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Stability Results for Special Solutions of Scalar-Field Equations with Variable Coeffcients

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"Stability Results for Special Solutions of Scalar-Field Equations with Variable Coefficients"
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Abstract

Title: *Stability Results for Special Solutions of Scalar-Field Equations with Variable Coefficients*

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We study the long-time behavior of general semilinear scalar-field equations on the real line with variable coefficients in the linear terms.

In the first part of the dissertation, we take the coefficients to be uniformly small, but slowly decaying, perturbations of a constant-coefficient operator. We are motivated by the question of how these perturbations of the equation may change the stability properties of kink solutions (one-dimensional topological solitons). We prove existence of a stationary kink solution in our setting, and perform a detailed spectral analysis of the corresponding linearized operator, based on perturbing the linearized operator around the constant-coefficient kink. We derive a formula that allows us to check whether a discrete eigenvalue emerges from the essential spectrum under this perturbation. Known examples suggest that this extra eigenvalue may have an important influence on the long-time dynamics in a neighborhood of the kink. We also establish orbital stability of solitary-wave solutions in the variable-coefficient regime, despite the possible presence of negative eigenvalues in the linearization.

In the second part, we address special solutions that are constant or perturbations of a constant state. For these solutions, we are able to prove asymptotic stability under an oddness assumption on the initial data, using a Virial argument based on defining suitable Lyapunov functionals.

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List of Notations

- $C^k(\Omega)$ is the space of k-times differentiable functions in Ω.

- $\|g\|_{C^k(\Omega)} := \sum_{n=0}^{k} \sup_{x\in\Omega} |g^{(n)}(x)|$.

- $C^0(\Omega) \equiv C(\Omega)$ denotes the space of continuous functions on Ω.

- $L^p(\Omega)(1 \leq p < +\infty)$ is the space of p-integrable functions in Ω.

- $\|g\|_{L^p(\Omega)} := \left(\int_\Omega |g(x)|^p dx\right)^{\frac{1}{p}}$.

- $L^p_{loc}(\Omega)$ is the space of locally p-integrable functions in Ω: the space of p-integrable functions on compact subsets of Ω.

- Weighted inner product
$$\langle f, g\rangle_\omega := \int_{\mathbb{R}} \omega f g \,\mathrm{d}y,$$
with a weight $\omega(y)$.

- $L^\infty(\Omega)$ is the spaces of essentially bounded measurable functions $g : \Omega \to \mathbb{R}$ endowed with the norm
$$\|g\|_{L^\infty} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |g(\mathbf{x})|.$$

- $\sigma(\mathcal{L}_T)$ spectrum of operator $\mathcal{L}_T$

- $\sigma_{ess}(\mathcal{L}_T)$ essential spectrum of operator $\mathcal{L}_T$

- $\sigma_d(\mathcal{L}_T)$ discrete spectrum of operator $\mathcal{L}_T$

- $W^{k,p}(\Omega), (1 \geq p \geq +\infty)$ is the time-independent Sobolev space of functions in $L^p(\Omega)$

- Finite Sobolev norm

$$\|g\|_{W^{k,p}(\Omega)} := \left(\sum_{i=0}^{k} \|g^{(i)}\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

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Dedication

My dissertation is dedicated to my beloved husband, my Father, Mother, brother, sisters, and children. I love them all, and may God bless them all.

Chapter 1

Introduction

1.1 General overview

This dissertation is concerned with scalar-field equations of the form

$$\partial_t^2 u - Lu + F'(u) = 0, \quad t \geq 0, x \in \mathbb{R},$$

where L is a second-order linear differential operator in x, and F is a potential. More precise conditions on L and F will be placed below. One important prototype is the (variable-coefficient) sine-Gordon equation, which corresponds to the potential $F(u) = 1 - \cos(u)$. For many of the results herein, the special case of sine-Gordon is considered first, before extending the result (as much as possible) to the general case.

The main concern is the existence and stability of special solutions, either of kink (topological soliton) type, which connect two different minima of F, constant stationary states corresponding to critical points of F, or near-constant stationary states that are perturbations of critical points of F. Orbital and linear stability can be studied in the context of a general F, but the more delicate property of asymptotic stability depends on the specific choice of F.

There are two main parts to the dissertation: First, Chapters 2 though 5[1] that deal with kink-type solutions. The main results are the existence and orbital stability of kinks, and a detailed study of the linearization around kinks. In this part, we do not make any oddness or evenness assumptions about F, u, or the coefficients of L. Second, Chapters 6 through 9 deal with constant and near-constant solutions. The main results are existence of near-constant steady states, and asymptotic stability of constant and near-constant steady states with respect to odd perturbations, for three important choices of potential $F(u)$. See below for more precise statements of all theorems.

1.2 Historical Review

The Klein-Gordon equation is one of the common classical equations of mathematical physics in one dimension. It is a linear model for the propagation of dispersive waves. The Klein-Gordon equation is given by

$$\phi_{tt} - \phi_{xx} + \phi = 0, \tag{1.2.1}$$

where $\phi(x,t)$ is a real wave field, t is the time and x is the spatial coordinate. Among these models related to the Klein-Gordon equation, the Sine-Gordon type models bear topological solutions called kink or antikink solutions. These solutions will be solitary waves which connect two minima of the potential. These solutions will not change their shape when the time changes with constant velocity. There are non-topological, oscillatory solutions called breathers which are states of a kink and antikink. There are also other solutions which can be interpreted as a nonlinear superposition of a kink and a breather called wobbles. The Sine-Gordon equation is given by

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0, \tag{1.2.2}$$

Another form of the one diminsional Sine-Gordon equations is written in the light-cone coordinates $\xi = \frac{x+t}{2}, \psi = \frac{x-t}{2}$ we have

$$\phi_{\xi\psi} = \sin\phi, \tag{1.2.3}$$

[1] Chapters 2 through 5 appeared in a slightly different form in [1].

The Sine-Gordon was known by Jacques Edmond Émile Bour [5] in 1862 as the Gauss-Codazzi equation describing 2-dimensional surfaces with constant negative curvature embedded into 3D Euclidean space. In the nineteenth century, Albert Victor Bäcklund [4] discovered the integrability of the one dimensional Sine-Gordon equation in the form of equation (1.2.3) in the differential geometry of pseudospheres. In 1939, the form of the continuous limit of Frenkel-Kontorova model of dislocations in solid state, which it was known as a discrete Sine-Gordon equation for lattice wave field $\phi_n(t)$ where n is the discrete coordinate.

$$\frac{d^2\phi_n}{dt^2} - C(\phi_{n+1} + \phi_{n-1} - 2\phi_n) + \sin\phi_n = 0 \tag{1.2.4}$$

where C is a constant of the intersite coupling. After that, a similar model was developed and used for a chain of adsorbed atoms on a metallic surface [39]. Many other physical applications of the Frenkel-Kontorova model have been known. In the 1970s, the Sine-Gordon equation has received great popularity as result of the discovery of its integrability by means of the Inverse Scattering Transform (IST) method and development understanding of the significance of this equation as model of many important physical systems. The name Sine-Gordon was known as a pun based on Klein-Gordon. The first paper with the name of Sine-Gordon was published by J. Rubinstein in 1970 [59]. In 1976, the double Sine-Gordon equation is defined as a separate nonintegrable model which takes the form

$$\phi_{tt} - \phi_{xx} + \sin\phi = \epsilon\sin(\frac{\phi}{2}) \tag{1.2.5}$$

where $\sin(\frac{\phi}{2})$ represents the fundamental harmonic while $\sin\phi$ is the second harmonic and ϵ is a small parameter [15].

The one dimensional Sine-Gordon equation is a fundamental model of modern mathematical physics because the Sine-Gordon equation is nonlinear, solvable by means of IST, and it is a universal model for media combining the wave dispersion (the same as in the Klein-Gordon equation) and the nonlinearity that is a periodic function of the field variable.

The interesting feature of Sine-Gordon is the existence of solitons and multisolitons. The

Sine-Gordon has 1-soliton solutions given by

$$\phi_{soliton}(x,t) = 4\arctan(e^{m\gamma(x-vt)+\delta})$$

where

$$\gamma^2 = \frac{1}{1-v^2}.$$

The general form of the equation is given by

$$\phi_{tt} - \phi_{xx} + m^2 \sin\phi = 0. \tag{1.2.6}$$

If we choose γ positive root, $\phi_{soliton}$ is called a kink and it represents a twist in the variable ϕ which take the system from one solution $\phi = 0$ to $\phi = 2\pi$. If we choose γ negative root, $\phi_{soliton}$ is called an antikink and the form can be represented by using a *Bäcklund* transform applied to the trivial constant vacuum solution and integration first order differentials.

$$\begin{aligned}
\phi'_u &= \phi_u + 2\beta\sin(\frac{\phi'+\phi}{2}),\\
\phi'_v &= -\phi_v + \frac{2}{\beta}\sin(\frac{\phi'-\phi}{2}), \qquad \text{with } \phi = \phi_0 = 0.
\end{aligned}$$

Multi-solitons can be acquired through continued application of the *Bäcklund* transform to the 1-soliton solution as known by Binachi lattice. The 2-soliton of the Sine-Gordon equation shows some of the characteristic features of solitons. The traveling Sine-Gordon kink and antikink pass through each other if it is permeable and the only observed effect is a phase shift. The colliding solitons regain their shape and velocity; such kind of interaction is called elastic collision. Another type of 2-soliton is known as a breather, and there are three types of breathers: standing, traveling large amplitude, and traveling small amplitude. 3-solitons interfere between a travailing kink or antikink and standing breather. The shift of breather is given by

$$\Delta_B = \frac{2\arctan\sqrt{(1-w^2)(1-v_k^2)}}{\sqrt{1-w^2}}$$

where v_k is velocity of the kink and w is the breather's frequency.

The most solution to the one dimension of Sine-Gordon are solitons. For example. the *Bäcklund* transform generates (n+1)-soliton solutions of the Sine-Gordon equation from the n-solitons ones so making it possible to generate an infinite hierarchy of solutions with increasing complexity starting of the trivial solution $\phi = 0$ which plays the role of the zero-soliton state.

1.3 Related work

1.3.1 ϕ^4 Constant-Speed Case

A classical nonlinear equation known as the ϕ^4 model in one dimension space is used in quantum field theory , statistical mechanics and other areas of physics. We refer the reader to see the instance [49, 54, 55, 68, 69, 71] for the phyical background. In the case where the propagation speed c is allowed to vary with position, the equation is given by:

$$\partial_t^2\phi - c^2(x)\partial_x^2\phi = \phi - \phi^3, (t, x) \in R \times R. \tag{1.3.1}$$

where c(x) is a uniformly positive function. The author interested in the case where c is even functions that are small deviations from the constant until speed $c(x) \equiv 1$. The case where $c(x) \equiv 1$, the stationary solution to (1.3.1) is known as explicitly and is given by:

$$H(x) := \tanh(\frac{x}{\sqrt{2}})$$

Where $H(x)$ is a time independent solution of (1.3.1) called the *kink* . It is the unique bounded, odd solution of the equation $-H''(X) = H(x) - H(x)^3$. Note that the important property of (1.3.1) that the energy is given by:

$$E(\phi, \partial_t\phi) := \int \frac{1}{c^2}\big(\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}c^2(\partial_x\phi)^2 + \frac{1}{4}(1-\phi^2)^2\big)dx$$

is formally conserved along the flow.

Since the energy of the *kink* of $E(H, 0)$ is finite, the pertubations of the form $(\phi, \partial_t \phi) = (H + \phi_0, \phi_1)$ are referred to as perturbations in the energy space where $(\phi_0, \phi_1) \in H^1 \times L^2$. The model (1.3.1) is locally well-posed for initial data $(\phi(0), \partial_t \phi(0))$ of the form $(H + \phi_0^{in}, \phi_1^{in})$ where $\phi^{in} = (\phi_0^{in}, \phi_1^{in}) \in H^1 \times L^2$, by standard arguments. Note that if we have odd initial data then the solution of (1.3.1) will be odd. Now for the long time behavior of solution (1.3.1), Henry, Perez and Werszinski [29] showed that the kink is orbitally stable of kink with respect to small perturbations in the energy space. In paper [42] Kowalczyk, Martel and Muñoz studied the case of odd perturbations (because the traveling wave will be fixed in place) and that can be improved to asymptotic stability. Their work is elementary and avoids the use of dispersive estimates. Their work depended on the work of Martel and Merle on the generlized Korteweg-de Vries equations [50, 51] and Merle and Raphael [52] on the mass-critical nonlinear Schrödinger equation but because of the exchange of energy between internal oscillations and radiation and the different decay rates of the corresponding components of the solution they get a difficult result. These difficulties were seen earlier in the context of general Klein-Gordon equations with potential given by Soffer and Weinstein [66]. They conjectured a similar mechanism was at work in ϕ^4 model. The assumption of odd perturbations has been seen in other works which studied asympototic stability of the solutions [40, 41]. In the ϕ^4 model with odd pertubations, the issues related to energy exchange appear. The authors [42] conjecture that the kink is asymptotically stable with respect to general perturbations in the energy space.

1.3.2 Variable-Speed case

Dr. Stanley Snelson [65] extended the result of [42] to (1.3.1) with a certain class of non-constant propagation speeds $c(x)$. He exchanged the second order coefficients in (1.3.1) for a small first order term by taking the variables $y = \int_0^x (\frac{1}{c(s)}) ds$. Defining $\Phi(t, y) = \phi(t, x(y))$. The equation is given by

$$\partial_t^2 \Phi - \partial_y^2 \Phi + b(y) \partial_y \Phi = \Phi - \Phi^3, \tag{1.3.2}$$

with $b(y) = \frac{1}{c(x(y))}\frac{d}{dy}c(x(y))$. The author deal with drift cofficents b that are odd and satisfied:

$$|b(y)| \le \delta e^{-\sqrt{2}|y|}, |b'(y)| \le \delta \tag{1.3.3}$$

for some constant $\delta > 0$. He assumed that in (1.3.1) $c(x) = 1 + c_\delta(x)$, with taken c_δ even , twice differentiable and satisfied

$$|c_\delta(x)| + |c_\delta'(x)| \le \delta e^{-c_1|x|}, |c_\delta''(x)| \le \delta,$$

with $c_1 = \frac{\sqrt{2}}{(1-\delta)}$. Note that the oddness in x is equivalent oddness in y and the solution of our equation (1.3.1), (1.3.2) are odd if the initial data are odd. The main result that the author found is the existence of a stationary solution and asymptotic stability in the variable-speed case.

Theorem 1.3.1. *Assume that b satisfies* (1.3.3) . *Then there exist an odd, bounded, time-independent solution K of* (1.3.2) . *Furthermore, for* $H(y) = \tanh(\frac{y}{\sqrt{2}})$, *the difference* $H_\delta = K\text{-}H$ *satisfies* $|H_\delta(y)| + |H_\delta'(y)| \le \delta e^{-\sqrt{2}|y|}$

In the proof, the author found a stationary solution to the equation (1.3.2) i.e an odd K solving

$$-\partial_y^2 K + b(y)\partial_y K = K - K^3 \tag{1.3.4}$$

He look for $H_\delta(y)$ such that $K(y) = H(y) + H_\delta(y)$ solve (1.3.4).

He got the equation:

$$-\partial_y^2 H_\delta + b(y)\partial_y H_\delta + (3H^2 - 1)H_\delta = -b(y)\partial_y H - H_\delta^3 - 3HH_\delta^2 \tag{1.3.5}$$

The author wrote (1.3.5) as:

$$\mathcal{L}_b H_\delta = -H_\delta^3 - 3HH_\delta^3 - b(y)\partial_y H$$

where

$$\mathcal{L}_b = -\partial_y^2 + b(y)\partial_y + (3H^2 - 1) = \mathcal{L} + b(y)\partial_y.$$

First he found a fundamental system for $\mathcal{L}Y = 0$ and $\mathcal{L}Z = 0$ is given by:

$$Y_0(y) = \frac{1}{2}\text{sech}^2(\frac{y}{\sqrt{2}})$$

$$Z_0(y) = \frac{-1}{32}\text{sech}^2(\frac{y}{\sqrt{2}})\left(12y + 8\sqrt{2}\sinh(\sqrt{2}y) + \sqrt{2}\sinh(2\sqrt{2}y)\right)$$

Then he found Y_b, Z_b with $\mathcal{L}_bY_b = \mathcal{L}_bZ_b = 0$. The second independent solution Z_b is given by:

$$Z_b(y) = Y_b(y)\int_0^y \frac{exp(\int_0^w b(s)ds)}{(Y_b(w))^2}dw.$$

Define the Green's function G_b for $\mathcal{L}_b$:

$$G_b(y,w) = \begin{cases} \frac{Y_b(y)Z_b(w)}{p(w)}, & 0 \le w < y, \\ \\ \frac{Y_b(w)Z_b(y)}{p(w)}, & 0 \le y < w, \end{cases}$$

The author wrote (1.3.5) as a nonlinear integral equation for $H_\delta(y)$:

$$H_\delta(y) = (\mathcal{T}H_\delta)(y) = h(y) - \int_0^\infty G_b(y,w)\left[H_\delta^3(w) + 3H(w)H_\delta^2(w)\right]dw. \tag{1.3.6}$$

where

$$h(y) = -\int_0^\infty G_b(y,w)b(w)\partial_w H(w)dw.$$

The author showed that $\mathcal{T}$ has a unique fixed point in a suitable class, so that $\|\mathcal{T}(\eta_1)(y) - \mathcal{T}(\eta_2)(y)\|_\sim \le C\delta\|\eta_1 - \eta_2\|_\sim$. If $\delta < \frac{1}{C}$ then $\mathcal{T}$ is a contraction in A_δ and a unique solution H_δ to (1.3.5) exist in A_δ. By differentiating (1.3.6), we get

$$|H_\delta(y)| + |H_\delta'(y)| \le \delta e^{-\sqrt{2}|y|}$$

For studying long-time asymptotics of odd perturbations of K(y) in the energy space, let $\phi(t) = (\phi_0(t), \phi_1(t)) \in H^1 \times L^2$ is odd in y and set $\Phi = K + \phi_0$, $\partial_t \Phi = \phi_1$ in (1.3.2) . Then the perturbation ϕ satisfies:

$$\begin{cases} \partial_t \phi_0 = \phi_1 \\ \partial_t \phi_1 = -\mathcal{L}_K \phi_0 - (3K\phi_0^2 + \phi_0^3) \end{cases} \tag{1.3.7}$$

where $\mathcal{L}_K$ is the linearized operator around K which is given by:

$$\mathcal{L}_K = -\partial_y^2 - b(y)\partial_y - 1 + 3K^2 = \mathcal{L} - b(y)\partial_y + d(y) \tag{1.3.8}$$

$\mathcal{L}$ is the linerarization around $H(y) = \tanh(\frac{y}{\sqrt{2})}$ and $d(y) = 3(K^2 - H^2)$ with the inner products which is :

$$\langle f, g \rangle = \int_R f(y)g(y)dy$$
$$\langle f, g \rangle_p = \int_R p(y)f(y)g(y)dy$$

where $p(y) = \exp(\int_0^y b(s))ds$. Note that $\mathcal{L}$ is self-adjoint with respect to $\langle ., . \rangle$ and $\mathcal{L}_K$ is self-adjoint with respect to $\langle ., . \rangle_p$. The main theorems of [65] were

Theorem 1.3.2. *There exist $\delta > 0$, $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$ and for any odd $\phi^{in} \in H^1 \times L^2$ with $\|\phi^{in}\|_{H^1 \times L^2} < \epsilon$ the solution ϕ of (1.3.7) with initial data $\phi(0) = \phi^{in}$ exist globally $\in H^1 \times L^2$ satisfies:*

$$\lim_{t \to \pm\infty} \|\phi(t)\|_{H^1(I) \times L^2(I)} = 0, \tag{1.3.9}$$

for any bounded interval $I \subset R$.

Theorem 1.3.3. *The operator $(\mathcal{L}_K)$ has real, simple eigenvalues corresponding λ_0, λ_1 such that $|\lambda_0| \leq \delta$ and $|\lambda_1 - \frac{3}{2}| \leq \delta$. The corresponding eigenfunction $\bar{Y_0}$ and $\bar{Y_1}$ are even and odd* *respectively and satisfy:*

$$|\bar{Y_0}(y) - Y_0(y)| + |\bar{Y_0}'(y) - Y_0'(y)| \le \delta e^{-\sqrt{2}|y|}$$
$$|\bar{Y_1}(y) - Y_1(y)| + |\bar{Y_1}'(y) - Y_1'(y)| \le \delta e^{-\frac{|y|}{\sqrt{2}}}$$

where Y_0 and Y_1 are the eigenfunction of $\mathcal{L}$ corresponding to 0 and $\frac{3}{2}$. Furthermore, the only discrete rigenvalue of $\mathcal{L}_K$ corresponding to odd eigenfunction and the continious spectrum $\sigma_c(\mathcal{L}) = [2, \infty)$.

Chapter 2

Kink-Type Solutions

2.1 Scalar-field equations

We consider a semilinear, variable-coefficient scalar field equation of the form

$$\partial_t^2 u - \big[a(x)\partial_x^2 u + b(x)\partial_x u + c(x)u\big] + F'(u) = 0, \quad x \in \mathbb{R}. \tag{2.1.1}$$

Our assumptions on the potential F are

$$\begin{gathered} F \in C^3(\mathbb{R}), \quad F(a_-) = F(a_+) = 0 \text{ for some } a_- < a_+, \\ F'(a_\pm) = 0, \quad F''(a_\pm) = m^2 > 0, \quad F(s) > 0, s \in (a_-, a_+). \end{gathered} \tag{2.1.2}$$

The linear operator $a(x)\partial_x^2 + b(x)\partial_x + c(x)$ is assumed to be a perturbation of the $1D$ Laplacian ∂_x^2. More precisely, for a small parameter $\delta > 0$, we assume

$$\||a-1| + |\partial_x a| + |b| + |c|\|_{L^1(\mathbb{R})} + \||a-1| + |\partial_x a| + |b| + |c|\|_{L^\infty(\mathbb{R})} \le \delta. \tag{2.1.3}$$

With $\omega(x) = \exp(\int_{-\infty}^x b(z)/a(z)\,\mathrm{d}z)$, the energy functional

$$E(u) := \int_{\mathbb{R}} \frac{\omega(x)}{a(x)} \left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}a(\partial_x u)^2 - \frac{1}{2}cu^2 + F(u)\right) \mathrm{d}x,$$

is formally conserved under the flow of (2.1.1). We note that equation (2.1.1) is not invariant under translations, and that we make no parity assumptions on F or the coefficients a, b, and c.

We are interested in the long-time behavior of solutions to (2.1.1). Our first result (Theorem 2.3.1) is the existence of a stationary solution T of *kink* or *solitary-wave* type, i.e. an increasing stationary solution with $T(x) \to a_\pm$ as $x \to \pm\infty$. Standard arguments then show that (2.1.1) is locally well-posed for initial data $(u, \partial_t u)|_{t=0} \in H^1_T(\mathbb{R}) \times L^2(\mathbb{R})$, where $H^1_T(\mathbb{R}) = \{\varphi : \varphi - T \in H^1(\mathbb{R})\}$. In this context, $H^1_T(\mathbb{R}) \times L^2(\mathbb{R})$ is referred to as the *energy space*, and indeed, it is not hard to see (using in particular that $|\omega/a - 1| \lesssim \delta$) that functions in this space have finite energy. Our goal is to study the stability of T with respect to small perturbations in the energy space.

2.2 Motivation

One-dimensional kinks such as $T(x)$ are the simplest examples of topological solitons, and thus are an important model for physical phenomena arising in areas such as quantum field theory, condensed matter physics, and cosmology, among others. (See [68, 48, 37, 49] for some physics-oriented discussions.) Understanding their stability has proven to be a difficult mathematical challenge. The majority of work focuses on the constant-coefficient version of (2.1.1),

$$\partial_t^2 u - \partial_x^2 u + F'(u) = 0. \tag{2.2.1}$$

In this constant-coefficient regime, it is standard that the assumptions (2.1.2) imply the existence of a kink solution connecting a_- and a_+. (Convenient proofs of this fact may be found in [44, Lemma 1.1] or [32, Proposition 2.1].) This constant-coefficient stationary kink, which we denote by S, satisfies

$$-S'' + F''(S) = 0, \quad \lim_{x\to-\infty} S(x) = a_-, \lim_{x\to\infty} S(x) = a_+. \tag{2.2.2}$$

We find in Theorem 2.3.1 that T and S are close in an appropriate norm.

Orbital stability of S in the constant-coefficient setting has been known for some time [29], and we extend this to our setting in Theorem 2.3.3. Asymptotic stability of kinks is more subtle, and depends on the specific choice of potential F. In particular, the two most studied versions of (2.2.1) are the ϕ^4 equation with nonlinearity $F'(u) = u^3 - u$, which is known to be asymptotically stable with respect to odd perturbations [42] and conjectured to be asymptotically stable in general; and the sine-Gordon equation with nonlinearity $F'(u) = \sin(u)$, which is not asymptotically stable, at least with respect to perturbations in the energy space. (See Section 2.4 for more on these examples.)

Our motivation is to understand the effect of *linear* perturbations of the equation (2.2.1) on the stability properties of kink solutions. On the one hand, given that (2.2.1) is in some sense an idealized model, it is important on physical grounds to understand whether stability properties of kink solutions persist under perturbations of the equation. There is also reason to expect such perturbations to have a nontrivial qualitative impact on the stability analysis (rather than simply adding a small error term) in some situations. As we explain below, this is connected with the possibility that a discrete eigenvalue may emerge from the essential spectrum of the linearized operator around the kink.

2.3 Main results

Before stating our main theorems, we make a technically convenient change of variables in (2.1.1). Letting $y = \int_0^x a^{-1/2}(z)\,\mathrm{d}z$, and abusing notation by writing $u(t, y) = u(t, x(y))$ and $b(y) = b(x(y)) - a^{-1/2}(x(y))\frac{d}{dy}a^{1/2}(x(y))$, we have

$$\partial_t^2 u - [\partial_y^2 u + b(y)\partial_y u + c(y)u] + F'(u) = 0. \tag{2.3.1}$$

The hypotheses (2.1.3) imply

$$\||b| + |c|\|_{L^1(\mathbb{R})} + \||b| + |c|\|_{L^\infty(\mathbb{R})} \leq C_0\delta, \tag{2.3.2}$$

for some $C_0 > 0$.

Our first result is the existence of a stationary kink:

Theorem 2.3.1. *For $\delta > 0$ sufficiently small, there exists a solution T to*

$$-T'' - b(y)T' - c(y)T + F'(T) = 0, \quad \lim_{y\to-\infty} T(y) = a_-, \lim_{y\to\infty} T(y) = a_+. \tag{2.3.3}$$

This solution can be written $T(y) = S(y) + S_b(y)$, where S solves (2.2.2) and

$$\|S_b\|_{W^{1,1}(\mathbb{R})} + \|S_b\|_{W^{1,\infty}(\mathbb{R})} \leq C\delta.$$

Unlike S, which satisfies $|S(x) - a_\pm| \lesssim e^{\mp mx}$ and $|S'(x)| \lesssim e^{-m|x|}$, our static kink T does not necessarily posess exponential tails. This behavior is reminiscent of some higher-order, constant-coefficient field theories that do not fit into the assumptions (2.1.2) (see e.g. [38]). Under additional exponential decay assumptions on b and c, it is possible to show T has exponential asymptotics at $\pm\infty$ as in [65, Theorem 1.1], but we do not explore the details here.

Our next result concerns the linearized operator around T. Writing $u(t,y) = T(y)+\varphi(t,y)$, the perturbation φ satisfies

$$\partial_t^2\varphi - \partial_y^2\varphi - b\partial_y\varphi - c\varphi = F'(T) - F'(T+\varphi).$$

Adding $F''(T)\varphi$ to both sides, and defining the linear operator $\mathcal{L}_T = -\partial_y^2 - b\partial_y - c + F''(T)$ and the nonlinearity $\mathcal{N}(T,\varphi) = F'(T) - F'(T+\varphi) + F''(T)\varphi = O(\varphi^2)$, the equation for φ can be written as a nonlinear Klein-Gordon equation:

$$\partial_t^2\varphi + \mathcal{L}_T\varphi = \mathcal{N}(T,\varphi). \tag{2.3.4}$$

We are most interested in situations where the spectrum of $\mathcal{L}_S = -\partial_y^2 + F''(S)$, the operator corresponding to the constant-coefficient kink, is known exactly. We then have $\mathcal{L}_T = \mathcal{L}_S - b\partial_y - c + F''(T) - F''(S)$, and we ask how the perturbation $-b\partial_y - c + F''(T) - F''(S)$ changes the spectral properties of $\mathcal{L}_S$.

The $L^2(\mathbb{R})$ spectrum of $\mathcal{L}_S$ is given by

$$\sigma(\mathcal{L}_S) = \{0, \lambda_1, \dots, \lambda_n\} \cup [m^2, \infty),$$

where $\lambda_1, \dots, \lambda_n$ is a possibly empty, increasing collection of positive, simple eigenvalues. The eigenfunction corresponding to 0 is exactly S', the translation invariance mode.

As expected, discrete eigenvalues λ_i will drift to nearby discrete eigenvalues λ_i' of $\mathcal{L}_T$ under the perturbation. A more delicate question is whether an extra discrete eigenvalue emerges from the essential spectrum. This aspect is especially relevant when $\mathcal{L}_S$ has a threshold resonance, i.e. a function $R \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ satisfying $\mathcal{L}_S R = m^2 R$, as is the case for both the ϕ^4 and sine-Gordon equations. We derive a criterion in terms of R and the coefficients b and c that governs whether the resonance drifts into a discrete eigenvalue.

Our results on the spectrum of $\mathcal{L}_T$ are collected in the following theorem:

Theorem 2.3.2. *Let $\mathcal{L}_T$ and $\mathcal{L}_S$ be as defined above. There exists a universal $c_0 > 0$ such that:*

(a) The spectrum $\sigma(\mathcal{L}_T)$ is real, the essential spectrum $\sigma_{ess}(\mathcal{L}_T) = \sigma_{ess}(\mathcal{L}_S) = [m^2, \infty)$, and $\sigma(\mathcal{L}_T)$ lies in the $c_0\delta$-neighborhood of $\sigma(\mathcal{L}_S)$.

(b) For every eigenvalue $\lambda \in \sigma_d(\mathcal{L}_S)$ with eigenvector Y_λ, there is a corresponding $\lambda' \in \sigma_d(\mathcal{L}_T)$. The eigenvalue λ' is real, simple, and satisfies $|\lambda - \lambda'| \le c_0\delta$. Also, if

$$A := \int_{\mathbb{R}} Y_\lambda[(F''(T) - F''(S) - c)Y_\lambda - b\partial_y Y_\lambda]\,\mathrm{d}y \neq 0,$$

then $\lambda' - \lambda$ has the same sign as A. The eigenfunction $Y_{\lambda'}$ of $\mathcal{L}_T$ corresponding to λ' satisfies $|Y_{\lambda'}(y)| + |Y_{\lambda'}'(y)| \lesssim e^{-\sqrt{m^2-\lambda'}|y|}$. Furthermore, for suitable normalizations of Y_λ and $Y_{\lambda'}$, we have

$$\|e^{\sqrt{m^2-\lambda'}|y|}Y_{\lambda'}(y) - e^{\sqrt{m^2-\lambda}|y|}Y_\lambda(y)\|_{L^\infty(\mathbb{R})} \le C\delta,$$

for a universal constant $C > 0$.

(c) If m^2 is a simple resonance of $\mathcal{L}_S$, and

$$\int_{\mathbb{R}} R[(F''(T) - F''(S) - c)R - b\partial_y R]\,\mathrm{d}y < 0,$$

then there exists a discrete eigenvalue λ of $\mathcal{L}_T$ with $0 < m^2 - \lambda < c_0\delta$. The eigenfunction Y_λ also satisfies $|Y_\lambda(y)| + |Y_\lambda'(y)| \lesssim e^{-\sqrt{m^2-\lambda}|y|}$ and

$$\|Y_\lambda(y) - e^{-\sqrt{m^2-\lambda}|y|}R(y)\|_{L^\infty(\mathbb{R})} \leq C(k+\delta)e^{-\sqrt{m^2-\lambda}|y|},$$

for suitable normalizations of Y_λ and R.

If

$$\int_{\mathbb{R}} R[(F''(T) - F''(S) - c)R - b\partial_y R]\,\mathrm{d}y > 0,$$

then there are no eigenvalues of $\mathcal{L}_T$ in $[m^2 - c_0\delta, m^2]$, and m^2 is non-resonant.

(d) If $\lambda = m^2$ is not a resonance or an embedded eigenvalue of $\mathcal{L}_S$, then the same is true of $\mathcal{L}_T$, and there are no eigenvalues of $\mathcal{L}_T$ in $[m^2 - c_0\delta, m^2]$.

Part (a) of this theorem is standard, and included for clarity of exposition. Part (b) is arguably not surprising, but its proof (see Section 4.2) is a useful warm-up for parts (c) and (d). We also remark that the formulas in this theorem may be replaced with (more cumbersome, but in some sense more elementary) formulas that depend only on F, S, b, and c, via a first-order approximation for $F''(T) - F''(S)$. (See (4.2.7) and (4.2.9).)

The possible extra eigenvalue as in Theorem 2.3.2(c) is one of our primary motivations for performing this perturbation analysis. In general, eigenvalues lying in between 0 and m^2 have a profound impact on the stability properties of the kink. At the very least, any proof of asymptotic stability or instability for T would likely need to account for this extra eigenvalue in some way.

It should be noted that we are outside the realm of analytic perturbation theory, since we do not assume any continuity of the coefficients b, c with respect to δ. Our spectral analysis is based on the well-known method of finding solutions $U^\lambda_{\pm\infty}$ to the eigenvalue equation $\mathcal{L}_T U^\lambda = \lambda U^\lambda$

which decay at $\pm\infty$, and studying the Evans function (see e.g. [23, 34, 53, 35, 36]) which is related to the Wronskian of U^λ_∞ and $U^\lambda_{-\infty}$. The key property is that the Wronskian is zero when λ is an eigenvalue or resonance of $\mathcal{L}_T$. The slow decay of our coefficients b and c (as well as $F''(T) - F''(S)$) rules out tools such as the Gap Lemma (see [3, 25]) which would allow one to analytically continue the Evans function past the threshold $\lambda = m^2$, but which requires exponential decay of the coefficients.

Our last main result establishes the orbital stability of T:

Theorem 2.3.3. *There exists an $\varepsilon > 0$, depending on δ, such that for any initial data $(u, \partial_t u)\Big|_{t=0} = (T + v_1, v_2)$ for $(v_1, v_2) \in L^2 \times H^1$ with*

$$\|(v_1, v_2)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \varepsilon,$$

the corresponding solution u to (2.3.1) *exists globally in time, and satisfies*

$$\|u - T\|_{H^1(\mathbb{R})} + \|\partial_t u\|_{L^2(\mathbb{R})} \leq C\varepsilon,$$

for some C depending on δ and F.

The proof is based on classical energy arguments, but must contend with the lack of translation invariance.

2.4 Examples

2.4.1 ϕ^4 model

The choice of a double-well potential $F(u) = \frac{1}{4}(1 - u^2)^2$ leads to the ϕ^4 model

$$\partial_t^2 u - \partial_x^2 u = u - u^3. \tag{2.4.1}$$

Standard references on this equation include [62, 12, 42, 58]. In this case, the kink solution $S(x) = \tanh(x/\sqrt{2})$ is known explicitly, and the linearization $\mathcal{L}_S = -\partial_x^2 + (3S^2 - 1)$ has spectrum equal to

$$\sigma(\mathcal{L}_S) = \left\{0, \frac{3}{2}\right\} \cup [2, \infty).$$

The odd eigenfunction $Y_{3/2} = \tanh(x/\sqrt{2})\ \text{sech}(x/\sqrt{2})$ corresponding to $\lambda = \frac{3}{2}$ is known as the *internal oscillation mode*. The operator $\mathcal{L}_S$ also possesses an even resonance $R = 2\tanh^2(x/\sqrt{2}) - \text{sech}^2(x/\sqrt{2})$ at the threshold $\lambda = 2$.

The ϕ^4 kink is asymptotically stable with respect to *odd* perturbations in the energy space, by the important work of Kowalczyk-Martel-Muñoz [42]. When working in the odd energy space, the even translation invariance mode at $\lambda = 0$ and the even resonance do not play any role, but the internal oscillation mode has a dramatic effect on the dynamics. The method of [42] involved projecting φ onto $Y_{3/2}$ and the continuous spectrum, and carefully tracking the interaction between these two parts induced by the nonlinear terms of (2.3.4). A delicate coupling between the internal oscillation mode and the continuous part leads to the dissipation of energy away from a neighborhood of the kink.

Asymptotic stability with respect to odd perturbations was extended to a variable-coefficient version of (2.4.1) by the second named author in [65], though the coefficients were less general than those considered here (only a second-order perturbation, which was taken to be even and exponentially decaying). The symmetry assumption means that any eigenvalue emerging from the essential spectrum would be even, and therefore can be ignored.

It remains an important open question whether this kink is asymptotically stable with respect to general perturbations. Our Theorem 2.3.2 implies that for certain choices of b, c in (2.3.1), the bottom of the continuous spectrum is non-resonant and there are no extra discrete eigenvalues. Such a version of (2.3.1) could serve as an interesting test case for the ϕ^4 asymptotic stability problem, especially if one is convinced that the threshold resonance is an important source of difficulties.

2.4.2 Sine-Gordon equation

The choice $F(u) = 1 - \cos(u)$ results in the sine-Gordon equation:

$$\partial_t^2 u - \partial_x^2 u = -\sin(u).$$

This equation arises in the study of superconductivity as well as of surfaces with constant negative curvature, among other areas. (See e.g. [31, 14, 15] for background on this equation.)

The explicit static kink is given by $S(x) = 4\arctan(e^x)$. The equation, which is completely integrable, possesses other special solutions including breathers and wobbling kinks [14, 61]. The presence of these wobbling kinks (periodic-in-time, spatially localized perturbations of the kink) implies that S is not asymptotically stable in the energy space. (However, see [11] for an asymptotic stability result in a different topology, and [2], which identified an infinite-codimensional manifold of initial data near the kink for which asymptotic stability in the energy space does hold.) With $\mathcal{L}_S = -\partial_x^2 + \cos(S)$ the linearization around S, it is known that

$$\sigma(\mathcal{L}_S) = \{0\} \cup [1, \infty),$$

The failure of asymptotic stability in the energy space is consistent with the absense of an internal oscillation mode, which rules out the mechanism of stability observed for the ϕ^4 model in [42]. However, there is an odd resonance $R(x) = \tanh(x)$ at the bottom of the continuous spectrum. Our Theorem 2.3.2 gives conditions under which the variable-coefficient version of sine-Gordon possesses a discrete eigenvalue λ with $0 < 1 - \lambda \ll 1$. In this case, one may ask whether the new odd eigenfunction behaves sufficiently like an internal oscillation mode that a stability mechanism like the one mentioned above comes into force. We plan to explore this question in a future article.

Somewhat different perturbed forms of the sine-Gordon equation have been considered in, e.g., [20, 21, 16, 24]. The general belief is that breathers and wobbles are non-generic phenomena, so one may conjecture that some dense set of coefficients satisfying (2.3.2) lead to asymptotic stability.

2.4.3 Other examples

Let us briefly mention some other models whose variable-coefficient counterparts are included in our setting: the $P(\phi)_2$ theory [48], the double-sine-Gordon equation [10], and certain higher-order field theories [37], i.e. potentials equal to a polynomial of even degree, which in some cases satisfies the assumptions (2.1.2) and other cases not.

2.5 Related work

The asymptotic stability of kinks in scalar field equations such as (2.2.1) is an active area of inquiry. In addition to the results mentioned above, we should mention the recent work of Kowalczyk-Martel-Muñoz-Van Den Bosch [44], which proved asymptotic stability for a general class of scalar-field models satisfying a condition on the potential F that, in particular, rules out internal oscillation modes and threshold resonances. In the setting of odd perturbations, Delort-Masmoudi [19] established explicit decay rates for odd perturbations of the ϕ^4 kink on time scales of order ε^{-4}, where ε is the size of the initial perturbation. Let us also mention asymptotic stability results by Komech-Kopylova [40, 41] for kink solutions of relativistic Ginzburg-Landau equations, which are of the form (2.2.1) with additional assumptions of the flatness of F at $a_{\pm}$.

This class of questions is a partial motivation for the closely related subject of scattering theory for NLKG equations similar to (2.3.4). See [17, 18, 47, 67, 46, 45, 26] and the references therein.

The operator $\mathcal{L}_S$ is (up to subtraction by $m^2 I$) a Schrödinger operator with rapidly decaying potential. There is a well-established theory of spectral perturbation of Schrödinger and related operators, see e.g. the review [64] for an overview. Works that specifically address perturbation of threshold resonances include [33, 9, 27, 56]. As mentioned above, aspects such as the slow decay of coefficients and lack of continuous dependence on δ make it convenient to perform the perturbation "by hand" in our setting, rather than apply an abstract theorem or existing result.

Chapter 3

Existence of Stationary Solution

3.1 Sine Gordon Case

In this section we extend the methods of [65] to the Sine-Gordon equation.

The Sine-Gordon equation is given by

$$\partial_t^2(u) - \partial_y^2(u) - b(y)\partial_y(u) = -\sin u \tag{3.1.1}$$

We will deal with drift coefficients b that are odd and satisfy

$$|b(y)| \leq \delta e^{-|y|}, |b'(y)| \leq \delta, \tag{3.1.2}$$

for some small constant $\delta > 0$. We prove Theorem (3.1.1). In the proof we solve integral equations of Fredholm type on the positive real line. For this, we use the standard lemma (3.1.1).

Lemma 3.1.1. *let $h \in L^\infty([0,\infty))$. If*

$$\nu = \sup_{0 \leq y < \infty} \int_0^\infty |G(y,w)| dw < 1,$$

then there is a unique solution to

$$f(y) = h(y) + \int_0^\infty G(y,w) f(w) dw \tag{3.1.3}$$

is given by

$$f(y) = h(y) + \sum_{n=1}^{\infty} \int_0^\infty .... \int_0^\infty \prod_{i=1}^{n} G(y_{i-1}, y_i) h(y_n) dy_n....dy_1, \tag{3.1.4}$$

with $y_0 = y$. *Furthermore, we have*

$$\|f\|_{L^\infty([0,\infty))} \leq \frac{1}{1-\nu} \|h\|_{L^\infty([0,\infty))}$$

Proof. we will check if the iteration (3.1.4) is convergence

$$\begin{aligned}
&\Big|\int_0^\infty .... \int_0^\infty \prod_{i=1}^{n} G(x_{i-1}, x_i) h(x_n) dx_n....dx_1\Big| \\
&\leq \|h\|_{L^\infty} \int_0^\infty .... \int_0^\infty \prod_{i=1}^{n-1} \big|G(x_{i-1}, x_i)\big| \int_0^\infty \big|G(x_{n-1}, x_n)\big| dx_n....dx_1 \\
&\leq \|h\|_{L^\infty} \nu \int_0^\infty .... \int_0^\infty \prod_{i=1}^{n-1} \big|G(x_{i-1}, x_i)\big| dx_{n-1}....dx_1 \\
&\leq \|h\|_{L^\infty} \nu^n
\end{aligned}$$

so the series is convergence, and $\|f\|_{L^\infty} \leq \frac{1}{1-\nu}\|h\|_{L^\infty}$. □

3.1.1 Main results

Theorem 3.1.1. *Assume that* b *satisfies* (3.1.2). *Then there exists an odd, bounded, time-independent solution* T *of* (3.1.1). *Furthermore, for* $S(y) = 4\arctan(e^y)$, *the difference* $S_b = T - S$ *satisfies*

$$|S_b(y)| + |S_b'(y)| \leq \delta e^{-|y|} \tag{3.1.5}$$

3.1.2 Proofs of the Main Results

Now we will find a stationary solution to our equation (3.1.1) i.e an odd T solving

$$-\partial_y^2 T - b(y)\partial_y T = -\sin T \tag{3.1.6}$$

Proof of Theorem 3.1.1. We look for $S_b(y)$ such that $T(y) = S(y) + S_b(y)$ solve (3.1.6) where $S(y) = 4\arctan(e^y)$ satisfies $-S_{yy} = -\sin S$. We have

$$\begin{aligned}
&-\partial_y^2(S+S_b) - b(y)\partial_y(S+S_b) = -\sin(S+S_b)\\
&-\partial_y^2 S - \partial_y^2 S_b - b(y)\partial_y S - b(y)\partial_y S_b = -\sin(S+S_b)\\
&-\partial_y^2 S - \partial_y^2 S_b - b(y)\partial_y S - b(y)\partial_y S_b = -\sin S\cos S_b - \sin S_b \cos S\\
&-\partial_y^2 S - \partial_y^2 S_b - b(y)\partial_y S - b(y)\partial_y S_b = -\sin S\cos S_b - \sin S_b \cos S + \sin S - \sin S\\
&-\partial_y^2 S_b - b(y)\partial_y S - b(y)\partial_y S_b = \sin S(1-\cos S_b) - \sin S_b\cos S\\
&-\partial_y^2 S_b - b(y)\partial_y S_b = b(y)\partial_y S + \sin S(1-\cos S_b) - \sin S_b\cos S + S_b\cos S - S_b\cos S\\
&-\partial_y^2 S_b - b(y)\partial_y S_b + S_b\cos S = b(y)\partial_y S + \sin S(1-\cos S_b) - (\sin S_b - S_b)\cos S\\
&-\partial_y^2 S_b - b(y)\partial_y S_b + \cos S S_b = b(y)\partial_y S - (\sin S_b - S_b)\cos S + \sin S(1-\cos S_b) \qquad (3.1.7)
\end{aligned}$$

and we have $S_b(0) = 0$, $S_b \to 0$ as $y \to \infty$. We will extend S_b to the real line by oddness we can write (3.1.7) as

$$\mathcal{L}_b S_b = -\partial_y^2 S_b - b(y)\partial_y S_b + \cos S S_b$$

where

$$\mathcal{L}_b = -\partial_y^2 - b(y)\partial_y + \cos S = \mathcal{L} - b(y)\partial_y$$

and

$$\mathcal{L} = -\partial_y^2 + \cos S$$

we will find S_b by computing a Green's function for $\mathcal{L}_b$ on $[0,\infty)$. A fundamental system for

$\mathcal{L}Y = 0$ and $\mathcal{L}Z = 0$ is given by

$$Y_0(y) = \frac{4e^y}{1+e^{2y}}$$

$$Z_0(y) = \frac{4e^y}{1+e^{2y}}\left[\frac{1}{2}e^{2y} - \frac{1}{2}e^{-2y} + 2y\right]$$

To find Y_b, Z_b with $\mathcal{L}_b Y_b = \mathcal{L}_b Z_b = 0$, frist we make the substitution $Y_b = Y_0 + V_b$ which leads to the equation

$$\mathcal{L}Y_b = \mathcal{L}(Y_0 + V_b) = \mathcal{L}Y_0 + \mathcal{L}V_b = \mathcal{L}V_b$$

$$\mathcal{L}_b Y_b = \mathcal{L}_b(Y_0 + V_b) = (\mathcal{L} - b\partial_y)(Y_0 + V_b) = \mathcal{L}(Y_0 + V_b) - b\partial_y(Y_0 + V_b) = \mathcal{L}V_b - b\partial_y(Y_0 + V_b)$$

which means

$$\mathcal{L}V_b = b(y)\partial_y(Y_0 + V_b)$$

for $V_b(y)$. This can be written as the integral equation

$$V_b(y) = g(y) + \int_0^\infty G_0(y,w)b(w)\partial_w V_b(w)dw, \quad y \geq 0, \tag{3.1.8}$$

where

$$g(y) = \int_0^\infty G_0(y,w)b(w)\partial_w Y_0(w)dw,$$

and

$$G_0(y,w) = \begin{cases} Y_0(y)Z_0(w), & 0 \leq w < y, \\ Y_0(w)Z_0(y), & 0 \leq y < w, \end{cases} \tag{3.1.9}$$

Using

$$|Y_0(y)| \leq 4e^{-|y|}, |Y_0'(y)| \leq 4e^{-|y|},$$

$$|Z_0(y)| \leq 2e^{|y|}, |Z_0'(y)| \leq 2e^{|y|}.$$

and the bound for b, $|b| \leq \delta e^{-|y|}$, $|b'| \leq \delta$. We see that

$$\begin{aligned}
g(y) &= \int_0^\infty G_0(y,w)b(w)\partial_w Y_0(w)dw, \qquad (3.1.10)\\
&= Y_0(y)\int_0^y Z_0(w)b(w)\partial_w Y_0(w)dw + Z_0(y)\int_y^\infty Y_0(w)b(w)\partial_w Y_0(w)dw\\
&\leq \delta\left(e^{-y}\int_0^y e^{-w}dw + e^y\int_y^\infty e^{-3w}dw\right)\\
&\leq \delta e^{-y}.
\end{aligned}$$

Now, integrate by parts in (3.1.8) to obtain the Fredholm equation

$$V_b(y) = g(y) - \int_0^\infty \partial_w[G_0(y,w)b(w)]V_b(w)dw$$

There are no boundary terms because $b(0) = 0$. By the bound of b and the above bounds on Y_0 and Z_0, we have

$$\begin{aligned}
&\sup_{[0,\infty)}\int_0^\infty \big|\partial_w[G_0(y,w)b(w)]\big|dw\\
&\leq \sup_{[0,\infty)}\left(|Y_0(y)|\int_0^y |Z_0'(w)b(w) + Z_0(w)b'(w)|dw + |Z_0(y)|\int_y^\infty |Y_0'(w)b(w) + Y_0(w)b'(w)|dw\right)\\
&\leq \sup_{[0,\infty)}\delta(8ye^{-y} - 4e^{-y} + 16) \leq 1 \qquad (3.1.11)
\end{aligned}$$

if δ is sufficiently small, so by lemma (3.1.1), a unique solution V_b exists, and $\|V_b\|_{L^\infty} \leq C\|g\|_{L^\infty} \leq C\delta$. It is clear from formula (3.1.4) in Lemma (3.1.1) and the decay of g that $|V_b| = |Y_b - Y_0| \leq \delta e^{-y}$.

Using reduction of order, we obtain a second independent solution Z_b given by

$$Z_b(y) = Y_b(y)\int_0^y \frac{exp(\int_0^w b(s)ds)}{(Y_b(w))^2}dw$$

We have $Z_b(0) = 0$, and $Z_b(y) \leq e^y$. let $p = Y_b Z_b' - Y_b' Z_b = exp(\int_0^y b(s)ds)$, and define the Green's function G_b for $\mathcal{L}_b$

$$G_b(y,w) = \begin{cases} \frac{Y_b(y)Z_b(w)}{p(w)}, & 0 \leq w < y, \\ \\ \frac{Y_b(w)Z_b(y)}{p(w)}, & 0 \leq y < w, \end{cases}$$

Note that $G_b(o,w) = 0$.

We can write (3.1.7) as a nonlinear integral equation for $S_b(y)$

$$S_b(y) = (\mathcal{T}S_b)(y) = h(y) - \int_0^\infty G_b(y,w)[(\sin S_b - S_b)\cos S - \sin S(1-\cos S_b)]dw. \quad (3.1.12)$$

where

$$\begin{aligned} h(y) &= \int_0^\infty G_b(y,w)b(w)\partial_w S dw \\ &= Y_b(y)\int_0^y \frac{Z_b(w)}{p(w)}\delta e^{-2w}dw + Z_b(y)\int_y^\infty \frac{Y_b(w)}{p(w)}\delta e^{-2w}dw \\ &= -8\delta e^{-2y} + 8\delta e^{-y} + \frac{2}{3}\delta e^{-2y} \leq C_1\delta e^{-y} \end{aligned}$$

We will show that $\mathcal{T}$ has a unique fixed point in a suitable class. Define the norm

$$\|\eta\|_\sim = \sup_{0\leq y<\infty} e^y|\eta(y)|.$$

Note first that, since $\partial_y S = \frac{4e^y}{1+e^{2y}}$, we have $|b(w)S'(w)| \leq \delta e^{-2|w|}$. We see that $|h(y)| \leq C_1\delta e^{-y}$ for some constant C_1. Let $C_0 = 2C_1$, and define the set $A_\delta = \{\eta \in C([0,\infty)) : \|\eta(y)\|_\sim \leq C_0\delta\}$

for $\eta \in A_\delta$, we check directly that

$$
\begin{aligned}
&|(\sin S_b - S_b)\cos S - \sin S(1-\cos S_b)| \\
&\leq |-\frac{\eta^3}{6}\cos S - \sin S\frac{\eta^2}{4}| \leq C_0^3\delta^2 e^{-3y}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_0^\infty G_b(y,w)[(\sin S_b - S_b)\cos S - \sin S(1-\cos S_b)]dw\Big| \\
&\leq \delta^2 C_0^3\left(|Y_b(y)|\int_0^y \frac{|Z_b(w)|}{p(w)}e^{-3w}dw + |Z_b(y)|\int_y^\infty \frac{|Y_b(w)|}{p(w)}e^{-3w}dw\right) \\
&\leq \frac{1}{2}C_0^3\delta^2 e^{-|y|}
\end{aligned}
$$

Then, if $\delta < \frac{1}{C_0^2}$, we have

$$
|\mathcal{T}\eta(y)| \leq C_1\delta e^{-|y|} + \frac{1}{2}C_0^3\delta^2 e^{-|y|} < 2C_1\delta e^{-|y|}
$$

so $\mathcal{T}\eta \in A_\delta$. Finally, for $\eta_1, \eta_2 \in A_\delta$, we have

$$
\begin{aligned}
&|(\sin\eta_1 - \eta_1)\cos S - \sin S(1-\cos\eta_1) - (\sin\eta_2 - \eta_2)\cos S + \sin S(1-\cos\eta_2)| \\
&\approx |-\frac{1}{6}(\eta_1^3 - \eta_2^3)\cos S - \frac{1}{2}\sin S(\eta_1^2 - \eta_2^2)| \\
&\leq |\eta_1 - \eta_2|\Big| - \frac{1}{6}(\eta_1^2 + \eta_1\eta_2 + \eta_2^2)\cos S - \frac{1}{2}\sin S(\eta_1 + \eta_2)\Big| \\
&\leq 5C_0\delta e^{-y}|\eta_1 - \eta_2|
\end{aligned}
$$

and proceeding as before we have

$$
\begin{aligned}
&|\mathcal{T}(\eta_1)(y) - \mathcal{T}(\eta_2)(y)| \\
&\leq 5C_0\delta\left(|Y_b(y)|\int_0^y \frac{|Z_b(w)|}{p(w)}e^{-w}|\eta_1(w) - \eta_2(w)|dw + |Z_b(y)|\int_y^\infty \frac{|Y_b(w)|}{p(w)}e^{-w}|\eta_1(w) - \eta_2(w)|dw\right) \\
&\leq \delta e^{-2y}\|\eta_1 - \eta_2\|_\sim
\end{aligned}
$$

so that $\|\mathcal{T}(\eta_1)(y) - \mathcal{T}(\eta_2)(y)\|_\sim \leq C\delta\|\eta_1 - \eta_2\|_\sim$. If $\delta < \frac{1}{C}$, then $\mathcal{T}$ is a contraction in A_δ and a unique solution S_b to (3.1.7) exist in A_δ. By differentiating (3.1.12), we get

$$|S_b(y)| + |S_b'(y)| \leq \delta e^{-|y|}$$

$\square$

3.2 General Case

First, we recall the existence of the static kink in the constant-coefficient case, which can be found by explicitly integrating the equation $S'' = F''(S)$. We quote from [44, Lemma 1.1]:

Lemma 3.2.1. *Under the assumptions* (2.1.2) *on* F, *there is a solution* $S \in C^4(\mathbb{R})$ *to the stationary equation*

$$-S'' + F'(S) = 0,$$

with $S' > 0$ *and* $S \to a_\pm$ *as* $y \to \pm\infty$. *Furthermore,* S *and* S' *satisfy*

$$|S(x) - a_\pm| \le Ce^{\mp my}, \quad |S'(x)| \le Ce^{-m|y|},$$

and the energy of S *is finite:*

$$\int_{\mathbb{R}} [S'(x)^2 + F(S(x))]\,\mathrm{d}x < \infty.$$

3.2.1 Proofs of the Main Results

We now prove the existence of a static kink $T(y)$ for our equation (2.3.1):

Proof of Theorem 2.3.1. Let S be the stationary solution to $-S'' + F'(S) = 0$ guaranteed by Lemma 3.2.1. Making the ansatz $T = S + S_b$, we have the following equation for S_b:

$$\begin{aligned} -S_b'' - bS_b' - cS_b &= bS' + cS - F'(S + S_b) + F'(S) \\ &= bS' + cS - F''(S)S_b - \mathcal{N}(S, S_b), \end{aligned} \tag{3.2.1}$$

where $\mathcal{N}(S, S_b) = F'(S + S_b) - F'(S) - F''(S)S_b$. Defining

$$\mathcal{L}_b = -\partial_y^2 - b(y)\partial_y - c(y) + F''(S)(y) = \mathcal{L}_S - b(y)\partial_y - c(y),$$

equation (3.2.1) becomes

$$\mathcal{L}_b S_b = bS' + cS - \mathcal{N}(S, S_b). \tag{3.2.2}$$

First, assume 0 is not an eigenvalue of $\mathcal{L}_b$.

In this case, we can find solutions $Y_{-\infty}, Y_\infty$ both satisfying $\mathcal{L}_b Y_{\pm\infty} = 0$, with $\lim_{y\to-\infty} Y_{-\infty} = 0$ and $\lim_{y\to\infty} Y_\infty = 0$. In more detail, $\mathcal{L}_b Y = 0$ may be written as the linear system $\mathbf{Y}' = (M_1 + M_2(y))\mathbf{Y}$, with $\mathbf{Y} = (Y, Y')$, and

$$M_1 = \begin{pmatrix} 0 & 1 \\ m^2 & 0 \end{pmatrix}, \quad M_2(y) = \begin{pmatrix} 0 & 0 \\ -c(y) + F''(S)(y) - m^2 & -b(y) \end{pmatrix}.$$

Lemma 9.0.5 below implies existence of Y_∞ and $Y_{-\infty}$. In particular, Y_∞ and $Y_{-\infty}$ are linearly independent, since otherwise there would be a nontrivial solution in L^2 to $\mathcal{L}_b Y = 0$, contradicting our assumption that 0 is not an eigenvalue.

Define the Green's function

$$G(y, w) := \frac{1}{W_{\mathbf{Y}}(y)} \begin{cases} Y_{-\infty}(y) Y_\infty(w), & y < w, \\ Y_\infty(y) Y_{-\infty}(w), & w \le y, \end{cases}$$

where $W_{\mathbf{Y}}(y) = \det(\mathbf{Y}_{-\infty}, \mathbf{Y}_\infty)$. Abel's formula implies $W_{\mathbf{Y}}(y) = W_{\mathbf{Y}}(0) \exp(\int_0^y b(z)\,\mathrm{d}z)$, which for $\delta > 0$ sufficiently small, is bounded uniformly away from 0.

For the inverse operator $\eta \mapsto \int_{\mathbb{R}} G(\cdot, w)\eta(w)\,\mathrm{d}w$, we have the following useful bounds. First,

$$\begin{aligned} \left| \int_{\mathbb{R}} G(y,w)\eta(w)\,\mathrm{d}w \right| &= \left| Y_\infty(y) \int_{-\infty}^{y} \frac{Y_{-\infty}(w)}{W_{\mathbf{Y}}(w)} \eta(w)\,\mathrm{d}w + Y_{-\infty}(y) \int_y^\infty \frac{Y_\infty(w)}{W_{\mathbf{Y}}(w)} \eta(w)\,\mathrm{d}w \right| \\ &\le C\|\eta\|_{L^\infty(\mathbb{R})} \left(e^{-my} \int_{-\infty}^{y} e^{mw}\,\mathrm{d}w + e^{my} \int_y^\infty e^{-mw}\,\mathrm{d}w \right) \\ &\le C\|\eta\|_{L^\infty(\mathbb{R})}, \end{aligned} \tag{3.2.3}$$

for all $y \in \mathbb{R}$. We also have

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} G(y,w)\eta(w)\,\mathrm{d}w\,\mathrm{d}y \right| \le C \left(\int_{\mathbb{R}} e^{-my} \int_{-\infty}^{y} e^{mw} |\eta(w)|\,\mathrm{d}w\,\mathrm{d}y + \int_{\mathbb{R}} e^{my} \int_y^\infty e^{-mw} |\eta(w)|\,\mathrm{d}w\,\mathrm{d}y \right). \tag{3.2.4}$$

For the first term on the right, we integrate by parts to obtain

$$\int_{\mathbb{R}} e^{-my} \int_{-\infty}^{y} e^{mw} |\eta(w)| \, \mathrm{d}w \, \mathrm{d}y = \int_{-\infty}^{y} \frac{e^{m(w-y)}}{-m} |\eta(w)| \, \mathrm{d}w \Big|_{y=-\infty}^{y=\infty} - \int_{\mathbb{R}} \frac{e^{-my}}{-m} e^{my} |\eta(y)| \, \mathrm{d}y.$$

If $\eta \in L^1(\mathbb{R})$, then since $e^{m(w-y)} \leq 1$, the boundary term at $-\infty$ vanishes, and the boundary term at ∞ is bounded by $\frac{1}{m}\|\eta\|_{L^1(\mathbb{R})}$. After applying a similar calculation to the last term in (3.2.4), we conclude

$$\left\| \int_{\mathbb{R}} G(\cdot, w)\eta(w) \, \mathrm{d}w \right\|_{L^1(\mathbb{R})} \leq C\|\eta\|_{L^1(\mathbb{R})}, \tag{3.2.5}$$

for a constant depending on m and the coefficients b, c. The estimates (3.2.3) and (3.2.5) clearly hold also if we replace $G(y, w)$ with $|G(y, w)|$.

In addition, using $|Y'_{\pm\infty}(y)| \lesssim e^{\mp my}$, estimates similar to (3.2.3) and (3.2.5) imply

$$\left\| \partial_y \int_{\mathbb{R}} G(y, w)\eta(w) \, \mathrm{d}w \right\|_{L^1(\mathbb{R})} \leq C\|\eta\|_{L^1(\mathbb{R})}, \quad \left\| \partial_y \int_{\mathbb{R}} G(y, w)\eta(w) \, \mathrm{d}w \right\|_{L^\infty(\mathbb{R})} \leq C\|\eta\|_{L^\infty(\mathbb{R})}. \tag{3.2.6}$$

Now we write the equation (3.2.2) for S_b as

$$S_b(y) = (\mathcal{T} S_b)(y) := g(y) - \int_{\mathbb{R}} G(y, w)\mathcal{N}(S, S_b) \, \mathrm{d}w, \tag{3.2.7}$$

where

$$g(y) = \int_{\mathbb{R}} G(y, w)[b(w)S'(w) + c(w)S(w)] \, \mathrm{d}w.$$

We want to find a fixed point for $\mathcal{T}$ in the space $X := L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with norm $\| \cdot \|_X := \| \cdot \|_{L^1(\mathbb{R})} + \| \cdot \|_{L^\infty(\mathbb{R})}$. From (3.2.3) and (3.2.5), we have

$$\|g\|_X \leq C \left(\|bS' + cS\|_{L^\infty(\mathbb{R})} + \|bS' + cS\|_{L^1(\mathbb{R})} \right) \leq C_0\delta,$$

since S and S' are bounded and $\|b+c\|_X \lesssim \delta$. For the nonlinear term, since F is C^3 on $[a_-, a_+]$, there is some $K > 0$ such that

$$|\mathcal{N}(S, \eta)| = |F'(S + \eta) - F'(S) - F''(S)\eta| \leq K\eta^2, \tag{3.2.8}$$

globally in y. This gives

$$\left|\int_{\mathbb{R}} G(y,w)\mathcal{N}(S,\eta)(w)\,\mathrm{d}w\right| \le K\int_{\mathbb{R}} |G(y,w)|\eta^2(w)\,\mathrm{d}w \tag{3.2.9}$$

and

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} G(y,w)\mathcal{N}(S,\eta)(w)\,\mathrm{d}w\,\mathrm{d}y\right| \le K\int_{\mathbb{R}} \left(e^{-my}\int_{-\infty}^{y} e^{mw}\eta^2(w)\,\mathrm{d}w + e^{my}\int_{y}^{\infty} e^{-mw}\eta^2(w)\,\mathrm{d}w\right)\mathrm{d}y,$$

so that the estimates (3.2.3) and (3.2.5) imply

$$\left\|\int_{\mathbb{R}} G(y,w)\mathcal{N}(S,\eta)(w)\,\mathrm{d}w\right\|_X \le CK\|\eta^2\|_X \le CK\|\eta\|_X^2,$$

after applying the standard interpolation $\|\cdot\|_{L^2}^2 \le \|\cdot\|_{L^\infty}\|\cdot\|_{L^1}$.

With C_0 such that $\|g\|_X \le C_0\delta$, define $\mathcal{A} := \{\eta \in X, \|\eta\|_X \le 2C_0\delta\}$. For any $\eta \in \mathcal{A}$, the above estimates imply

$$\|\mathcal{T}\eta\|_X = \left\|g - \int_{\mathbb{R}} G(\cdot,w)\mathcal{N}(S,\eta)(w)\,\mathrm{d}w\right\|_X \le \|g\|_X + CK\|\eta\|_X^2 \le C_0\delta + C\delta^2,$$

so for $\delta < C_0/C$, we have $T\eta \in \mathcal{A}$. Next, for $\eta_1, \eta_2 \in \mathcal{A}$, we have from Taylor's Theorem that

$$F'(S+\eta_1) = F'(S+\eta_2) + F''(S+\eta_1)(\eta_1-\eta_2) + \frac{1}{2}F'''(\xi_y)(\eta_1-\eta_2)^2,$$

for some $\xi_y \in [a_-, a_+]$ depending on y. Using this in $\mathcal{N}(S,\eta_1) - \mathcal{N}(S,\eta_2)$, we have

$$\begin{aligned}
|\mathcal{N}(S,\eta_1) - \mathcal{N}(S,\eta_2)| &= |F'(S+\eta_1) - F'(S+\eta_2) - F''(S)(\eta_1-\eta_2)| \\
&= \left|[F''(S+\eta_1) - F''(S)](\eta_1-\eta_2) + \frac{1}{2}F'''(\xi_y)(\eta_1-\eta_2)^2\right| \\
&\le |\max_{[a_-,a_+]} |F'''(s)||\eta_1||\eta_1-\eta_2| + \frac{1}{2}|F'''(\xi_y)|(\eta_1-\eta_2)^2 \\
&\le K\delta|\eta_1-\eta_2|,
\end{aligned}$$

for some $K > 0$. By (3.2.3) and (3.2.5) we have

$$\begin{aligned}\|(\mathcal{T}\eta_1)(y) - (\mathcal{T}\eta_2)(y)\|_X &= \left\| \int_{\mathbb{R}} G(y,w)[\mathcal{N}(S,\eta_1) - \mathcal{N}(S,\eta_2)](w)\,\mathrm{d}w \right\|_X \\ &\leq CK\delta\|\eta_1 - \eta_2\|_X,\end{aligned} \tag{3.2.10}$$

as above. The constant $CK > 0$ depends on m and the C^3 norm of F. For δ sufficiently small, we conclude $\mathcal{T}$ is a contraction on $\mathcal{A}$, and a unique solution S_b to (3.2.7) exists in $\mathcal{A}$.

To derive the bounds on S_b', we differentiate equation (3.2.7) and use the derivative bounds (3.2.6) and the Taylor estimate (3.2.8):

$$\|S_b'\|_X = \left\|\partial_y \int_{\mathbb{R}} G(y,w)[bS' + cS - N(S,S_b)]\,\mathrm{d}w\right\|_X \leq \|bS' + cS - N(S,S_b)\|_X \lesssim \delta + K\|S_b^2\|_X \lesssim \delta.$$

Next, consider the case that 0 is an eigenvalue of $\mathcal{L}_b$, i.e. there is some nonzero $Y \in L^2(\mathbb{R})$ with $\mathcal{L}_b Y = 0$ in the sense of distributions. Then, let $\lambda < 0$ be a small parameter to be chosen later. We will solve the following equivalent formulation of (3.2.1):

$$\mathcal{L}_b S_b - \lambda S_b = bS' + cS - \lambda S_b - \mathcal{N}(S,S_b). \tag{3.2.11}$$

Let $k = \sqrt{m^2 - \lambda}$. From Lemma 9.0.5, we can construct solutions $Y_{-\infty}^\lambda$, Y_∞^λ to $\mathcal{L}_b Y_{\pm\infty}^\lambda = \lambda Y_{\pm\infty}^\lambda$ with $|Y_{-\infty}^\lambda(y)| \leq Ce^{ky}$ and $|Y_\infty^\lambda(y)| \leq Ce^{-ky}$ for all $y \in \mathbb{R}$. Our Green's function for $\mathcal{L}_b - \lambda I$ is defined by

$$G_\lambda(y,w) = \frac{1}{W_{\mathbf{Y}}^\lambda(y)} \begin{cases} Y_{-\infty}^\lambda(y) Y_\infty^\lambda(w), & y < w, \\ Y_\infty^\lambda(y) Y_{-\infty}^\lambda(w), & w \leq y, \end{cases}$$

where $W_{\mathbf{Y}}^\lambda(y) = Y_\infty^\lambda(y)(Y_{-\infty}^\lambda)'(y) - Y_{-\infty}^\lambda(y)(Y_\infty^\lambda)'(y)$. Clearly, G_λ satisfies the same estimates (3.2.3), (3.2.5), and (3.2.6) with constants that are uniform for λ in a small neighborhood of 0.

For $\eta \in X$, let us define

$$\mathcal{T}_\lambda \eta := g_\lambda(y) - \lambda \int_{\mathbb{R}} G_\lambda(y,w)\eta(w)\,\mathrm{d}w - \int_{\mathbb{R}} G_\lambda(y,w)\mathcal{N}(S,\eta)(w)\,\mathrm{d}w,$$

where

$$g_\lambda(y) = \int_{\mathbb{R}} G_\lambda(y,w)[b(w)S'(w) + c(w)S(w)]\,\mathrm{d}w.$$

With $\|\cdot\|_X$ defined as above, we have

$$\|\mathcal{T}_\lambda \eta\|_X \le \|g_\lambda\|_X + C\lambda\|\eta\|_X + C\|\eta\|_X^2,$$

with C depending on the C^3 norm of F as above. With C_0 such that $\|g_\lambda\|_\sim \le C_0\delta$ and $\mathcal{A}_\lambda = \{\eta : \|\eta\|_\sim \le 2C_0\delta\}$, for $|\lambda|$ sufficiently small, we see $\mathcal{T}_\lambda$ maps from $\mathcal{A}_\lambda$ to itself. The estimate

$$\|\mathcal{T}_\lambda\eta_1 - \mathcal{T}_\lambda\eta_2\|_X \le K(\delta+\lambda)\|\eta_1 - \eta_2\|_X,$$

now follows from calculations similar to (3.2.10). For a fixed $\lambda < 0$ sufficiently close to 0, we see T_λ has a unique fixed point S_b. The $W^{1,1}$ and $W^{1,\infty}$ bounds follow as above. □

The proof of Theorem 2.3.1 also provides the following approximation for $S_b = T - S$: since $S_b = g + \int_{\mathbb{R}} G(y,w)\mathcal{N}(S,S_b)(w)\,\mathrm{d}w$, the estimates (3.2.3), (3.2.5) imply

$$\begin{aligned}\left\|S_b - \int_{\mathbb{R}} \tilde{G}(\cdot,w)[bS' + cS](w)\,\mathrm{d}w\right\|_X &\le \left\|\int_{\mathbb{R}} \tilde{G}(\cdot,w)\mathcal{N}(S,S_b)(w)\,\mathrm{d}w\right\|_X \\ &\lesssim \|S_b\|_X^2 \lesssim \delta^2,\end{aligned} \tag{3.2.12}$$

with $\tilde{G} = G$ if 0 is not an eigenvalue of $\mathcal{L}_b = \mathcal{L}_S - b\partial_y - c$, and $\tilde{G} = G_\lambda$ otherwise, with λ chosen such that $|\lambda| \lesssim \delta$, so that $\lambda S_b - \lambda\int_{\mathbb{R}} G_\lambda(y,w)S_b(w)\,\mathrm{d}w$ are $O(\delta^2)$.

Chapter 4

Perturbation Of the Spectrum

4.1 Sine Gordon Case

In this section we will analyze the spectrum of the operator $\mathcal{L}_T = -\partial_y^2 - b(y)\partial_y + cosT$. We can write our operator as $\mathcal{L}_T = \mathcal{L} - b(y)\partial_y + d(y)$ where $\mathcal{L} = -\partial_y^2 + cosS$ and $d(y) = cosT - cosS$. We find that the $L^2 - spectrum$ $\sigma(\mathcal{L}_T)$ of $\mathcal{L}_T$ is qualitatively similar to the spectrum of $\mathcal{L}$ in the following Theorem.

Theorem 4.1.1. *The operator $(\mathcal{L}_T)$ has real, simple eigenvalue λ_0 such that $|\lambda_0| \leq \delta$. The corresponding eigenfunction $\bar{Y}_0$ is even and satisfy:*

$$\|Y_0 - \bar{Y}_0\|_{L^\infty} \leq \delta$$

where Y_0 is eigenfunction of $\mathcal{L}$ corresponding to 0, and the continuous spectrum $\sigma_c(\mathcal{L}_T) = [1, \infty)$.

Definition 1. *Let $T \in \mathcal{L}(X)$. A complex number λ is said to be in the resolvent set $\rho(T)$ of T if $\lambda I - T$ is a bijection with a bounded inverse. $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ. if $\lambda \notin \rho(T)$, then λ is said to be in the spectrum $\sigma(T)$ of T.*

4.1.1 Proofs of the Main Results

Proof of Theorem 4.1.1. The operator $\mathcal{L}_T$ is self-adjoint with respect to the $\langle .,. \rangle_p$ where $\langle f, g \rangle_p = \int p(y)f(y)g(y)dy$, and $d(y) = exp \int_0^y b(s)ds$. Therefore, $\sigma(\mathcal{L}_T) \subset \mathbb{R}$. Now, by using the general theory the continuous spectrum of $\mathcal{L}$ is stable under the relatively compact perturbation $-b\partial_y + d$. Thus $\sigma_c(\mathcal{L}_T) = \sigma_c(\mathcal{L}) = [1, \infty)$.

We show $\sigma_c(\mathcal{L}_T)$ is inside the $C_0\delta$ neighborhood of $\sigma(\mathcal{L})$ for some constant C_0. Assume that $\lambda \in \rho(\mathcal{L}) \cap \sigma(\mathcal{L}_T)$, where $\rho(\mathcal{L})$ is the resolvent set of $\mathcal{L}$, and we will take $d_* = dist(\lambda, \sigma(\mathcal{L}))$. Since $\lambda \in \rho(\mathcal{L})$, so for any $y \in L^2(\mathbb{R})$ we have $\|\mathcal{L} - \lambda I)^{-1} y\| \leq \frac{\|y\|}{d_*}$ is equivalent to $\|\mathcal{L} - \lambda I)w\| \geq d_* \|w\|$ for all $w \in D(\mathcal{L})$ where $\|.\|$ denote the norm in $L^2(\mathbb{R})$.

Since $\lambda \in \sigma(\mathcal{L}_T)$, there exist a sequence $w_n \in D(\mathcal{L}_T) = D(\mathcal{L})$ satisfies $\|w_n\| = 1$, and $(\mathcal{L}_T - \lambda I)w_n \to 0$ in $L^2(\mathbb{R})$ we have

$$\|(\mathcal{L}_T - \mathcal{L})w_n\| = \|bw_n' + dw_n\| \geq \frac{d_*}{2}$$

for n sufficiently large. We have

$$\|bw_n'\| \leq \delta \|w_n'\|$$

$$\begin{aligned}
\|w_n'\| &= \int (w_n')^2 = \int w_n(-w_n'') \\
&= \int [w_n \left((\mathcal{L}_T - \lambda I)w_n + bw_n' - (\cos S + d - \lambda I)w_n \right)] \\
&\leq \|w_n\| \left[\|(\mathcal{L}_T - \lambda I)w_n\| - \int b' w_n + C\|w_n\| \right] \\
&\leq \|(\mathcal{L}_T - \lambda I)w_n\| + C\|w_n\|
\end{aligned}$$

from our assumption $\|(\mathcal{L}_T - \lambda I)w_n\| \to 0$ for n is sufficiently large we have

$$\begin{aligned}
\frac{d_*}{2} &\leq \|bw_n'\| + \|dw_n\| \\
&\leq \delta \|w_n'\| + \delta \|w_n\| \leq \delta
\end{aligned}$$

Therefore, $d_* = dist(\lambda, \sigma(\mathcal{L})) \leq C_0\delta$

Now, we will show that $\mathcal{L}_T$ has exactly one eigenvalue in $[-C_0\delta, C_0\delta]$. let take λ_T satisfies $\lambda_T \geq C_0\delta$ and $|\lambda_T| \leq \delta$.

Also we will take $\lambda \in [-\lambda_T, \lambda_T]$, and we will look for $\tilde{Y_0}(y) \in L^2$ satisfying $\mathcal{L}_T\tilde{Y_0} = \lambda\tilde{Y_0}$. Let take $\tilde{Y_0} = Y_0 + U_\lambda$.

We obtain the following equation for U_λ :

$$\mathcal{L}_T\tilde{Y_0} = \mathcal{L}_T(Y_0 + U_\lambda) = \mathcal{L}_T Y_0 + \mathcal{L}_T U_\lambda$$

We will get

$$\mathcal{L}U_\lambda = bY_0' + (\lambda - d)Y_0 + bU_\lambda' + (\lambda - d)U_\lambda \tag{4.1.1}$$

We can write (4.1.1) on $[0, \infty)$ as the integral equation:

$$U_\lambda(y) = g_0(y) + \int_0^\infty G_0(y, w)\left[b(w)U_\lambda'(w) + (\lambda - d(w))U_\lambda(w)\right] dw, \tag{4.1.2}$$

where G_0 is the Green's function defined as before, and

$$\begin{aligned} g_0(y) = &\int_0^\infty G_0(y, w)\left[b(w)Y_0'(w) + (\lambda - d(w))Y_0(w)\right] dw \\ &= Y_0(y)\int_0^y Z_0(w)\left[b(w)Y_0'(w) + (\lambda - d(w))Y_0(w)\right] dw \\ &+ Z_0(y)\int_y^\infty Y_0(w)\left[b(w)Y_0'(w) + (\lambda - d(w))Y_0(w)\right] dw \end{aligned}$$

from our assumption of Y_0 and Z_0, we have $|g_0(y)| \leq \delta y e^{-y}$. To solve (4.1.2), we check that

$$\int_0^\infty |\partial_w G_0(y, w)b(w)| + (b'(w) - \lambda + d(w))G_0(y, w)|dw \leq \delta$$

By using lemma (3.1.1) there exist U_λ on $[0, \infty)$ for each λ, $\|U_\lambda\|_{L^\infty} \leq \|g_0\|_{L^\infty} \leq \delta$ and

$|U_\lambda| \leq \delta e^{-y}$. To extend by evenness to the real line , we need $U'_\lambda(0) = 0$. (4.1.2) implies

$$U'_\lambda(0) = \int_0^\infty Y_0 \left[b(Y_0 + U_\lambda)' + (\lambda - d)(Y_0 + U_\lambda)\right] dw \tag{4.1.3}$$

$$= \lambda \int_0^\infty Y_0(Y_0 + U_\lambda)dw - \int_0^\infty \left[(d + b')Y_0 + bY_0'\right](Y_0 + U_\lambda)dw. \tag{4.1.4}$$

Since $\int_0^\infty Y_0(Y_0 + U_\lambda) \geq 8 - C\delta \geq \frac{1}{4}$ and $\|Y_0 + U_\lambda\|_{L_\infty} \leq 1$, we choose

$$\lambda_T = max\left(C_0\delta, \int_0^\infty |(d + b')Y_0 + bY_0'|dw\right),$$

Therefore, $U'_{\lambda_T}(0) > 0, U'_{-\lambda_T}(0) < 0$, and $|\lambda_T| \leq \delta$. We will now show that $U'_\lambda(0)$ depends on λ in a continues and monotonic way. For $\lambda, \mu \in [-\lambda_T, \lambda_T]$, and let take $U = U_\lambda - U_\mu$ satisfies

$$U(y) = g_\gamma(y) + \int_0^\infty G_0(y, w)\left[b(U'_\lambda - U'_\mu) - d(U_\lambda - U_\mu)\right] dw + \int_0^\infty G_0(y, w)(\lambda - \mu)Y_0. \tag{4.1.5}$$

$$g_\gamma(y) = \int_0^\infty G_0(y, w)(\lambda U_\lambda - \mu U_\mu)dw. \tag{4.1.6}$$

Since $|\lambda U_\lambda - \mu U_\mu| \leq \delta e^{-y}$. using lemma (3.1.1) U is exist and satisfies $\|U\|_{L^\infty} \leq \|g_\gamma\|_{L^\infty}$. We have

$$\begin{aligned}\|g_\gamma\|_{L^\infty} &\leq \|\lambda U_\lambda - \mu U_\mu\|_{L^\infty} \\ &= \|\lambda U_\lambda - \mu U_\lambda + \mu U_\lambda - \mu U_\mu\|_{L^\infty} \\ &= \|(\lambda - \mu)U_\lambda + \mu U\|_{L^\infty} \\ &\leq C_1|\lambda - \mu| + C_2\delta\|U\|_{L^\infty}\end{aligned}$$

By combining them we will have

$$\begin{aligned}&\|U\|_{L^\infty} \leq \|g_\gamma\|_{L^\infty} \leq C_1|\lambda - \mu| + C_2\delta\|U\|_{L^\infty} \\ &(1 - C_2\delta)\|U\|_{L^\infty} \leq C_1|\lambda - \mu| \\ &\|U\|_{L^\infty} \leq \frac{C_1}{1 - C_2\delta}|\lambda - \mu|\end{aligned}$$

We conclude if δ is sufficiently small we have;

$$\|U_\lambda - U_\mu|_{L^\infty} \leq |\lambda - \mu| \tag{4.1.7}$$

Now, let $\lambda > \mu$ we have,

$$U'_\lambda(0) - U'_\mu(0) = (\lambda - \mu)\int_0^\infty Y_0^2 dw + \int_0^\infty Y_0(\lambda U_\lambda - \mu U_\mu) - (dY_0 + b'Y_0 + bY_0')(U_\lambda - U_\mu)dw. \tag{4.1.8}$$

Since $\|U_\lambda - U_\mu\|_{L^\infty} \leq |\lambda - \mu|$ and $\|\lambda U_\lambda - \mu U_\mu\|_{L^\infty} \leq \delta|\lambda - \mu|$. The second integral will be bounded given by:

$$\int_0^\infty Y_0(\lambda U_\lambda - \mu U_\mu) - (dY_0 + b'Y_0 + bY_0')(U_\lambda - U_\mu)dw. \leq C\delta|\lambda - \mu|$$

This implies that $U'_\lambda(0) > U'_\mu(0)$ and that $U'_\lambda(0)$ depends continuously on λ. We conclude $U'_\lambda(0) = 0$ for a unique $\lambda_0 \in [-\lambda_T, \lambda_T]$. This λ_0 is eigenvalue of $\mathcal{L}_T$ corresponding to the even eigenfunction $\tilde{Y_0} = Y_0 + U_{\lambda_0}$. Differentiating (4.1.2) , we will have $|\tilde{Y_0}'(y) - \tilde{Y_0}(y)| \leq \delta e^{-|y|}$. $\square$

4.2 General Case

We consider the spectrum of

$$\mathcal{L}_T := -\partial_y^2 - b\partial_y - c + F''(T), \tag{4.2.1}$$

where T is the stationary solution guaranteed by Theorem 2.3.1. Defining $d = -c + F''(T) - F''(S)$, we have $\mathcal{L}_T = \mathcal{L}_S - b\partial_y + d$. By the C^3 regularity of F, we have $|F''(T) - F''(S)| \le K|S_b|$, and Theorem 2.3.1 implies $\|d\|_{L^1(\mathbb{R})} + \|d\|_{L^\infty(\mathbb{R})} \lesssim \delta$. With (3.2.12), we can also write a first-order approximation for d as follows:

$$d(y) = -c(y) + F'''(S)(y) \int_{\mathbb{R}} \tilde{G}(y,w)[bS' + cS](w)\,\mathrm{d}w + \varepsilon(y), \tag{4.2.2}$$

with $\tilde{G}$ as in (3.2.12) and $\varepsilon(y) = o(S_b(y))$.

Our goal is to investigate how the spectrum of $\mathcal{L}_S$ changes under the perturbation $-b\partial_y + d$. Since $\mathcal{L}_T$ is self-adjoint with respect to the inner product

$$\langle f, g\rangle_\omega := \int_{\mathbb{R}} \omega f g \,\mathrm{d}y,$$

with $\omega = \int_{-\infty}^y b(z)\,\mathrm{d}z$, the spectrum $\sigma(\mathcal{L}_T)$ is real. Since the perturbation is relatively $\mathcal{L}$-compact, we have $\sigma_{ess}(\mathcal{L}_T) = \sigma_{ess}(\mathcal{L}_S)$. (See, e.g. [30, Chapter 14].) Given our upper bounds on b and d, it is standard that $\sigma(\mathcal{L}_T)$ lies in the $c_0\delta$ neighborhood of $\sigma(\mathcal{L}_S)$, for some $c_0 > 0$. (An elementary argument to this effect can be found in the proof of Theorem 3.1 in [65]. This already establishes part (a) of Theorem 2.3.2.

To analyze the eigenvalue problem, we write the equation $(\mathcal{L}_S - \lambda)Y^\lambda = 0$ in vector form:

$$(\mathbf{Y}^\lambda)'(y) = \left(\begin{pmatrix} 0 & 1 \\ m^2 - \lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ F''(S) - m^2 & 0 \end{pmatrix} \right) \mathbf{Y}^\lambda(y).$$

For any $\lambda \leq m^2$, Lemma 9.0.5(a) implies there exist $Y^\lambda_\infty, Y^\lambda_{-\infty} \in L^\infty_{\text{loc}}(\mathbb{R})$ satisfying $(\mathcal{L}_S - \lambda)Y^\lambda_{\pm\infty} = 0$, and

$$\lim_{y\to\pm\infty} e^{\pm ky}\mathbf{Y}^\lambda_{\pm\infty}(y) = \begin{pmatrix} 1 \\ \mp k \end{pmatrix}, \tag{4.2.3}$$

with $k = \sqrt{m^2-\lambda}$. For $\lambda < m^2$, we also obtain the integral representations

$$\begin{aligned} e^{ky}\mathbf{Y}^\lambda_\infty &= \begin{pmatrix} 1 \\ -k \end{pmatrix} - \frac{1}{2}\int_y^\infty (F''(S)-m^2)Y^\lambda_\infty(w)e^{kw}\begin{pmatrix} (e^{2k(y-w)}-1)/k \\ e^{2k(y-w)}+1 \end{pmatrix}\mathrm{d}w \\ e^{-ky}\mathbf{Y}^\lambda_{-\infty} &= \begin{pmatrix} 1 \\ k \end{pmatrix} - \frac{1}{2}\int_{-\infty}^y (F''(S)-m^2)Y^\lambda_{-\infty}(w)e^{-kw}\begin{pmatrix} (e^{2k(y-w)}-1)/k \\ e^{2k(y-w)}+1 \end{pmatrix}\mathrm{d}w. \end{aligned} \tag{4.2.4}$$

For the operator $\mathcal{L}_T$, we similarly apply Lemma 9.0.5(a) with $V = F''(S) - m^2 + d$ to obtain $U^\lambda_{\pm\infty}$ solving $(\mathcal{L}_T - \lambda)U^\lambda_{\pm\infty} = 0$, with the same boundary conditions (4.2.3), and for $\lambda < m^2$,

$$\begin{aligned} e^{ky}\mathbf{U}^\lambda_\infty &= \begin{pmatrix} 1 \\ -k \end{pmatrix} - \frac{1}{2}\int_y^\infty [(F''(S)-m^2+d)U^\lambda_\infty - b(U^\lambda_\infty)']e^{kw}\begin{pmatrix} (e^{2k(y-w)}-1)/k \\ e^{2k(y-w)}+1 \end{pmatrix}\mathrm{d}w \\ e^{-ky}\mathbf{U}^\lambda_{-\infty} &= \begin{pmatrix} 1 \\ k \end{pmatrix} - \frac{1}{2}\int_{-\infty}^y [(F''(S)-m^2+d)U^\lambda_{-\infty} - b(U^\lambda_{-\infty})']e^{-kw}\begin{pmatrix} (e^{2k(y-w)}-1)/k \\ e^{2k(y-w)}+1 \end{pmatrix}\mathrm{d}w. \end{aligned} \tag{4.2.5}$$

First, we prove a suitable approximation lemma for $Y^\lambda_{\pm\infty}$ and $U^\lambda_{\pm\infty}$ for nearby values of λ:

Lemma 4.2.1. *Assume $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \leq 1$. For any compact subset B of $(-\infty, m^2)$, there exists a constant $C > 0$ such that for any $\lambda_1, \lambda_2 \in B$, there holds*

$$\begin{aligned} \|e^{k_1 y}\mathbf{Y}^{\lambda_1}_\infty - e^{k_2 y}\mathbf{U}^{\lambda_2}_\infty\|_{L^\infty([0,\infty),\mathbb{R}^2)} &\leq C(|\lambda_1-\lambda_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}), \\ \|e^{-k_1 y}\mathbf{Y}^{\lambda_1}_{-\infty} - e^{-k_2 y}\mathbf{U}^{\lambda_2}_{-\infty}\|_{L^\infty((-\infty,0],\mathbb{R}^2)} &\leq C(|\lambda_1-\lambda_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}), \end{aligned}$$

where $k_i = \sqrt{m^2-\lambda_i}$.

Proof. We prove only the first estimate, as the second follows by a similar argument.

From (4.2.4) and (4.2.5) we have

$$
\begin{aligned}
&e^{k_1 y}\mathbf{Y}_\infty^{\lambda_1}(y) - e^{k_2 y}\mathbf{U}_\infty^{\lambda_2}(y) \\
&\quad= \begin{pmatrix} 0 \\ k_2 - k_1 \end{pmatrix} - \frac{1}{2}\int_y^\infty \Bigg[(F''(S) - m^2)\begin{pmatrix} (e^{2k_1(y-w)} - 1)/k_1 \\ e^{2k_1(y-w)} + 1 \end{pmatrix} Y_\infty^{\lambda_1}(w)e^{k_1 w} \\
&\qquad\qquad - \begin{pmatrix} (e^{2k_2(y-w)} - 1)/k_2 \\ e^{2k_2(y-w)} + 1 \end{pmatrix} [(F''(S) - m^2 + d)U_\infty^{\lambda_2}(w) - b(U_\infty^{\lambda_2})' e^{k_2 w}]\Bigg] \,\mathrm{d}w \\
&\quad= \mathbf{J}_1(y) + \mathbf{J}_2(y) \\
&\qquad - \frac{1}{2}\int_y^\infty (F''(S) - m^2)\begin{pmatrix} (e^{2k_1(y-w)} - 1)/k_1 \\ e^{2k_1(y-w)} + 1 \end{pmatrix} (e^{k_1 w}Y_\infty^{\lambda_1}(w) - e^{k_2 w}U_\infty^{\lambda_2}(w)) \,\mathrm{d}w,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{J}_1(y) &:= \begin{pmatrix} 0 \\ k_2 - k_1 \end{pmatrix} \\
&\quad - \frac{1}{2}\int_y^\infty (F''(S) - m^2)e^{k_2 w}U_\infty^{\lambda_2}(w)\begin{pmatrix} (e^{2k_1(y-w)} - 1)/k_1 - (e^{2k_2(y-w)} - 1)/k_2 \\ e^{2k_1(y-w)} - e^{2k_2(y-w)} \end{pmatrix} \mathrm{d}w \\
\mathbf{J}_2(y) &:= \frac{1}{2}\int_y^\infty \begin{pmatrix} (e^{2k_2(y-w)} - 1)/k_2 \\ e^{2k_2(y-w)} + 1 \end{pmatrix} [dU_\infty^{\lambda_2} - b(U_\infty^{\lambda_2})']e^{k_2 w} \,\mathrm{d}w.
\end{aligned}
$$

Since $y - w \le 0$, the mean value theorem applied to $x \mapsto e^{2x(y-w)}$ and $x \mapsto (e^{2x(y-w)} - 1)/x$ implies, after a straightforward calculation, the inequalities

$$
\left| \begin{pmatrix} (e^{2k_1(y-w)} - 1)/k_1 - (e^{2k_2(y-w)} - 1)/k_2 \\ e^{2k_1(y-w)} - e^{2k_2(y-w)} \end{pmatrix} \right| \le C(1 + |y - w|)|k_1 - k_2|,
$$

for a constant C depending on B. Since $|F''(S) - m^2| \le e^{-m|w|}$ and $e^{k_2 w}U_\infty^{\lambda_2}(w)$ is uniformly bounded on $[0, \infty)$, we therefore have $\|\mathbf{J}_1\|_{L^\infty([0,\infty),\mathbb{R}^2)} \le C|k_1 - k_2|$.

For $\mathbf{J}_2$, since $e^{k_2 w}\mathbf{U}_\infty^{\lambda_2}$ is bounded on $[0, \infty)$, we have $\|\mathbf{J}_2\|_{L^\infty([0,\infty),\mathbb{R}^2)} \le C\|d + b\|_{L^1(\mathbb{R})}$, for a constant depending only on k_2.

Define the integral kernel

$$K(y,w) = -\frac{1}{2}(F''(S) - m^2)\begin{pmatrix} (e^{2k_1(y-w)} - 1)/k_1 & 0 \\ e^{2k_1(y-w)} + 1 & 0 \end{pmatrix},$$

From the exponential decay of $F''(S) - m^2$ we see that

$$\int_0^\infty \sup_{0<y<w} \|K(y,w)\| \, dw$$

is bounded by a constant depending only on k_1 and m. Lemma 9.0.4 then implies

$$\|e^{k_1 y}\mathbf{Y}_\infty^{\lambda_1}(y) - e^{k_2 y}\mathbf{U}_\infty^{\lambda_2}(y)\|_{L^\infty([0,\infty),\mathbb{R}^2)} \leq C\|\mathbf{J}_1 + \mathbf{J}_2\|_{L^\infty([0,\infty),\mathbb{R}^2)} \leq C(|k_1 - k_2| + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}). \tag{4.2.6}$$

Since $|k_1 - k_2| \leq C|\lambda_1 - \lambda_2|$ for a constant depending only on K, the proof is complete. $\square$

Now, we are ready to derive a result that governs the direction in which eigenvalues of $\mathcal{L}_S$ drift under the perturbation:

Theorem 4.2.1. *Assume $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \leq \delta \ll 1$. For any eigenvalue $\lambda_* < m^2$ of $\mathcal{L}_S$ with eigenfunction Y_*, there exists a simple, real eigenvalue λ of $\mathcal{L}_T$ with $|\lambda - \lambda_*| \leq C\delta$. Furthermore, we have the following expansion for λ:*

$$\lambda = \lambda_* + \frac{\int_{\mathbb{R}} Y_*[dY_* - bY_*'] \, dy}{\int_{\mathbb{R}} (Y_*)^2 \, dy} + O(\delta^2).$$

In particular, if

$$A := \int_{\mathbb{R}} Y_*[dY_* - bY_*'] \, dy \neq 0,$$

then $\lambda - \lambda_*$ has the same sign as A.

Remark 1. *Using the formula* (4.2.2), *one can show that* A *has the same sign as*

$$\int_{\mathbb{R}} \left[Y_*^2(y) \left(-c(y) + F'''(S)(y) \int_{\mathbb{R}} \tilde{G}(y,w)[bS' + cS](w)\, \mathrm{d}w \right) - b(y) Y_*'(y) Y_*(y) \right] \mathrm{d}y. \quad (4.2.7)$$

Now, we analyze the threshold resonance R, which is an L^∞ function solving $\mathcal{L}_S R - m^2 R = 0$. First, we prove a modified version of Lemma 4.2.1 for the borderline case m^2. This will be useful in tracking how a threshold resonance of $\mathcal{L}_S$ translates to the spectrum of $\mathcal{L}_T$. Writing $(\mathcal{L}_S - m^2)Y^\lambda = 0$ in vector form as above, Lemma 9.0.5(b) implies

$$\mathbf{Y}_\infty^{m^2}(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_y^\infty (F''(S) - m^2) Y_\infty^{m^2} \begin{pmatrix} y-w \\ 1 \end{pmatrix} \mathrm{d}w. \quad (4.2.8)$$

Lemma 4.2.2. *Assume* $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \le 1$. *For any* $\lambda_0 < m^2$, *there exists a constant* $C > 0$ *such that for any* $\lambda \in (\lambda_0, m^2)$, *there holds*

$$\|\mathbf{Y}_\infty^{m^2} - e^{ky}\mathbf{U}_\infty^\lambda\|_{L^\infty([0,\infty),\mathbb{R}^2)} \le C(k + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}),$$
$$\|\mathbf{Y}_{-\infty}^{m^2} - e^{-ky}\mathbf{U}_{-\infty}^\lambda\|_{L^\infty((-\infty,0],\mathbb{R}^2)} \le C(k + \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}),$$

where $k = \sqrt{m^2 - \lambda}$.

Proof. The proof is similar to Lemma 4.2.1, with the difference that $\mathbf{Y}_\infty^{m^2}$ satisfies the modified integral equation (4.2.8). From (4.2.8) and (4.2.5), we have

$$\begin{aligned} \mathbf{Y}_\infty^{m^2}(y) - e^{ky}\mathbf{U}_\infty^\lambda(y) = \begin{pmatrix} 0 \\ k \end{pmatrix} - \frac{1}{2}\int_y^\infty \Bigg[& 2(F''(S) - m^2) \begin{pmatrix} y-w \\ 1 \end{pmatrix} Y_\infty^{m^2}(w) \\ & - \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} [(F''(S) - m^2 + d)U_\infty^\lambda(w) - b(U_\infty^\lambda)' e^{kw}] \Bigg] \mathrm{d}w \end{aligned}$$

$$= \mathbf{J}_1(y) + \mathbf{J}_2(y) -$$

$$\frac{1}{2}\int_y^\infty (F''(S) - m^2) \begin{pmatrix} (e^{2k(y-w)} - 1)/k \\ e^{2k(y-w)} + 1 \end{pmatrix} (Y_\infty^{m^2}(w) - e^{kw} U_\infty^\lambda(w))\,\mathrm{d}w,$$

with

$$\mathbf{J}_1(y) := \begin{pmatrix} 0 \\ k \end{pmatrix} - \frac{1}{2}\int_y^\infty (F''(S) - m^2) Y_\infty^{m^2}(w) \begin{pmatrix} 2(y-w) - (e^{2k(y-w)} - 1)/k \\ 2 - (e^{2k(y-w)} + 1) \end{pmatrix} \mathrm{d}w,$$

and $\mathbf{J}_2(y)$ defined as in the proof of Lemma 4.2.1, with λ replacing λ_2.

We claim that $\|\mathbf{J}_1\|_{L^\infty([0,\infty),\mathbb{R}^2)} \leq C|k|$. Indeed, applying the mean value theorem to $f(x) = e^{2x(y-w)}$ gives

$$|f(k) - f(0)| \leq |k| \sup_{0<x<k} |f'(x)| \leq |k| 2|y-w| e^{2x(y-w)} \leq |k| 2|y-w|,$$

or $|e^{2k(y-w)} - 1| \leq 2|k||y-w|$. Next, Taylor's Theorem implies $f(k) = f(0) + f'(0)k + \varepsilon$, with $|\varepsilon| \leq \frac{1}{2}k^2 \sup_{0<x<k} |f''(x)|$. We have $f'(0) = 2(y-w)$ and $f''(x) = 4(y-w)^2 e^{2x(y-w)} \leq 4(y-w)^2$, since $y - w < 0$. This gives $e^{2k(y-w)} - 1 = 2k(y-w) + \varepsilon$, with $|\varepsilon| \leq 2k^2(y-w)^2$, or

$$|(e^{2k(y-w)} - 1)/k - 2(y-w)| \leq 2k(y-w)^2.$$

Plugging these inequalities into the definition of $\mathbf{J}_1$ and using decay of $F''(S) - m^2$ gives the desired estimate.

The same calculation as in the proof of Lemma 4.2.1 implies that the boundedness property (9.0.11) is satisfied for this integral equation, and Lemma 9.0.4 establishes the conclusion of the lemma. □

In the following theorem, we make the (mild) assumption that the limits at $\pm\infty$ of $R(y)$ are nonzero.

Theorem 4.2.2. (a) *Assume that m^2 is a simple resonance for $\mathcal{L}_S$, i.e. that there exists*

$R \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ with $\mathcal{L}_S R = m^2 R$. Then there exists $\delta > 0$ depending only on the function R, such that if $\|b\|_{L^1(\mathbb{R})} + \|d\|_{L^1(\mathbb{R})} \lesssim \delta$ and

$$\int_{\mathbb{R}} R[dR - bR']\,\mathrm{d}w < 0,$$

then there exists a discrete eigenvalue λ of $\mathcal{L}_T$ with $0 < m^2 - \lambda < C\delta$. If

$$\int_{\mathbb{R}} R[dR - bR']\,\mathrm{d}w > 0,$$

then there is no discrete eigenvalue of $\mathcal{L}_T$ in a neighborhood of the essential spectrum, i.e. the discrete spectrum $\sigma_d(\mathcal{L}_T)$ consists of the same number of eigenvalues as $\sigma_d(\mathcal{L}_S)$.

(b) *On the other hand, if m^2 is nonresonant and not an eigenvalue of $\mathcal{L}_S$, then for δ is sufficiently small, m^2 cannot be a resonance or an eigenvalue of $\mathcal{L}_T$, and there is no eigenvalue of $\mathcal{L}_T$ in a neighborhood of the essential spectrum.*

Remark 2. *As above, using (4.2.2), the quantity $\int_{\mathbb{R}} R[dR - bR']\,\mathrm{d}w$ has the same sign as*

$$\int_{\mathbb{R}} \left[R^2(y) \left(-c(y) + F'''(S)(y) \int_{\mathbb{R}} \tilde{G}(y,w)[bS' + cS](w)\,\mathrm{d}w \right) - b(y)R'(y)R(y) \right] \mathrm{d}y \qquad (4.2.9)$$

4.2.1 Proofs of the Main Results

Proof of Theorem (4.2.1). With $Y_{\pm\infty}^{\lambda_*}$ solving (4.2.4), since λ_* is a simple eigenvalue, we have $Y_{\pm\infty} = c_\pm Y_*$, for constants $c_\pm$. Let $k_* = \sqrt{m^2 - \lambda_*}$. From our construction, it is clear that Y_* decays exponentially at a rate $Y_*(y) \lesssim e^{-k_*|y|}$.

For λ near λ_* let $k = \sqrt{m^2 - \lambda}$ and let $U_{\pm\infty}^{\lambda}$ be the solutions to (4.2.5) as above. From Lemma 4.2.1 and our assumption that $\|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \lesssim \delta$, we can write

$$\mathbf{U}_{\pm\infty}^{\lambda}(y) = e^{\pm(k_* - k)y}\mathbf{Y}_{\pm\infty}^{\lambda_*}(y) + e^{\mp ky}\mathbf{E}_{\pm\infty}(y), \qquad (4.2.10)$$

with $\|\mathbf{E}_\infty\|_{L^\infty([0,\infty),\mathbb{R}^2)} + \|\mathbf{E}_{-\infty}\|_{L^\infty((-\infty,0],\mathbb{R}^2)} \lesssim \delta$. Denote the Wronskian

$$W_{\mathbf{U}}(\lambda, y) = \det(\mathbf{U}^\lambda_\infty(y), \mathbf{U}^\lambda_{-\infty}(y)).$$

By Abel's formula, $\exp(\int_{-\infty}^y b(z)\,\mathrm{d}z)W_{\mathbf{U}}(\lambda, y)$ is independent of y. We focus on $y = 0$ and apply (4.2.10) to obtain

$$\begin{aligned} W_{\mathbf{U}}(\lambda, 0) &= \det(\mathbf{Y}^{\lambda_*}_\infty(0) + \mathbf{E}_\infty(0), \mathbf{Y}^{\lambda_*}_{-\infty}(0) + \mathbf{E}_{-\infty}(0)) \\ &= \det(\mathbf{Y}^{\lambda_*}_\infty(0), \mathbf{E}_{-\infty}(0)) + \det(\mathbf{E}_\infty(0), \mathbf{Y}^{\lambda_*}_{-\infty}(0)) + \det(\mathbf{E}_\infty(0), \mathbf{E}_{-\infty}(0)) \\ &= \det(\mathbf{Y}^{\lambda_*}_\infty(0), \mathbf{U}^\lambda_{-\infty}(0)) + \det(\mathbf{U}^\lambda_\infty(0), \mathbf{Y}^{\lambda_*}_{-\infty}(0)) + O(\delta^2). \end{aligned} \tag{4.2.11}$$

In the second line, we used that $\mathbf{Y}^{\lambda_*}_\infty$ and $\mathbf{Y}^{\lambda_*}_{-\infty}$ are parallel, and in the last line, we used $\mathbf{E}_{\pm\infty}(0) = \mathbf{U}^\lambda_{\pm\infty}(0) - \mathbf{Y}^{\lambda_*}_{\pm\infty}(0)$ and $|\mathbf{E}_{\pm\infty}(0)| \lesssim \delta$. Since

$$\det(\mathbf{Y}^{\lambda_*}_{\pm\infty}, \mathbf{U}^\lambda_{\mp\infty})'(y) = Y^{\lambda_*}_{\pm\infty}\left[(d + \lambda_* - \lambda)U^\lambda_{\mp\infty} - b(U^\lambda_{\mp\infty})'\right],$$

we can use (4.2.10) again to write

$$\begin{aligned} \det(\mathbf{Y}^{\lambda_*}_\infty(0), \mathbf{U}^\lambda_{-\infty}(0)) &= \int_{-\infty}^0 Y^{\lambda_*}_\infty\left[(d + \lambda_* - \lambda)U^\lambda_{-\infty} - b(U^\lambda_{-\infty})'\right]\mathrm{d}w \\ &= \int_{-\infty}^0 e^{(k-k_*)w}Y^{\lambda_*}_\infty\left[(d + \lambda_* - \lambda)Y^{\lambda_*}_{-\infty} - b(Y^{\lambda_*}_{-\infty})'\right]\mathrm{d}w \\ &\quad + \int_{-\infty}^0 e^{kw}Y^{\lambda_*}_\infty(w)[(d + \lambda_* - \lambda)E^1_{-\infty} - bE^2_{-\infty}]\,\mathrm{d}w, \end{aligned}$$

and

$$\begin{aligned} \det(\mathbf{U}^\lambda_\infty, \mathbf{Y}^{\lambda_*}_{-\infty})(0) &= \int_0^\infty Y^{\lambda_*}_{-\infty}\left[(d + \lambda_* - \lambda)U^\lambda_\infty - b(U^\lambda_\infty)'\right]\mathrm{d}w \\ &= \int_0^\infty e^{(k_*-k)w}Y^{\lambda_*}_{-\infty}\left[(d + \lambda_* - \lambda)Y^{\lambda_*}_\infty - b(Y^{\lambda_*}_\infty)'\right]\mathrm{d}w \\ &\quad + \int_0^\infty e^{-kw}Y^{\lambda_*}_{-\infty}(w)[(d + \lambda_* - \lambda)E^1_\infty - bE^2_\infty]\,\mathrm{d}w. \end{aligned}$$

Feeding these expressions into (4.2.11), we obtain

$$W_{\mathbf{U}}(\lambda,0)=c_+c_-\int_{-\infty}^{\infty}e^{(k_*-k)|w|}[(d+\lambda_*-\lambda)(Y_*(w))^2-bY_*(w)Y_*'(w)]\,\mathrm{d}w+O(\delta^2).$$

From the approximation $|e^{(k_*-k)|w|}-1|\leq|k_*-k||w|e^{|k_*-k||w|}$ and the exponential decay of Y_* we have, for λ an eigenvalue of $\mathcal{L}_T$,

$$0=e^{\int_{-\infty}^{y}b(z)\,\mathrm{d}z}c_+c_-\int_{-\infty}^{\infty}[(d+\lambda_*-\lambda)(Y_*(w))^2-bY_*(w)Y_*'(w)]\,\mathrm{d}w+O(\delta^2),$$

which implies the first-order expansion for λ in the statement of the theorem. □

Proof of Theorem (4.2.2). With $Y_{\pm\infty}^{m^2}$ solving (4.2.8), we have $Y_{\pm\infty}^{m^2}=c_\pm R$, for constants $c_\pm$.

Our first step is to analyze the unperturbed Wronskian $W_{\mathbf{Y}}(\lambda,y)=\det(\mathbf{Y}_{\infty}^{\lambda},\mathbf{Y}_{-\infty}^{\lambda})$. With λ near m^2 and $k=\sqrt{m^2-\lambda}$, by abuse of notation, we write $W_{\mathbf{Y}}(k,y)=W_{\mathbf{Y}}(m^2-k^2,y)$. Applying Lemma 4.2.2 with $b=d=0$, we may write

$$\mathbf{Y}_{\pm\infty}^{\lambda}(y)=e^{\mp ky}(Y_{\pm\infty}^{m^2}(y)+\mathbf{E}_{\pm\infty}),\quad y\in\mathbb{R},$$

with $\|\mathbf{E}_{\infty}\|_{L^{\infty}([0,\infty),\mathbb{R}^2)}+\|\mathbf{E}_{-\infty}\|_{L^{\infty}((-\infty,0],\mathbb{R}^2)}\lesssim k$. By the equation satisfied by $Y_{\pm\infty}^{\lambda}$, the Wronskian $W_{\mathbf{Y}}(\lambda,y)$ is independent of y. Proceeding as in the proof of Theorem 4.2.1, we have as before (see (4.2.11))

$$W_{\mathbf{Y}}(k,0)=\det(\mathbf{Y}_{\infty}^{m^2}(0),\mathbf{Y}_{-\infty}^{\lambda}(0))+\det(\mathbf{Y}_{\infty}^{\lambda}(0),\mathbf{Y}_{-\infty}^{m^2}(0))+O(k^2).\tag{4.2.12}$$

A direct calculation shows $\det(\mathbf{Y}^{m^2}_{\pm\infty}, \mathbf{Y}^{\lambda}_{\mp\infty})'(y) = (m^2-\lambda)Y^{m^2}_{\pm\infty}Y^{\lambda}_{\mp\infty}$, which gives, since $k^2 = m^2 - \lambda$,

$$
\begin{aligned}
W_{\mathbf{Y}}(k,0) &= k^2 \int_{-\infty}^{0} Y^{m^2}_{\infty} Y^{\lambda}_{-\infty} \,\mathrm{d}w + k^2 \int_{0}^{\infty} Y^{m^2}_{-\infty} Y^{\lambda}_{\infty} \,\mathrm{d}w + O(k^2) \\
&= k^2 \int_{-\infty}^{0} Y^{m^2}_{\infty} e^{kw}(Y^{m^2}_{-\infty} + E^1_{-\infty}) \,\mathrm{d}w + k^2 \int_{0}^{\infty} Y^{m^2}_{-\infty} e^{-kw}(Y^{m^2}_{\infty} + E^1_{\infty}) \,\mathrm{d}w + O(k^2) \\
&= k^2 c_+ c_- \int_{-\infty}^{\infty} e^{-k|w|} R^2 \,\mathrm{d}w \\
&\quad + k^2 \int_{-\infty}^{\infty} R(w) e^{-k|w|} \left(1_{\{w<0\}} c_+ E^1_{-\infty} + 1_{\{w\geq 0\}} c_- E^1_{\infty}\right) \,\mathrm{d}w + O(k^2).
\end{aligned}
\tag{4.2.13}
$$

Note that all integrals converge, since $Y^{m^2}_{\pm\infty}$, $e^{\pm ky}Y^{\lambda}_{\pm\infty}$, and $e^{\pm ky}E^1_{\pm\infty}$ are all uniformly bounded.

In the last expression of (4.2.13), we note that the first term is proportional to k. Indeed, since R has non-zero limits as $y \to \pm\infty$, there exist $\zeta, M > 0$ (independent of k) such that $R^2(y) \geq \zeta$ if $|y| \geq M$. As a result, for any $k \in (0,1)$, one has $\int_{\mathbb{R}} e^{-k|w|}R^2 \,\mathrm{d}w \geq 2\zeta e^{-M}/k$. On the other hand, we have $\int_{\mathbb{R}} R^2 e^{-k|w|} \,\mathrm{d}w \leq 2\|R\|^2_{L^\infty(\mathbb{R})}/k$. It is also clear that, since $\|\mathbf{E}_{\pm\infty}\|_{L^\infty(\mathbb{R})} \lesssim k$, the second term on the right in (4.2.13) is $O(k^2)$. To sum up, we have shown

$$
W_{\mathbf{Y}}(k,0) = c_+ c_- A(k) + O(k^2), \tag{4.2.14}
$$

with $A(k) \geq A_0 k$ for some $A_0 > 0$ independent of k.

Now we turn to the perturbed operator $\mathcal{L}_T$. Let $U^{\lambda}_{\pm\infty}$ be the solutions to (4.2.5) as above. Applying Lemma 4.2.2 with $\lambda_1 = \lambda_2 = \lambda$, we have

$$
\mathbf{U}^{\lambda}_{\pm\infty}(y) = \mathbf{Y}^{\lambda}_{\pm\infty}(y) + \tilde{\mathbf{E}}_{\pm\infty}(y), \tag{4.2.15}
$$

with $\|e^{ky}\tilde{\mathbf{E}}_{\infty}\|_{L^\infty([0,\infty),\mathbb{R}^2)} + \|e^{-ky}\tilde{\mathbf{E}}_{-\infty}\|_{L^\infty((-\infty,0],\mathbb{R}^2)} \lesssim \|d\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} \lesssim \delta$.

With the Wronskian $W_{\mathbf{U}}(\lambda, y)$ defined as in the proof of Theorem 4.2.1, we again write

$W_{\mathbf{U}}(k, y) = W_{\mathbf{U}}(m^2 - k^2, y)$, and obtain

$$\begin{aligned} W_{\mathbf{U}}(\lambda, 0) &= \det(\mathbf{Y}^{\lambda}_{\infty}(0), \mathbf{Y}^{\lambda}_{-\infty}(0)) + \det(\tilde{\mathbf{E}}_{\infty}(0), \mathbf{Y}^{\lambda}_{-\infty}(0)) \\ &\quad + \det(\mathbf{Y}^{\lambda}_{\infty}(0), \tilde{\mathbf{E}}_{-\infty}(0)) + \det(\tilde{\mathbf{E}}_{\infty}(0), \tilde{\mathbf{E}}_{-\infty}(0)) \\ &= W_{\mathbf{Y}}(\lambda, 0) + \det(\tilde{\mathbf{E}}_{\infty}(0), \mathbf{Y}^{\lambda}_{-\infty}(0)) + \det(\mathbf{Y}^{\lambda}_{\infty}(0), \tilde{\mathbf{E}}_{-\infty}(0)) + O(\delta^2). \end{aligned} \tag{4.2.16}$$

Since $\tilde{\mathbf{E}}_{\pm\infty} = U^{\lambda}_{\pm\infty} - Y^{\lambda}_{\pm\infty}$ satisfy $\tilde{\mathbf{E}}''_{\pm\infty} = (F''(S) - m^2 - \lambda)\tilde{\mathbf{E}}_{\pm\infty} - b(U^{\lambda}_{\pm\infty})' + dU^{\lambda}_{\pm\infty}$, we have

$$\det(\tilde{\mathbf{E}}_{\pm\infty}, \mathbf{Y}^{\lambda}_{\mp\infty})'(y) = [dU^{\lambda}_{\pm\infty} - b(U^{\lambda}_{\pm\infty})']Y^{\lambda}_{\mp\infty}.$$

Because $b, d \in L^1(\mathbb{R})$, $|U^{\lambda}_{\infty}| \lesssim e^{-ky}$, and $|Y^{\lambda}_{-\infty}| \lesssim e^{ky}$, the expression $\left[dU^{\lambda}_{\infty} - b(U^{\lambda}_{\infty})'\right] Y^{\lambda}_{-\infty}$ is integrable on $[0, \infty)$, and we can use (4.2.15) to write

$$\begin{aligned} \det(\tilde{\mathbf{E}}_{\infty}(0), \mathbf{Y}^{\lambda}_{-\infty}(0)) &= \int_0^{\infty} \left[dU^{\lambda}_{\infty} - b(U^{\lambda}_{\infty})'\right] Y^{\lambda}_{-\infty}\, \mathrm{d}w \\ &= \int_0^{\infty} [dY^{\lambda}_{\infty} - b(Y^{\lambda}_{\infty})']Y^{\lambda}_{-\infty}\, \mathrm{d}w + \int_0^{\infty} [d\tilde{E}^1_{\infty} - b\tilde{E}^2_{\infty}]Y^{\lambda}_{-\infty}\, \mathrm{d}w, \end{aligned}$$

where the last integral converges and is $O(\delta^2)$ since $\|\tilde{\mathbf{E}}_{\infty}\| \lesssim \delta e^{-ky}$ and $Y^{\lambda}_{-\infty} \lesssim e^{ky}$. For the first integral on the right, we use Lemma 4.2.1 with $d = b = 0$ and obtain

$$\int_0^{\infty} [dY^{\lambda}_{\infty} - b(Y^{\lambda}_{\infty})']Y^{\lambda}_{-\infty}\, \mathrm{d}w = \int_0^{\infty} [dY^{m^2}_{\infty} - b(Y^{m^2}_{\infty})']Y^{m^2}_{-\infty}\, \mathrm{d}w + O(\delta k).$$

After applying a similar analysis to $\det(\mathbf{Y}^{\lambda}_{\infty}(0), \tilde{\mathbf{E}}_{-\infty}(0))$, the expression (4.2.16) becomes

$$W_{\mathbf{U}}(k, 0) = W_{\mathbf{Y}}(k, 0) + c_+c_- \int_{-\infty}^{\infty} [d(R(w))^2 - bR(w)R'(w)]\, \mathrm{d}w + O(\delta k) + O(\delta^2).$$

With (4.2.14), this implies

$$W_{\mathbf{U}}(k, 0) = c_+c_- \left(A(k) + \int_{-\infty}^{\infty} [d(R(w))^2 - bR(w)R'(w)]\, \mathrm{d}w\right) + O(\delta k) + O(\delta^2).$$

For δ small enough, the expression inside the parentheses determines whether any zeroes of

$W_{\mathbf{U}}(k, 0)$ are present for $k > 0$. The bound $A(k) \geq A_0 k$ with $A_0 > 0$ implies statement (a) of the theorem.

For statement (b), the assumption that m^2 is not a resonance or eigenvalue implies $W_{\mathbf{Y}}(0, 0) \neq 0$. The approximation (4.2.15) easily implies $W_{\mathbf{U}}(0, 0) = W_{\mathbf{Y}}(0, 0) + O(\delta) \neq 0$ for δ small enough. $\square$

Chapter 5

Orbital Stability Of Kink Solutions

We prove orbital stability, i.e. that solutions starting close to T are always close to some shifted version of T.

Proof of Theorem 2.3.3. For any solution u of (2.3.1), the energy

$$E(u) = \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}(\partial_y u)^2 - \frac{1}{2}cu^2 + F(u) \right] \omega(y)\, \mathrm{d}y,$$

is conserved, where $\omega(y) = \exp(\int_{-\infty}^{y} b(z)\, \mathrm{d}z) = 1 + O(\delta)$, uniformly in y. We also define the potential energy

$$E_p(u) = \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_y u)^2 - \frac{1}{2}cu^2 + F(u) \right] \omega(y)\, \mathrm{d}y.$$

A simple computation shows that

$$|E_p(\psi) - \tilde{E}_p(\psi)| \le C\delta \left(\|\psi\|^2_{L^\infty(\mathbb{R})} + \tilde{E}_p(\psi) \right), \quad \psi \in H^1_T(\mathbb{R}), \tag{5.0.1}$$

where $\tilde{E}_p$ is the potential energy corresponding to the constant coefficient equation (2.2.1):

$$\tilde{E}_p(\psi) := \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_t \psi)^2 + \frac{1}{2}(\partial_y \psi)^2 + F(\psi) \right] \mathrm{d}y$$

The idea is to use the (known) property that $\tilde{E}_p(\psi) - \tilde{E}_p(S)$ controls the distance between ψ and S, to show the corresponding fact for E_p and T. In more detail, for $q > 0$, define

$$d_q(\psi, T)^2 := \inf_{\xi \in \mathbb{R}} \int_{\mathbb{R}} [(\partial_y \psi(y) - T'(y+\xi))^2 + q(\psi(y) - T(y+\xi))^2] \, \mathrm{d}y,$$

for any ψ in the energy space. We define $d_q(\psi, S)$ in the analogous way. Proposition 1 of [29] proves the following: There exist $C, r, q > 0$ such that

$$d_q(\psi, S)^2 \leq C(\tilde{E}_p(\psi) - \tilde{E}_p(S)),$$

whenever $d_q(\psi, S) \leq r$.[1] Note that

$$\begin{aligned} |\tilde{E}_p(T) - \tilde{E}_p(S)| &= \left| \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_y S_b)^2 + \partial_y S \partial_y S_b + F(S + S_b) - F(S) \right] \mathrm{d}y \right| \\ &\leq \|S_b\|^2_{H^1(\mathbb{R})} + \|\partial_y S\|^2_{L^2(\mathbb{R})} + \|F\|_{C^1([a_-, a_+]} \|S_b\|_{L^1(\mathbb{R})} \\ &\lesssim \delta, \end{aligned}$$

by Theorem 2.3.1. Using (5.0.1) twice, we then have

$$\begin{aligned} d_q(\psi, S)^2 &\leq C\left(E_p(\psi) - E_p(S)\right) + C\delta \left(\||\psi| + |T|\|^2_{L^\infty(\mathbb{R})} + \tilde{E}_p(\psi) + \tilde{E}_p(S) \right) \\ &\leq C\left(E_p(\psi) - E_p(T)\right) + C\delta \left(1 + \tilde{E}_p(\psi) + \tilde{E}_p(S) \right). \end{aligned} \tag{5.0.2}$$

To get to the last line, we used Sobolev embedding to write $\|\psi\|_{L^\infty(\mathbb{R})} \leq C_q d_q(\psi, S)$, and combined this term into the left-hand side.

Since $\int_{\mathbb{R}} (u(t,y) - S(y+\xi))^2 \, \mathrm{d}y \to \infty$ as $\xi \to \pm\infty$, there is some $\xi_0 \in \mathbb{R}$ where the infimum defining $d_q(u(t,\cdot), S)$ is achieved. To save space, write $T_\xi = T(y + \xi_0)$, and similarly for S_ξ and

[1] Proposition 1 in [29] is stated for $u(t, \cdot)$ where u is a solution of (2.2.1), but an examination of the proof shows that the conclusion holds for any $\psi(y)$ satisfying the hypotheses stated here.

$S_{b,\xi}$. We then have

$$\begin{aligned}
d_q(\psi,T)^2 &\le \int_{\mathbb{R}} \left[(\partial_y\psi - T_\xi')^2 + q(\psi - T_\xi)^2\right]\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \left[(\partial_y\psi - S_\xi')^2 + q(\psi - S_\xi)^2\right]\,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}} \left[(S_{b,\xi}')^2 + qS_{b,\xi}^2 - 2(\partial_y\psi - S_\xi')\partial_y S_{b,\xi} - 2q(\psi - S_\xi)S_{b,\xi}\right]\,\mathrm{d}y \\
&\le d_q(\psi,S)^2 + Cq\|S_b\|_{H^1(\mathbb{R})}^2 + \|S_b\|_{H^1(\mathbb{R})}\|\psi - S\|_{H^1(\mathbb{R})} \\
&\le d_q(\psi,S)^2 + Cq\|S_b\|_{H^1(\mathbb{R})}^2 + \|S_b\|_{H^1(\mathbb{R})} d_q(\psi,S)^2 \\
&\le 2d_q(\psi,S)^2 + C\delta^2.
\end{aligned} \tag{5.0.3}$$

For $\delta > 0$ small enough compared to r and q, this implies $d_q(\psi,T)^2 \le 2d_q(\psi,S)^2 + r/2$. By exchanging the roles of T and S in this calculation, we also obtain $d_q(\psi,S)^2 \le 2d_q(\psi,T) + r/2$.

Next, combining (5.0.2) and (5.0.3),

$$d_q(\psi,T)^2 \le C\left(E_p(\psi) - E_p(T)\right) + C\delta\left(1 + \tilde{E}_p(\psi) + \tilde{E}_p(S)\right). \tag{5.0.4}$$

This inequality holds for ψ such that $d_q(\psi,S) \le r$. By above, we can ensure this condition by choosing $d_q(\psi,T) \le r/4$.

Now, for a solution u to (2.3.1) with $d_q(u(0,\cdot),T) \le r/4$ and $\partial_t u(0,\cdot)$ sufficiently small in $L^2(\mathbb{R})$, (5.0.4) implies that

$$d_q(u(t,\cdot),T)^2 \le C(E(u(t,\cdot)) - E(T)) + C\delta\left(1 + \tilde{E}_p(u(t,\cdot)) + \tilde{E}_p(S)\right).$$

The quantity $E(u(t,\cdot))$ is conserved in time. Calculations similar to (5.0.1) show that $\tilde{E}_p(u) \le 2E_p(u) + C\delta\|u\|_{L^\infty(\mathbb{R})}^2 \le E(u) + C\delta d_q(u,T)^2$, and the last term may be combined into the left side. We finally have

$$d_q(u(t,\cdot),T)^2 \le C(E(u_0) - E(T)) + C\delta(1 + \tilde{E}_p(S)).$$

This right-hand side is independent of t, which implies the solution u never leaves the neighborhood of T as long as it exists. As above, for every t, there is some $\xi = \xi(t)$ at which the infimum defining $d_q(u, T)$ is achieved. By standard arguments, this time-independent bound on $d_q(u(t, \cdot), T)$ combined with energy conservation implies the solution u exists for all $t \in [0, \infty)$. $\square$

Chapter 6

Constant and near-constant solutions: overview

In the next several chapters, we focus on stationary solutions to (2.3.1) that are constant in y, or uniformly close to a constant.

We assume $F'(0) = 0$, so that $u \equiv 0$ is a stationary solution to (2.3.1). If $F'(\xi) = 0$ for some nonzero $\xi \in [a_-, a_+]$, we will also consider this constant solution in the case $c \equiv 0$, or stationary solutions close to zero when c is nonzero.

We always assume

$$b, c \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),$$

but in general b and c may not be small in any norm. For certain results, we will place extra conditions on b, c, and F.

Compared to the study of kink solutions, constant and near-constant states are more amenable to asymptotic stability proofs because the Virial functionals take a simpler form, and the modulation analysis is much simpler.

Our first result in this direction is the existence of near-constant states:

Theorem 6.0.1. *With b, c, and F as above, assume in addition that $F'(\xi) = 0$ and*

$F''(\xi) > 0$. *Assume that either*

(i) $|\xi| \le \delta$, *or*

(ii) $\|c(y)\|_X \le \delta$.

Then, for $\delta > 0$ *sufficiently small, there exists a stationary solution* $U(y)$ *to* (2.3.1), *with*

$$\|U - \xi\|_{L^1(\mathbb{R})} + \|U - \xi\|_{L^\infty(\mathbb{R})} + \|U'\|_{L^1(\mathbb{R})} + \|U'\|_{L^\infty(\mathbb{R})} \lesssim \delta.$$

If, in addition, $c(y)$ *satisfies* $|c(y)| \le Ke^{-k|y|}$ *for some* $K > 0$ *and some* $0 < k < F''(\xi)$, *and either*

(i) $|\xi| \le \delta$, *or*

(ii) $K \le \delta$,

then we have the improved decay estimates

$$\|e^{k|y|}(U(y) - \xi)\|_{L^\infty(\mathbb{R})} + \|e^{k|y|}U'(y)\|_{L^\infty(\mathbb{R})} \lesssim \delta.$$

We emphasize that, in case (i), we do not make any smallness assumption on the coefficients b and c.

After showing orbital stability under some additional assumptions in Theorem 8.0.1, we prove our last main result: asymptotic stability of consant and near-consant solutions, with respect to odd perturbations:

Theorem 6.0.2. *Let* $\xi = 0$, *and let* $U(y)$ *be the near-constant state guaranteed by Theorem 6.0.1. Assume there exists* $\mu > 0$ *such that*

$$c(y) \le F''(0) - \mu, \quad y \in \mathbb{R}, \tag{6.0.1}$$

(so that, by Theorem 8.0.1, $U(y)$ *is orbitally stable). Assume in addition that* F *and* b *are odd,* c *is even, and*

$$|b(y)| + |c(y)| \lesssim \delta e^{-|y|},$$

for some (sufficiently small) $\delta > 0$*. (This implies* $U(y)$ *is also odd and exponentially decaying.)*

Furthermore, assume that one of the following three cases is in effect:

(1) $c(y) \equiv 0$*, which implies* $U \equiv 0$*.*

(2) $F'(u) = u - u^3$*.*

(3) $F'(u) = \sin(u)$*.*

Then there exists $\varepsilon > 0$ *such that if the initial data* (u_0, u_1) *is odd in* y*, and satisfies*

$$\|u_0 - U\|_{H^1(\mathbb{R})} + \|u_1\|_{L^2} < \varepsilon,$$

then the solution is asymptotically stable in the sense that

$$\lim_{t\to\infty} \left(\|u(t) - U\|_{H^1(I)} + \|\partial_t u(t)\|_{L^2(I)}\right) = 0,$$

for any bounded interval $I \subset \mathbb{R}$*.*

Note that the conclusion of Theorem 6.0.2 cannot be improved by replacing I with $\mathbb{R}$, because such a conclusion would violate energy conservation. (Note that (6.0.1) implies the energy of u is finite.)

We should also note that the oddness assumption is necessary, at least in the case of constant-speed sine-Gordon, $F'(u) = \sin(u)$, because of the presence of even breather solutions that oscillate near the zero solution and do not decay as $t \to \infty$. The explicit formula for these breathers is

$$B(t, x) = 4 \arctan\left(\frac{\beta}{\alpha} \frac{\cos(\alpha t)}{\cosh(\beta x)}\right).$$

for parameters α, β with $\alpha^2 + \beta^2 = 1$. These solutions may have arbitrarily small norm by choosing $|\beta|$ small.

The proof strategy for Theorem 6.0.2, inspired by [42, 43] is based on a Virial functional of the form

$$\mathcal{I}(v) = \int_{\mathbb{R}} \left(\tanh(y/\lambda)\partial_y v + \frac{1}{2}\tanh'(y/\lambda)v\right) \partial_t v \,\mathrm{d}y,$$

where v is the difference between the solution u and the steady-state U. The functional $\mathcal{I}$ is a Lyaponov-type functional satisfying two important properties:

(a) Coercivity: the time derivative of $\mathcal{I}$ controls the (weighted) H^1 norm of v from above.

(b) Boundedness: $\mathcal{I}$ can be bounded above by $H^1 \times L^2$ norms of v in a straightforward way.

These two properties make $\mathcal{I}$ a valuable tool in controlling the norm of v for large time, once one has already shown that v exists globally and remains close to 0 for all time.

The novelty of our proof compared to [42, 43, 65] consists in the extra terms coming from the near-constant states $U(y)$. These terms change drastically with the choice of F, which is why our proof must proceed in a case-by-case manner with different potentials of interest.

Chapter 7

Existence of near-constant stationary solutions

We consider nonzero number $\xi \in [a_-, a_+]$ such that $F'(\xi) = 0$. Then, if the zeroth-order coefficient $c(y)$ is identically 0, $u(y) \equiv \xi$ is obviously a stationary solution to (2.3.1). On the other hand, if c is nonzero, then $u \equiv \xi$ is no longer a solution, but following methods similar to Theorem 2.3.1, we can prove the existence of a stationary state $U(y)$ uniformly close to ξ:

Proof of Theorem 6.0.1. Write $U = \xi + U_\delta$, plug in to $-U'' - bU' - cU = -F'(U)$ to obtain

$$-U_\delta'' - bU_\delta' - (c - F''(\xi))U_\delta = c\xi + \mathcal{N}(U_\delta), \tag{7.0.1}$$

where $\mathcal{N}(U_\delta) = -F'(\xi + U_\delta) + F''(\xi)U_\delta$. We would like to find solutions $Y_{\pm\infty}$ to the linear equation $-Y'' - bY' - (c - F''(\xi))Y = 0$, which can be written in system form, with $\mathbf{Y} = \langle Y, Y' \rangle$,

$$\mathbf{Y}' = \left(\begin{pmatrix} 0 & 1 \\ F''(\xi) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -c(y) & -b(y) \end{pmatrix} \right) \mathbf{Y}.$$

Let $m = \sqrt{F''(\xi)} > 0$. By Lemma 9.0.5, there exist solutions $\mathbf{Y}_{\pm\infty}$ to this system, with

$$\lim_{y \to \pm\infty} e^{\pm my} \mathbf{Y}_{\pm\infty}(y) = \begin{pmatrix} 1 \\ \mp m \end{pmatrix}.$$

Define $\mathcal{L}_{b,c} = -\partial_y^2 - b\partial_y - (c - F''(\xi))$. First, we assume 0 is not an eigenvalue of $\mathcal{L}_{b,c}$. Then, $\mathbf{Y}_{\pm\infty}$ are linearly independent, and we can define the Green's function

$$G(y,w) := \frac{1}{W_{\mathbf{Y}}(y)} \begin{cases} Y_{-\infty}(y) Y_{\infty}(w), & y < w, \\ Y_{\infty}(y) Y_{-\infty}(w), & w \leq y, \end{cases}$$

where $W_{\mathbf{Y}}(y) = \det(\mathbf{Y}_{-\infty}, \mathbf{Y}_{\infty})$. Abel's formula implies $W_{\mathbf{Y}}(y) = W_{\mathbf{Y}}(0) \exp(\int_0^y b(z)\,\mathrm{d}z)$. Since $b \in L^1(\mathbb{R})$, this exponential is bounded away from zero.

We recall an estimate from the proof of Theorem 2.3.1: with $\|\cdot\|_X = \|\cdot\|_{L^1(\mathbb{R})} + \|\cdot\|_{L^\infty(\mathbb{R})}$,

$$\left\| \int_{\mathbb{R}} G(\cdot, w)\eta(w)\,\mathrm{d}w \right\|_X + \left\| \partial_y \int_{\mathbb{R}} G(\cdot, w)\eta(w)\,\mathrm{d}w \right\|_X \leq C\|\eta\|_X, \tag{7.0.2}$$

for a constant C depending on b, c, and m. The Green's function G allows us to write (7.0.1) as

$$U_\delta(y) = (\mathcal{T}U_\delta)(y) := \xi \int_{\mathbb{R}} G(y,w)c(w)\,\mathrm{d}w + \int_{\mathbb{R}} G(y,w)N(U_\delta)(w)\,\mathrm{d}w, \tag{7.0.3}$$

Regarding the first term, we have

$$\left\| \xi \int_{\mathbb{R}} G(\cdot, w)c(w)\,\mathrm{d}w \right\|_X \leq C|\xi|\|c(w)\|_X.$$

Recall that in case (i), $|\xi| \leq \delta$, and in case (ii), $\|c(w)\|_X \leq \delta$. In either case, this term is bounded by a constant C_0 times δ.

For the second term in (7.0.3), since F is C^3, we have, for $\eta \in X$,

$$N(\eta) = |-F'(\xi + \eta) + F''(\xi)\eta| \leq K\eta^2, \tag{7.0.4}$$

for some $K > 0$ depending on the C^3 norm of F. Therefore,

$$\begin{aligned}\|\mathcal{T}\eta\|_X &\leq C_0\delta + K\left\|\int_{\mathbb{R}} G(\cdot, w)\eta^2(w)\,\mathrm{d}w\right\|_X \\ &\leq C_0\delta + K\|\eta^2\|_X \\ &\leq C_0\delta + K\|\eta\|_X^2,\end{aligned}$$

where the value of K changes line-by-line, and the last inequality followed by the interpolation $\|h\|_{L^2} \leq \|h\|_{L^1}^{1/2}\|h\|_{L^\infty}^{1/2}$.

Defining $\mathcal{A}_\delta := \{\eta \in X : \|\eta\|_X \leq 2C_0\delta\}$, we claim $\mathcal{T}$ maps $\mathcal{A}_\delta$ into itself, if δ is small enough. Indeed, our estimates imply that for $\eta \in \mathcal{A}_\delta$,

$$\|\mathcal{T}\eta\|_X \leq C_0\delta + K\|\eta\|_X^2 \leq C_0\delta + 4C_0^2K\delta^2.$$

Choosing $\delta < 1/(4C_0K)$, we see $\mathcal{T}\eta \in \mathcal{A}_\delta$, as claimed.

Next, for $\eta_1, \eta_2 \in \mathcal{A}_\delta$, repeating the calculation from the proof of Theorem 2.3.1, with ξ replacing $S(y)$, we obtain

$$|\mathcal{N}(\eta_1) - \mathcal{N}(\eta_2)| \leq C\delta|\eta_1 - \eta_2|,$$

for a constant $C > 0$. Our estimates for G now imply

$$\|\mathcal{T}\eta_1 - \mathcal{T}\eta_2\|_X \leq \left\|\int_{\mathbb{R}} G(\cdot, w)[\mathcal{N}(\eta_1) - N(\eta_2)]\,\mathrm{d}w\right\|_X \leq C\delta\|\eta_1 - \eta_2\|_X.$$

For $\delta > 0$ small enough, $\mathcal{T}$ is a contraction on $\mathcal{A}_\delta$, and a unique solution U_δ to (7.0.3) exists in $\mathcal{A}_\delta$.

To obtain the bound on U_δ', we differentiate (7.0.3) and use (7.0.2) and (7.0.4):

$$\begin{aligned}\|U_\delta'\|_X &= \left\|\partial_y \int_{\mathbb{R}} G(y, w)[\xi c(w) + \mathcal{N}(U_\delta)(w)]\,\mathrm{d}w\right\|_X \\ &\leq C\left(|\xi|\|c(w)\|_X + \|\mathcal{N}(U_\delta)\|_X\right) \\ &\leq C\delta + K\|U_\delta^2\|_X \lesssim \delta,\end{aligned} \tag{7.0.5}$$

where, as above, we have used that $|\xi|\|c(w)\|_X \lesssim \delta$ in both case (i) and case (ii).

It remains to analyze the case where 0 is an eigenvalue of $\mathcal{L}_{b,c}$. In this case, since $F''(\xi) = m^2 > 0$, standard Sturm-Liouville theory implies that the essential spectrum of $\mathcal{L}_{b,c}$ is exactly $[m, \infty)$, meaning that 0 is an isolated point of the spectrum. Therefore, we let $\lambda < 0$ be a spectral parameter sufficiently close to 0, but in the resolvent set of $\mathcal{L}_{b,c}$. The equation (7.0.1) is equivalent to

$$-U_\delta'' - bU_\delta' - (c - F''(\xi) - \lambda)U_\delta = c\xi + \lambda U_\delta + \mathcal{N}(U_\delta).$$

Now, since the operator $\mathcal{L}_\lambda := -\partial_y^2 - b\partial_y - (c - F''(\xi) - \lambda)$ is invertible, we may construct a Green's function for $\mathcal{L}_\lambda$ and find U_δ via contraction as above. The details, which are similar to the proof of Theorem 2.3.1, are omitted.

Next, assume that $c(y)$ has exponential decay of order $\lesssim e^{-k|y|}$ for some $k > 0$. For fixed k, define

$$\|h\|_\sim := \sup_{y\in\mathbb{R}} e^{k|y|}|h(y)|.$$

In the case that 0 is not an eigenvalue of $\mathcal{L}_{b.c}$, we claim the following additional estimate for the operator $\eta \mapsto \int_\mathbb{R} G(\cdot, w)\eta(w)\,\mathrm{d}w$:

$$\left\| \int_\mathbb{R} G(\cdot, w)\eta(w)\,\mathrm{d}w \right\|_\sim \le C\|\eta\|_\sim, \qquad \text{if } 0 < k < m, \tag{7.0.6}$$

for some $C > 0$ depending on k and m. Indeed, since $Y_{\pm\infty}(y) \sim e^{\mp my}$ as $y \to \pm\infty$, we have

$$\begin{aligned}\int_\mathbb{R} G(y, w)\eta(w)\,\mathrm{d}w &= \frac{1}{W_\mathbf{Y}(y)}\left[Y_\infty(y)\int_{-\infty}^y Y_{-\infty}(w)\eta(w)\,\mathrm{d}w + Y_{-\infty}(y)\int_y^\infty Y_\infty(w)\eta(w)\,\mathrm{d}w\right] \\ &\le C\|\eta\|_\sim \left[e^{-my}\int_{-\infty}^y e^{mw}e^{-k|w|}\,\mathrm{d}w + e^{my}\int_y^\infty e^{-mw}e^{-k|w|}\,\mathrm{d}w\right].\end{aligned}$$

Focusing on the first term on the right, when $y < 0$ we have

$$e^{-my}\int_{-\infty}^y e^{mw}e^{kw}\,\mathrm{d}w \lesssim e^{ky} = e^{-k|y|},$$

and when $y \geq 0$ we have

$$e^{-my}\int_{-\infty}^{y} e^{mw}e^{-k|w|}\,\mathrm{d}w \lesssim e^{-my}\left(\int_{-\infty}^{0} e^{(m+k)w}\,\mathrm{d}w + \int_{0}^{y} e^{(m-k)w}\,\mathrm{d}w\right) = e^{-k|y|},$$

since $k < m$. After applying a symmetric argument to the integral over $[y, \infty)$, we have shown (7.0.6).

We now apply a contraction mapping argument in the $\|\cdot\|_\sim$ norm. With $\mathcal{T}$ defined as in (7.0.3), the estimate (7.0.6) and $c(w) \lesssim e^{-k|w|}$ imply

$$\left\|\xi \int_{\mathbb{R}} G(\cdot, w)c(w)\,\mathrm{d}w\right\|_\sim \leq C|\xi|\|c(w)\|_\sim.$$

Since either $|\xi| \leq \delta$ or $\|c(w)\|_\sim \leq \delta$, we conclude this term is bounded by $C_0\delta$ for some constant $C_0 > 0$. By (7.0.4), we have

$$\|\mathcal{T}\eta\|_\sim \leq C_0\delta + K\left\|\int_{\mathbb{R}} G(\cdot, w)\eta^2(w)\,\mathrm{d}w\right\|_\sim \leq C_0\delta + K\|\eta^2\|_\sim \leq C_0\delta + K\|\eta\|_\sim^2,$$

and we may apply a contraction argument similar to above to conclude $\|U_\delta\|_\sim$ is finite, as desired.

To establish the bounds on $\partial_y U_\delta$ in the $\|\cdot\|_\sim$ norm, we first show

$$\left\|\partial_y \int_{\mathbb{R}} G(y, w)\eta(w)\,\mathrm{d}w\right\|_\sim \leq C\|\eta\|_\sim,$$

following the proof of (7.0.6) exactly (since $Y'_{\pm\infty}(y) \lesssim e^{\mp m|y|}$). The estimate $\|U'_\delta\|_\sim \lesssim \delta$ now follows from differentiating (7.0.3) and proceeding as in (7.0.5).

Finally, we must extend these arguments to the case in which 0 is an eigenvalue of $\mathcal{L}_{b,c}$. As above, we omit the details of this straightforward modification, which follows ideas similar to the proof of Theorem 2.3.1. □

Chapter 8

Orbital stability of constant and near-constant states

With ξ such that $F'(\xi) = 0$, we will assume (by shifting the potential F, which does not affect the equation of motion (2.3.1)) that $F(\xi) = 0$, so that the energy of the constant state is finite.

Theorem 8.0.1. *Assume there exists $\mu > 0$ such that*

$$c(y) \leq F''(\xi) - \mu, \quad y \in \mathbb{R}.$$

Then there exist $C, \varepsilon_0 > 0$ such that if $\|u(0,\cdot) - \xi\|_{H^1(\mathbb{R})} + \|\partial_t u(0,\cdot)\|_{L^2(\mathbb{R})} < \varepsilon$ with $\varepsilon \leq \varepsilon_0$, then the corresponding solution $u(t,x)$ exists for all time, with

$$\|u(t,\cdot)\|_{H^1(\mathbb{R})} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})} \leq C\varepsilon.$$

A typical application of this result would be in the case where $F''(\xi) > 0$ and $c(y) \leq 0$ everywhere, but the hypotheses are also satisfied if $F''(\xi) < 0$ and $c(y)$ is negative enough to counterbalance $F''(\xi)$.

Proof of Theorem 8.0.1. Recall that the energy

$$E(u(t,\cdot)) = \int_{\mathbb{R}} \left[\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}(\partial_y u)^2 - \frac{1}{2}c(y)u^2 + F(u)\right] \omega(y)\,\mathrm{d}y,$$

is conserved along the flow of (2.3.1), where $\omega(y) = \exp(\int_{-\infty}^{y} b(z)\,\mathrm{d}z)$. Since $c \in L^\infty$ and $F \in C^2$, we have

$$E(u(t,\cdot)) \le C_1(\|u(t,\cdot)\|_{H^1(\mathbb{R})}^2 + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})}^2), \tag{8.0.1}$$

for any t in the time domain of u. On the other hand, a Taylor expansion for F shows that

$$\frac{1}{2}(F(u) - cu^2) \ge \frac{1}{2}(F''(\xi) - c)u^2 - Ku^3 \ge \frac{\mu}{2}u^2 - Ku^3,$$

for any $y \in \mathbb{R}$, where $K > 0$ depends on the C^3 norm of F. By Sobolev embedding, $\|\psi\|_{L^\infty} \le C_0\|\psi\|_{H^1}$, so for any t with $\|u(t,\cdot)\|_{H^1} \le r_0 := \mu/(2C_0K)$, we have

$$\|u(t,\cdot)\|_{H^1(\mathbb{R})}^2 + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})}^2 \le C_2 E(u(t,\cdot)), \quad \text{if } \|u(t,\cdot)\|_{H^1(\mathbb{R})} \le r_0. \tag{8.0.2}$$

Now, let $\varepsilon_0 := \sqrt{r_0/(1 + C_1C_2)}$, and assume $\|u(0,\cdot)\|_{H^1(\mathbb{R})} + \|\partial_t u(0,\cdot)\|_{L^2(\mathbb{R})} < \varepsilon$ for $\varepsilon < \varepsilon_0$, as in the statement of the theorem. Let $E_0 = E(u_0)$ be the energy of $(u(0,\cdot), \partial_t u(0,\cdot))$. Thanks to (8.0.1) applied at $t = 0$, we must have $E_0 \le C_1\varepsilon^2$. By a standard fixed-point argument, a solution to (2.3.1) exists on $[0, T] \times \mathbb{R}$ for some $T > 0$ depending only on ε.

We claim $\|u(t,\cdot)\|_{H^1(\mathbb{R})} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})} < r_0$ for all $t \in [0, T]$. If not, let t_0 be the first time where $\|u(t_0,\cdot)\|_{H^1(\mathbb{R})} = r_0$. By energy conservation and (8.0.2), we have

$$\|u(t_0,\cdot)\|_{H^1}^2 + \|\partial_t u(t_0,\cdot)\|_{L^2}^2 \le C_2E_0 \le C_1C_2\varepsilon^2 < r_0,$$

a contradiction.

Therefore, the solution can be extended up to a time $T + T_1$, with $T_1 > 0$ depending on r_0. Repeating this argument, we extend the solution for all time. □

We can also address the orbital stability of the near-constant states $U(y) = \xi + U_\delta(y)$

constructed in Theorem 6.0.1, in the case $F''(\xi) > 0$. Using $F(\xi) = F'(\xi) = 0$ and $\|U_\delta\|_X + \|\partial_y U_\delta\|_X \lesssim \delta$, it is not hard to see that the energy of $U(y)$ is finite and bounded above by a constant times δ.

Furthermore, if $\delta > 0$ is sufficiently small depending on ε, then initial data which is δ-close to $(\xi, 0)$ in the $H^1 \times L^2$ norm also satisfies the hypotheses of Theorem 8.0.1. This implies near-constant states are also orbitally stable, since solutions staying close to the constant state ξ are also δ-close to $U(y)$.

Chapter 9

Asymptotic stability

Assume F is even, b is odd, and c is even. We focus on the case $\xi = 0$.

We define the following weighted norms:

$$\|h\|_{L^2_\omega}^2 := \int_{\mathbb{R}} h^2 \operatorname{sech}(y)\,\mathrm{d}y, \quad \|h\|_{H^1_\omega}^2 := \int_{\mathbb{R}} ((\partial_y h)^2 + h^2) \operatorname{sech}(y)\,\mathrm{d}y.$$

Lemma 9.0.1. *For $\lambda > 0$, define $\psi(y) = \tanh(y/\lambda)$. Define the bilinear form*

$$\mathcal{B}(v) := \int_{\mathbb{R}} \psi' (\partial_y v)^2\,\mathrm{d}y - \frac{1}{4}\int_{\mathbb{R}} \psi''' v^2\,\mathrm{d}y. \tag{9.0.1}$$

For any odd function $v \in H^1(\mathbb{R})$ and any $\lambda > 0$, there holds

$$\mathcal{B}(v) \geq \frac{3}{4}\int_{\mathbb{R}} (\partial_y w)^2\,\mathrm{d}y, \qquad \text{where } w(y) = \operatorname{sech}(y/\lambda) v(y).$$

Furthermore, with the choice $\lambda = 100$ and $w(y) = \operatorname{sech}(y/\lambda)v(y)$, there holds

$$\|v\|_{H^1_\omega} \lesssim \|\partial_y w\|_{L^2(\mathbb{R})}, \tag{9.0.2}$$

and therefore $\mathcal{B}$ satisfies the following estimate:

$$\mathcal{B}(v) \geq \kappa \|v\|_{H^1_\omega}^2,$$

for some constant $\kappa > 0$, and any odd $v \in H^1_\omega$.

Proof. The first estimate is exactly [43, Lemma 2.1]. The inequality (9.0.2) and the second estimate for $\mathcal{B}$ are proven in Formula (2.20) in [43]. □

The first step in proving asymptotic stability is the following integral estimate for solutions staying close to the profile U for all time.

Lemma 9.0.2. *Let F be as in Theorem 6.0.2. For any odd (in y) global solution $u(t, y)$ of* (2.3.1) *with*

$$\|u(t,\cdot) - U\|_{H^1(\mathbb{R})}^2 + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})}^2 \lesssim \varepsilon^2, \quad t \in [0, \infty)$$

there holds

$$\int_0^\infty (\|u(t,\cdot) - U\|_{H^1_\omega}^2 + \|\partial_t u(t,\cdot)\|_{L^2_\omega}^2)\, \mathrm{d}t \lesssim \varepsilon^2.$$

Proof. Let $v(y) = u(y) - U(y)$. Then v satisfies

$$\partial_t^2 v - [\partial_y^2 v + b\partial_y v + cv] = F'(U) - F'(U + v) = -\mathcal{N}(U, v),$$

with $\mathcal{N}(U, v) = F'(U + v) - F'(U)$. Note that, since F' is odd, the equation for v preserves oddness.

For convenience, we rewrite this as a first-order system, with $(v_1, v_2) = (v, \partial_t v)$:

$$\begin{aligned} \partial_t v_1 &= v_2, \\ \partial_t v_2 &= \partial_y^2 v_1 + b\partial_y v_1 + cv_1 + \mathcal{N}(U, v_1). \end{aligned} \tag{9.0.3}$$

Let $\psi(y) = \tanh(y/\lambda)$ for some $\lambda > 0$ to be chosen later. Define the Virial functional

$$\mathcal{I}(v) := \int_{\mathbb{R}} \psi(\partial_y v_1) v_2 \,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}} \psi' v_1 v_2 \,\mathrm{d}y = \int_{\mathbb{R}} \left(\psi\partial_y v_1 + \frac{1}{2}\psi' v_1\right) v_2 \,\mathrm{d}y. \tag{9.0.4}$$

For any $w \in H^1(\mathbb{R})$, we have the identity

$$\int_{\mathbb{R}} \left(\psi \partial_y w + \frac{1}{2}\psi' w \right) w \, \mathrm{d}y = 0.$$

Using this identity with $w = v_2$, we differentiate $\mathcal{I}(v)$ for (v_1, v_2) a solution of (9.0.3), and integrate by parts several times:

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}(v) &:= \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1 \right) \partial_t^2 v_1 \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1 \right) \left(\partial_y^2 v_1 + b\partial_y v_1 + c v_1 + \mathcal{N}(U, v_1) \right) \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(-\psi'(\partial_y v_1)^2 - \frac{1}{2}\psi'' v_1 \partial_y v_1 + \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1 \right) (b\partial_y v_1 + c v_1 + \mathcal{N}(U, v_1)) \right) \, \mathrm{d}y \\
&= -\int_{\mathbb{R}} \psi'(\partial_y v_1)^2 + \frac{1}{4}\int_{\mathbb{R}} \psi''' v_1^2 + \int_{\mathbb{R}} \left(\psi b (\partial_y v_1)^2 + \frac{1}{2}\psi' c v_1^2 \right) \, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}} \left(\frac{1}{2}\psi' b + \psi c \right) (\partial_y v_1) v_1 \, \mathrm{d}y + \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1 \right) \mathcal{N}(U, v_1) \, \mathrm{d}y.
\end{aligned} \tag{9.0.5}$$

The first two terms in the last expression are equal to $-\mathcal{B}(v_1)$, where $\mathcal{B}$ is defined as in (9.0.1). With $w(y) = \operatorname{sech}(y/\lambda) v_1(y)$ with $\lambda = 100$, Lemma 9.0.1 implies

$$\mathcal{B}(v_1) \geq \frac{3}{4}\|\partial_y w\|_{L^2_\omega}^2 \gtrsim \|v_1\|_{H^1_\omega}^2. \tag{9.0.6}$$

Next, by the exponential decay of b and c and Young's inequality, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} \left(\psi b (\partial_y v_1)^2 + \frac{1}{2}\psi' c v_1^2 \right) \, \mathrm{d}y \right| + \left| \int_{\mathbb{R}} \left(\frac{1}{2}\psi' b + \psi c \right) (\partial_y v_1) v_1 \, \mathrm{d}y \right| \\
\leq \frac{1}{2} \int_{\mathbb{R}} \left(\frac{1}{2}\psi' b + \psi c \right) \left[(\partial_y v_1)^2 + v_1^2 \right] \, \mathrm{d}y \\
\lesssim \|e^{|y|}(|b(y)| + |c(y)|)\|_{L^\infty} \|v_1\|_{H^1_\omega}^2 \\
\lesssim \delta \|v_1\|_{H^1_\omega}^2.
\end{aligned}$$

The analysis of the term involving $\mathcal{N}(U, v_1) = F'(U) - F'(U + v_1)$ in (9.0.5) is contained in

Lemma 9.0.3 below. That lemma states

$$\left|\int_{\mathbb{R}}(F'(U)-F'(U+v_1))(\psi\partial_y v_1+\frac{1}{2}\psi' v_1)\,\mathrm{d}y\right|\le\varepsilon\|\partial_y w\|_{L^2}^2+\delta\|v_1\|_{H^1_\omega}^2.$$

Since ε and δ are small, by (9.0.2), we finally obtain

$$-\frac{d}{dt}\mathcal{I}(u)\ge C_1\|v_1\|_{H^1_\omega}. \tag{9.0.7}$$

Next, we notice that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}\operatorname{sech}(y)v_1v_2\,\mathrm{d}y &= \int_{\mathbb{R}}\operatorname{sech}(y)(v_2^2+v_1\partial_t v_2)\,\mathrm{d}y\\
&= \int_{\mathbb{R}}\operatorname{sech}(y)[v_2^2+v_1(\partial_y^2 v_1+b\partial_y v_1+cv_1+\mathcal{N}(U,v_1))]\,\mathrm{d}y\\
&= \int_{\mathbb{R}}\left[\operatorname{sech}(y)(v_2^2-(\partial_y v_1)^2+v_1\mathcal{N}(U,v_1))-\operatorname{sech}'(y)v_1\partial_y v_1\right]\,\mathrm{d}y\\
&\quad+\int_{\mathbb{R}}\operatorname{sech}(y)\left[bv_1\partial_y v_1+cv_1^2\right]\,\mathrm{d}y\\
&= \int_{\mathbb{R}}\left[\operatorname{sech}(y)(v_2^2-(\partial_y v_1)^2+v_1\mathcal{N}(U,v_1))+\frac{1}{2}\operatorname{sech}''(y)v_1^2\right]\,\mathrm{d}y\\
&\quad+\int_{\mathbb{R}}\operatorname{sech}(y)\left[bv_1\partial_y v_1+cv_1^2\right]\,\mathrm{d}y\\
&\ge \|v_2\|_{L^2_\omega}^2-\|v_1\|_{H^1_\omega}^2\\
&\quad+\int_{\mathbb{R}}\left(\operatorname{sech}(y)v_1\mathcal{N}(U,v_1)+\frac{1}{2}\operatorname{sech}''(y)v_1^2\right)\,\mathrm{d}y-C\delta\|v_1\|_{H^1_\omega}^2.
\end{aligned}$$

To bound the last expression from below, note that $|\operatorname{sech}''(y)|=\operatorname{sech}(y)|\tanh^2(y)-\operatorname{sech}^2(y)|\lesssim\operatorname{sech}(y)$. Also, since $\mathcal{N}(U,v_1)=F'(U)-F'(U+v_1)=F''(z)v_1$ for some $z\in[a_-,a_+]$, we have $|v_1\mathcal{N}(U,v_1)|\lesssim v_1^2$. We then have

$$\frac{d}{dt}\int_{\mathbb{R}}\operatorname{sech}(y)v_1v_2\,\mathrm{d}y\ge\|v_2\|_{L^2_\omega}^2-C\|v_1\|_{H^1_\omega}^2, \tag{9.0.8}$$

for $\delta>0$ sufficiently small. Combining this with (9.0.7), we conclude in the following way: integrate (9.0.7) from 0 to t_0 and use the assumption that $\|v_1\|_{H^1}+\|v_2\|_{L^2}\lesssim\varepsilon$ to bound $\mathcal{I}$

from above, since $\mathcal{I}(v) \lesssim \|v_1\|_{H^1}\|v_2\|_{L^2} \lesssim \varepsilon^2$:

$$\int_0^{t_0} \|u_1\|_{H^1_\omega}^2 \leq -\mathcal{I}(t_0) + \mathcal{I}(0) \lesssim \varepsilon^2.$$

Sending $t_0 \to \infty$, we have

$$\int_0^\infty \|v_1\|_{H^1_\omega}^2 \lesssim \varepsilon^2.$$

Next, we similarly integrate (9.0.8) from 0 to t_0, use the bound $\|v_1\|_{H^1} + \|v_2\|_{L^2} \lesssim \varepsilon$ to write $\int_{\mathbb{R}} \operatorname{sech}(y) v_1 v_2 \,\mathrm{d}y \lesssim \|v_1\|_{L^2}\|v_2\|_{L^2} \lesssim \varepsilon^2$, and send $t_0 \to \infty$:

$$\int_0^\infty \|v_2\|_{L^2_\omega}^2 \,\mathrm{d}t \lesssim \int_0^\infty \|u_1\|_{H^1_\omega}^2 \,\mathrm{d}t + \varepsilon^2 \lesssim \varepsilon^2,$$

as claimed. □

Lemma 9.0.3. *For each of the cases (1), (2), and (3) as in the statement of Theorem 6.0.2, there holds*

$$\left| \int_{\mathbb{R}} (F'(U) - F'(U + v_1))(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1) \,\mathrm{d}y \right| \lesssim \varepsilon \|\partial_y w\|_{L^2}^2 + \delta \|v_1\|_{H^1_\omega}^2. \tag{9.0.9}$$

Proof. We begin with a useful inequality quoted from [43, Formula (2.22)]: for any odd v_1, let $\lambda = 100$ and $w = \zeta v_1 = \operatorname{sech}(y/\lambda) v_1$. Then for any $q > 0$, there holds

$$\int_{\mathbb{R}} \psi' |v_1|^{2+q} \,\mathrm{d}y \lesssim \|v_1\|_{L^\infty}^q \|\partial_y w\|_{L^2}^2. \tag{9.0.10}$$

Now we are ready to prove lemma for each of the three cases. For brevity, let J refer to the left-hand side of (9.0.9).

Case 1: $c(y) \equiv 0$ and $U(y) \equiv 0$. Then, by our assumption $F'(0) = 0$, we have

$$\begin{aligned} J = -\int_{\mathbb{R}} F'(v_1)(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1) \,\mathrm{d}y &= -\int_{\mathbb{R}} (\psi \partial_y (F(v_1)) + \frac{1}{2}\psi' v_1 F'(v_1)) \,\mathrm{d}y \\ &= \int_{\mathbb{R}} (\psi'(F(v_1) - v_1 F'(v_1)) \,\mathrm{d}y. \end{aligned}$$

Since $F \in C^3$ and $F(0) = F'(0) = 0$, a Taylor expansion shows

$$\begin{aligned}|F(v_1) - \frac{1}{2}v_1 F'(v_1)| &= \Big|F(0) + F'(0)v_1 + \frac{1}{2}F''(0)v_1^2 + F'''(z_1)v_1^3 - \frac{1}{2}F'(0)v_1 \\ &\qquad - \frac{1}{2}v_1^2 F''(0) - \frac{1}{4}v_1^3 F'''(z_2)\Big| \\ &\lesssim |v_1|^3.\end{aligned}$$

We therefore have

$$|J| \le \int_{\mathbb{R}} \psi' |v_1|^3 \,\mathrm{d}y,$$

and we may apply (9.0.10) to obtain the conclusion of the lemma.

Case 2: $F'(u) = u - u^3$. In this case, $F'(U) - F'(U + v_1) = U - U^3 - (U^3 - (U + v_1)^3) = (3U^2 - 1)v_1 + 3Uv_1^2 + v_1^3$. Since $F''(0) = 1$, Theorem 6.0.1 implies $|U(y)| + |U'(y)| \lesssim \delta e^{-|y|}$. We write $J = J_1 + J_2$, with

$$\begin{aligned}J_1 &:= \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1\right)(3U^2 - 1)v_1 \,\mathrm{d}y \\ &= \int_{\mathbb{R}} \left(\frac{1}{2}\psi \partial_y (v_1^2) + \frac{1}{2}\psi' v_1^2\right)(3U^2 - 1)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}} \partial_y(\psi v_1^2)(3U^2 - 1)\,\mathrm{d}y \\ &= -3\int_{\mathbb{R}} \psi v_1^2 U U' \,\mathrm{d}y.\end{aligned}$$

We have $|J_1| \lesssim \delta^2 \int_{\mathbb{R}} \psi v_1^2 e^{-2|y|}\,\mathrm{d}y \lesssim \delta^2 \|v_1\|_{L^2_\omega}^2$.

The remaining part of the integral J reads

$$\begin{aligned}J_2 &:= \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2}\psi' v_1\right)(3Uv_1^2 + v_1^3)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \left[\psi(U\partial_y(v_1^3) + \frac{1}{4}\partial_y(v_1^4)) + \frac{1}{2}\psi' v_1^3(3 + v_1)\right]\mathrm{d}y \\ &= -\int_{\mathbb{R}} \left(\psi'(v_1^3 U + \frac{1}{4}v_1^4) + \psi U' v_1^3\right) + \frac{1}{2}\int_{\mathbb{R}} \psi' v_1^3(3 + v_1)\,\mathrm{d}y.\end{aligned}$$

Therefore,

$$|J_2| \lesssim \int_{\mathbb{R}} \psi' |v_1|^3 \,\mathrm{d}y,$$

and we may apply (9.0.10) to obtain the desired conclusion.

Case 3: $F'(u) = \sin(u)$. We now have $F'(U) - F'(U + v_1) = \sin U(1 - \cos v_1) - \cos U \sin v_1$, and $F''(0) = 1$, so that

$$|\sin U(y)| \leq |U(y)| \lesssim \delta e^{-|y|}, \quad |U'(y)| \lesssim \delta e^{-|y|}.$$

Similar to Case 2, we write $J = J_1 + J_2$, with

$$\begin{aligned} |J_1| &:= \left| \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2} \psi' v_1 \right) \sin U(1 - \cos v_1) \, \mathrm{d}y \right| \\ &\lesssim \delta \int_{\mathbb{R}} e^{-|y|} (|\partial_y v_1| v_1^2 + v_1^3) \, \mathrm{d}y \\ &\lesssim \delta\varepsilon \int_{\mathbb{R}} \operatorname{sech}(y)((\partial_y v_1)^2 + v_1^2) \, \mathrm{d}y \\ &\lesssim \delta \|v_1\|_{H^1_\omega}^2, \end{aligned}$$

and

$$\begin{aligned} J_2 &:= \int_{\mathbb{R}} \left(\psi \partial_y v_1 + \frac{1}{2} \psi' v_1 \right) \cos U \sin v_1 \, \mathrm{d}y \\ &= \int_{\mathbb{R}} \left(\psi \cos U \partial_y (1 - \cos v_1) + \frac{1}{2} \psi' v_1 \cos U \sin v_1 \right) \, \mathrm{d}y \\ &= \int_{\mathbb{R}} \psi' \cos U \left(\cos(v_1) - 1 + \frac{1}{2} v_1 \sin v_1 \right) \, \mathrm{d}y + \int_{\mathbb{R}} \psi U' \sin U(1 - \cos v_1) \, \mathrm{d}y. \end{aligned}$$

Standard Taylor expansions show $|1 - \cos v_1| \lesssim v_1^2$ and $|\cos(v_1) - 1 + \frac{1}{2} v_1 \sin v_1| \lesssim v_1^4$. Therefore, with $|\sin U(y)| + |U'(y)| \lesssim \delta e^{-|y|}$,

$$|J_2| \lesssim \int_{\mathbb{R}} \psi' v_1^4 \, \mathrm{d}y + \delta^2 \int_{\mathbb{R}} e^{-2|y|} v_1^2 \, \mathrm{d}y \leq \int_{\mathbb{R}} \psi' v_1^4 \, \mathrm{d}y + \delta^2 \|v_1\|_{L^2_\omega}^2.$$

Applying (9.0.10) a final time, the proof is complete. □

The following can be called a "no breathers" result–i.e. there are no solutions that remain close to $U(y)$ for all time, but do not converge. It applies even in situations where orbital stability is not necessarily known.

Theorem 9.0.1. *With F as in Theorem 6.0.2, for any odd (in y) global solution $u(t,y)$ of* (2.3.1) *with*

$$\|u(t,\cdot)-U\|_{H^1(\mathbb{R})}^2+\|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})}^2 \lesssim \varepsilon^2, \quad t\in[0,\infty)$$

there holds

$$\lim_{t\to\infty}\big(\|u(t)-U\|_{H^1(I)}+\|\partial_t u(t)\|_{L^2(I)}\big)=0,$$

for any bounded interval $I\subset\mathbb{R}$.

Proof of Theorem 9.0.1. Following the proof of [43, Theorem 1.1], we define

$$\mathcal{H}(t):=\int_{\mathbb{R}} \operatorname{sech}(y)[(\partial_y v_1)^2+v_1^2+v_2^2]\,\mathrm{d}y.$$

Differentiating and using the equation (9.0.3) satisfied by (v_1,v_2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}&=2\int_{\mathbb{R}} \operatorname{sech}(y)[\partial_y v_1\partial_t\partial_y v_1+v_1\partial_t v_1+v_2\partial_t v_2]\,\mathrm{d}y\\&=2\int_{\mathbb{R}} \operatorname{sech}(y)[\partial_y v_1\partial_y v_2+v_1v_2+v_2(\partial_y^2 v_1+b\partial_y v_1+cv_1+\mathcal{N}(U,v_1))]\,\mathrm{d}y\\&=2\int_{\mathbb{R}} \operatorname{sech}(y)v_2((1+c)v_1+b\partial_y v_1+\mathcal{N}(U,v_1))\,\mathrm{d}y+\int_{\mathbb{R}} \operatorname{sech}(y)\partial_y(v_2\partial_y v_1)\,\mathrm{d}y\\&=2\int_{\mathbb{R}} \operatorname{sech}(y)v_2((1+c)v_1+b\partial_y v_1+\mathcal{N}(U,v_1))\,\mathrm{d}y-\int_{\mathbb{R}} \operatorname{sech}'(y)v_2\partial_y v_1\,\mathrm{d}y.\end{aligned}$$

Applying Young's inequality and $|\mathcal{N}(U,v_1)|\lesssim v_1$, we have

$$\left|\frac{d}{dt}\mathcal{H}\right|\lesssim\int_{\mathbb{R}} \operatorname{sech}(y)[v_1^2+v_2^2+(\partial_y v_1)^2]\,\mathrm{d}y\le\|v_1\|_{H^1_\omega}^2+\|v_2\|_{L^2_\omega}^2.$$

By Lemma 9.0.2, there is a sequence $t_n\to\infty$ such that $\mathcal{H}(t_n)\to 0$. For any $t\ge 0$, we can integrate our bound for $|\frac{d}{dt}\mathcal{H}|$ from t to t_n to obtain

$$|\mathcal{H}(t_n)-\mathcal{H}(t)|\lesssim\int_t^{t_n}(\|v_1(t')\|_{H^1_\omega}^2+\|v_2(t')\|_{L^2_\omega}^2)\,\mathrm{d}t'.$$

Sending $n \to \infty$, we obtain

$$\mathcal{H}(t) \lesssim \int_t^\infty (\|v_1(t')\|_{H^1_\omega}^2 + \|v_2(t')\|_{L^2_\omega}^2)\, \mathrm{d}t'.$$

Since $(\|v_1(t')\|_{H^1_\omega}^2 + \|v_2(t')\|_{L^2_\omega}^2)$ is integrable for large times thanks to Lemma 9.0.2, we must have $\mathcal{H}(t) \to 0$ as $t \to \infty$, which implies the statement of the theorem. □

Combining Theorem 9.0.1 with the orbital stability result of Theorem 8.0.1 implies Theorem 6.0.2.

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Appendix: ODE Methods

In this section, we collect some convenient facts about the solvability and asymptotics of 2×2 first-order systems on $\mathbb{R}$.

First, we have a standard lemma on vector-valued integral equations of Volterra type:

Lemma 9.0.4. *For $a \in \mathbb{R}$ and $\mathbf{U} \in L^\infty([a,\infty),\mathbb{R}^2)$, the Volterra equation*

$$\mathbf{Z}(y) = \mathbf{U}(y) + \int_y^\infty K(y,w)\mathbf{Z}(w)\,\mathrm{d}w,$$

has a unique solution in $L^\infty([a,\infty),\mathbb{R}^2)$, provided

$$\mu := \int_a^\infty \sup_{a<y<w} \|K(y,w)\|\,\mathrm{d}w < \infty, \tag{9.0.11}$$

where $\|\cdot\|$ is the operator norm of the matrix $K(y,w)$. This solution is given by the iteration

$$\mathbf{Z}(y) = \mathbf{U}(y) + \sum_{n=1}^\infty \int_a^\infty \cdots \int_a^\infty \prod_{i=1}^n 1_{\{y_{i-1}<y_i\}} K(y_{i-1},y_i)\mathbf{U}(y_n)\,\mathrm{d}y_n \cdots \mathrm{d}y_1, \tag{9.0.12}$$

with $y_0 = y$. This solution satisfies

$$\|\mathbf{Z}\|_{L^\infty([a,\infty),\mathbb{R}^2)} \le e^\mu \|\mathbf{U}\|_{L^\infty([a,\infty),\mathbb{R}^2)}.$$

Proof. See [60, Lemma 2.4] for a proof of the corresponding fact for scalar-valued Volterra equations. The proof in the present vector-valued case is essentially the same, so we omit it. □

Next, we address a class of linear systems that arise from the eigenvalue problems in Sections 3.2 and 4.2:

Lemma 9.0.5. *(a) For $k > 0$, consider the system*

$$\mathbf{Y}'(y) = (M_1 + M_2(y))\mathbf{Y}(y), \tag{9.0.13}$$

where

$$M_1 = \begin{pmatrix} 0 & 1 \\ k^2 & 0 \end{pmatrix}, \quad M_2(y) = \begin{pmatrix} 0 & 0 \\ V(y) & -b(y) \end{pmatrix}$$

with $V, b \in L^1(\mathbb{R})$. There exist solutions $\mathbf{Y}_{-\infty}$, $\mathbf{Y}_\infty$ defined on $\mathbb{R}$, such that

$$\lim_{y\to\infty} e^{ky}\mathbf{Y}_\infty(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix}, \quad \lim_{y\to-\infty} e^{-ky}\mathbf{Y}_{-\infty}(y) = \begin{pmatrix} 1 \\ k \end{pmatrix}, \tag{9.0.14}$$

and the bound $|\mathbf{Y}_\infty(y)| \le Ce^{-ky}$ holds for all $y \in \mathbb{R}$, where the constant depends on k and $\|V + b\|_{L^1(\mathbb{R})}$. These solutions also satisfy the integral equations

$$\begin{aligned} \mathbf{Y}_\infty(y) &= \begin{pmatrix} 1 \\ -k \end{pmatrix} e^{-ky} - \frac{1}{2}\int_y^\infty (V(w)Y_\infty - bY_\infty'(w)) \begin{pmatrix} \frac{1}{k}(e^{k(y-w)} - e^{-k(y-w)}) \\ e^{k(y-w)} + e^{-k(y-w)} \end{pmatrix} \mathrm{d}w, \\ \mathbf{Y}_{-\infty}(y) &= \begin{pmatrix} 1 \\ k \end{pmatrix} e^{ky} + \frac{1}{2}\int_{-\infty}^y (V(w)Y_{-\infty} - bY_{-\infty}'(w)) \begin{pmatrix} \frac{1}{k}(e^{k(y-w)} - e^{-k(y-w)}) \\ e^{k(y-w)} + e^{-k(y-w)} \end{pmatrix} \mathrm{d}w. \end{aligned} \tag{9.0.15}$$

(b) For $k = 0$, assume in addition that $(1 + |y|^2)^{1/2}V$ and $(1 + |y|^2)^{1/2}b$ lie in $L^1(\mathbb{R})$. Then there exist solutions $\mathbf{Y}_{-\infty}, \mathbf{Y}_\infty$ to (9.0.13) satisfying

$$\lim_{y\to\infty} \mathbf{Y}_{\pm\infty}(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

as well as the integral equations

$$
\begin{aligned}
\mathbf{Y}_\infty(y) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_y^\infty (V(w)Y_\infty - bY_\infty'(w)) \begin{pmatrix} y-w \\ 1 \end{pmatrix} \mathrm{d}w, \\
\mathbf{Y}_{-\infty}(y) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^y (V(w)Y_{-\infty} - bY_{-\infty}'(w)) \begin{pmatrix} y-w \\ 1 \end{pmatrix} \mathrm{d}w.
\end{aligned} \tag{9.0.16}
$$

Proof. (a) Note that the eigenvalues of M_1 are $\pm k$ corresponding to eigenvectors $\begin{pmatrix} 1 \\ \mp k \end{pmatrix}$. We will find a solution to the integral equation

$$
\mathbf{Y}_\infty(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix} e^{-ky} - \int_y^\infty e^{M_1(y-w)} M_2(w) \mathbf{Y}_\infty(w)\,\mathrm{d}w, \quad y \in \mathbb{R}, \tag{9.0.17}
$$

satisfying $|\mathbf{Y}_\infty(y)| \leq Ce^{-ky}$ and $\lim_{y\to\infty} e^{ky}\mathbf{Y}_\infty = \begin{pmatrix} 1 \\ -k \end{pmatrix}$. By direct calculation, such $\mathbf{Y}_\infty$ also solves (9.0.13), as well as the first integral equation in (9.0.15). Letting $\mathbf{Z}(y) = e^{ky}\mathbf{Y}_\infty(y)$, (9.0.17) is equivalent to

$$
\mathbf{Z}(y) = \begin{pmatrix} 1 \\ -k \end{pmatrix} - \int_y^\infty e^{M_1(y-w)} M_2(w) e^{k(y-w)} \mathbf{Z}(w)\,\mathrm{d}w, \quad y \in \mathbb{R}. \tag{9.0.18}
$$

By diagonalizing M_1, we obtain

$$
e^{M_1(y-w)} M_2 = \frac{1}{2} \begin{pmatrix} \frac{1}{k} V(e^{k(y-w)} - e^{-k(y-w)}) & -\frac{1}{k} b(e^{k(y-w)} - e^{-k(y-w)}) \\ V(e^{k(y-w)} + e^{-k(y-w)}) & -b(e^{k(y-w)} + e^{-k(y-w)}) \end{pmatrix}.
$$

With $K(y,w) := e^{M_1(y-w)} M_2(w) e^{k(y-w)}$, we therefore have

$$
\|K(y,w)\| \leq C(1 + e^{2k(y-w)})(|V(w)| + |b(w)|),
$$

and that

$$\int_0^\infty \sup_{0<y<w} \|K(y,w)\| \, \mathrm{d}w \le C(\|V\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}).$$

Lemma 9.0.4 now implies a solution to (9.0.18) exists on $[0,\infty)$, and $\|\mathbf{Z}\|_{L^\infty([0,\infty),\mathbb{R}^2)}$ is bounded by a constant, which implies the boundary condition (9.0.14) holds for $\mathbf{Y}_\infty$, as well as the upper bound

$$|\mathbf{Y}_\infty(y)| \le Ce^{-ky}, \quad y \ge 0,$$

where $\mathbf{Y}_\infty = (Y_\infty, Y'_\infty)$. Applying a similar argument with $-y$ replacing y, we can obtain a solution $\mathbf{Y}_{-\infty}$ defined on $\mathbb{R}$ with

$$\lim_{y\to-\infty} e^{-ky}\mathbf{Y}_{-\infty}(y) = \begin{pmatrix} 1 \\ k \end{pmatrix} \quad \text{and} \quad |Y_{-\infty}(y)| + |Y'_{-\infty}(y)| \le Ce^{ky}, \quad y \le 0.$$

For $y < 0$, we can write

$$Y_\infty(y) = c_0 Y_{-\infty}(y) \left(\int_0^y \frac{\exp\left(-\int_{-\infty}^w b(z)\,\mathrm{d}z\right)}{Y_{-\infty}^2(w)} \mathrm{d}w + c_1 \right),$$

with c_0, c_1 chosen so that $Y_\infty(0)$ and $Y'_\infty(0)$ match our previous definition. This formula implies $\mathbf{Y}_\infty = (Y_\infty, Y'_\infty)$ solves (9.0.13) and satisfies $|\mathbf{Y}_\infty(y)| \le Ce^{-ky}$ for negative y also. By a similar method, we extend $\mathbf{Y}_{-\infty}$ to the real line and obtain $|\mathbf{Y}_{-\infty}(y)| \le Ce^{ky}$ for all $y \in \mathbb{R}$.

(b) In the case $k = 0$, we have

$$e^{M_1(y-w)}M_2(w) = \begin{pmatrix} 1 & y-w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ V(y) & -b(y) \end{pmatrix} = \begin{pmatrix} (y-w)V(y) & -(y-w)b(y) \\ V(y) & -b(y) \end{pmatrix},$$

and the integral equation (9.0.17) reduces to

$$\mathbf{Y}_\infty(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_y^\infty (V(w)Y_\infty - bY'_\infty(w)) \begin{pmatrix} y-w \\ 1 \end{pmatrix} \mathrm{d}w.$$

Defining $K(y, w) = e^{M_1(y-w)} M_2(w)$, we have

$$\|K(y, w)\| \leq C\sqrt{1 + (y - w)^2}(|V(w) + |b(w)|),$$

and

$$\int_0^\infty \sup_{0<y<w} \|K(y, w)\| \,\mathrm{d}w \leq C(\|(1 + |y|^2)^{1/2} V\|_{L^1(\mathbb{R})} + \|(1 + |y|^2)^{1/2} b\|_{L^1(\mathbb{R})}).$$

By Lemma 9.0.4, a solution $\mathbf{Y}_\infty$ exists on $[0, \infty)$, which also solves (9.0.13) by a direct calculation. Applying a similar method for $\mathbf{Y}_{-\infty}$ and extending both solutions to the real line proceeds as in the proof of (a). □