Inverse Problems,*. Springer, 2005.

[61] C Kenig, J Sjöstrand, and G Uhlmann. The Calderon problem with partial data. *Annals of Mathematics*, 165:567–591, 2007.

[62] C Kenig, J Sjöstrand, and G Uhlmann. The Calderon problem with partial data on manifolds and aplications. *Analysis and PDE*, 6:2003–2048, 2013.

[63] MV Klibanov, J Li, and W Zhang. Convexification of electrical impedance tomography with restricted dirichlet-to-neumann map data. *Inverse Problems*, 35:33, 2019.

[64] K Knudsen, M Lassas, JL Mueller, and S Siltanen. D-bar method for electrical impedance tomography with discontinuous conductivities. *SIAM Journal on Applied Mathematics*, 67:893–913, 2007.

[65] K Knudsen, M Lassas, JL Mueller, and S Siltanen. Reconstructions of piecewise constant conductivities by the d-bar method for electrical impedance tomography. *Journal of Physics: Conference Series*, 124, 2008.

[66] K Knudsen, M Lassas, JL Mueller, and S Siltanen. Regularized d-bar method for the inverse conductivity problem. *Inverse Problems and Imaging*, 3:599–624, 2009.

[67] RV Kohn and M Vogelius. Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.*, 37:289–298, 1984.

[68] RV Kohn and M Vogelius. Determining conductivity by boundary measurements. ii. interior results. *Comm. Pure Appl. Math.*, 38:643–667, 1985.

[69] V Kolehmainen, M Lassas, P Ola, and S Siltanen. Recovering boundary shape and conductivity in electrical impedance tomography, inverse problems and imaging. *Inverse Problems and Imaging*, 7:217–242, 2013.

[70] O Kwon, JK Seo, and JR Yoon. A real-time algorithm for the location search of discontinuous conductivities with one measurement. *Comm. Pure Appl. Math.*, 55:1–29, 2002.

[71] OA Ladyzhenskaya. *The boundary value problems of mathematical physics*. Springer Science & Business Media, 2013.

[72] OA Ladyzhenskaya and NN Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc., 1968.

[73] S Lasanen, JML Huttunen, and L Roininen. Whittle-matern priors for bayesian statistical inversion with applications in electrical impedance tomography. *Inverse problems and Imaging*, pages 561–586, 2014.

[74] M Lassas, E Saksman, and S Siltanen. Discretization-invariant bayesian inversion and besov space priors. *Inverse Problems and Imaging*, pages 87–122, 2009.

[75] M Lassas and G Uhlmann. On determining a riemannian manifold from the dirichlet-to-neumann map. *Ann. Sci. Ecole Norm. Sup.*, 34:771–787, 2001.

[76] M Lassas, G Uhlmann, and M Taylor. The dirichlet-to-neumann map for complete riemannian manifolds with boundary. *Comm. Anal. Geom.*, 11(2):207–221, 2003.

[77] S Laufer, A Ivorra, VE Reuter, B Rubinsky, and SB Solomon. Electrical impedane characterization of normal and cancerous human hepatic tissue. *Physiological Measurements*, 31:995–1009, 2010.

[78] A Lechleiter and A Rieder. Newton regularization for impedance tomography: convergence by local injectivity. *Inverse problems*, 24(6):18, 2008.

[79] JM Lee and G Uhlmann. Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.*, 42:1097–1112, 1989.

[80] WRB Lionheart. Conformal uniqueness results in anisotropic electrical impedance imaging. *Inverse Problems*, 13:125–134, 1997.

[81] JL Lions. Optimal control of systems governed by partial differential equations problèmes aux limites. 1971.

[82] JL Lions. Decomposition methods for variational inequalities. In J.L. Menaldi, E. Rofman, and A. Sulem, editors, *Optimal Control and Partial Differential Equations: In Honour of Professor Alain Bensoussan's 60th Birthday*, pages 38–46. IOS Press, 2001.

[83] G Matthias, J Bangti, and X Lu. An analysis of finite element approximation in electrical impedance tomography. *Inverse Problems*, 30, 2014.

[84] S May, R Rannacher, and B Vexler. Error analysis for a finite element approximation of elliptic dirichlet boundary control problems. *SIAM J Control Optim.*, 51(3):2585–2611, 2013.

[85] AI Nachman. Reconstructions from boundary measurements. *Annals of Mathematics*, 128(3):531–576, 1988.

[86] AI Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Annals of Mathematics*, 143(2):71–96, 1996.

[87] SM Nikol'skii. *Approximation of Functions of Several Variables and Imbedding Theorems*. Springer-Verlag, New York-Heidelberg, 1975.

[88] JK Seo, J Lee, SW Kim, H Zribi, and EJ Woo. Frequency-difference electrical impedance tomography: algorithm development and feasibility study. *Phys. Meas.*, 29:929–941, 2008.

[89] JK Seo and EJ Woo. Magnetic resonance electrical impedance tomography. *SIAM Review*, 53:40–68, 2011.

[90] E Somersalo, M Cheney, and D Isaacson. Existence and uniqueness for electrode models for electric current computed tomography. *SIAM Journal on Applied Mathematics*, 52(4):1023–1040, 1992.

[91] J Sylvester. An anisotropic inverse boundary value problem. *Comm. Pure Appl. Math.*, 43(2):201–232, 1990.

[92] J Sylvester and G Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, pages 153–169, 1987.

[93] AN Tikhonov and FP Vasil'ev. Methods for the solution of ill-posed extremal problems. In *Mathematical models and numerical methods*, number 3 in Banach Center Publications, pages 297–342, Warsaw, 1978. Stefan banach Internat. Math. Center.

[94] F Tröltzsch. *Optimal control of partial differential equations: theory, methods, and applications*, volume 112. American Mathematical Soc., 2010.

[95] FP Vasil'ev. Methods for solving extremal problems. minimization problems in function spaces, regularization, approximation. *Moscow, Nauka*, 1981.

[96] T Widlak and O Scherzer. Hybrid tomography for conductivity imaging. *Inverse Problems*, 28, 2012.