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Time Sensitive Functionals of Marked Random Measures in Real Time

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by

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ABSTRACT

Title:

Time Sensitive Functionals of Marked Random Measures

in Real Time

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In this dissertation, we study marked random measures that model stochastic networks (under attacks), status of queueing systems during vacation modes, responses to cancer treatments (such as chemotherapy and radiation), hostile actions in economics and warfare. We extend the recently developed time sensitivity technique for investigating the processes' behavior about a fixed threshold to a novel time sensitive technique in three important directions: (1) real-time monotone stochastic processes; (2) two-dimensional signed random measures; and (3) antagonistic stochastic games with two active players and one passive player. The need for the time sensitive feature in our study (i.e., an analytical association with real-time parameter) allows stochastic control implementation in sharp contrast with time insensitive analysis very often occurring in the literature. To reach our objectives, we proceed with the classical approach of fluctuation analysis of a particle running through a random grid of a convex set that the particle is trying to escape using stand-alone techniques of stochastic expansion and Laplace transform. We investigate the status of the processes upon as well as the statuses at each time in a given observation time interval. For the monotone process, we target the first passage time, pre-first passage time, the status of the associated continuous time parameter process between these two epochs, and the status of the process upon these two epochs. We obtain analytically tractable formulas and demonstrate them on special cases of marked Poisson processes. Inspired by the monotone result, we embellish it to a two-dimensional signed random measure with position dependent marking. The real-valued component of the associated marked point process is non-monotone presenting an analytical challenge. We manage to investigate various characteristics of that component, including the nth drop or a sharp surge that find applications to finance (like option trading) and risk theory. Finally, we apply the technique to a class of antagonistic stochastic games of three players A,B, and T, of whom the first two are active and the third is a passive player. The active players exchange hostile attacks of random magnitudes with each other and also with player T exerted at random times. At some point (ruin time), one of the two active players is ruined, when the cumulative damages become unsustainable. We obtain the a closed form of the joint distribution functional representing the status of all players upon and also at each time prior to . We illustrate the game on a number of practical models, including stock option trading and queueing systems with vacations and (N,T)-policy.

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List of Keywords

random walk

independent and stationary increments processes

fluctuations of stochastic processes

marked point processes

first passage time

Poisson process

exit time

renewal theory

signed marked random measures

time sensitive analysis

List of Abbreviations

i.i.d.:	independently and identically distributed
r.v.:	random variable
a.s.:	almost surely
LST:	Laplace-Stieljes transform
pgf:	probability generating function

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Chapter 1

Introduction

1.1 Overview

Several studies have been done on the first passage time in fluctuation theory and its applications to queuing, stochastic games, seismology, and finance (cf. [4,6,14,15,20,16,21,23,30,31,,36,41-43,45,47,53]). Fluctuation theory pertains to the behavior of an underlying process around a critical threshold and more generally, when a process escapes from a fixed manifold. The time when that passage takes place is referred to as the first passage time. Another critical value of that situation is the new location of the process upon its escape. Besides the original topics mentioned above, fluctuation theory has become a stand-alone subject in numerous articles appeared through the decades of intense research, cf. [9-11,13,17,32,38,39,45,50,54].

Consider a piecewise constant process N_t (valued in $\mathbb{R}$) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with independent and stationary increments such that $\{t_n\}$ is a point process on $\mathbb{R}_+$ of stopping times relative to $(\mathcal{F}_t)$ and N_t is constant between t_{n-1} and $t_n, n = 1, 2, \ldots$. If N_t is monotone non-decreasing, often of interest is to determine (probabilistically) the crossing a fixed threshold M by N_t at some time t_n and the value N_{t_n} at the crossing. To continue, let us introduce some more notation. Define

$$\nu = min\{n = 0, 1, \ldots : N_{t_n} \geq M\}. \tag{1.1.1}$$

Then, N_t crosses M at some moment t_ν, referred to as the first passage time. Note that because the paths of N_t are not continuous, the value N_{t_ν} is likely to exceed M rather than take M at the crossing, even if N_t and M are integer-valued. Consequently, N_{t_ν} is called the first excess of M.

In the past work, Dshalalow et. al [3,4,6,13-17,20-26,30-32] studied transforms of various classes of such processes in connection with stochastic games, queueing, stochastic networks, and finance, targeting functionals like $Ee^{i\phi N_{t_\nu}} e^{-\theta t_\nu}$ and their embellishments. They were called time insensitive functionals, because the associated reference values were not related to real time parameter t. In some way, this is a common shortcoming of embedded processes compared to those with continuous time parameter. There were efforts made to revive lost information on the behavior of N_t around the first passage time. In all of them, Dshalalow and his coauthors [20,16,32] studied N_t observed over a sequence $\{\tau_m\}$ (being independent of filtration $(\mathcal{F}_t)$). For that matter, the interpolation of N_t (pertaining to time sensitivity) was referred to the interval $(\tau_{\rho-1}, \tau_\rho]$, where

$$\rho = min\{m = 0, 1, \ldots : N_{\tau_m} \geq M\}. \tag{1.1.2}$$

It is understood that the real crossing of M at t_ν takes place at an earlier time than τ_ρ, but in various applications, data collection is impossible in real time. This class of problems makes associated modeling more realistic, but there are still many applications where real time information is possible or when changes of N_t between t_{n-1} and t_n can be neglected. Note that even if there are no changes between t_{n-1} and t_n, it is still of great importance to find the distribution of N_t when t is from a specific time interval. In Chapters 2 and 3 we are focus our attention to the underlying process N_t, where t is from the interval $(t_{\nu-1}, t_\nu]$. Specifically, in chapter 2 we analyze the time sensitive functional:

$$\Phi := Ee^{i(\delta N_t + \phi N_{t_{\nu-1}} + \xi N_{t_\nu})} e^{-\vartheta t_{\nu-1} - \theta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t). \tag{1.1.3}$$

This is a joint transform of pre-first passage time $t_{\nu-1}$, the first passage time t_ν, pre-excess value $N_{t_{\nu-1}}$, the excess value N_{t_ν}, and the value of N_t continuously observed between $t_{\nu-1}$ and t_ν, being very noteworthy reference points. Again even though N_t is not supposed to alter between these two moments, the t-sensitivity a substantial refinement of the functional Φ compared to a more limited $Ee^{i\phi N_{t_\nu}} e^{-\theta t_\nu}$. (We will refer it to real time sensitivity as opposed to a delayed time sensitivity of associated functionals related to the interval $(\tau_{\rho-1}, \tau_\rho]$.)

While the process N_t can be real- or integer-valued, in Chapter 2, we focus on the latter. Firstly, we are eager to explore and demonstrate benefits of so-called discrete operational calculus related to fluctuations of N_t. Secondly, the integer-valued nature of N_t can be easily refined with an arbitrarily small multiple factor coming close to the usual continuous topology. (On occasion, a real-valued version of N_t may still be worth considering.). Thus in Chapter 2, we deal with non-negative random measures and increment processes.

In Chapter 3, we study a class of signed marked random measures $(\mathcal{A}, \Pi, \mathcal{T}) = \sum_{n=0}^{\infty} (x_n, \pi_n) \varepsilon_{t_n}$ with position dependent marking, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Whereas marks X_n's are non-negative, marks π_n's are real-valued. This is a solid upgrade from Dshalalow and Nandyose [27] discussed in Chapter 2, because not only do we add yet another component, but this component is non-monotone. Studies of non-monotone components are very few in the literature on fluctuations. Most prominent of them was by Takács [50]. Yet the results in [50] were not tractable.

As it is common in the theory of fluctuations, in Chapter 3 we focus on the behavior of $(\mathcal{A}, \Pi, \mathcal{T})$ around a fixed threshold $M > 0$ with respect to its first component $\mathcal{A}$, referred to as an active component. With

$$A_n = x_0 + x_1 + \ldots + x_n \tag{1.1.4}$$

we have $\{A_n\}$ monotone non-decreasing, whereas

$$P_n = \pi_0 + \pi_1 + \ldots + \pi_n \tag{1.1.5}$$

is non-monotone, with π_k's being real-valued marks. Our interest is in an extreme behavior of the marginal process $(\Pi, \mathcal{T}) = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}$ that is difficult to analyze due to the non-monotone nature of its marks. For that reason we introduce active mark x_n being non-negative and integer-valued that is to oversee π_n. For instance, we might be curious when the process $(\mathit{\Pi}, \mathcal{T})$ alters its monotonicity or when it experiences its first extreme drop or a surge. For example, we set $x_0 = x_1 = \ldots = x_{n-1} = 0, x_n = 1$, if $\pi_0 > a, \pi_1 > a, \ldots, \pi_{n-1} > a$, and $\pi_n \leq a$. In the general case, the increments x_i's need not be constant, but they can be random variables with particular marginal distributions. For a fixed positive integer M, we define the exit index as

$$\nu := inf \{n = 0, 1, \ldots : A_n \geq M\}. \tag{1.1.6}$$

Then, t_ν is called the first passage time of process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$. It is the first epoch when the crossing of M occurs. Obviously, t_ν is a stopping time relative to filtration $\mathcal{F}_t$. The respective excess values of A_ν and P_ν representing active and passive components, $\mathcal{A}$ and $\mathit{\Pi}$, respectively, are also of interest. We further assume that

A1 the increments $\{x_n, \pi_n, \Delta_n = t_n - t_{n-1}\}$ for $n = 0, 1, 2, \ldots$, $t_{-1} = 0$, of the process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ are independent (position dependent marking), that is, x_n and π_n are dependent only on Δ_n.

A2 for $n = 1, 2, \ldots$, $\{x_n, \pi_n, \Delta_n\}$ are identically distributed.

Associated with $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ is the "time sensitive" counting process

$$(N_t, \mathit{\Pi}_t) = (\mathcal{A}, \mathit{\Pi}) [0, t] = \sum_{n=0}^{\infty} (x_n, \pi_n) \varepsilon_{t_n} [0, t], t \geq 0. \tag{1.1.7}$$

We will be interested in the value of $(N_t, \mathit{\Pi}_t)$ of some t enclosed between $t_{\nu-1}$ and t_ν providing us with the information about $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ between two key reference points as well as $(N_t, \mathit{\Pi}_t)$ for $t \in [0, \tau_\nu)$ (that we will discuss later on, in section

3.5).

So we target the joint Laplace- and Fourier-Stieltjes transform of the above r.v.'s:

$$\Phi_\nu(t) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t),$$

$$\|z\| \le 1, \|u\| \le 1, \|v\| \le 1, \mathrm{Re}\vartheta_0 \ge 0, \mathrm{Re}\vartheta \ge 0, \eta \in \mathbb{R}, \varphi \in \mathbb{R}, \phi \in \mathbb{R}. \qquad (1.1.8)$$

Note that because we manage to observe the process in real time, i.e., upon $t_0, t_1, t_2, \ldots$ (meaning that there are no changes between those epochs), it raises a natual question about a need in the continuous time interpolation. Indeed, in some past work (cf. Dshalalow and White [31]), when a process was observed over arbitrary time epochs (i.e., unrelated to $t_0, t_1, t_2, \ldots$), its continuous revival made perfect sense. In our case, however, it is more about associating the point process $t_0, t_1, t_2, \ldots$, especially the reference points $t_{\nu-1}, t_\nu$, with time t, than anything else. Its key benefit is to know about the process over time related intervals like $[0, t]$ which was impossible to obtain with time insensitive versions. From a practical stand point, observing the process over arbitrary time epochs is more realistic than in real time. However, whenever it is possible to get, its second benefit lies in far more tractable results compared to delayed observations that additionally require the named point process to be Poisson or alike. Furthermore, we also obtain explicit characteristics of the continuous time parameter process in interval $[0, t_\nu)$ giving us a broad spectrum of information about process N_t. The associated functional will read

$$Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t), \|z\| \le 1, \|v\| \le 1, \mathrm{Re}\vartheta \ge 0, \eta \in \mathbb{R}, \phi \in \mathbb{R}. \qquad (1.1.9)$$

As regards the random measure $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$, we recall that the passive component $\mathit{\Pi}$ is real-valued making this random measure signed.

In Chapter 4, we study a class of stochastic games that pertain to stock option trading as well as queueing systems with vacations and T and NT policies. To serve this objective, we introduce a generic game of three players, A, P, and T of whom A and T are "active players" who are involved in a periodic exchange of hostile actions aimed at inflicting damages upon each other, while player P is passive who occasionally receives punches, recovers from them, but does not respond to the actions from the other players. In a nutshell, the game can be described as follows. At random times $t_0, t_1, t_2 \ldots$, players A and T exchange with simultaneous hostile actions inflicting damages of random magnitudes, $x_0, x_1, x_2, \ldots$, and $\Delta_0, \Delta_1, \Delta_2, \ldots$, respectively, until one of the two players becomes ruined. The latter is stipulated by two sustainability thresholds, M and T, so that once (that is, upon one of the epochs t_j's) those thresholds are crossed due to excessive damages to the players or at least one of the players, the game ends. Whereas players A and T are fighting with each other, player P collects hits of random magnitudes and while player P does not respond to the attacks at $t_0, t_1, t_2, \ldots$, he recovers at some of these epochs. It is manifested by a series $\pi_0, \pi_1, \pi_2, \ldots$ of real-valued increments, some of which are damages (with π's positive) and some recoveries (with other π's being negative). The status of player P upon the game end is often of main interest.

1.2 Motivation

The stochastic modeling considered in this work is driven by problems arising in real-life situations such as cyber security, cancer treatment, stock markets, finance, and queuing systems. Below, we expand more on these applications.

(i) ***Finance***

In the situation related to stock option trading, suppose a trader buys a long option for a volatile stock with maturity at time T. One of the trading strategies would be to exercise the option at a highest possible price for the stock prior to the expiration date. Suppose the trader observes the stock randomly at times $t_0, t_1, t_2, \ldots,$ priced at $P_0, P_1, P_2, \ldots,$ respectively. We introduce an auxiliary process of marks $x_0, x_1, x_2, \ldots$ such that $x_k = 0$ when $\pi_k = P_k - P_{k-1} \geq 0$ and $x_k = 1$ when $\pi_k < 0$. Because the trader is supposed to acquire the option for an appreciating stock, he/she hopes that x_k's are zero for a while until x_ν becomes strictly negative at some t_ν. The trader opts to exercise the option at t_ν believing that the trend changes thereafter for worse. Even a better strategy is to exercise the option at $t_{\nu-1}$ prior to the first drop at time t_ν. Notice that t_ν is defined as the first moment when the stock drops for the first time or when the option expires earlier. Hence, the epoch $t_{\nu-1}$ will guarantee that $t_{\nu-1} < T$ and at the same time, the stock price will still be reasonably high. The latter may not be true though, but this needs to be investigated and evaluated predicting the status of such game upon $t_{\nu-1}$ and t_ν. Perhaps even more reasonable would be to predict the time of a stock's sharp drop rather than just turning negative.

(ii) ***Queueing***

***a*) N-Policy in M/G/1 System.** In queuing theory an often studied-to scenario is when a server waits or vacates until the queue accumulates to a certain level (cf., N-Policy), and until then, service is suspended. Once threshold N is crossed by the contents of the waiting room, the server resumes his service. This situation can be treated using fluctuation theory and further interpolated if there is a need to work on a continuous time parameter process. (Cf. Al-Matar and Dshalalow [6].)

***b*) Modified T-Policy in GI/G/1 Queueing System with Multiple Vacations.** Suppose a single server, when he is done with a queue, goes on multiple vacations. During his absence, newly arriving customers are placed in a buffer waiting for the server to resume his service. Suppose customers arrivals form a (delayed) renewal process $t_0, t_1, t_2, \ldots$,whereas the server's vacations form an independent Poisson point process upon the times $\tau_1, \tau_2, \ldots$ The server's vacations (i.e., vacation segments) are monitored upon times $t_0, t_1, t_2, \ldots$, so that the number of server's vacations at time t_k is $A_k = x_0 + \ldots + x_k$, with x_i being the number of server's vacations in interval $[t_{i-1}, t_i)$ $(t_{-1} = 0)$. (The latter information can be easily obtained.) The server's absence is limited by two parameters: the number of vacations, say M, and the total duration of server's absence from the system, limited by some $T > 0$. Thus, the server must return to the system upon one of the epochs $t_0, t_1, t_2, \ldots$, say t_ν, whichever of the two events takes place earlier: $A_\nu \geq M$ or $t_\nu \geq T$. This is a modified T-Policy in a game-theoretical setting. A basic T-Policy was introduced in the late seventies by Heyman [35] and then periodically shown up in the literature [49,50,53]. The present setting applies to GI/G/1/∞-type queues with multiple vacations, in particular, to a GI/M/1/∞

system.

***c*) NT-Policy in M/G/1 System with Multiple Vacations.** Here is another variant of the modified T-Policy. Suppose customers arrive in the system according to a marked Poisson process in an M/G/1/∞-type queue. After the queue of available customers drops below some r, the server leaves the system on some maintenance. Suppose the status of the system is monitored upon a delayed renewal process $t_0, t_1, t_2, \ldots$. The entire absence of the server is limited by two parameters N and T, that is, if upon one of the epochs, say t_ν, the total number of customers (including those left behind on server's departure) has accumulated to N or more ($N \geq r$) or if at t_ν, server's vacation time exceeded T, the server must return to the system. There, in the system, the server will possibly rest waiting, if the queue length is still below N. The two policies combined are known as an NT-Policy model. There are quite a few papers on NT-Policy (cf., [1,2,37,40,53]). While it is of interest to find a probabilistic information about the first passage time t_ν and status of the system at t_ν, it is of further importance to obtain the distribution of the queueing process at any time t during the interval $[0, t_\nu)$. The latter is needed for the formation of the semi-regenerative kernel [5] in order to obtain the probability distribution of the time dependent parameter queueing process $Q(t)$. This method is generally superior to a more common supplementary variables technique that, along with other limitations, requires the existence of densities for generally distributed random variables. The same need in the continuous time parameter process would be warranted for the other two models, although it is obvious that a more refined information about the underlying processes (namely, for each for $t \in [0, t_\nu)$) does not require any further apologetics and it is a stand-alone research.

(iii) *Cyberattacks*

High-profile cyberattacks all over the world have amplified fears and led to heavy monetary, computer system, and information losses. For instance, in 2004, a German College student Sven Jaschan, released a computer virus that disabled Delta Airline's computer system, resulting in many flight cancellations and over $500 million dollars in losses. In yet another high-profile attack, during the 2008 presidential election, Chinese and Russian Hackers hacked into Barack Obama and John McCain's campaign computers systems gaining access to sensitive data. The computers were subsequently confiscated by the FBI.

Cyberattacks have become capable of far more than stealing consumer information or embarrassing politicians and business executives. Whether conducted by lone intruders or nation-states, they can compromise the safety of medical food and water systems, disrupt transportation, and destabilize nuclear power plants. Such attacks can undermine democratic institutions or encourage violence by spreading false information. The cyber threat has become existential. (Cf., The Wall Street Journal, July 12, 2017.)

Very recently we witnessed an escapade of cyberattacks on multiple infrastructure and industry throughout the world, such as Faux Ransomware (cf. The Wall Street Journal, June 30, 2017) and the infamous breach in Equifax credit institution compromising more than 145 millions personal files. Among many other places affected by Fauz Ransomware was Princeton Community Hospital in West Virginia, USA. Here the attack froze the hospital's electronic medical record system leaving doctors unable to review patient's medical history or transmit laboratory and pharmacy orders. Officials were unable to restore services, and found there was no way to pay a ransom for the return of their system. The cyberattack almost left Princeton Community Hospital without even paper templates, which were stored on a computer file, to be printed. Surgeons at the Heritage Valley Hospital in Beaver, Pennsylvania, canceled elective surgeries for two days.

These kinds of economic, social, and privacy violations raise important stochastic analysis questions concerning cyberattacks proofing and worst case scenarios. Once some systems fail, how long would it take before the attack spreads to the entire network or the attack spreads to a point of no return? What level of risk is posed by failures of individual components based on network connectivity? These and many other questions can be answered by modeling an underlying computer network as a marked point process where the points are the attack times and the marks the number of downed computers, whereas a given threshold is a minimal number of nodes by whose crossing the network becomes totally compromised. Whereas our modeling does not prevent those attacks, nor is it a firewall in any sense, it aims at containing damages by predicting the time and caliber of casualties and thereby allowing a surviving part of the network to be separated from an infected subnetwork that is to be quarantined.

(iv) ***Cancer Treatment***

When diagnosed with cancer, a patient is often prescribed an aggressive treatment such as radiation or chemo therapy. In quite a few cases, an underlying cancer does not respond well to either chemo or radiation, and this is a bad news for a patient, not only because he may run out of options, but also because much time is wasted that could have been used for alternative treatments. Knowing this, one key challenge is to decide ahead of the time whether the cancer is going to be treatable shortly after the therapy starts. One approach is to model the response to the therapy by a marked point process. The idea is to predict the cancer progression (or regression) ahead of the time and if needed give the patient another treatment before patient's condition deteriorates.

(v) ***Risk Theory***

Our methodology also applies to the classical risk problem originally posed by Filip Lundberg (see [46]). Assume that an insurance company starts at zero with the initial capital u and let the premium be a linear function with a constant premium rate c, so that the premium income of the company at time t is $u+ct$. Assume that the aggregate claims form a marked point process $\mathcal{Y} = \sum_{k=0}^{\infty} y_k \varepsilon_{t_k}$, with t_k being the time of the kth claim and y_k - the amount of claim. Now Lundberg postulated that $\mathcal{Y}$ was a marked Poisson process with position independent marking. We relax either condition by assuming that neither is $\mathcal{Y}$ Poisson, nor is it with position independent marking.

If $\Delta_k = t_k - t_{k-1}$,we have $c\Delta_k$ premiums' increase from t_{k-1} to t_k. The mark $\pi_k = c\Delta_k - y_k$ is the change of company's asset from t_{k-1} to t_k. Now,

$$\mathit{\Pi} = \sum_{k=0}^{\infty} \pi_k \varepsilon_{t_k} \tag{1.2.1}$$

is a purely signed marked random measure and

$$P_t = \mathit{\Pi}\,[0,t] \tag{1.2.1}$$

is the process describing the asset changes of the insurance company on interval $[0,t]$. Notice that P_t does not give us the true value of the company's asset at time t, because P_t is a piecewise constant interpolation of the true asset value process

$$R_t = u + ct - \sum_{k=0}^{\infty} y_k \varepsilon_{t_k}\,[0,t] \tag{1.2.3}$$

known as the risk process. They coincide upon times $t_0, t_1, t_2, \ldots$ which is exactly what we need. Our process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ is defined through the active component

$$x_k = \begin{cases} 0, & \pi_k > 0 \\ 1, & \pi_k \leq 0 \end{cases}. \tag{1.2.4}$$

So we are interested in the moment when P_k becomes negative or zero for the first time (which would trigger $x_k = 1$). Thus, $\pi_0, \pi_1, \ldots, \pi_{\nu-1}$ are positive, while π_ν is negative or zero. $\{t_{\nu_n}\}$ is the embedded sequence of consecutive drops of P_t. Then obviously, the risk process R_t will become negative or zero only upon one of the epochs $\{t_{\nu_n}\}$, known as the ruin time of R_t.

Let $\mathcal{F}_t$ be the natural filtration with respect to the risk process R_t. Then, $\{t_{\nu_n}\}$ is a sequence of stopping times relative $\mathcal{F}_t$ that are also locally strong Markov points, that is either R_t and P_t have a locally strong Markov property at each point t_{ν_n}. Therefore, R_t and P_t conditionally regenerate upon these epochs. We can slightly modify P_t to make it semi-regenerative with respect to $\{t_{\nu_n}\}$.

While a further discussion on the risk process and its study as a semi-regenerative process is beyond the scope of this dissertation, the time of the first or the second or the nth drop of the risk process is of interest for statistics purposes and it is often raised by insurance companies.

(vi) Other Applications

Other applications of fluctuation theory can also be found in stochastic signals such as time continuous readings for automated seizure detection and quantification using EEGs, and heart attack activity monitoring through detection by EKGs, real time blood pressure monitoring.

1.3 Significance of Time Sensitive Analysis

In some of the above applications we can use time insensitive analysis such as [13,17]. However, the real time interpolation allows one to employ stochastic control making it a very useful embellishment. For example, suppose the main process N_t gives the status of a patient measured in integer units and $h(k)$ is a weight function of k units of such measurement. If the underlying indicator is of the white blood cell count, say k, $h(k)$ is the hemoglobin level (that can be determined by using regression analysis). Then,

$$\mathrm{Q}[0,t] = E\int_{y=0}^{t}(N_y)dy \tag{1.3.1}$$

gives the mean hemoglobin level in interval $[0,t]$.

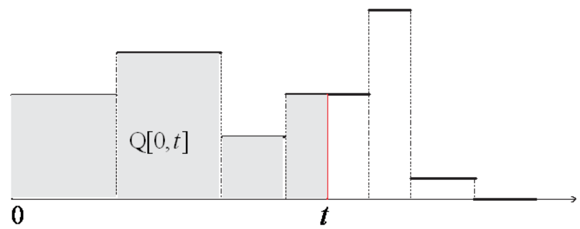

Figure 1.1: Mean process level in $[0, t]$

We can represent Q$[0, t]$ as follows, by using Fubini's theorem:

$$\mathrm{Q}[0,t] = \sum_{k\geq 0} E\left[\int_{y=0}^{t} \mathbf{1}_{\{k\}}(N_y)h(N_y)dy\right] = \sum_{k\geq 0} h(k)\int_{y=0}^{t} P\{N_y = k\}dy. \quad (1.3.2)$$

Now we assume that we know the stationary distribution of N_t

$$\pi_k = lim_{t\to\infty} P\{N_t = k\} = lim_{t\to\infty} \frac{1}{t}\int_{y=0}^{t} P\{N_y = k\}dy, k = 0, 1, \ldots \quad (1.3.4)$$

Applying (1.3.1) to (1.3.2) and using the monotone convergence theorem we have

$$\mathrm{Q} = \lim_{t\to\infty} \frac{\mathrm{Q}[0,t]}{t} = \sum_{k\geq 0} h(k)\pi_k \quad (1.3.5)$$

the mean hemoglobin level per unit time calculated over the time $[0, \infty)$. □

1.4 Related Literature

Our work topically falls into fluctuations of stochastic processes, which is a widely referred to area of random walk, renewal processes, random measures, recurrent processes, Markov processes, and semi-Markov processes, to name a few [9-11,38, 45, 50] and their applications to astronomy and biology [36], queueing [5,18,19], stochastic games [20], finance and economics [15,28,41-43,53], physics [36, 45], and earthquakes [47]. The main focus of our work is on time sensitive fluctuations, meaning that at least some of the component processes are with continuous time parameter. Some prior work on time sensitive fluctuations is due to Dshalalow [20] (pertaining to antagonistic games)and White [32] on two-dimensional processes. The work in the present work by Dshalalow and Nandyose is in [27,28,29] and encompasses renewal and recurrent processes. A very interesting survey paper is by Bingham [9] and a seminal work on fluctuations belongs to Takács [50] and it relates to fluctuations of recurrent and semi-Markov processes. The main keywords (some synonymous) associated with fluctuations are first passage time, first exit time, first excess level, and level crossing [7,8,10-13,17,15,26,33,39,44,45,47,53,55]. These papers focus on determining probability distributions of the first passage (exit/crossing) time and the process value upon crossing a critical threshold or manifold or departing from a compact set. As previously mentioned, just a handful of work (to the best of our knowledge, mainly by the first co-author), belongs to time sensitive analysis [3,4,6,22,20,16,32]. Fluctuation analysis is a powerful method that finds applications in stochastic games [4,20,16,23-25], stochastic networks [16,31], finance [14,15,41-43,53], queueing [4,6], physics and astronomy [36,45], earthquakes [47], and general stochastic processes [3], in particular, with independent and stationary increments [8,10,11,13,32,33,55]. As far as analytical tools, the Laplace-Carson transform is being used for real-valued processes [20,16,32] and discrete operational calculus for integer-valued processes [13,14,15,21,23, 30,31] (developed by the first co-author).

Studies related to signed random measures have previously been done in various topological and stochastic analysis contexts. In [34] Hellmund extended the idea of completely random measures to completely random signed measure and gave a characterization of this class of signed random measures. He demonstrated that the classes of Lévy random measures (utilized in Lévy adaptive regression kernel models) and Lévy bases (utilized in spatio-temporal modeling) are natural extensions of completely random signed measures and that independence is a fundamental concept in defining Lévy random measures and Lévy bases. Other concepts related to signed random measures are in the work by Smorodina and Faddeev [48] who studied symmetric stable signed measures and showed that they are limit measures of sums of independent random variables. Now Dshalalow and White [32] studied random measures of two active nonnegative components competing against each other. However, the game was observed over arbitrary times (not adapted to the process) and consequently, the information on the status of the game was delayed. Nor did the authors of [32] consider a third, passive, non monotone component. The delayed factor yielded less tame functionals than in the present work. Finally, our processes relate to a class of signed random measures introduced in section 2 that we rarely see in the literature (cf. [28, 48]).

1.5 Organization of the Dissertation

Chapter 2 consists of the work in [27] and is laid out as follows. Section 2.1 deals with a more rigorous description of the underlying marked monotone random process, reference stopping times, and a related functional. Section 2.2 introduces the notion of piecewise constant interpolation of a marked random measure and proves several preliminary lemmas and a proposition. Section 2.3 deals with a stochastic expansion of an underlying joined functional in series that allows us to obtain a fully tractable expression of that functional. Section 2.4 aims to demonstrate analytic tractability by an important special case of a marked Poisson process.

Next we proceed with Chapter 3 comprising the work in [28]. In section 3.1 we begin with a further formalism of our model and introduce basics of discrete operational calculus developed earlier by Dshalalow [13,17] and Dshalalow and Iwezulu [23]. In section 3.2, we use the method of stochastic decomposition previously developed in work Dshalalow and Nandyose [27] and Dshalalow and White [31,32], only now embellished for non-monotone components. We establish a key formula for the functional $\Phi_\nu(t)$ of (3.1.5) that we claim is analytically tractable. This claim is justified throughout section 3.3 in a number of examples and special cases. In section 3.4, we continue with time sensitive analysis where time t runs interval $[0, \tau_\nu)$ and find the joint transform of N_t, P_t, N_ν,and the first passage time t_ν in a fully closed form.

We continue with the work in [29] in Chapter 4. In section 4.1, we proceed with a rigorous formalism of the game in the context of the functional (joint transform)

$$\Phi_\nu(t) = Ez^{A(t)}e^{-i\eta P(t)}v^{A_{\nu-1}}e^{-i\varphi P_{\nu-1}}e^{-\psi t_{\nu-1}}u^{A_\nu}e^{-i\phi P_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t). \qquad (1.5.1)$$

This functional provides us with the probabilistic status of the game for players A, P, and T at any time t from the inception of the game until its end (exit) as well as the status of the players upon the exit t_ν and one step prior to the exit

$t_{\nu-1}$ from the game. Section 4.2 undergoes a detailed analysis of this functional ending with a closed-form expression, whereas section 4.3 discusses special cases to support our claim of analytic tractability. We turn our attention to the stock option trading (section 1.2 application (i)) and, among other valuable data, predict the time of the first drop of a stock, then validate the results obtained by simulation. In particular, the special cases we test are $Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}(t)$ and $P(t_\nu > t)$. We show that independent simulation results of these cases agree with our closed form solution for player's A threshold levels $M = 1$ and $M = 2$. These threshold levels represent the first and second drop of the stock price process $P(t)$ respectively. The MATLAB code for the Chapter 4 comparison results is in Appendix B.

Finally in Chapter 5, we summarize our work and outline a few present and future research directions.

Chapter 2

Continuous Time Interpolation of Monotone Marked Random Measures with Applications

2.1 Formalism

Let $(\Omega, \mathcal{F}, P)$ be a probability space and

$$\mathcal{A} = \sum_{k=0}^{\infty} x_k \varepsilon_{t_k} \;\; (\varepsilon_a \text{ is the point mass at } a) \tag{2.1.1}$$

a marked random measure, with position dependent marking, that is, $x_k \otimes (t_k - t_{k-1}) : \Omega \to \mathbb{N} \times \mathbb{R}_+$ being independent and all but $x_0 \otimes t_0$ identically distributed. The underlying support counting measure $\sum_{k=0}^{\infty} \varepsilon_{t_k}$ is a delayed renewal process. One of the key questions that arise in applications for processes like $\mathcal{A}$ is the behavior of $\mathcal{A}$ around some threshold, say M. We assume that the marks x_k's are non-negative, that $t_n \to \infty$, and, without loss of generality, the sequence of sums A_n defined as

$$A_n = x_0 + \ldots + x_n, n = 0, 1, \ldots,$$

runs to ∞ a.s. as $n \to \infty$. Thus the total of the marks A_n will a.s. cross M at some point t_n. Obviously, such an A_n will be equal to or greater than M. The integer-valued r.v.

$$\nu = \inf \{n = 0, 1, \ldots : A_n \geq M\} \tag{2.1.2}$$

is called the exit index. The r.v. A_ν is the excess (level) over M and t_ν is the first passage time (a standard terminology from fluctuation theory).

In various past work (cf. Dshalalow [13,17]), the fluctuations of $\mathcal{A}$ around M were thoroughly investigated and the joint distribution of the key components A_ν, t_ν,and ν, along with $A_{\nu-1}, t_{\nu-1}$, were found. Many other tools were implemented to refine the results. In some of them the authors observed $\mathcal{A}$ over a third-party point process $\mathcal{T} = \{\tau_0, \tau_1, \ldots\}$ [29,30] to get an additional information on $\mathcal{A}$ and include some auxiliary thresholds lower than M that $\mathcal{A}$ was to cross prior to crossing M [25].

Here we attempt to refine the results by introducing the continuous time parameter process

$$N_t = \mathcal{A}\,[0, t]\,, t \geq 0, \tag{2.1.3}$$

that gives us more information about $\mathcal{A}$ which we plan to obtain around the key reference points $t_{\nu-1}$ and t_ν. More significantly, as we will see it, the presence of

N_t makes $\mathcal{A}$ time sensitive (as it relates to real time t) allowing us to implement control. We want to focus on interval $(t_{\nu-1}, t_\nu]$ just before the crossing of M at t_ν takes place. It thus stands for reason to investigate the joint functional

$$Ez^{N_t}u^{A_{\nu-1}}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\,, \tag{2.1.4}$$

$$\|z\| \leq 1,\ \|u\| \leq 1,\ \|v\| \leq 1,\ \mathrm{Re}\vartheta_0 \geq 0, \mathrm{Re}\vartheta \geq 0,$$

of process N_t observed in interval $(t_{\nu-1}, t_\nu]$. In this particular case, we assume that the marks are also integer-valued that as mentioned, works in various applications and can accurately approximate a continuous topology when using a small multiplier. Whereas the tools we use here do not pertain to discrete-valued r.v.'s x_k only, there are benefits of discrete operational calculus (that goes with the discrete marks) we are going to employ that yield analytically more tractable formulas compared to their continuous counterparts.

2.2 Time-Sensitive Analysis Preliminaries

To work on functional (2.1.4) we begin with the following assertions.

Lemma 2.2.1. *Suppose (A, T) and (U, Δ) are random vectors on probability space $(\Omega, \mathcal{F}, P)$ each valued in $(\mathbb{N}_0, \mathbb{R}_+)$ and with a joint probability distribution $P_{A\otimes V\otimes T\otimes\Delta}$ on the product space*

$$(\mathbb{N}_0 \times \mathbb{R}_+ \times \mathbb{N}_0 \times \mathbb{R}_+, \mathcal{P}(\mathbb{N}_0) \otimes \mathrm{B}_+ \otimes \mathcal{P}(\mathbb{N}_0) \otimes \mathrm{B}_+)\,,$$

where $B_+ = B(\mathbb{R}_+)$ *is the Borel* σ*-algebra. Then the following formula holds.*

$$\Gamma\left(u,\xi,v,\vartheta;\theta\right) := \int_{t\geq 0} e^{-\theta t} E u^A e^{-\xi T} v^U e^{-\vartheta\Delta} \mathbf{1}_{t\in(T,T+\Delta]} dt$$

$$= \frac{1}{\theta}\left[E u^A v^U e^{-(\xi+\theta)T} e^{-\vartheta\Delta} - E u^A v^U e^{-(\xi+\theta)T} e^{-(\vartheta+\theta)\Delta}\right],$$

$$|u|\leq 1, |v|\leq 1, \text{ Re}\xi\geq 0, \text{Re}\vartheta\geq 0, \text{Re}\theta\geq 0. \tag{L1}$$

Corollary 2.2.2. *In the event that* (A,T) *are independent of* (U,Δ)*, the functional* Γ *of Lemma 1 reduces to*

$$\Gamma\left(u,\xi,v,\vartheta;\theta\right) = \frac{1}{\theta} E u^A e^{-(\xi+\theta)T}\left[E v^U e^{-\vartheta\Delta} - E v^U e^{-(\vartheta+\theta)\Delta}\right]. \tag{C2}$$

Proof of Lemma 2.2.1 and Corollary 2.2.2. Unfolding the expectation we have the following chain of equations.

$$\int_{t\geq 0} e^{-\theta t} E u^A e^{-\xi T} v^U e^{-\vartheta\Delta} \mathbf{1}_{t\in(T,T+\Delta]} dt$$
$$= \sum_{k=0}^{\infty} u^k \sum_{j=0}^{\infty} v^j \int_{t=0}^{\infty} e^{-\theta t} \int_{s\geq 0} e^{-\xi s} \int_{\delta\geq 0} e^{-\vartheta\delta} \mathbf{1}_{t\in(s,s+\delta]} dP_{A\otimes U\otimes T\otimes\Delta}\left(k,j,s,\delta\right) dt$$
$$= \sum_{k=0}^{\infty} u^k \sum_{j=0}^{\infty} v^j \int_{s\geq 0} e^{-(\xi+\theta)s} \int_{\delta\geq 0} e^{-\vartheta\delta} \int_{t=s}^{s+\delta} e^{-\theta(t-s)} dt dP_{A\otimes U\otimes T\otimes\Delta}\left(k,j,s,\delta\right)$$

due to the translation invariance of the Borel-Lebesgue measure and by the change of variables

$$= \frac{1}{\theta} \sum_{k=0}^{\infty} u^k \sum_{j=0}^{\infty} v^j \int_{s \geq 0} e^{-(\xi+\theta)s} \int_{\delta \geq 0} \left[e^{-\vartheta \delta} - e^{-(\vartheta+\theta)\delta} \right] dP_{A \otimes U \otimes T \otimes \Delta} (k, j, s, \delta)$$

$$= \frac{1}{\theta} \left[Eu^A v^U e^{-(\xi+\theta)T} e^{-\vartheta \Delta} - Eu^A v^U e^{-(\xi+\theta)T} e^{-(\vartheta+\theta)\Delta} \right]$$

that proves formula (L1). If (A, T) and (U, Δ) are independent, we easily arrive at formula (C2) stated in Corollary 2.2.2. □

In the context of section 2.1, with $\mathcal{A}$ being a delayed marked renewal process, we introduce the following notation.

$$A_n = x_0 + \ldots + x_n, \Delta_n = t_n - t_{n-1}, n = 0, 1, \ldots, t_{-1} = 0 \tag{2.2.1}$$

$$\gamma_0 (u, \vartheta) = Eu^{x_0} e^{-\vartheta t_0}, \, \gamma (u, \vartheta) = Eu^{x_i} e^{-\vartheta \Delta_i}, i = 1, 2, \ldots . \tag{2.2.2}$$

As per (2.1.3), $\mathcal{A} [0, t] = N_t = \sum_{k=0}^{\infty} \mathbf{1}_{[0,t]} (t_k)$ is the counting process associated with the point process $\sum_{k=0}^{\infty} \varepsilon_{t_k}$. In light of the figure below,

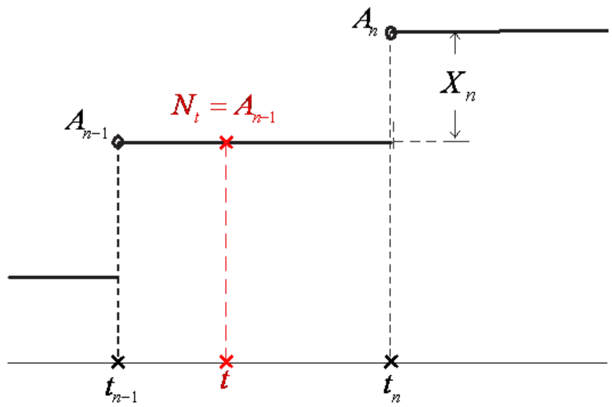

Figure 2.1: 1-D marked point process

consider the functional

$$F_n(t) = Ez^{N_t}u^{A_{n-1}}v^{A_n}e^{-\vartheta_0 t_{n-1}-\vartheta t_n}\mathbf{1}_{\{t_{n-1}\le t<t_n\}}, n = 1, 2, \dots \tag{2.2.3}$$

unfolded as

$$\begin{aligned} F_n(z) &= E(zu)^{A_{n-1}}v^{A_{n-1}+x_n}e^{-(\vartheta_0+\vartheta)t_{n-1}-\vartheta(t_n-t_{n-1})}\mathbf{1}_{\{t_{n-1}\le t<t_n\}} \\ &= E(uvz)^{A_{n-1}}e^{-(\vartheta_0+\vartheta)t_{n-1}}v^{x_n}e^{-\vartheta\Delta_n}\mathbf{1}_{\{t_{n-1}\le t<t_n\}}, n = 1, 2 \dots \end{aligned} \tag{2.2.4}$$

Due to the independence of (A_{n-1}, t_{n-1}) and (X_n, Δ_n) [(A, T) and (U, Δ) in the context of Lemma 2.2.1], applying Corollary 2.2.2 to the Laplace transform of F_n

we have

$$\begin{aligned}F_n^*(\theta) &= \int_{t=0}^{\infty} e^{-\theta t} F_n(t)\, dt \\ &= \int_{t\geq 0} e^{-\theta t} E(uvz)^{A_{n-1}} e^{-(\vartheta_0+\vartheta)t_{n-1}} v^{X_n} e^{-\vartheta \Delta_n} \mathbf{1}_{t\in(t_{n-1},t_n]} dt \\ &= \frac{1}{\theta} \mathit{\Gamma}_{n-1}(uvz, \vartheta_0+\vartheta+\theta)[\gamma(v,\vartheta) - \gamma(v,\vartheta+\theta)]\end{aligned}$$

where

$$\mathit{\Gamma}_{n-1}(uvz, \vartheta_0+\vartheta+\theta) = \gamma_0(uvz, \vartheta_0+\vartheta+\theta)\, \gamma^{n-1}(uvz, \vartheta_0+\vartheta+\theta) \text{ for } n \geq 1. \tag{2.2.5}$$

Summing up F_n for all $n = 1, 2, \ldots$, with (2.2.5) in mind, we formally arrive at the expression

$$\sum_{n=1}^{\infty} F_n^*(\theta) = \frac{1}{\theta}\gamma_0(uvz, \vartheta_0+\vartheta+\theta)[\gamma(v,\vartheta) - \gamma(v,\vartheta+\theta)] \frac{1}{1-\gamma(uvz, \vartheta_0+\vartheta+\theta)}. \tag{2.2.6}$$

In Proposition A.1 (see the Appendix), we show that $\|\gamma(uvz, \vartheta_0+\vartheta+\theta)\| < 1$.

Proposition 2.2.3. *Let* $F_0(t) = Ez^{N_t} v^{A_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0]}(t)$. *Then*

$$F_0^*(\theta) = \frac{1}{\theta}[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)].$$

Proof. From the figure below

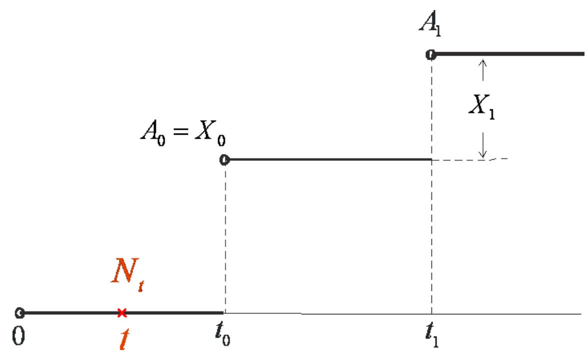

Figure 2.2: Marked point process: initial r.v.s

we readily deduce that

$$F_0(t) = Ev^{u_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0]}(t).$$

The statement follows from Corollary 2.2.2 with $T = 0$, $U = u_0$, and $\Delta = \Delta_0 = t_0$. □

With Proposition 2.2.3, we can adhere F_0^* to the series $\sum_{n=1}^{\infty} F_n^*$ of formula (2.2.6):

$$\sum_{n=0}^{\infty} F_n^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)\right]$$

$$+\frac{1}{\theta}\gamma_0(uvz,\vartheta_0+\vartheta+\theta)\left[\gamma(v,\vartheta)-\gamma(v,\vartheta+\theta)\right]\frac{1}{1-\gamma(uvz,\vartheta_0+\vartheta+\theta)}. \quad (2.2.7)$$

Formula (2.2.7) will be used in section 2.3.

2.3 First Passage Time of $\mathcal{A}$ and its Ramifications

Now we return to the formalism of section 2.1 about random measure $\mathcal{A}$, the associated continuous time parameter jump process $N_t = \mathcal{A}[0,t]$, the exit index ν, and the first passage time t_ν. Of equal interest are A_ν, the first excess value of $\{A_n\}$ of threshold M upon t_ν. We also target $t_{\nu-1}$ (pre-first passage time) and $A_{\nu-1}$ (pre-first excess value). Because X_k's are non-negative, $A_{\nu-1} < M$.

As previously noted, not only do we want to screen N_t from $t_{\nu-1}$ to t_ν (providing us with a refined information about $\mathcal{A}$ between the two key reference points), but even more importantly, we want to connect underlying fluctuation parameters with real time. So, we target the joint distribution of the introduced r.v.'s under the following transform

$$\begin{aligned}\Phi_\nu(t) &= E z^{N_t} u^{A_{\nu-1}} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{(t_{\nu-1}, t_\nu]}(t), \\ &\|z\| \le 1, \|u\| \le 1, \|v\| \le 1, \mathrm{Re}\vartheta_0 \ge 0, \mathrm{Re}\vartheta \ge 0.\end{aligned} \tag{2.3.1}$$

We notice that Φ_ν cannot be treated directly by applying Lemma 2.2.1 or Corollary 2.2.2, simply because we do not know the distribution of $(A_\nu - A_{\nu-1}, t_\nu - t_{\nu-1})$, nor is the latter independent of $(A_{\nu-1}, t_{\nu-1})$. The method of dealing with functional Φ_ν will include several steps. In step 1, we introduce the auxiliary sequence $\{\nu(p)\}$ of exit indices relative to the sequence $\{0, 1, \ldots\}$ of thresholds to be crossed by A_n, of which $\nu = \nu(M-1)$. Namely, let

$$\nu(p) = \inf\{n = 0, 1, \ldots : A_n > p\}, p = 0, 1, \ldots$$

Given a fixed p, we have

$$\Phi_{\nu(p)}(t) = Ez^{N_t} u^{A_{\nu(p)-1}} v^{A_{\nu(p)}} e^{-\vartheta_0 t_{\nu(p)-1} - \vartheta t_{\nu(p)}} \mathbf{1}_{(t_{\nu(p)-1}, t_{\nu(p)}]}(t).$$

In step 2, we apply to $\Phi_{\nu(p)}$ transformation D_p defined as

$$D_p\{f(p)\}(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), \ ||x|| < 1,$$

where f is a real-valued function with the domain $\mathbb{N}_0 = \{0, 1, \ldots\}$. The inverse of D_p is the so-called $\mathcal{D}$-operator defined in Dshalalow [17] as

$$\mathcal{D}_x^k \varphi(x,y) = \begin{cases} \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x}\varphi(x,y)\right], & k \geq 0 \\ 0, & k < 0 \end{cases}$$

(φ is analytic at zero in variable x).

From $\Phi_{\nu(p)}(t) = \sum_{n=0}^{\infty} \Phi_{\nu(p)}(t) \mathbf{1}_{\{v(p)=n\}}$, we have

$$\begin{aligned} \Phi(t,x) := D_p\left[\Phi_{\nu(p)}(t)\right](x) &= \sum_{n=0}^{\infty} \Phi_{\nu(p)}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x) \\ &= \sum_{n=0}^{\infty} \Phi_{\nu(p)=n}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x), \end{aligned}$$

with $\Phi_{\nu(p)=n}(t) = Ez^{N_t} u^{A_{n-1}} v^{A_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \leq t < t_n\}} = F_n(t)$.

From $1_{\{v(p)=n\}} = \mathbf{1}_{\{A_{n-1}\leq p\}}\mathbf{1}_{\{A_n>p\}}$,

$$D_p\mathbf{1}_{\{v(p)=n\}}(x) = (1-x)\sum_{p=0}^{\infty} x^p \mathbf{1}_{\{A_{n-1}\leq p\}}\mathbf{1}_{\{A_n>p\}}$$
$$= (1-x)\sum_{p=A_{n-1}}^{A_n-1} x^p = x^{A_{n-1}} - x^{A_n}$$

that yields

$$\Phi(t,x) = \sum_{n=0}^{\infty} F_n(t)\left(x^{A_{n-1}} - x^{A_n}\right)$$
$$= \sum_{n=0}^{\infty} F_n(ux, v, z, \vartheta_0, \vartheta, t) - F_n(u, vx, z, \vartheta_0, \vartheta, t), \text{ where } A_{-1} = 0.$$

Finally, applying the Laplace transform to

$\Phi(t,x)$, in notation $\Phi^*(\theta,x) = \int_{t=0}^{\infty} e^{-\theta t}\Phi(t,x)\,dt$, we have

$$\theta\Phi^*(\theta,x) = \sum_{n=0}^{\infty}[F_n^*(ux, v, z, \vartheta_0, \vartheta, t) - F_n^*(u, vx, z, \vartheta_0, \vartheta, t)].$$

We can copy the results for $\sum_{n=1}^{\infty} F_n^*$ from (2.2.6). As far as F_0^*, we have to proceed with caution, since $x^{A_{n-1}} - x^{A_n} = 1 - x^{X_0}$, if $n = 0$. Thus in this case we have,

$$F_0^*(v,\vartheta,t) - F_0^*(vx,\vartheta,t)$$
$$= \frac{1}{\theta}\left[\gamma_0(v,\vartheta) - \gamma_0(v,\vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx,\vartheta) - \gamma_0(vx,\vartheta+\theta)\right]. \quad (2.3.2)$$

Furthermore,

$$\sum_{n=1}^{\infty} F_n^*\left(ux,v,z,\vartheta_0,\vartheta,t\right)-F_n^*\left(u,vx,z,\vartheta_0,\vartheta,t\right)$$

$$\begin{aligned}
&= \frac{1}{\theta}\gamma_0\left(uvzx,\vartheta_0+\vartheta+\theta\right)\left[\gamma\left(v,\vartheta\right)-\gamma\left(v,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)}\\
&-\frac{1}{\theta}\gamma_0\left(uvzx,\vartheta_0+\vartheta+\theta\right)\left[\gamma\left(vx,\vartheta\right)-\gamma\left(vx,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)}\\
&=\frac{1}{\theta}\gamma_0\left(uvxz,\vartheta_0+\vartheta+\theta\right)\frac{1}{1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)}\\
&\qquad\times\left[\gamma\left(v,\vartheta\right)-\gamma\left(vx,\vartheta\right)+\gamma\left(vx,\vartheta+\theta\right)-\gamma\left(v,\vartheta+\theta\right)\right]. \qquad (2.3.3)
\end{aligned}$$

Hence combining (2.3.2) and (2.3.3), summing up over all $n \geq 0$ yields:

$$\begin{aligned}
\theta\Phi^*\left(\theta,x\right) &= \gamma_0\left(v,\vartheta\right)-\gamma_0\left(v,\vartheta+\theta\right)-\gamma_0\left(vx,\vartheta\right)+\gamma_0\left(vx,\vartheta+\theta\right)\\
&\quad+\gamma_0\left(uvxz,\vartheta_0+\vartheta+\theta\right)\frac{1}{1-\gamma\left(uvzx,\vartheta_0+\vartheta+\theta\right)}\\
&\quad\times\left[\gamma\left(v,\vartheta\right)-\gamma\left(vx,\vartheta\right)+\gamma\left(vx,\vartheta+\theta\right)-\gamma\left(v,\vartheta+\theta\right)\right]. \qquad (2.3.4)
\end{aligned}$$

Applying the $\mathcal{D}$-operator to Φ^* of (2.3.4) we get

$$\theta\Phi^*\left(\theta\right)=\mathcal{D}_x^{M-1}\theta\Phi^*\left(\theta,x\right). \qquad (2.3.5)$$

$\Phi_\nu\left(t\right) = Ez^{N_t}u^{A_{\nu-1}}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}\left(t\right)$ can be extracted from (2.3.4-2.3.5) by applying the inverse Laplace transform (subject to our discussion in section 2.4).

Without a "delay" of $\mathcal{A}$, $\gamma_0 = 1$ and thus (2.3.4) is simplified to

$$\theta\Phi^*(\theta, x) = \frac{1}{1-\gamma(uvzx, \vartheta_0+\vartheta+\theta)} \times [\gamma(v,\vartheta) - \gamma(vx,\vartheta) + \gamma(vx,\vartheta+\theta) - \gamma(v,\vartheta+\theta)]. \tag{2.3.6}$$

Finally, applying the $\mathcal{D}$-operator to Φ^* version (2.3.6) we get

$$\theta\Phi^*_\nu(\theta) = \mathcal{D}_x^{M-1}\frac{1}{1-\gamma(uvzx, \vartheta_0+\vartheta+\theta)} \times [\gamma(v,\vartheta) - \gamma(vx,\vartheta) + \gamma(vx,\vartheta+\theta) - \gamma(v,\vartheta+\theta)]. \tag{2.3.7}$$

2.4 Special Case: Marked Poisson Process with Position Independent Marking

To illustrate tractability of the results obtained in (2.3.6-2.3.7), let $\mathcal{A} = \sum_{n=1}^{\infty} X_n \varepsilon_{t_n}$ be a marked Poisson measure with position independent marking and support counting measure $\sum_{n=1}^{\infty} \varepsilon_{t_n}$ of intensity λ. We assume that the marks $X_1, X_2, \ldots \in [\text{Geo}_1(p)]$ are independent and identically distributed (iid), thus with the common pgf

$$a(z) = \frac{pz}{1-qz}, \tag{2.4.1}$$

and that interrenewal times $\Delta_n = t_n - t_{n-1}$, $n = 1, 2, 3, \ldots$ (assuming no delay and $t_0 = 0$) are iid with the common LST

$$\gamma(\theta) = Ee^{-\Delta_1\theta} = \frac{\lambda}{\lambda+\theta},\ \text{Re}\theta \geq 0. \tag{2.4.2}$$

So, due to position independent marking,

$$\gamma(u,\theta) = Eu^{X_1}e^{-\Delta_1\vartheta} = a(u)\,\gamma(\vartheta) = \frac{pu}{1-qu}\frac{\lambda}{\lambda+\theta}. \tag{2.4.3}$$

We will work on formula (2.3.7) substituting there (2.4.3) for γ. Furthermore, from

$$1-\gamma(uvzx,\vartheta_0+\vartheta+\theta) = 1-\frac{\lambda}{\lambda+\vartheta_0+\vartheta+\theta}\frac{puvzx}{1-quvzx},$$

after some algebra,

$$\frac{1}{1-\gamma(uvzx,\vartheta_0+\vartheta+\theta)} = \frac{A-Bx}{A-Cx} = \left[\frac{B}{C}+\frac{A(C-B)}{C(A-Cx)}\right]$$

$$A = \lambda+\vartheta_0+\vartheta+\theta$$

$$B = (qvuz)(\lambda+\vartheta_0+\vartheta+\theta)$$

$$C = (qvuz)(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz = B+\lambda pvuz$$

Then we have

$$\gamma(v,\vartheta)-\gamma(vx,\vartheta)+\gamma(vx,\vartheta+\theta)-\gamma(v,\vartheta+\theta)$$

$$= G\left(\frac{1}{1-qv}-\frac{1}{1-qvx}\right),\ qv \neq 1,\ qvx \neq 1,$$

where

$$G = \frac{\lambda p\theta}{q(\vartheta+\lambda)(\vartheta+\lambda+\theta)}$$

Then, using Dshalalow [17],

$$\theta \Phi_{\nu}^{*}(\theta) = D_{x}^{M-1}\left[G\left(\frac{B}{C} + \frac{A(C-B)}{C(A-Cx)}\right)\left(\frac{1}{1-qv} - \frac{1}{1-qvx}\right)\right]$$

$$= \frac{BG}{C(1-qv)} - \frac{BG}{C}\frac{1-(qv)^{M}}{1-qv} + \frac{G(C-B)}{C(1-qv)}\frac{1-\left(\frac{C}{A}\right)^{M}}{1-\frac{C}{A}}$$

$$- \frac{G(C-B)}{C(1-qv)}\left(\frac{1-\left(\frac{C}{A}\right)^{M}}{1-\frac{C}{A}} - (qv)^{M}\frac{1-\left(\frac{C}{Aqv}\right)^{M}}{1-\frac{C}{Aqv}}\right)$$

$$= \frac{G}{C(1-qv)}\left[(qv)^{M}B + (C-B)(qv)^{M}\frac{1-\left(\frac{C}{Aqv}\right)^{M}}{1-\frac{C}{Aqv}}\right]$$

or returning to the original expressions

$$\Phi_{\nu}^{*}(\theta) = \frac{\lambda p}{q(\vartheta+\lambda)(\vartheta+\lambda+\theta)(1-qv)\left[q(\lambda+\vartheta_0+\vartheta+\theta)+\lambda p\right]}$$
$$\times\left[q(qv)^{M}(\lambda+\vartheta_0+\vartheta+\theta) + \lambda p(qv)^{M}\frac{1-\left(\frac{qvuz(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz}{(\lambda+\vartheta_0+\vartheta+\theta)qv}\right)^{M}}{1-\frac{qvuz(\lambda+\vartheta_0+\vartheta+\theta)+\lambda pvuz}{(\lambda+\vartheta_0+\vartheta+\theta)qv}}\right]. \tag{2.4.4}$$

We are interested in various marginal versions of functional

$$\Phi_{\nu}(t) = Ez^{N_t}u^{A_{\nu-1}}v^{A_{\nu}}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_{\nu}}\mathbf{1}_{(t_{\nu-1},t_{\nu}]}(t).$$

For brevity we will write

$$\varPhi_\nu(t) = \varPhi_\nu(t; z, u, v, \vartheta_0, \vartheta),$$

which can be derived by taking the Laplace inverse of $\varPhi^*_\nu(\theta) = \varPhi^*_\nu(\theta; z, u, v, \vartheta_0, \vartheta)$.

(i) First, consider the marginal functional $\varPhi_\nu(t; z, 1, 1, 0, 0) = Ez^{N_t}\mathbf{1}_{(t_{\nu-1}, t_\nu]}(t)$ of the Poisson counting process N_t. Here,

$$\varPhi^*_\nu(\theta; z, 1, 1, 0, 0) = \frac{\lambda}{q(\lambda)(\lambda+\theta)[q(\lambda+\theta)+\lambda p]} \times \left[q^{M+1}(\lambda+\theta) + \lambda p q^M \frac{1 - \left(\frac{qz(\lambda+\theta)+\lambda pz}{(\lambda+\theta)q}\right)^M}{1 - \frac{qz(\lambda+\theta)+\lambda pz}{(\lambda+\theta)q}} \right]$$

that can be reduced to

$$\varPhi^*_\nu(\theta; z, 1, 1, 0, 0) = \frac{((\lambda+\theta)q)^{M-1} + \lambda p \sum_{k=1}^{M-1} ((\lambda+\theta)q)^{k-1} z^{M-k} (\lambda + q\theta)^{M-1-k}}{(\lambda+\theta)^M}$$

after some algebra, and then further to

$$\varPhi^*_\nu(\theta; z, 1, 1, 0, 0) = \frac{q^{M-1}}{\lambda+\theta} + \lambda p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} z^{M-k} q^{k+i-1} (\lambda p)^{M-1-k-i} \left(\frac{1}{\lambda+\theta}\right)^{M+1-k-i}.$$

Thus, denoting $\mathcal{L}_\theta^{-1}$ for the Laplace inverse operator in variable θ we have

$$
\begin{aligned}
\Phi_\nu(t;z,1,1,0,0) &= Ez^{N_t}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)(t) = \mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*(\theta;z,1,1,0,0)\right\}(t) \\
&= \mathcal{L}_\theta^{-1}\left\{\frac{q^{M-1}}{\lambda+\theta}\right\}(t) \\
&\quad + \lambda p\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i} z^{M-k}q^{k+i-1} \\
&\qquad \times(\lambda p)^{M-1-k-i}\,\mathcal{L}_\theta^{-1}\left\{\left(\frac{1}{\lambda+\theta}\right)^{M+1-k-i}\right\}(t) \\
&= q^{M-1}e^{-\lambda t} \\
&\quad + \sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i} z^{M-k}q^{k+i-1}(\lambda p)^{M-k-i}e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}.
\end{aligned}
\tag{2.4.5}
$$

In particular, for $M=1$,$M=2$,and $M=3$ we get:

$$M=1:\Phi_\nu(t;z,1,1,0,0) = e^{-\lambda t} \tag{2.4.6}$$

$$M=2:\Phi_\nu(t;z,1,1,0,0) = qe^{-\lambda t} + \lambda pzte^{-\lambda t} = (q+\lambda pzt)e^{-\lambda t} \tag{2.4.7}$$

$$M=3:\ \Phi_\nu(t;z,1,1,0,0) = q^2e^{-\lambda t} + \lambda pqzt(1+z)e^{-\lambda t} + \frac{(\lambda pzt)^2}{2}e^{-\lambda t}. \tag{2.4.8}$$

We confirm the results for $Ez^{N_t}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ obtained from (2.4.5) for $M=1,2$, and 3 using direct probability arguments.

When $M=1$, the first passage time occurs in the event that $X_1\geq 1$. Thus for

$M = 1$ and $z = 1$,

$$\Phi_\nu\,(t; 1, 1, 1, 0, 0) = E\mathbf{1}_{(t_{\nu-1}, t_\nu]}\,(t) = P\{t_{\nu-1} \leq t \leq t_\nu\}\, P\,\{X_1 \geq 1\}$$
$$= P\{0 \leq t \leq t_1\} \cdot 1 = e^{-\lambda t} \qquad (2.4.9)$$

which agrees with (2.4.6).

When $M = 2$, the first passage time could occur in the event that $X_1 \geq 2$ (interval $[0, t_1]$) or in the event that $X_1 = 1$ and $X_2 \geq 1$ (interval $[t_1, t_2]$). Thus, when $M = 2$ and $z = 1$,

$$\Phi_\nu\,(t; 1, 1, 1, 0, 0) = E\mathbf{1}_{(t_{\nu-1}, t_\nu]}\,(t) \; = P\{t_{\nu-1} \leq t \leq t_\nu\}$$
$$= P\{0 \leq t \leq t_1\} P\,\{X_1 \geq 2\} + \; P\{t_1 \leq t \leq t_2\} P\,\{X_1 = 1\}\, P\,\{X_2 \geq 1\}$$
$$= P\{N_t = 0\} P\,\{X_1 \geq 2\} + \; P\{N_t = 1\} P\,\{X_1 = 1\}\, P\,\{X_2 \geq 1\}$$
$$= qe^{-\lambda t} \; + \; \lambda pte^{-\lambda t}, \qquad (2.4.10)$$

which agrees with (2.4.7).

Similarly, when $M = 3$, the $first passage$ time occurs when $X_1 \geq 3$, or $X_1 = 1$ and $X_2 > 1$, or when $X_1 = 1$, $X_2 = 1$, and $X_3 \geq 1$ in their respective intervals.

Therefore, when $M = 3$ and $z = 1$,

$$\begin{aligned}
\varPhi_\nu(t;1,1,1,0,0) &= E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = P\{t_{\nu-1} \le t \le t_\nu\} \\
=&P\{0 \le t \le t_1\}P\{X_1 \ge 3\} \\
&+ P\{t_1 \le t \le t_2\}[P\{X_1 = 1\}P\{X_2 \ge 2\} + P\{X_1 = 2\}P\{X_2 \ge 1\}] \\
&+ P\{t_2 \le t \le t_3\}P\{X_1 = 1\}P\{X_2 = 1\}P\{X_3 \ge 1\} \\
=&P\{N_t = 0\}P\{X_1 \ge 3\} \\
&+ P\{N_t = 1\}[P\{X_1 = 1\}P\{X_2 \ge 2\} + P\{X_1 = 2\}P\{X_2 \ge 1\}] \\
&+ P\{N_t = 2\}P\{X_1 = 1\}P\{X_2 = 1\}P\{X_3 \ge 1\} \\
=&q^2e^{-\lambda t} + 2pq\lambda te^{-\lambda t} + \frac{(\lambda pt)^2}{2}e^{-\lambda t}
\end{aligned} \tag{2.4.11}$$

which agrees with (2.4.8).

(ii) Now, considering the marginal functional $\varPhi_\nu(t;1,1,1,0,\vartheta) = Ee^{-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ of t_ν, the first passage time (i.e., when the first threshold crossing occurs), we have

$$\begin{aligned}
\varPhi^*_\nu(\theta;1,1,1,0,\vartheta) = &\frac{\lambda}{q(\vartheta+\lambda)(\vartheta+\lambda+\theta)[q(\vartheta+\lambda+\theta)+\lambda p]} \\
&\times\left[q^{M+1}(\vartheta+\lambda+\theta) + \lambda pq^M\frac{1-\left(\frac{q(\vartheta+\lambda+\theta)+\lambda p}{(\vartheta+\lambda+\theta)q}\right)^M}{1-\frac{q(\vartheta+\lambda+\theta)+\lambda p}{(\vartheta+\lambda+\theta)q}}\right]
\end{aligned}$$

reducing it to

$$\Phi_\nu^*(\theta;1,1,1,0,\vartheta) = \frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)} \times \left[1 + \lambda p \sum_{k=1}^{M-1} \frac{q^{k-M}}{(\vartheta+\lambda+\theta)^{M-k}} \left(q(\vartheta+\lambda+\theta)+\lambda p\right)^{M-k-1}\right]$$

and then to

$$\Phi_\nu^*(\theta;1,1,1,0,\vartheta) = \frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)} + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{\vartheta+\lambda} \times (\lambda p)^{M-k-1-i} \left(\frac{1}{\vartheta+\lambda+\theta}\right)^{M-k+1-i}.$$

Thus,

$$\begin{aligned}\Phi_\nu(t;1,1,1,0,\vartheta) &= \mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*(\theta;1,1,1,0,\vartheta)\right\}(t)\\ &= \mathcal{L}_\theta^{-1}\left\{\frac{\lambda q^{M-1}}{(\vartheta+\lambda)(\vartheta+\lambda+\theta)}\right\}(t) + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{(\vartheta+\lambda)}\\ &\quad \times (\lambda p)^{M-k-1-i} \mathcal{L}_\theta^{-1}\left\{\left(\frac{1}{\vartheta+\lambda+\theta}\right)^{M-k+1-i}\right\}(t).\end{aligned}$$

Hence,

$$\begin{aligned}\Phi_\nu(t;1,1,1,0,\vartheta) &= Ee^{-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t).\\ &= \frac{\lambda q^{M-1}}{\vartheta+\lambda} e^{-(\vartheta+\lambda)t} + \lambda^2 p \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} \frac{q^{k+i-1}}{\vartheta+\lambda}\\ &\quad \times (\lambda p)^{M-k-1-i} e^{-(\vartheta+\lambda)t} \frac{t^{M-k-i}}{(M-k-i)!}.\end{aligned}$$

Specifically for

$$
\begin{aligned}
M=1: \Phi_\nu\left(t;1,1,1,0,\vartheta\right) &= \frac{\lambda}{\vartheta+\lambda}e^{-(\vartheta+\lambda)t} \\
M=2: \Phi_\nu\left(t;1,1,1,0,\vartheta\right) &= \frac{\lambda q}{\vartheta+\lambda}e^{-(\vartheta+\lambda)t} + \lambda^2 pt\frac{1}{\vartheta+\lambda}e^{-(\vartheta+\lambda)t} \\
&= \frac{\lambda}{\vartheta+\lambda}\left(q+\lambda pt\right)e^{-(\vartheta+\lambda)t},
\end{aligned}
$$

which agree with (5.9) and (5.10), respectively, when $\vartheta = 1$.

Furthermore, for general M, applying the inverse Laplace transform in variable ϑ

$$
\begin{aligned}
&\frac{d}{dx}\left[P\left\{t_\nu \le x, t\in\left(t_{\nu-1},t_\nu\right]\right\}\right] \\
&\quad = \mathcal{L}_{\vartheta}^{-1}\left\{\Phi_\nu\left(t;1,1,1,0,\vartheta\right)\right\}(x) \\
&\quad = \lambda q^{M-1}e^{-\lambda x}\mathbf{1}_{(t,\infty)}(x) \\
&\quad + \lambda^2 p\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{k+i-1}\left(\lambda p\right)^{M-k-1-i}\frac{t^{M-k-i}}{(M-k-i)!}e^{-\lambda x}\mathbf{1}_{(t,\infty)}(x).
\end{aligned}
$$

(iii) Thirdly, for the marginal functional $\Phi_\nu\left(\theta;1,1,v,0,0\right) = Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ of the first excess level A_ν, setting $z=1$, $u=1$, $\vartheta_0=0$, $\vartheta=0$ in $\Phi_\nu^*\left(\theta;z,u,v,\vartheta_0,\vartheta\right)$ we arrive at:

$$
\begin{aligned}
\Phi_\nu^*\left(\theta;1,1,v,0,0\right) &= \frac{pv^M q^{M-1}}{(\lambda+\theta)(1-qv)} \\
&+ v^M\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{1-qv}\left(\lambda p\right)^{M-k-1-i}\left(\frac{1}{\lambda+\theta}\right)^{M-k+1-i}
\end{aligned}
$$

after a similar algebraic routine. Thus,

$$\begin{aligned}
&\mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*\left(\theta;1,1,v,0,0\right)\right\}(t)\\
&\quad=\mathcal{L}_\theta^{-1}\left\{\frac{pv^M q^{M-1}}{(\lambda+\theta)(1-qv)}\right\}(t)+v^M\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{(1-qv)}\\
&\qquad\qquad\times(\lambda p)^{M-k-1-i}\,\mathcal{L}_\theta^{-1}\left\{\left(\frac{1}{\lambda+\theta}\right)^{M-k+1-i}\right\}(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Phi_\nu\left(t;1,1,v,0,0\right)=Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\\
&\quad=\frac{pv^M q^{M-1}}{1-qv}e^{-\lambda t}+v^M\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}\frac{q^{k+i-1}}{(1-qv)}\\
&\qquad\qquad\times(\lambda p)^{M-k-1-i}\,e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}.
\end{aligned}$$

Notice that expanding $Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ in Taylor series in v we have

$$\begin{aligned}
&Ev^{A_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\\
&=\sum_{n=0}^{\infty}\Bigg(pv^M q^{M-1}e^{-\lambda t}+v^M\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{k+i-1}\\
&\qquad\times(\lambda p)^{M-k-1-i}\,e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}\Bigg)(qv)^n\\
&=\sum_{n=M}^{\infty}\Bigg(pq^{n+M-1}e^{-\lambda t}+\lambda p^2\sum_{k=1}^{M-1}\sum_{i=0}^{M-1-k}\binom{M-1-k}{i}q^{n+k+i-1}\\
&\qquad\times(\lambda p)^{M-k-1-i}\,e^{-\lambda t}\frac{t^{M-k-i}}{(M-k-i)!}\Bigg)v^n.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&P\left\{A_\nu = n, t \in (t_{\nu-1}, t_\nu]\right\} \\
&= \left[\lambda p^2 \sum_{k=1}^{M-1} \sum_{i=0}^{M-1-k} \binom{M-1-k}{i} q^{n+k+i-1} (\lambda p)^{M-k-1-i} e^{-\lambda t} \frac{t^{M-k-i}}{(M-k-i)!}\right. \\
&\qquad \left. + pq^{n+M-1} e^{-\lambda t}\right] \mathbf{1}_{\{M,M+1,\ldots\}}(n), n = 0, 1, \ldots
\end{aligned}
$$

which agrees with the fact that $A_\nu \geq M$ a.s.

Specifically, for $M = 1$,

$$\Phi_\nu(t; 1, 1, v, 0, 0) = \frac{pv}{1-qv} e^{-\lambda t}$$

and for $M = 2$,

$$\Phi_\nu(t; 1, 1, v, 0, 0) = \frac{pqv^2}{1-qv} e^{-\lambda t} + \lambda p^2 v^2 t \frac{1}{1-qv} e^{-\lambda t} = (q+\lambda pt) e^{-\lambda t} \frac{v^2 p}{1-qv}$$

which agree with (2.4.9) and (2.4.10) respectively when $v = 1$.

Chapter 3

Real Time Analysis of Signed Marked Random Measures with Applications to Finance and Insurance

3.1 Formalism and Notation

We now return to the functional Φ_ν. Note that we do not know the distribution of the random vector $(A_\nu - A_{\nu-1}, t_\nu - t_{\nu-1})$ nor is the latter independent of $(A_{\nu-1}, t_{\nu-1})$. The remedy for this predicament is the use of stochastic expansion that will include several steps. In the first step, we introduce the auxiliary sequence $\{\nu(p)\}$ of exit indices relative to the sequence $\{0, 1, \ldots\}$ of thresholds to be crossed by A_n, of which $\nu = \nu\,(M-1)$ was introduced in (1.1.6). Namely, let

$$\nu\,(p) = inf\,\{n = 0, 1, \ldots : A_n > p\}\,, p = 0, 1, \ldots. \tag{3.1.1}$$

With p fixed, we have the sequence of functionals

$$\varPhi_{\nu(p)}\,(t) = Ez^{N_t} e^{-i\eta \varPi_t} e^{-i\varphi P_{\nu(p)-1}} u^{A_{\nu(p)-1}} e^{-i\phi P_{\nu(p)}} v^{A_{\nu(p)}} e^{-\vartheta_0 t_{\nu(p)-1} - \vartheta t_{\nu(p)}} \mathbf{1}_{[t_{\nu(p)-1}, t_{\nu(p)})}\,(t)\,. \tag{3.1.12}$$

In our second step, we apply to $\varPhi_{\nu(p)}$ of (3.1.2) the transformation D_p defined as

$$D_p\{f(p)\}\,(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), ||x|| < 1, \tag{3.1.3}$$

where f is a real-valued function with the domain $\mathbb{N}_0 = \{0, 1, \ldots\}$. The inverse of D_p is the so-called $\mathcal{D}$-operator previously introduced in Dshalalow [13,17]:

$$\mathcal{D}_x^k \varphi(x,y) = \begin{cases} \lim_{x\to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x}\varphi(x,y)\right], & k \geq 0 \\ 0, & k < 0. \end{cases} \tag{3.1.4}$$

From $\varPhi_{\nu(p)}\,(t) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}\,(t)\,\mathbf{1}_{\{v(p)=n\}}$, we have

$$\varPhi\,(t,x) := D_p\left[\varPhi_{\nu(p)}\,(t)\right](x) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}\,(t)\,D_p \mathbf{1}_{\{\nu(p)=n\}}\,(x)$$

$$= \sum_{n=0}^{\infty} \varPhi_{\nu(p)=n}\,(t)\,D_p \mathbf{1}_{\{\nu(p)=n\}}\,(x)\,,$$

with

$$\varPhi_{\nu(p)=n}(t) = Ez^{N_t}u^{A_{n-1}}e^{-i\eta\varPi_t}e^{-i\varphi P_{n-1}}v^{A_n}e^{-i\phi P_n}e^{-\vartheta_0 t_{n-1}-\vartheta t_n}\mathbf{1}_{\{t_{n-1}\le t<t_n\}} = F_n(t). \tag{3.1.5}$$

From $\mathbf{1}_{\{v(p)=n\}} = \mathbf{1}_{\{A_{n-1}\le p\}}\mathbf{1}_{\{A_n>p\}}$,

$$D_p\mathbf{1}_{\{v(p)=n\}}(x) = (1-x)\sum_{p=0}^{\infty}x^p\mathbf{1}_{\{A_{n-1}\le p\}}\mathbf{1}_{\{A_n>p\}}$$

$$= (1-x)\sum_{p=A_{n-1}}^{A_n-1}x^p$$

$$= (1-x)\left(\sum_{p=0}^{A_n-1}x^p - \sum_{p=0}^{A_{n-1}-1}x^p\right) = (1-x)\left(\frac{1-x^{A_n}}{1-x} - \frac{1-x^{A_{n-1}}}{1-x}\right) = x^{A_{n-1}} - x^{A_n}$$

that yields

$$\varPhi(t,x) = \sum_{n=0}^{\infty}F_n(t)\left(x^{A_{n-1}} - x^{A_n}\right)$$

$$= \sum_{n=0}^{\infty}\left[F_n(ux,v,z,\vartheta_0,\vartheta,t) - F_n(u,vx,z,\vartheta_0,\vartheta,t)\right], where A_{-1} = 0. \tag{3.1.6}$$

Finally, applying the Laplace transform to $\varPhi(t,x)$ of (3.1.6) we have

$$\varPhi^*(\theta,x) = \int_{t=0}^{\infty}e^{-\theta t}\varPhi(t,x)\,dt = \sum_{n=0}^{\infty}\left[F_n^*(ux,v,z,\vartheta_0,\vartheta,t) - F_n^*(u,vx,z,\vartheta_0,\vartheta,t)\right]. \tag{3.1.7}$$

Now functionals F_n and their transforms F_n^* are subject to our scrutiny in Section 3.2.

3.2 Analysis of F_n

With $n = 1, 2, \ldots$, we work on

$$F_n(t) = Ez^{N_t} u^{A_{n-1}} e^{-i\eta \Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}, \tag{3.2.1}$$

(defined in (3.1.5)). (3.2.1) can be brought to the expression

$$F_n(t) = E[(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta)t_{n-1}} v^{A_n - A_{n-1}}$$

$$\times e^{-i\phi(P_n - P_{n-1})} e^{-\vartheta(t_n - t_{n-1})} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}]$$

$$= E(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta)t_{n-1}} v^{X_n} e^{-i\phi \pi_n} e^{-\vartheta \Delta_n} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}, n = 1, 2 \ldots. \tag{3.2.2}$$

in light of:

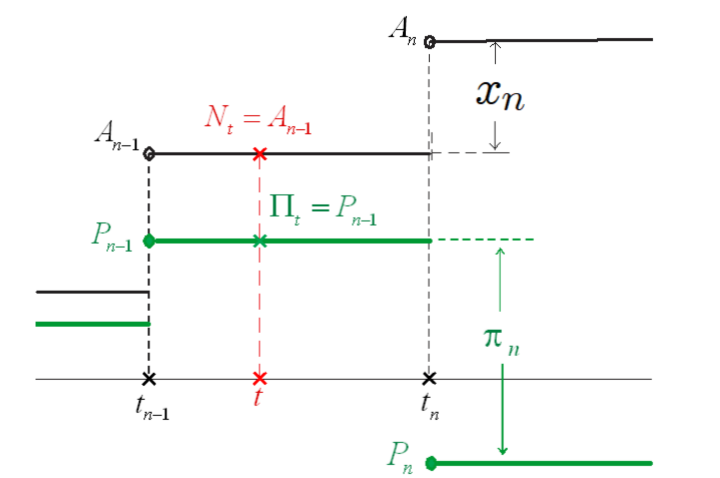

Figure 3.1: Non-monotone passive and monotone active processes

The Laplace transform of F_n with the expectation unfolded reads

$$F_n^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t} F_n(t)\, dt$$

$$= \sum_{k=0}^{\infty} (zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\geq 0} e^{-(\vartheta_0+\vartheta)s} e^{-\theta s}$$

$$\times \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\geq 0} e^{-\vartheta\delta} \int_{t-s=0}^{\delta} e^{-\theta(t-s)} dt$$

$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k, j, dp, ds, dq, d\delta)$$

$$= \frac{1}{\theta} \sum_{k=0}^{\infty} (zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\geq 0} e^{-(\vartheta_0+\vartheta+\theta)s} \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\geq 0} e^{-\vartheta\delta}$$

$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k, j, dp, ds, dq, d\delta)$$

$$-\frac{1}{\theta} \sum_{k=0}^{\infty} (zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\geq 0} e^{-(\vartheta_0+\vartheta+\theta)s} \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\geq 0} e^{-(\vartheta+\theta)\delta}$$

$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k, j, dp, ds, dq, d\delta)$$

due to independence of $A_{n-1} \otimes P_{n-1} \otimes t_{n-1}$ and $X_n \otimes \pi_n \otimes \Delta_n$

$$= \frac{1}{\theta} E[(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta+\theta)t_{n-1}}]$$

$$\times \left[E v^{X_n} e^{-i\phi\pi_n} e^{-\vartheta\Delta_n} - E v^{X_n} e^{-i\phi\pi_n} e^{-(\vartheta+\theta)\Delta_n} \right]$$

$$= \frac{1}{\theta} \Gamma_{n-1}(zuv, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta) \left[\gamma(v, \phi, \vartheta) - \gamma(v, \phi, \vartheta+\theta) \right], \qquad (3.2.3)$$

where

$$\Gamma_{n-1}(zuv, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)$$

$$= \gamma_0 \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right) \gamma^{n-1} \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right) \text{ for } n \geq 1 \tag{3.2.4}$$

and

$$\gamma_0 \left(u, \varphi, \vartheta \right) = E u^{X_0} e^{-i\varphi\pi_0} e^{-\vartheta t_0}, \, \gamma \left(u, \varphi, \vartheta \right) = E u^{X_k} e^{-i\varphi\pi_k} e^{-\vartheta \Delta_k}, k = 1, 2, \ldots. \tag{3.2.5}$$

Summing up F_n for all $n = 1, 2, \ldots,$ with (3.2.3-3.2.4) in mind, we formally arrive at the expression

$$\sum_{n=1}^{\infty} F_n^* \left(\theta \right) = \frac{1}{\theta} \gamma_0 \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right)$$

$$\times \left[\gamma \left(v, \phi, \vartheta \right) - \gamma \left(v, \phi, \vartheta + \theta \right) \right] \frac{1}{1 - \gamma \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right)}. \tag{3.2.6}$$

To warrant the convergence of the geometric series $\sum_{n=1}^{\infty} F_n^* \left(\theta \right)$, in the proposition below, we show that the norm $\left\| \gamma \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right) \right\| < 1$.

Proposition 3.2.1 *The series*

$$\sum_{n=1}^{\infty} F^* \left(\theta \right) = \sum_{n=1}^{\infty} \int_{t=0}^{\infty} e^{-\theta t} E [(zuv)^{A_{n-1}} e^{-i(\eta + \varphi + \phi) P_{n-1}}$$

$$\times e^{-(\vartheta_0 + \vartheta) t_{n-1}} v^{X_n} e^{-i\phi\pi_n} e^{-\vartheta \Delta_n} \mathbf{1}_{\{t_{n-1} \leq t < t_n\}}] dt$$

converges to

$$\sum_{n=1}^{\infty} F_n^* \left(\theta \right) = \frac{1}{\theta} \gamma_0 \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right)$$

$$\times \left[\gamma \left(v, \phi, \vartheta \right) - \gamma \left(v, \phi, \vartheta + \theta \right) \right] \frac{1}{1 - \gamma \left(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta \right)},$$

with

$$\|\gamma(zuv, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta)\| < 1,$$

provided one of the following conditions is met:

$$\mathrm{Re}\vartheta_0 > 0, or \mathrm{Re}\vartheta > 0, or \mathrm{Re}\,\theta > 0 or \; \|u\| < 1, or \; \|v\| < 1, or \; \|z\| < 1.$$

Proof. The first part of the proposition is due to the above steps that formally ended in formula (3.2.6). Inequality (3.2.7) holds due to the following arguments:

$$\left\|\gamma(uvz, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta)\right\| \leq E\left\|(uvz)^{X_1} e^{-i(\eta+\varphi+\phi)\pi_n} e^{-(\vartheta_0+\vartheta+\theta)\Delta_1}\right\|$$

$$= \sum_{k=0}^{\infty} \left\|uvz\right\|^k \int_{t=0}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$= \int_{t=0}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(0, dt) + \sum_{k=1}^{\infty} \left\|uvz\right\|^k \int_{t=0}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$= \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(0, dt) + \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(0, dt)$$

$$+ \sum_{k=1}^{\infty} \left\|uvz\right\|^k \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$+ \sum_{k=1}^{\infty} \left\|uvz\right\|^k \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1 \otimes \Delta_1}(k, dt)$$

$$\leq \int_{t=0}^{1} P_{X_1 \otimes \Delta_1}(0, dt) + e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1 \otimes \Delta_1}(0, dt)$$

$$+ \left\|uvz\right\| \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_1 \otimes \Delta_1}(k, dt) + \left\|uvz\right\| \sum_{k=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1 \otimes \Delta_1}(k, dt),$$

since $\|uvz\| \geq \|uvz\|^k$ for $\|uvz\| \leq 1$ and $k > 1$. Let

$$a := \int_{t=0}^{1} P_{X_i \otimes \Delta_i}(0, dt),\ b := \int_{t=1}^{\infty} P_{X_i \otimes \Delta_i}(0, dt)$$

$$c := \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_i \otimes \Delta_i}(k, dt),\ d := \sum_{k=1}^{\infty} \int_{t=1}^{\infty} P_{X_i \otimes \Delta_i}(k, dt).$$

Then clearly, $a + b + c + d = 1$ and thus,

$$a + e^{-\mathrm{Re}(\vartheta_0 + \vartheta + \theta)} b + \|uvz\| c + \|uvz\| e^{-\mathrm{Re}(\vartheta_0 + \vartheta + \theta)} d < 1$$

whenever $\|uvz\| < 1$ or $\mathrm{Re}(\vartheta_0 + \vartheta + \theta) > 0$ and we are done with the proof. □

We continue with F_n for $n = 0$. F_0 is the functional of the underlying process on interval $[0, t_0)$. With $N_t = \Pi_t = A_{-1} = P_{-1} = t_{-1} = 0$ we have

$$F_0(t) = E z^{N_t} u^{A_{n-1}} e^{-i\eta \Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{0 \leq t < t_0\}}$$

$$= E v^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0, t_0)}(t).$$

The following is easy to prove.

Proposition 3.2.2 *Let* $F_0(t) = E v^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0)}(t)$. *Then*

$$F_0^*(\theta) = \frac{1}{\theta} \left[\gamma_0(v, \phi, \vartheta) - \gamma_0(v, \phi, \vartheta + \theta) \right]. \tag{3.2.7}$$

With Proposition 3.2.1, we can augment the series $\sum_{n=1}^{\infty} F_n^*$ of formula (3.2.6) to include F_0^*:

$$\sum_{n=0}^{\infty} F_n^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right] + \frac{1}{\theta}\gamma_0(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)$$

$$\times\left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta)\right]\frac{1}{1-\gamma(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)}. \tag{3.2.8}$$

From (3.2.7) and (3.2.8) we arrive at

$$\Phi^*(\theta,x) = \sum_{n=0}^{\infty}\left[F_n^*(ux,v,z,\vartheta_0,\vartheta,t) - F_n^*(u,vx,z,\vartheta_0,\vartheta,t)\right]$$

$$= \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx,\phi,\vartheta) - \gamma_0(vx,\phi,\vartheta+\theta)\right]$$

$$+\frac{1}{\theta}\gamma_0(zuvx,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\frac{1}{1-\gamma(zuvx,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)}$$

$$\times\left[\gamma(v,\phi,\vartheta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta) - \gamma(v,\phi,\vartheta+\theta)\right]. \tag{3.2.9}$$

The Laplace transform $\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}\Phi_\nu(t)\,dt$ of the functional

$$\Phi_\nu(t) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)$$

can be extracted from $\Phi^*(\theta,x)$ of (3.2.9) using the $\mathcal{D}$-operator.

The entire effort in this section can be reduced to the following.

Theorem 3.2.3 *Let $\Phi_\nu(\theta)$ denote the Laplace transform of the functional*

$$\Phi_\nu(t) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) \tag{3.2.10}$$

$$\|z\| \leq 1, \|u\| \leq 1, \|v\| \leq 1, Re\vartheta_0 \geq 0, Re\vartheta \geq 0, \eta, \varphi, \phi \in \mathbb{R},$$

Then, with

$$\|u\| < 1, or \ \|v\| < 1, or \ \|z\| < 1, \ or \ Re\vartheta_0 > 0, or \ Re\vartheta > 0, \ or \ Re\,\theta > 0, \quad (3.2.11)$$

$$\begin{aligned} &\varPhi_\nu^*(\theta) \\ &= \mathcal{D}_x^{M-1}\Bigg\{\frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx,\phi,\vartheta) - \gamma_0(vx,\phi,\vartheta+\theta)\right] \\ &\quad + \frac{1}{\theta}\gamma_0(zuvx,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\frac{1}{1-\gamma(zuvx,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)} \\ &\qquad \times\left[\gamma(v,\phi,\vartheta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta) - \gamma(v,\phi,\vartheta+\theta)\right]\Bigg\}. \quad (3.2.12) \end{aligned}$$

3.3 Applications to Option Trading

For an illustration, consider the following special case. Suppose that we observe a constantly fluctuating stock price of some company over the times $t_0 = 0, t_1, t_2, \ldots$ that starts off at time zero with a price π_0.

Case 1. Observation of process P_i upon the first drop.

1a. Suppose we are interested in the characteristics of the process around the period when the stock price drops for the first time. Because the stock prices cannot be modeled by a monotone process, we have the observed prices upon t's as the passive component, and introduce the active component

$$X_n = \begin{cases} 0, & \pi_n \geq 0 \\ 1, & \pi_n < 0 \end{cases}. \tag{3.3.1}$$

Suppose π_0 is a nonnegative r.v. with some specified distribution and let $X_0 = \tau_0 = 0$. So,

$$\gamma_0 \left(z, \phi, \theta\right) = Ee^{-i\phi\pi_0} \ (innotation) = \gamma_0 \left(\phi\right).$$

Next, with M=1 according to our assumption about the first drop, formula (3.2.12) further reduces to

$$\theta \Phi_\nu^* \left(\theta\right) = \gamma_0 \left(\eta + \varphi + \phi\right) \frac{1}{1 - \gamma \left(0, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta\right)}$$

$$\times \left[\gamma \left(v, \phi, \vartheta\right) - \gamma \left(0, \phi, \vartheta\right) + \gamma \left(0, \phi, \vartheta + \theta\right) - \gamma \left(v, \phi, \vartheta + \theta\right)\right]. \tag{3.3.2}$$

Because the active component is merely auxiliary, we are less interested in any information about $N_t, A_{\nu-1}, A_\nu$, as well as $P_{\nu-1}, t_{\nu-1}$, so we set $z = u = v = 1$ and $\varphi = \vartheta_0 = 0$ restricting the Laplace transform of Φ_ν to the marginal transform

$$\int_{t=0}^{\infty} e^{-\theta t} Ee^{-i\eta \Pi_t} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)} \left(t\right) dt$$

$$= \frac{1}{\theta} \gamma_0 \left(\eta + \phi\right) \frac{1}{1 - \gamma \left(0, \eta + \phi, \vartheta + \theta\right)}$$

$$\times \left[\gamma \left(1, \phi, \vartheta\right) - \gamma \left(0, \phi, \vartheta\right) + \gamma \left(0, \phi, \vartheta + \theta\right) - \gamma \left(1, \phi, \vartheta + \theta\right)\right], \tag{3.3.3}$$

where

$$\gamma(z,\phi,\theta) = Ez^{X_1}e^{-i\phi\pi_1}e^{-\Delta_1\theta} \; and \gamma(0,\phi,\theta) = Ez^{X_1}e^{-i\phi\pi_1}e^{-\Delta_1\theta}\Big|_{z=0}.$$

From

$$Ez^{X_1}\Big|_{z=0} = P\left\{X_1=0\right\} + zP\left\{X_1=1\right\}\Big|_{z=0} = P\left\{X_1=0\right\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1\geq 0\}}$$

we have

$$\gamma\left(0,\phi,\theta\right) = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}e^{-\Delta_1\theta}.$$

Suppose now that Δ's and π's are independent, that is, the observation epochs and stock price changes are independent. This may not always apply, but it would simplify establishing of γ. Then

$$\gamma\left(0,\phi,\theta\right) = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta},$$

if the observation epochs occur according to a Poisson point process of intensity γ. Our next assumption is that the marginal distribution of π_1 is Laplace with parameter μ and zero shift. That being said, the PDF of π_1 is

$$f_{\pi_1}\left(x\right) = \frac{1}{2}\mu e^{-\mu|x|}, x \in \mathbb{R}. \tag{3.3.4}$$

Then

$$\gamma\left(0,\phi,0\right) = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1} = \int_{x=0}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx = \frac{1}{2}\frac{\mu}{\mu+i\phi}.$$

Because $Ee^{-i\phi\pi_1} = Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq 0\}} + \mathbf{1}_{\{\pi_1<0\}}\right)$,we have

$$Ee^{-i\phi\pi_1} = \frac{1}{2}\frac{\mu}{\mu+i\phi} + \int_{x=-\infty}^{0} e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx$$

$$= \frac{1}{2}\frac{\mu}{\mu+i\phi} + \frac{1}{2}\frac{\mu}{\mu-i\phi} = \frac{1}{2}\mu\frac{2\mu}{\mu^2+\phi^2} = \frac{\mu^2}{\mu^2+\phi^2}.$$

Thus,

$$\gamma(1,\phi,\theta) = Ee^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = Ee^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta} = \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}.$$

Next the following two further marginals are of interest.

(i) With $\eta = \phi = 0$ in (3.3.3),the functional

$$\int_{t=0}^{\infty} e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \frac{1}{\theta}\frac{1}{1-\gamma(0,0,\vartheta+\theta)}\left[\gamma(1,0,\vartheta) - \gamma(0,0,\vartheta) + \gamma(0,0,\vartheta+\theta) - \gamma(1,0,\vartheta+\theta)\right] \tag{3.3.5}$$

represents the Laplace transform of the first passage time t_ν's marginal functional at the first drop with the time t falling between the pre-first passage time $t_{\nu-1}$ and t_ν. Here

$$\gamma(1,0,\vartheta+\theta) = \frac{\gamma}{\gamma+\vartheta+\theta}$$

$$\gamma(0,\phi,\theta) = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = \frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta}$$

$$\gamma(0,0,\vartheta) = \frac{1}{2}\frac{\gamma}{\gamma+\vartheta}.$$

Therefore,

$$\int_{t=0}^{\infty} e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \left(1 + \frac{\gamma}{\gamma + 2(\vartheta + \theta)}\right) \frac{\gamma}{2} \frac{1}{(\gamma + \vartheta)(\gamma + \vartheta + \theta)} \tag{3.3.1}$$

implying that the inverse of the Laplace transform is

$$Ee^{-\vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t) = \mathcal{L}_\theta^{-1} \left\{ \int_{t=0}^{\infty} e^{-\theta t} Ee^{-\vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t)\, dt \right\} = \frac{\gamma}{2(\gamma + \vartheta)} e^{-\frac{t}{2}(\gamma + 2\vartheta)}. \tag{3.3.7}$$

(ii) With $\eta = \vartheta = 0$ in (3.3.3),we have the Laplace transform of the P_ν's marginal functional upon the first passage time t_ν jointly with the time t running between $t_{\nu-1}$ and t_ν.

$$\int_{t=0}^{\infty} e^{-\theta t} Ee^{-i\phi P_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t)\, dt$$

$$= \frac{1}{\theta} \gamma_0(\phi) \frac{1}{1 - \gamma(0, \phi, \theta)} \left[\gamma(1, \phi, 0) - \gamma(0, \phi, 0) + \gamma(0, \phi, \theta) - \gamma(1, \phi, \theta)\right]. \tag{3.3.8}$$

Because

$$\gamma_0(\phi) = e^{-i\phi p_0}$$

(assuming the initial price $\pi_0 = p_0$ a.s. where p_0 is a constant)

and

$$\gamma(1, \phi, \theta) = \frac{\mu^2}{\mu^2 + \phi^2} \frac{\gamma}{\gamma + \theta},$$

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} Ee^{-\Delta_1 \theta} = \frac{1}{2} \frac{\mu}{\mu + i\phi} \frac{\gamma}{\gamma + \theta},$$

$$1 - \gamma(0, \phi, \theta) = \frac{2(\mu + i\phi)(\gamma + \theta) - \mu\gamma}{2(\mu + i\phi)(\gamma + \theta)},$$

and

$$\frac{1}{1 - \gamma(0, 0, \vartheta + \theta)} = 1 + \frac{\mu\gamma}{2(\mu + i\phi)(\gamma + \theta) - \mu\gamma},$$

we have that

$$\int_{t=0}^{\infty} e^{-\theta t} E e^{-i\phi P_\nu} \mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\, dt$$

$$= \frac{1}{\theta} e^{-i\phi p_0} \left(1 + \frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right)$$

$$\times \left[\frac{\mu^2}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu+i\phi} + \frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta} - \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}\right]$$

$$= e^{-i\phi p_0}\left[\frac{\mu^2}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu+i\phi}\right]\left(1 + \frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right)\frac{1}{\gamma+\theta}. \tag{3.3.9}$$

Thus,

$$E e^{-i\phi P_\nu} \mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t} E e^{-i\phi P_\nu} \mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\right\}$$

$$= \left[\frac{\mu^2}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu+i\phi}\right] e^{-\left(\frac{\gamma t}{2}\left(\frac{\mu+2i\phi}{\mu+i\phi}\right)+i\phi p_0\right)} \tag{3.3.10}$$

and

$$E P_\nu \mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = i \lim_{\phi\to 0}\frac{\partial}{\partial\phi} E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = \frac{1}{2\mu}\left(\frac{\gamma t}{2} + \mu p_0 - 1\right) e^{-\frac{\gamma t}{2}} \tag{3.3.11}$$

$$E P_\nu^2 \mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = -\lim_{\phi\to 0}\frac{\partial^2}{\partial\phi^2} E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)$$

$$= \frac{1}{2\mu^2}\left(2 + \left(\frac{\gamma t}{2}\right)^2 + (\mu p_0)^2 + 2\mu p_0\frac{\gamma t}{2} - 2\mu p_0\right) e^{-\frac{\gamma t}{2}}. \tag{3.3.12}$$

So

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\{t_{\nu-1} \le t < t_\nu\} = \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t} E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\, dt\right\} = \frac{e^{-\frac{\gamma}{2}t}}{2}. \tag{3.3.13}$$

1b. One could be interested in when the passive component drops lower than

R,for some $R < 0$. Thus the active component reads now

$$X_n = \begin{cases} 0, & \pi_n \geq R \\ 1, & \pi_n < R \end{cases}. \tag{3.3.2}$$

With $M = 1$ assumed and because

$$Ez^{X_1}\Big|_{z=0} = P\{X_1 = 0\} + zP\{X_1 = 1\}\Big|_{z=0} = P\{X_1 = 0\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1 \geq R\}},$$

we have

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq R\}} e^{-i\phi\pi_1} e^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1 \geq R\}} e^{-i\phi\pi_1} \frac{\gamma}{\gamma + \theta},$$

$$\gamma(0, \phi, 0) = E\mathbf{1}_{\{\pi_1 \geq R\}} e^{-i\phi\pi_1} = \int_{x=R}^{0} e^{-i\phi x} \frac{1}{2} \mu e^{\mu x} dx + \int_{x=0}^{\infty} e^{-i\phi x} \frac{1}{2} \mu e^{-\mu x} dx$$
$$= \frac{1}{2} \frac{\mu}{\mu - i\phi} \left[1 - e^{(\mu - i\phi)R}\right] + \frac{1}{2} \frac{\mu}{\mu + i\phi}.$$

Since

$$Ee^{-i\phi\pi_1} = Ee^{-i\phi\pi_1} \left(\mathbf{1}_{\{\pi_1 \geq R\}} + \mathbf{1}_{\{\pi_1 < R\}}\right),$$

we have

$$Ee^{-i\phi\pi_1} = \frac{1}{2} \frac{\mu}{\mu - i\phi} \left[1 - e^{(\mu - i\phi)R}\right] + \frac{1}{2} \frac{\mu}{\mu + i\phi} + \int_{x=-\infty}^{R} e^{-i\phi x} \frac{1}{2} \mu e^{\mu x} dx$$

$$= \frac{\mu^2}{\mu^2 + \phi^2}.$$

Thus,

$$\gamma(1, \phi, \theta) = Ee^{-i\phi\pi_1} Ee^{-\Delta_1\theta} = Ee^{-i\phi\pi_1} \frac{\gamma}{\gamma + \theta} = \frac{\mu^2}{\mu^2 + \phi^2} \frac{\gamma}{\gamma + \theta}$$

and with $\eta = \phi = 0 = \vartheta$ in (3.3.3),the functional

$$\int_{t=0}^{\infty} e^{-\theta t} E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$
$$= \frac{1}{\theta}\frac{1}{1-\gamma(0,0,\theta)}\left[\gamma(1,0,0)-\gamma(0,0,0)+\gamma(0,0,\theta)-\gamma(1,0,\theta)\right]$$
$$= e^{\mu R}\frac{1}{2\theta+\gamma e^{\mu R}}$$

and

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\left\{t_{\nu-1} \le t < t_\nu\right\}$$
$$= \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t} E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\} = \frac{1}{2}e^{-\left(\frac{\gamma t e^{\mu R}}{2}-\mu R\right)} \tag{3.3.15}$$

which reduces to (3.4.13) when $R = 0$.

Case 2. Observation of process P_i upon general Mth drop.

2a. For the general threshold level M (when the stock price drops Mth times), since the active process increments X_n are Bernoulli with $p = 0.5$ due to the symmetric Laplace PDF of π_n defined in (3.3.4) above with zero shift and with

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = \Phi_\nu(t)\Big|_{z,v,u,\vartheta=1,\eta,\varphi,\phi,\vartheta_0,\vartheta=0},$$

$$\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t} E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0(0)\frac{1}{1-\gamma(x,0,\theta)} \times \left[\gamma(1,0,0)-\gamma(x,0,0)+\gamma(x,0,\theta)-\gamma(1,0,\theta)\right], \tag{3.3.16}$$

where

$$\gamma(1,0,\theta) = \frac{\gamma}{\gamma+\theta},$$

$$\gamma(x,0,0) = \frac{1+x}{2},\ \gamma(x,0,\theta) = (\frac{1+x}{2})\frac{\gamma}{\gamma+\theta}.$$

Therefore,

$$\Phi_\nu^*(\theta) = \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[1-\frac{1+x}{2}+\frac{1+x}{2}\frac{\gamma}{\gamma+\theta}-\frac{\gamma}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[\frac{1-x}{2}\frac{\theta}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\} - \mathcal{D}_x^{M-2}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\}$$

$$= \frac{1}{(\gamma+2\theta)}\left(\frac{\gamma}{\gamma+2\theta}\right)^{M-1} = \frac{\gamma^{M-1}}{(\gamma+2\theta)^M}.$$

So

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = P\{t_{\nu-1} \le t < t_\nu\} = \mathcal{L}_\theta^{-1}\{\Phi_\nu(\theta)\} = \frac{1}{2}\frac{\left(\frac{\gamma t}{2}\right)^{M-1}}{(M-1)!}e^{-\frac{\gamma}{2}t}\ . \qquad (3.3.17)$$

2b. Next we obtain the result for $E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ for general M and general shift parameter a in our model such that

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x-a|}, x \in \mathbb{R}.$$

After some algebra we have

$$\gamma(0,\phi,0) = E\mathbf{1}_{\{\pi_1\ge 0\}}e^{-i\phi\pi_1} = \int_{x=0}^{a} e^{-i\phi x}\frac{1}{2}\mu e^{\mu(x-a)}dx + \int_{x=a}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu(x-a)}dx$$

$$= \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}\left(\mu + i\phi\right)}{\mu^2 + \phi^2}$$

$$\gamma(0,\phi,\theta) \;=\; \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}\left(\mu + i\phi\right)}{\mu^2 + \phi^2}\frac{\gamma}{\gamma+\theta}$$

and

$$\gamma(0,\phi,0) = \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}\left(\mu + i\phi\right)}{\mu^2 + \phi^2}$$

$$Ee^{-i\phi\pi_1} \;=\frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[\left(\mu + i\phi\right)e^{-\mu a} + \left(\mu - i\phi\right)e^{\mu a}\right]$$

$$\gamma(1,\phi,\theta) = \left[\frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[\left(\mu + i\phi\right)e^{-\mu a} + \left(\mu - i\phi\right)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}$$

$$\gamma(x,\phi,\theta) = (p+qx)\left[\frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[\left(\mu + i\phi\right)e^{-\mu a} + \left(\mu - i\phi\right)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}.$$

Hence

$$\Phi_\nu^*\left(\theta\right) = \int_{t=0}^{\infty} e^{-\theta t} E\mathbf{1}_{\left(t_{\nu-1},t_\nu\right]}\left(t\right)dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0\left(0\right)\frac{1}{1-\gamma\left(x,0,\theta\right)} \times \left[\gamma\left(1,0,0\right) - \gamma\left(x,0,0\right) + \gamma\left(x,0,\theta\right) - \gamma\left(1,0,\theta\right)\right]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\theta}\frac{\left(\gamma+\theta\right)}{\gamma+\theta-p\gamma\left(2-\cosh(\mu a)\right) - q\gamma\left(2-\cosh(\mu a)\right)x}\right.$$

$$\times\left[\left(2-\cosh(\mu a)\right) - \left(p+qx\right)\left(2-\cosh\mu a)\right) + \left(p+qx\right)\left(2-\cosh(\mu a)\right)\frac{\gamma}{\gamma+\theta}\right.$$

$$\left.\left. - \left(2-\cosh(\mu a)\right)\frac{\gamma}{\gamma+\theta}\right]\right\}$$

$$= \frac{\gamma^{M-1}(q\left(2-\cosh(\mu a)\right))^M}{\left(\gamma+\theta-p\gamma\left(2-\cosh(\mu a)\right)\right)^M} \tag{3.3.18}$$

by the $\mathcal{D}$-operator inversion formulas from [21].

$$E\mathbf{1}_{\left[t_{\nu-1},t_\nu\right)}\left(t\right) = P\left\{t_{\nu-1} \le t < t_\nu\right\} = \mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*\left(\theta\right)\right\}$$

$$= \gamma^{M-1}(q\,(2-\cosh(\mu a)))^M \frac{t^{M-1}}{(M-1)!} e^{-(\gamma - p\gamma(2-\cosh(\mu a)))t}$$

$$= q\,(2-\cosh(\mu a)) \frac{(\gamma q\,(2-\cosh(\mu a))\,t)^{M-1}}{(M-1)!} e^{-(\gamma - p\gamma(2-\cosh(\mu a)))t}. \qquad (3.3.19)$$

Notice that when $a = 0$ (in the symmetric case), (3.3.19) reduces to (3.3.17) and the value of μ is irrelevant given it is finite.

3.4 Continuous Time Parameter Process on Interval $[0, t_\nu)$

Now consider the functional of passive process P being observed over the period $[0, t_\nu)$, jointly with the active process A_ν, the first passage time t_ν, and the counting processes N_t and Π_t. The functional satisfies the formula:

$$\overset{\wedge}{\Phi}_\nu\,(t) = E z^{N_t} e^{-i\eta \Pi_t} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t)$$

$$= \sum_{k=0}^{\infty} E z^{N_t} e^{-i\eta \Pi_t} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \mathbf{1}_{\{\nu=k\}}.$$

Since $\sum_{j=0}^{\nu} E\mathbf{1}_{[t_{\nu-j-1}, t_{\nu-j})}(t) = E\mathbf{1}_{[0,t_\nu)}(t)$,

$$\overset{\wedge}{\Phi}_\nu\,(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} E[z^{A_{k-j-1}} v^{A_{k-j-1}} v^{\sum_{i=k-j}^{k} X_i} e^{-i\eta \Pi_{k-j-1} - i\phi P_{k-j-1}}$$

$$\times e^{-i\phi \sum_{i=k-j}^{k} \pi_i} e^{-\vartheta t_{k-j-1}} e^{-\vartheta \sum_{i=k-j}^{k} \Delta_i} \mathbf{1}_{[t_{k-j-1}, t_{k-j})}(t)],$$

and applying the transformation D_p to $\overset{\wedge}{\Phi}_\nu(t)$ we have:

$$D_p\left[\overset{\wedge}{\Phi}_\nu(t)\right](x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} F_{jk}(t)\, x^{X_{k-j+1}+\cdots+X_{k-1}} \times E(vx)^{X_{k-j+1}+\cdots+X_{k-1}}\, e^{-i\phi\left(\pi_{k-j+1}+\cdots+\pi_{k-1}\right)} e^{-\vartheta\left(\Delta_{k-j+1}+\cdots+\Delta_{k-1}\right)} \times E\left(1-x^{X_k}\right) e^{-i\phi\pi_k} e^{-\vartheta\Delta_k} v^{X_k},$$

where

$$F_{jk}(t) = E(zvx)^{A_{k-j-1}}\, e^{-i(\eta+\phi)P_{k-j-1}} e^{-\vartheta t_{k-j-1}} \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,(vx)^{X_{k-j}}\, e^{-i\phi\left(\pi_{k-j}\right)} e^{-\vartheta\left(\Delta_{k-j}\right)}.$$

Let

$$\widetilde{F}(t) = Ez^A e^{-i\eta P} e^{-\vartheta T} \mathbf{1}_{[T,T+\Delta)}(t)\, v^X e^{-i\phi\pi} e^{-\vartheta\Delta}$$

under the assumptions that random vectors $A\otimes P\otimes T$ and $X\otimes\pi\otimes\Delta$ are independent. Then

$$\widetilde{F}^*(\theta) = \sum_r z^r \sum_m v^m \int_p e^{-i\eta p} \int_w e^{-i\phi w} \int_{s\geq 0} e^{-\vartheta s} e^{-\theta s}. \times \frac{1}{\theta}\int_\delta \left(e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta}\right) P_{A\otimes P\otimes T\otimes X\otimes\pi\otimes\Delta}(r,m,dp,ds,dw,d\delta)$$

and because $A \otimes P \otimes T$ and $X \otimes \pi \otimes \Delta$ are independent,

$$= \frac{1}{\theta} E\left[z^{A} e^{-i\eta P} e^{-(\vartheta+\theta)T}\right]\left[\gamma\left(v,\phi,\vartheta\right) - \gamma\left(v,\phi,\vartheta+\theta\right)\right].$$

Thus

$$F_{jk}^{*}\left(\theta\right) = \frac{1}{\theta}\gamma_0 \delta^{1j-1}\left[\delta - \delta^1\right]\gamma^{k-j-1}\left[\delta^1 - \delta^{13}\right] \tag{3.4.1}$$

and

$$(i) \quad \sum_{k>0}\sum_{j=1}^{k-1} F_{jk}^{*}\left(\theta\right) = \frac{1}{\theta}\gamma_0 \Psi\delta \sum_{k>0}\gamma^{k-2}\sum_{j=1}^{k-1}\left(\frac{\delta^1}{\gamma}\right)^{j-1} = \frac{1}{\theta}\gamma_0 \frac{\Psi\delta}{\left(1-\gamma\right)\left(1-\delta^1\right)}, \tag{3.4.2}$$

with notation $\gamma := \gamma\left(zvx, \eta+\phi, \vartheta\right)$ and $\gamma_0 := \gamma_0\left(zvx, \eta+\phi, \vartheta\right)$, and further

$$\delta^1 = \gamma\left(vx,\phi,\vartheta\right), \delta_0^1 = \gamma_0\left(vx,\phi,\vartheta\right), \delta = \gamma\left(v,\phi,\vartheta\right),$$

$$\delta^3 = \gamma\left(v,\phi,\vartheta+\theta\right), \delta^{13} = \gamma\left(vx,\phi,\vartheta+\theta\right)$$

$$\delta_0 = \gamma_0\left(v,\phi,\vartheta\right), \delta_0^{13} = \gamma_0\left(vx,\phi,\vartheta+\theta\right),$$

$$\Gamma\delta = \delta - \delta^3 - \delta^1 + \delta^{13},\ \Lambda\delta = \frac{\Psi\delta}{1-\delta^1} + \Gamma\delta, \Psi\delta = \left(\delta - \delta^1\right)\left(\delta^1 - \delta^{13}\right).$$

(ii) Consider $j = k = 0$. $A_{-1} = t_{-1} = P_{-1} = 0$ for $t \in [0,t_0)$ and $N_t = A_{-1} = \Pi_t = 0$.

$$F_{00}\left(t\right) = E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_0} v^{A_0} e^{-i\phi P_0}\left(1 - x^{A_0}\right).$$

$$F_{00}^{*}\left(\theta\right) = \sum_r v^r \int_p e^{-i\phi p}\int_s e^{-\vartheta s}\int_{t=0}^{s} e^{-\theta t} dt P_{A_0\otimes P_0\otimes t_0}\left(r,dp,ds\right)$$

$$= \sum_r v^r \int_p e^{-i\phi p}\int_s e^{-\vartheta s}\frac{1}{\theta}\left[e^{-\vartheta s} - e^{-(\vartheta+\theta)s}\right] P_{A_0\otimes P_0\otimes t_0}\left(r,dp,ds\right) = \frac{1}{\theta}\Gamma\delta_0. \tag{3.4.3}$$

(iii) Consider $j = 0, k > 0$.

$$F_{0k}(t) = Ez^{N_t} v^{A_k} e^{-i\eta P_{k-1}} e^{-i\phi P_k} e^{-\vartheta t_k} \mathbf{1}_{[t_{k-1}, t_k)}(t) \left(x^{A_{k-1}} - x^{A_k} \right)$$

$$= E(zvx)^{A_{k-1}} e^{-i(\eta+\phi)P_{k-1}} e^{-\vartheta t_{k-1}} \mathbf{1}_{[t_{k-1}, t_k)}(t) v^{X_k} \left(1 - x^{X_k} \right) e^{-i\phi \pi_k} e^{-\vartheta \Delta_k}.$$

$$F_{0k}^*(\theta) = \int_t e^{-\theta t} \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p} \sum_m v^m \int_q e^{-i\phi q} \int_s e^{-\vartheta s}$$

$$\times \int_\delta e^{-\vartheta \delta} \mathbf{1}_{[s, s+\delta)}(t) \, dt P_{A_{k-1} \otimes P_{k-1} \otimes T_{k-1} \otimes X_k \otimes \pi_k \otimes \Delta_k}(r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p} \sum_m v^m \int_q e^{-i\phi q} \int_s e^{-\vartheta s} e^{-\theta s}$$

$$\times \int_{t-s=0}^{\delta} e^{-\theta(t-s)} dt P_{A_{k-1} \otimes P_{k-1} \otimes T_{k-1} \otimes X_k \otimes \pi_k \otimes \Delta_k}(r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p} \sum_m v^m \int_q e^{-i\phi q} \int_\delta \left[e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta} \right]$$

$$\times P_{A_{k-1} \otimes P_{k-1} \otimes X_k \otimes \pi_k \otimes \Delta_k}(r, dp, m, dq, d\delta)$$

$$= \frac{1}{\theta} \gamma^{k-1}(zvx, \eta+\phi, \vartheta) \gamma_0(zvx, \eta+\phi, \vartheta)$$

$$\times \left[\gamma(v, \phi, \vartheta) - \gamma(v, \phi, \vartheta+\theta) - \gamma(vx, \phi, \vartheta) + \gamma(vx, \phi, \vartheta+\theta) \right]$$

and

$$\sum_{k>0} F_{0k}^*(\theta) = \frac{1}{\theta} \gamma_0(zvx, \eta+\phi, \vartheta) \frac{1}{1 - \gamma(zvx, \eta+\phi, \vartheta)}$$

$$\times \left[\gamma(v, \phi, \vartheta) - \gamma(v, \phi, \vartheta+\theta) - \gamma(vx, \phi, \vartheta) + \gamma(vx, \phi, \vartheta+\theta) \right] = \frac{1}{\theta} \frac{\gamma_0}{1-\gamma} \mathit{\Gamma} \delta. \tag{3.4.4}$$

(iv) Consider $j = k > 0$.

$$F_{kk}(t) = E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} v^{A_k} e^{-i\phi P_k} \left(x^{A_{k-1}} - x^{A_k}\right)$$

$$= E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} (vx)^{X_1+\cdots+X_{k-1}} e^{-i\phi(\pi_1+\cdots+\pi_{k-1})} e^{-\vartheta(\Delta_1+\cdots+\Delta_{k-1})}$$

$$\times \left[v^{X_k} - (vx)^{X_k}\right] e^{-i\phi\pi_k} e^{-\vartheta\Delta_k}$$

$$= E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \gamma^{k-1}(vx,\phi,\vartheta)\left[\gamma(v,\phi,\vartheta) - \gamma(vx,\phi,\vartheta)\right].$$

So,

$$\sum_{k>0} F^*_{kk}(\theta) = \frac{1}{\theta}\frac{\Psi\delta_0}{1-\delta^1}. \tag{3.4.5}$$

Altogether, from (i) through (iv) we have

$$\begin{aligned} \overset{\wedge}{\Phi}{}^*_\nu(\theta) &= \int_{t=0}^{\infty} e^{-\theta t}\Phi_\nu(t)\,dt \\ &= \mathcal{D}_x^{M-1}\left\{\sum_{k>0}\sum_{j=1}^{k-1} F^*_{jk}(\theta) + F^*_{00}(t) + \sum_{k>0} F^*_{0k}(\theta) + \sum_{k>0} F^*_{kk}(\theta)\right\} \\ &= \mathcal{D}_x^{M-1}\left\{\frac{1}{\theta}\left(\Lambda\delta_0 + \frac{\gamma_0}{1-\gamma}\Lambda\delta\right)\right\} \end{aligned} \tag{3.4.6}$$

where

$$\Lambda\alpha = \Gamma\alpha + \frac{\Psi\alpha}{1-\delta^1} \text{ and } \alpha = \delta \text{ or } \delta_0.$$

The Laplace inverse of (3.4.6) will permit the recovery of $\overset{\wedge}{\Phi}_\nu(t)$.

Chapter 4

Time Dependent Analysis of Stochastic Games of Three Players with Applications

4.1 Problem Setting

A generic game under consideration is driven by the following bivariate signed random measure on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$

$$\mathcal{A} \otimes \mathcal{P} \otimes \mathcal{T} = \sum_{n=0}^{\infty} (x_n, \pi_n)\, \varepsilon_{t_n} \ (\varepsilon_a \text{ is the Dirac point mass at point } a), \qquad (4.1.1)$$

where the marks x_n's are non-negative integer- and π_n's are real-valued r.v.'s (random variables). The associated support counting measure $\sum_{n=0}^{\infty} \varepsilon_{t_n}$ forms a delayed renewal process. We further assume that the random measure $\mathcal{A} \otimes \mathcal{P} \otimes \mathcal{T}$ is position dependent, that is $x_n, \pi_n, \Delta_n = t_n - t_{n-1}$ are dependent. However, $(x_n, \pi_n, \Delta_n = t_n - t_{n-1})$ is a tri-variate sequence of independent random vectors and for $n = 1, 2, \ldots$, they are identically distributed. That being said, we assume that their joint distributions are known and given by the transforms

$$\gamma_0 (u, \phi, \vartheta) = Eu^{x_0} e^{-i\phi\pi_0} e^{-\vartheta\Delta_0}, \ \|u\| \leq 1, \phi \in \mathbb{R}, \mathrm{Re}\vartheta > 0. \tag{4.1.2}$$

$$\gamma (u, \phi, \vartheta) \ = Eu^{x_1} e^{-i\phi\pi_1} e^{-\vartheta\Delta_1}, \ \|u\| \leq 1, \phi \in \mathbb{R}, \mathrm{Re}\vartheta > 0. \tag{4.1.3}$$

Next,

$$A_n := \sum_{k=0}^{n} x_k, \ P_n = \sum_{k=0}^{n} \pi_k, \ t_n = \sum_{k=0}^{n} \Delta_k. \tag{4.1.4}$$

Let

$$\nu_1 := \min \{n = 0, 1, \ldots : A_n \geq M\}, \ \nu_2 := \min \{n = 0, 1, \ldots : t_n \geq T\} \tag{4.1.5}$$

and

$$\nu := \nu_1 \wedge \nu_2 \tag{4.1.6}$$

called the *exit index* of the game. The r.v. t_ν is referred to as the *first passage time* of the game.

The game is played between two *active* hostile players A and T, where T represents the time-related point process $\{t_0, t_1, \ldots\}$. The third player, P is *passive*. The game runs until one of the two active players (or both) are defeated on one of the epochs t_k's and it takes place at t_ν.

Now

$$(A(t), P(t)) := \mathcal{A} \otimes \mathcal{P} \otimes \mathcal{T}[0, t] \tag{4.1.7}$$

is the associated continuous time parameter process. The following functional of the game is of interest.

$$\varPhi_\nu(t) = Ez^{A(t)} e^{-i\eta P(t)} v^{A_{\nu-1}} e^{-i\varphi P_{\nu-1}} e^{-\psi t_{\nu-1}} u^{A_\nu} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t), \tag{4.1.8}$$

that is, $\varPhi_\nu$ gives the status of the game at any time t prior to t_ν, along with the output variables A_ν, P_ν, and those one step prior to the exit at time $t_{\nu-1}$, $A_{\nu-1}$ and $P_{\nu-1}$.

For the sequel, we set

$$A_{-1} = t_{-1} = P_{-1} = 0. \tag{4.1.9}$$

We further notice that $(A(t), P(t))$ is piece-wise linear jump process and it is constant on each time subinterval $[t_{k-1}, t_k)$.

4.2 Analysis of $\varPhi_\nu$

To deal with $\varPhi_\nu$, we break it into three functionals $\varPhi_{\nu_1<\nu_2}, \varPhi_{\nu_1>\nu_2}$, and $\varPhi_{\nu_1=\nu_2}$ treated in this section separately.

1. Player T Defeats Player A. In this subsection, we consider the game adapted to the confined filtration $\mathcal{F}_t \cap \{\nu_1 < \nu_2\}$. The associated functional reads

$$\begin{aligned}
&\Phi_{\nu_1<\nu_2}(t) = \\
&\quad Ez^{A(t)}e^{-i\eta P(t)}v^{A_{\nu_1-1}}e^{-i\varphi P_{\nu_1-1}}e^{-\psi t_{\nu_1-1}}u^{A_{\nu_1}}e^{-i\phi P_{\nu_1}}e^{-\vartheta t_{\nu_1}}\mathbf{1}_{[0,t_{\nu_1})}(t)\,\mathbf{1}_{\{\nu_1<\nu_2\}} \\
&\qquad = \sum_{k=0}^{\infty}\sum_{n>k} E\Big[z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}} \\
&\qquad\qquad \times u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}\mathbf{1}_{[0,t_k)}(t)\,\mathbf{1}_{\{k\}}(\nu_1)\,\mathbf{1}_{\{n\}}(\nu_2)\Big]. \qquad (4.2.1)
\end{aligned}$$

We partition the interval $[0,t_k)$ into $[0,t_k) = \bigcup_{j=0}^{k}[t_{k-j-1}, t_{k-j})$ under (4.1.7) to yield

$$\begin{aligned}
\Phi_{\nu_1<\nu_2}(t) = \sum_{k=0}^{\infty}\sum_{n>k}\sum_{j=0}^{k} E\Big[&z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k} \\
&\times \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,\mathbf{1}_{\{k\}}(\nu_1)\,\mathbf{1}_{\{n\}}(\nu_2)\Big]. \qquad (4.2.2)
\end{aligned}$$

In our next efforts to unfold $\Phi_{\nu_1<\nu_2}$ we will apply methods of discrete and continuous operational calculus. We will use an approach previously developed by Dshalalow [20] and further explored in Dshalalow and White [32] and referred to as a *stochastic expansion*. To this end, we define two sequences of exit indices

$$\nu_1(p) = \inf\{n = 0,1,\ldots : A_n > p\}, p = 0,1,\ldots, \qquad (4.2.3)$$

and

$$\nu_2(q) = \inf\{n = 0,1,\ldots : t_n > q\},\ q \geq 0. \qquad (4.2.4)$$

Then, for any pair $(p,q) \in \mathbb{N}_0 \times \mathbb{R}_+$, let

$$\nu(p,q) = \nu_1(p) \wedge \nu_2(q). \tag{4.2.5}$$

Note that $\nu_1 = \nu_1(M-1)$ and $\nu_2 = \nu_2(T)$. Now consider the associated extended family of functionals $\left\{\Phi_{\nu_1(p)<\nu_2(q)}(t) : (p,q) \in \mathbb{N}_0 \times \mathbb{R}_+\right\}$ in the form of

$$\begin{aligned}\Phi_{\nu_1(p)<\nu_2(q)}(t) = E\Big[& z^{A(t)} e^{-i\eta P(t)} v^{A_{\nu_1(p)-1}} e^{-i\varphi P_{\nu_1(p)-1}} e^{-\psi t_{\nu_1(p)-1}} \\ & \times u^{A_{\nu_1(p)}} e^{-i\phi P_{\nu_1(p)}} e^{-\vartheta t_{\nu_1(p)}} \mathbf{1}_{[0,t_{\nu_1(p)})}(t) \, \mathbf{1}_{\{\nu_1(p)<\nu_2(q)\}}\Big]\end{aligned} \tag{4.2.6}$$

to which we apply the transform D_p and $\mathcal{LC}_q$ defined as follows.

$$f^*(x,q) = D_p\{f(p,q)\}(x,q) := \sum_{p=0}^{\infty} x^p f(p,q)(1-x), ||x|| < 1, \tag{4.2.7}$$

where f is a complex-valued analytic function with the domain $\mathbb{N}_0 \times \mathbb{R}_+$. The inverse of D_p is the so-called $\mathcal{D}$-operator previously introduced in Dshalalow [18,21]:

$$f(p,q) = \mathcal{D}_x^k f^*(x,q) = \begin{cases} \lim_{x\to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x} f^*(x,q)\right], & k \geq 0 \\ 0, & k < 0. \end{cases} \tag{4.2.8}$$

The $\mathcal{LC}$ denotes for the Laplace-Carson transform defined as

$$\mathcal{LC}_q \varphi(y) = y \int_{q=0}^{\infty} e^{-yq} \varphi(q)\, dq, \mathrm{Re} y > 0. \tag{4.2.9}$$

The Laplace-Carson transform follows the familiar pattern of the Laplace transform whose inverse is

$$\mathcal{LC}_y^{-1}(\cdot) = \mathcal{L}_y^{-1}\left[\frac{1}{y}\cdot\right], \text{with } \mathcal{L} \text{ standing for the Laplace transform.} \tag{4.2.10}$$

Together, they form the composition

$$\mathcal{LC}_q D_p := \mathcal{LC}_q \circ D_p. \tag{4.2.11}$$

Thus if

$$\hat{f}(x,y) = \mathcal{LC}_q D_p f(p,q)(x,y), \tag{4.2.12}$$

then

$$f(p,q) = \mathcal{D}_x^k \mathcal{LC}_y^{-1}\left(\hat{f}(x,y)\right)(p,q). \tag{4.2.13}$$

Note that the transforms can be applied in any order.

Now with (4.2.7-4.2.13) in mind, the functional $\Phi_{\nu_1<\nu_2}(t)$, after the (p,q)-expansion and the use of operator $\mathcal{LC}_q D_p$ turns to

$$\begin{aligned}\overset{\wedge}{\Phi}_{\nu_1<\nu_2}(t,x,y) &= \mathcal{LC}_q D_p \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{n>k} E\Big[z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}} \\ &\qquad \times u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,\mathbf{1}_{\{k\}}(\nu_1)\,\mathbf{1}_{\{n\}}(\nu_2)\Big](x,y) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{n>k} E\Big[z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k} \\ &\qquad \times \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,\mathcal{LC}_q D_p\left[\mathbf{1}_{\{k\}}(\nu_1)\,\mathbf{1}_{\{n\}}(\nu_2)\right](x,y)\Big].\end{aligned} \tag{4.2.14}$$

By Fubini's theorem and from

$$\mathcal{L}\mathcal{C}_q D_p\left[\mathbf{1}_{\{k\}}(\nu_1)\,\mathbf{1}_{\{n\}}(\nu_2)\right](x,y) = \left(x^{A_{k-1}} - x^{A_k}\right)\left(e^{-yt_{n-1}} - e^{-yt_n}\right) \tag{4.2.15}$$

(cf., Dshalalow [21]), we have from (4.2.14),

$$\overset{\wedge}{\varPhi}_{\nu_1<\nu_2}(t,x,y) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\varPhi_{jk}(t,x,y) \tag{4.2.16}$$

where

$$\varPhi_{jk}(t,x,y) = \sum_{n>k} E\Big[z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k} \times \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\left(x^{A_{k-1}} - x^{A_k}\right)\left(e^{-yt_{n-1}} - e^{-yt_n}\right)\Big]. \tag{4.2.17}$$

To further continue with $\overset{\wedge}{\varPhi}_{\nu_1<\nu_2}(t,x,y)$ we break (4.2.16-4.2.17) into four separate cases. We start with

Case (i)**.** $0 < j < k$**.**

$$\Phi_{jk}(t,x,y) = \sum_{n>k} E\Big[z^{A_{k-j-1}} e^{-i\eta P_{k-j-1}} v^{A_{k-1}} e^{-i\varphi P_{k-1}} e^{-\psi t_{k-1}} u^{A_k} e^{-i\phi P_k} e^{-\vartheta t_k} \times \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t) \left(x^{A_{k-1}} - x^{A_k}\right)\left(e^{-yt_{n-1}} - e^{-yt_n}\right)\Big]$$
$$= E\Big[\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)(uvzx)^{A_{k-j-1}} e^{-i(\varphi+\phi+\eta)P_{k-j-1}} e^{-(\vartheta+\psi+y)t_{k-j-1}} \times (uvx)^{x_{k-j}+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_{k-j}+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_{k-j}+\ldots+\Delta_{k-1})}\Big]$$
$$\times E\Big[u^{x_k} e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}(1-x^{x_k})\Big] \sum_{n>k} E\Big[e^{-y(\Delta_{k+1}+\ldots+\Delta_{n-1})}\left(1-e^{-y\Delta_n}\right)\Big]. \tag{4.2.18}$$

From (4.2.3),

$$\sum_{n>k} E\Big[e^{-y(\Delta_{k+1}+\ldots+\Delta_{n-1})}\left(1-e^{-y\Delta_n}\right)\Big] = \sum_{n>k} \gamma^{n-k-1}(1,0,y)\left[1-\gamma(1,0,y)\right] = 1. \tag{4.2.19}$$

The latter holds true, because $\|\gamma(1,0,y)\| < 1$ due to the restriction $\mathrm{Re} y > 0$ in (4.2.9) (imposed on y for different reasons). Thus, from (4.2.19),

$$\Phi_{jk}(t,x,y) = E\Big[\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)(uvzx)^{A_{k-j-1}} e^{-i(\varphi+\phi+\eta)P_{k-j-1}} \times e^{-(\vartheta+\psi+y)t_{k-j-1}} (uvx)^{x_{k-j}} e^{-i(\varphi+\phi)\pi_{k-j}} e^{-(\vartheta+\psi+y)\Delta_{k-j}}\Big]$$
$$\times E\Big[(uvx)^{x_{k-j+1}+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_{k-j+1}+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_{k-j+1}+\ldots+\Delta_{k-1})}\Big]$$
$$\times E\Big[u^{x_k} e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}(1-x^{x_k})\Big] \tag{4.2.20}$$

(such factorization is due to Δ_{k-j}'s dependence on $[t_{k-j-1}, t_{k-j})$ and x_{k-j} and π_{k-j}

dependence on Δ_{k-j}) arriving at

$$\mathit{\Phi}_{jk}\left(t,x,y\right)=\mathit{\Phi}_{jk}\left(t,x,y\right)=F_{jk}\left(t,x,y\right)f_{j}\left(x,y\right). \tag{4.2.21}$$

Here

$$\begin{aligned}F_{jk}\left(t,x,y\right)=E\Big[&\mathbf{1}_{[t_{k-j-1},t_{k-j-1}+\Delta_{k-j})}\left(t\right)\left(uvzx\right)^{A_{k-j-1}}\\&\times e^{-i(\varphi+\phi+\eta)P_{k-j-1}}e^{-(\vartheta+\psi+y)t_{k-j-1}}\left(uvx\right)^{x_{k-j}}e^{-i(\varphi+\phi)\pi_{k-j}}e^{-(\vartheta+\psi+y)\Delta_{k-j}}\Big]\end{aligned} \tag{4.2.22}$$

and, with (4.2.3),

$$f_{j}\left(x,y\right)=\gamma^{j-1}\left(uvx,\varphi+\phi,\vartheta+\psi+y\right)\left[\gamma\left(u,\phi,\vartheta+y\right)-\gamma\left(ux,\phi,\vartheta+y\right)\right]. \tag{4.2.23}$$

At this point, we turn to a further transform applied to $\overset{\wedge}{\mathit{\Phi}}_{\nu}\left(t,x,y\right)$ and subsequently to $\overset{\wedge}{\mathit{\Phi}}_{\nu_1<\nu_2}\left(t,x,y\right)$ and all pertinent terms and factors containing time t, namely, the Laplace transform with respect to t.

$$\mathit{\Phi}_{\nu}^{*}\left(\theta,x,y\right):=\int_{t=0}^{\infty}e^{-\theta t}\overset{\wedge}{\mathit{\Phi}}_{\nu}\left(t,x,y\right)dt. \tag{4.2.24}$$

Hence, from (4.2.22),

$$
\begin{aligned}
F^{*}_{jk}(\theta,x,y) &= \int_{t=0}^{\infty} e^{-\theta t} F_{kj}(t,x,y)\,dt \\
&= \sum_{a=0}^{\infty} (uvzx)^{a} \sum_{\xi=0}^{\infty} (uvx)^{\xi} \int_{p=-\infty}^{\infty} e^{-i(\varphi+\phi+\eta)p} \int_{q=-\infty}^{\infty} e^{-i(\varphi+\phi)q} \int_{s=0}^{\infty} e^{-(\vartheta+\psi+y)s} \\
&\quad \times \int_{\delta=0}^{\infty} e^{-(\vartheta+\psi+y)\delta} \left(\int_{t=s}^{s+\delta} e^{-\theta t} dt \right) \\
&\quad \times P_{A_{k-j-1}\otimes P_{k-j-1}\otimes t_{k-j-1}\otimes x_{k-j}\otimes \pi_{k-j}\otimes \Delta_{k-j}}(a,dp,ds,\xi,ds,d\delta), \qquad (4.2.25)
\end{aligned}
$$

and after straightforward calculations, with (4.1.2-4.1.3),

$$
\begin{aligned}
F^{*}_{jk}(\theta,x,y) = \frac{1}{\theta}\gamma_0 &(uvzx,\varphi+\phi+\eta,\theta+\vartheta+\psi+y) \\
&\times \gamma^{k-j-1}(uvzx,\varphi+\phi+\eta,\theta+\vartheta+\psi+y) \\
&\times \left[\gamma(uvx,\varphi+\phi,\vartheta+\psi+y) - \gamma(uvx,\varphi+\phi,\theta+\vartheta+\psi+y)\right]. \qquad (4.2.26)
\end{aligned}
$$

For now and in the sequel, it will be easier to operate under the following abbreviations:

$$\gamma := \gamma(uvzx,\varphi+\phi+\eta,\theta+\vartheta+\psi+y), \qquad (4.2.27)$$

$$\gamma_0 := \gamma_0(uvzx,\varphi+\phi+\eta,\theta+\vartheta+\psi+y), \qquad (4.2.28)$$

$$\overset{\wedge}{\gamma} := \gamma(uvx,\varphi+\phi,\vartheta+\psi+y), \overset{\wedge}{\gamma}(\theta) := \gamma(uvx,\varphi+\phi,\theta+\vartheta+\psi+y), \qquad (4.2.29)$$

$$g = \gamma(ux,\phi,\vartheta+y), g_1 = \gamma(u,\phi,\vartheta+y). \qquad (4.2.30)$$

Thus, (4.2.26) reads

$$F_{jk}^{*}(\theta,x,y)=\frac{1}{\theta}\gamma_0\gamma^{k-j-1}\left[\overset{\wedge}{\gamma}-\overset{\wedge}{\gamma}(\theta)\right] \text{ and } f_j(x,y)=\overset{\wedge}{\gamma}^{j-1}[g_1-g]. \tag{4.2.30}$$

Now summing up $F_{jk}^{*}(\theta,x,y)\,f_j(x,y)$ for all $0<j<k$ we get

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\Phi_{jk}^{*}(\theta,x,y)=\gamma_0\frac{g_1-g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma, \tag{4.2.31}$$

where

$$\Phi_{jk}^{*}(\theta,x,y):=\int_{t=0}^{\infty}e^{-\theta t}\Phi_{jk}(t,x,y)\,dt, \tag{4.2.32}$$

and

$$\Gamma\gamma:=\frac{1}{\theta}\left[\overset{\wedge}{\gamma}-\overset{\wedge}{\gamma}(\theta)\right]. \tag{4.2.33}$$

Now formula (4.2.31) is due to the convergence of associated geometric series guaranteed by

$$\|\gamma\|<1| \text{ and } \left\|\overset{\wedge}{\gamma}\right\|<1 \tag{4.2.34}$$

(cf. Dshalalow and Nandyose [27,28]).

Case (ii)**.** $j=0,k>0.$

$$\Phi_{0k}(t,x,y)=\sum_{n>k}E\Big[z^{A(t)}e^{-i\eta P(t)}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}\mathbf{1}_{[t_{k-1},t_k)}(t)$$
$$\times\left(x^{A_{k-1}}-x^{A_k}\right)\left(e^{-yt_{n-1}}-e^{-yt_n}\right)\Big]$$
$$=E\Big[z^{A_{k-1}}e^{-i\eta P_{k-1}}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}\mathbf{1}_{[t_{k-1},t_k)}(t)\left(x^{A_{k-1}}-x^{A_k}\right)$$
$$\times e^{-yt_{k-1}}e^{-y\Delta_k}\Big]\sum_{n>k}E\Big[e^{-y\Delta_{k+1}+\ldots+\Delta_{n-1}}\left(1-e^{-y\Delta_n}\right)\Big]$$
$$=E\Big[(uvzx)^{A_{k-1}}e^{-i(\varphi+\phi+\eta)P_{k-1}}e^{-(\psi+\vartheta+y)t_{k-1}}$$
$$\times u^{x_k}\left(1-x^{x_k}\right)e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}\mathbf{1}_{[t_{k-1},t_{k-1}+\Delta_k)}(t)\Big].\tag{4.2.35}$$

Then applying the Laplace transform and unfolding the expectation, from (4.2.35),

$$\Phi^*_{0k}(\theta,x,y)=\sum_{a=0}^{\infty}(uvzx)^a\sum_{\xi=0}^{\infty}\left[u^{\xi}-(ux)^{\xi}\right]\int_{p=-\infty}^{\infty}e^{-i(\varphi+\phi+\eta)p}$$
$$\times\int_{q=-\infty}^{\infty}e^{-i\phi q}\int_{s=0}^{\infty}e^{-(\vartheta+\psi+y)s}\int_{\delta=0}^{\infty}e^{-(\vartheta+y)\delta}\int_{t=s}^{s+\delta}e^{-\theta t}dt$$
$$\times P_{A_{k-1}\otimes P_{k-1}\otimes t_{k-1}\otimes x_k\otimes\pi_k\otimes\Delta_k}(a,dp,ds,\xi,ds,d\delta)=\frac{1}{\theta}\gamma_0\gamma^{k-1}\left[g_1-g-g_1(\theta)+g(\theta)\right],$$

with

$$g(\theta)=\gamma(ux,\phi,\theta+\vartheta+y)\ \text{ and }\ g_1(\theta)=\gamma(u,\phi,\theta+\vartheta+y).\tag{4.2.36}$$

Summing up $\Phi^*_{0k}(\theta, x, y)$ over all $k = 1, 2, \ldots,$ with (4.2.34) gives

$$\sum_{k=1}^{\infty} \Phi^*_{0k}(\theta, x, y) = \frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_1 - g - g_1(\theta) + g(\theta)\right]. \tag{4.2.37}$$

Case $(iii)\,.\, j = k > 0.$

$$\begin{aligned}\Phi_{kk}(t, x, y) &= \sum_{n>k} E\Big[z^{A(t)} e^{-i\eta P(t)} v^{A_{k-1}} e^{-i\varphi P_{k-1}} e^{-\psi t_{k-1}} u^{A_k} e^{-i\phi P_k} e^{-\vartheta t_k} \\ &\qquad \times \mathbf{1}_{[t_{-1},t_0)}(t)\left(x^{A_{k-1}} - x^{A_k}\right)\left(e^{-yt_{n-1}} - e^{-yt_n}\right)\Big] \\ &= E\Big[\mathbf{1}_{[t_{-1},t_0)}(t)\,(uvzx)^{A_{-1}} e^{-i(\varphi+\phi+\eta)P_{-1}} e^{-(\vartheta+\psi+y)t_{-1}} \\ &\quad \times (uvx)^{x_0+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_0+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_0+\ldots+\Delta_{k-1})}\Big] \\ &\times E\Big[u^{x_k} e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}\left(1 - x^{x_k}\right)\Big] \sum_{n>k} E\Big[e^{-y(\Delta_{k+1}+\ldots+\Delta_{n-1})}\left(1 - e^{-y\Delta_n}\right)\Big].\end{aligned} \tag{4.2.38}$$

Next with $t_{-1} = A_{-1} = P_{-1} = 0$ and from (4.2.19), $\sum_{n>k} Ee^{-y(\Delta_{k+1}+\ldots+\Delta_{n-1})}\left(1 - e^{-y\Delta_n}\right) =$ 1 implying that

$$\begin{aligned}\Phi_{kk}(t, x, y) &= E\mathbf{1}_{[0,t_0)}(t)\,(uvx)^{x_0+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_0+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_0+\ldots+\Delta_{k-1})} \\ &\qquad \times E u^{x_k} e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}\left(1 - x^{x_k}\right) \\ &= E\mathbf{1}_{[0,t_0)}(t)\,(uvx)^{A_0} e^{-i(\varphi+\phi)P_0} e^{-(\vartheta+\psi+y)t_0} \\ &\quad \times E\Big[(uvx)^{x_1+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_1+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_1+\ldots+\Delta_{k-1})} \\ &\qquad \times e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}\left(u^{x_k} - (ux)^{x_k}\right)\Big].\end{aligned} \tag{4.2.39}$$

So

$$\Phi^*_{kk}(t,x,y) = \sum_a (uvx)^a \int_{p=0}^{\infty} e^{-i(\varphi+\phi)p} \int_{s=0}^{\infty} e^{-(\vartheta+\psi+y)s} \int_0^s e^{-\theta t} dt P_{A_0\otimes P_0\otimes t_0}(a,dp,ds)$$

$$\times E\Big[(uvx)^{x_1+\ldots+x_{k-1}} e^{-i(\varphi+\phi)(\pi_1+\ldots+\pi_{k-1})} e^{-(\vartheta+\psi+y)(\Delta_1+\ldots+\Delta_{k-1})}$$

$$\times e^{-i\phi\pi_k} e^{-(\vartheta+y)\Delta_k}\left(u^{x_k}-(ux)^{x_k}\right)\Big]$$

$$= \frac{1}{\theta}\left[\gamma_0(uvx,\varphi+\phi,\theta+\vartheta+\psi+y) - \gamma_0(uvx,\varphi+\phi,\vartheta+\psi+y)\right]$$

$$\times \gamma^{k-1}(uvx,\varphi+\phi,\vartheta+\psi+y)\left[\gamma(u,\phi,\vartheta+y) - \gamma(ux,\phi,\vartheta+y)\right]$$

$$= \frac{1}{\theta}\left[\hat{\gamma}_0 - \hat{\gamma}_0(\theta)\right]\hat{\gamma}^{k-1}\left[g_1 - g\right]. \tag{4.2.40}$$

By (4.2.34) and summing up $\Phi^*_{kkn}(\theta,x,y)$ over all pertinent k's gives

$$\sum_{k=1}^{\infty} \Phi^*_{kk}(\theta,x,y) = \frac{g_1-g}{1-\hat{\gamma}}\left[\hat{\gamma}_0 - \hat{\gamma}_0(\theta)\right] = \frac{g_1-g}{1-\hat{\gamma}}\Gamma\gamma_0$$

and so we are done with case (iii).

Case (iv)**.**$j=k=0$.

$$\Phi_{00}(t,x,y) = \sum_{n>0} E\Big[z^{A(t)} e^{-i\eta P(t)} v^{A_{-1}} e^{-i\varphi P_{-1}} e^{-\psi t_{-1}} u^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[t_{-1},t_0)}(t)$$

$$\times \left(x^{A_{-1}} - x^{A_0}\right)\left(e^{-yt_{n-1}} - e^{-yt_n}\right)\Big]$$

$$= E\Big[z^{A_{-1}} e^{-i\eta P_{-1}} v^{A_{-1}} e^{-i\varphi P_{-1}} e^{-\psi t_{-1}} u^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[t_{-1},t_0)}(t)\left(x^{A_{-1}} - x^{A_0}\right)$$

$$\times e^{-yt_{-1}} e^{-y\Delta_0}\Big] \sum_{n>0} E\left[e^{-y\Delta_1+\ldots+\Delta_{n-1}}\left(1-e^{-y\Delta_n}\right)\right] \tag{4.2.41}$$

where $\Delta_0 = t_0$. Thus,

$$\Phi_{00}(t,x,y) = E\left[u^{A_0} e^{-i\phi P_0} e^{-(\vartheta+y)t_0} \mathbf{1}_{[0,t_0)}(t)\left(1-x^{A_0}\right)\right] \tag{4.2.42}$$

and

$$\Phi_{00}^*(t,x,y) = \sum_a \left(u^a - (ux)^a\right) \int_{p=0}^{\infty} e^{-i\phi p} \int_{s=0}^{\infty} e^{-(\vartheta+y)s} \int_0^s e^{-\theta t} dt P_{A_0\otimes P_0 \otimes t_0}(a,dp,ds)$$

$$= \frac{1}{\theta}\left[g_{01} - g_0 - g_{01}(\theta) + g_0(\theta)\right]. \tag{4.2.43}$$

Finally, summing up all respective components in the four cases we have

$$\Phi_{\nu_1<\nu_2}^*(\theta,x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} \Phi_{jk}^*(\theta,x,y) + \sum_{k=1}^{\infty} \Phi_{0k}^*(\theta,x,y) + \sum_{k=1}^{\infty} \Phi_{kk}^*(\theta,x,y) + \Phi_{00}^*(t,x,y)$$

$$= \gamma_0 \frac{g_1 - g}{(1-\gamma)\left(1-\hat{\gamma}\right)} \Gamma\gamma + \frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_1 - g - g_1(\theta) + g(\theta)\right]$$

$$+ \frac{g_1 - g}{1-\hat{\gamma}} \Gamma\gamma_0 + \frac{1}{\theta}\left[g_{01} - g_0 - g_{01}(\theta) + g_0(\theta)\right]. \tag{4.2.44}$$

2. Player A Defeats Player T. In the functional

$$\Phi_{\nu_2<\nu_1}(t) =$$

$$E z^{A(t)} e^{-i\eta P(t)} v^{A_{\nu_2-1}} e^{-i\varphi P_{\nu_2-1}} e^{-\psi t_{\nu_2-1}} u^{A_{\nu_2}} e^{-i\phi P_{\nu_2}} e^{-\vartheta t_{\nu_2}} \mathbf{1}_{[0,t_{\nu_2})}(t)\, \mathbf{1}_{\{\nu_2<\nu_1\}} \tag{4.2.45}$$

and its Laplace transform we need to reverse the roles of x and y, but only in those terms where the representation of x and y is asymmetric, e.g., $g(1,y)$, $g(1,y,\theta)$ and subsequently Γg and $\Gamma g(\theta)$, with

$$g_2 = g(x,0) = \gamma(ux,\phi,\vartheta),\ g_2(\theta) = g(x,0,\theta) = \gamma(ux,\phi,\theta+\vartheta),$$

thus, arriving at

$$\Phi^*_{\nu_1<\nu_2}(\theta,x,y) = \gamma_0 \frac{g_2 - g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)} \Gamma\gamma + \frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_2 - g - g_2(\theta) + g(\theta)\right]$$

$$+ \frac{g_2 - g}{1-\overset{\wedge}{\gamma}} \Gamma\gamma_0 + \frac{1}{\theta}\left[g_{02} - g_0 - g_{02}(\theta) + g_0(\theta)\right]. \tag{4.2.46}$$

3. Either player is ruined upon the first passage time. Here we have

$$\Phi_{\nu_1=\nu_2}(t)$$
$$= E z^{A(t)} e^{-i\eta P(t)} v^{A_{\nu_1-1}} e^{-i\varphi P_{\nu_1-1}} e^{-\psi t_{\nu_1-1}} u^{A_{\nu_1}} e^{-i\phi P_{\nu_1}} e^{-\vartheta t_{\nu_1}} \mathbf{1}_{[0,t_{\nu_1})}(t)\, \mathbf{1}_{\{\nu_1=\nu_2\}} \tag{4.2.47}$$

and

$$\overset{\wedge}{\Phi}_{\nu_1=\nu_2}(t,x,y) = D_p \mathcal{L}\mathcal{C}_q \Phi_{\nu_2=\nu_1}(t) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \Phi_{jk}(t,x,y),$$

where

$$\Phi_{jk}(t,x,y) = E\Big[z^{A(t)} e^{-i\eta P(t)} v^{A_{k-1}} e^{-i\varphi P_{k-1}} e^{-\psi t_{k-1}} u^{A_k} e^{-i\phi P_k} e^{-\vartheta t_k}$$

$$\times \mathbf{1}_{[t_{k-j-1},t_{k-j})}(t) \left(x^{A_{k-1}} - x^{A_k}\right)\left(e^{-y t_{k-1}} - e^{-y t_k}\right)\Big]. \tag{4.2.48}$$

Then we break (4.2.48) into four cases below.

Case (*i*). $0 < j < k$. Here,

$$\Phi_{jk}(t,x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} E\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)$$
$$\times (uvzx)^{A_{k-j-1}}(uvx)^{x_{k-j}}(uvx)^{x_{k-j+1}+\ldots+x_{k-1}}u^{x_k}\left(1-x^{x_k}\right)$$
$$\times e^{-i(\varphi+\phi+\eta)P_{k-j-1}}e^{-i(\varphi+\phi)\pi_{k-j}}e^{-i(\varphi+\phi)(\pi_{k-j+1}+\ldots+\pi_{k-1})}e^{-i\phi\pi_k}$$
$$\times e^{-(\vartheta+\psi+y)t_{k-j-1}}e^{-(\vartheta+\psi+y)(\Delta_{k-j}+\ldots+\Delta_{k-1})}e^{-\vartheta\Delta_k}\left(1-e^{-y\Delta_k}\right)$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} F_{jk}(t,x,y)\, f_j(x,y) \tag{4.2.49}$$

where

$$F_{jk}(t,x,y) = E\Bigg[\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,(uvzx)^{A_{k-j-1}}e^{-i(\varphi+\phi+\eta)P_{k-j-1}}e^{-(\vartheta+\psi+y)t_{k-j-1}}$$
$$\times(uvx)^{x_{k-j}}e^{-i(\varphi+\phi)\pi_{k-j}}e^{-(\vartheta+\psi+y)\Delta_{k-j}}\Bigg] \tag{4.2.50}$$

$$f_j(x,y) = E(uvx)^{x_{k-j+1}+\ldots+x_{k-1}}e^{-i(\varphi+\phi)(\pi_{k-j+1}+\ldots+\pi_{k-1})}e^{-(\vartheta+\psi+y)(\Delta_{k-j+1}+\ldots+\Delta_{k-1})}$$
$$\times Eu^{x_k}\left(1-x^{x_k}\right)e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}\left(1-e^{-y\Delta_k}\right)$$
$$= \overset{\wedge}{\gamma}^{j-1}\left[g(1,0)-g(1,y)-g(x,0)+g(x,y)\right]. \tag{4.2.51}$$

$F^{*}_{jk}(\theta,x,y)$ can be copied from Case (i), equation (4.2.31), of $\Phi_{\nu_1<\nu_2}$ as

$$F^{*}_{jk}(\theta,x,y) = \frac{1}{\theta}\gamma_0\gamma^{k-j-1}\left[\overset{\wedge}{\gamma}-\overset{\wedge}{\gamma}(\theta)\right] \tag{4.2.52}$$

to get

$$\Phi_{jk}^{*}(\theta,x,y)=\frac{1}{\theta}\gamma_0\gamma^{k-j-1}\left[\overset{\wedge}{\gamma}-\overset{\wedge}{\gamma}(\theta)\right]\overset{\wedge}{\gamma}^{j-1}\left[g(1,0)-g(1,y)-g(x,0)+g(x,y)\right] \tag{4.2.53}$$

yielding

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\Phi_{jk}^{*}(\theta,x,y)=\gamma_0\frac{g_{12}-g_1-g_2+g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma. \tag{4.2.54}$$

Case (ii). **With** $j=0,\ k>0.$

$$\Phi_{0k}(t,x,y)=Ez^{A_{k-1}}e^{-i\eta P_{k-1}}v^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}$$

$$\times\mathbf{1}_{[t_{k-1},t_k)}(t)\left(x^{A_{k-1}}-x^{A_k}\right)\left(e^{-yt_{k-1}}-e^{-yt_k}\right)$$

$$=\Phi_{0k}^{(1)}(t,x,y)-\Phi_{0k}^{(2)}(t,x,y), \tag{4.2.55}$$

where

$$\Phi_{0k}^{(1)}(t,x,y)=E\Bigg[\mathbf{1}_{[t_{k-1},t_k)}(t)(uvzx)^{A_{k-1}}e^{-i(\eta+\varphi+\phi)P_{k-1}}e^{-(\psi+\vartheta+y)t_{k-1}}$$

$$\times\left(u^{x_k}e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}-(ux)^{x_k}e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}\right)\Bigg] \tag{4.2.56}$$

and

$$\Phi_{0k}^{(2)}(t,x,y)=E\Bigg[\mathbf{1}_{[t_{k-1},t_k)}(t)(uvzx)^{A_{k-1}}e^{-i(\eta+\varphi+\phi)P_{k-1}}e^{-(\psi+\vartheta+y)t_{k-1}}$$

$$\times\left(u^{x_k}e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}-(ux)^{x_k}e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}\right)\Bigg]. \tag{4.2.57}$$

Here $\Phi_{0k}^{(2)}(t,x,y)$ is identical to that of Case (ii) of $\Phi_{\nu_1<\nu_2}(t)$ implying that

$$\Phi_{0k}^{*(2)}(\theta,x,y)=\gamma_0\gamma^{k-1}\left[\Gamma g_1-\Gamma g_1(\theta)\right]=\frac{1}{\theta}\left[g_1-g-g_1(\theta)+g(\theta)\right], \qquad (4.2.58)$$

whereas

$$\Phi_{0k}^{*(1)}(t,x,y)=\sum_{a=0}^{\infty}(uvzx)^a\sum_{\xi=0}^{\infty}\left[u^{\xi}-(ux)^{\xi}\right]\int_{p=-\infty}^{\infty}e^{-i(\varphi+\phi+\eta)p}$$

$$\times\int_{q=-\infty}^{\infty}e^{-i\phi q}\int_{s=0}^{\infty}e^{-(\vartheta+\psi+y)s}\int_{\delta=0}^{\infty}e^{-\vartheta\delta}\int_{t=s}^{s+\delta}e^{-\theta t}dt$$

$$\times P_{A\otimes P\otimes t_{k-j-1}\otimes x_{k-j}\otimes\pi\otimes\Delta}(a,dp,ds,\xi,ds,d\delta)$$

$$=\frac{1}{\theta}\gamma_0\gamma^{k-1}\left[g_{12}-g_2-g_{12}(\theta)+g_2(\theta)\right].(4.2.59)$$

Hence

$$\Phi_{0k}^{*}(t,x,y)=\frac{1}{\theta}\gamma_0\gamma^{k-1}\left[g_{12}-g_2-g_{12}(\theta)+g_2(\theta)-g_1+g+g_1(\theta)-g(\theta)\right] \qquad (4.2.60)$$

and from (4.2.55),

$$\sum_{k=1}^{\infty}\Phi_{0k}^{*}(\theta,x,y)=\frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_{12}-g_2-g_{12}(\theta)+g_2(\theta)-g_1+g+g_1(\theta)-g(\theta)\right]. \qquad (4.2.61)$$

Case $(\boldsymbol{iii})$. $j=k>0$. Here we have

$$\Phi_{kk}(t,x,y)$$

$$=Ev^{A_{k-1}}e^{-i\varphi P_{k-1}}e^{-\psi t_{k-1}}u^{A_k}e^{-i\phi P_k}e^{-\vartheta t_k}\mathbf{1}_{[0,t_0)}(t)\left(x^{A_{k-1}}-x^{A_k}\right)\left(e^{-yt_{k-1}}-e^{-yt_k}\right) \qquad (4.2.62)$$

$$= E\mathbf{1}_{[0,t_0)}(t)(uvx)^{A_0}e^{-i(\varphi+\phi)P_0}e^{-(\vartheta+\psi+y)t_0}$$

$$\times E(uvx)^{x_1+\ldots+x_{k-1}}e^{-i(\varphi+\phi)(\pi_1+\ldots+\pi_{k-1})}e^{-(\vartheta+\psi+y)(\Delta_1+\ldots+\Delta_{k-1})}$$

$$\times E\left(u^{x_k}-(ux)^{x_k}\right)e^{-i\phi\pi_k}\left(e^{-\vartheta\Delta_k}-e^{-(\vartheta+y)\Delta_k}\right)$$

$$= F_{kk}(t,x,y)\, f_k(x,y), \tag{4.2.63}$$

where

$$F_{kk}(t,x,y) = E\mathbf{1}_{[0,t_0)}(t)(uvx)^{A_0}e^{-i(\varphi+\phi)P_0}e^{-(\vartheta+\psi+y)t_0} \tag{4.2.64}$$

and

$$f_k(x,y) = E(uvx)^{x_1+\ldots+x_{k-1}}e^{-i(\varphi+\phi)(\pi_1+\ldots+\pi_{k-1})}e^{-(\vartheta+\psi+y)(\Delta_1+\ldots+\Delta_{k-1})}$$

$$\times E\left(u^{x_k}e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}-u^{x_k}e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}\right.$$

$$\left.-(ux)^{x_k}e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}+(ux)^{x_k}e^{-i\phi\pi_k}e^{-(\vartheta+y)\Delta_k}\right)$$

$$= \overset{\wedge}{\gamma}^{k-1}\left[g(1,0)-g(1,y)-g(x,0)+g(x,y)\right], \tag{4.2.65}$$

so that

$$F^*_{kk}(\theta,x,y) = \mathit{\Gamma}\gamma_0. \tag{4.2.66}$$

Thus, due to the validity of (4.2.34),

$$\sum_{k=1}^{\infty}\mathit{\Phi}^*_{0k}(\theta,x,y) = \frac{g_{12}-g_1-g_2+g}{1-\overset{\wedge}{\gamma}}\mathit{\Gamma}\gamma_0 \tag{4.2.67}$$

and so we are done with Case (iii).

Case $(iv)\,.\, j = k = 0$**.**

$$\Phi_{00}\left(t,x,y\right) = Eu^{A_0}e^{-i\phi P_0}e^{-\vartheta t_0}\mathbf{1}_{[0,t_0)}\left(t\right)\left(1-x^{A_0}\right)\left(1-e^{-yt_0}\right)$$

$$= Ee^{-i\phi P_0}\mathbf{1}_{[0,t_0)}\left(t\right)\left(u^{A_0}-\left(ux\right)^{A_0}\right)\left(e^{-\vartheta t_0}-e^{-(\vartheta+y)t_0}\right) \tag{4.2.68}$$

$$= E\mathbf{1}_{[0,t_0)}\left[t\right)\left(u^{A_0}e^{-i\phi P_0}e^{-\vartheta t_0}-\left(ux\right)^{A_0}e^{-i\phi P_0}e^{-\vartheta t_0}\right.$$

$$\left.+u^{A_0}e^{-i\phi P_0}e^{-(\vartheta+y)t_0}-\left(ux\right)^{A_0}e^{-i\phi P_0}e^{-(\vartheta+y)t_0}\right]$$

and

$$\Phi_{00}^{*}\left(\theta,x,y\right) = \Phi_{00}^{*(1)}\left(\theta,x,y\right) - \Phi_{00}^{*(2)}\left(\theta,x,y\right), \tag{4.2.69}$$

with $\Phi_{00}^{*(2)}\left(\theta,x,y\right)$ exactly the same as of Case (iv) under $\Phi_{\nu_2<\nu_1}$, that is

$$\Phi_{00}^{*(2)}\left(\theta,x,y\right) = \frac{1}{\theta}\left[g_{01}-g_0-g_{01}\left(\theta\right)+g_0\left(\theta\right)\right]. \tag{4.2.70}$$

Unlike $\Phi_{00}^{(2)}\left(t,x,y\right)$, the functional $\Phi_{00}^{(1)}\left(t,x,y\right)$ has $y=0$ in its third component.Thus

$$\Phi_{00}^{*(1)}\left(\theta,x,y\right) = \frac{1}{\theta}\left[g_{012}-g_{02}-g_{012}\left(\theta\right)+g_{02}\left(\theta\right)\right] \tag{4.2.71}$$

implying that

$$\Phi_{00}^{*}\left(\theta,x,y\right) = \Phi_{00}^{*(1)}\left(\theta,x,y\right) - \Phi_{00}^{*(2)}\left(\theta,x,y\right)$$

$$= \frac{1}{\theta}\left[g_{012}-g_{02}-g_{012}\left(\theta\right)+g_{02}\left(\theta\right)-g_{01}+g_0+g_{01}\left(\theta\right)-g_0\left(\theta\right)\right] \tag{4.2.72}$$

and completing Case(iv). Now summing up Φ^{*}'s from all four cases and under provisions of (4.2.34),gives

$$\Phi^*_{\nu_1=\nu_2}(\theta,x,y) = \gamma_0 \frac{g_{12}-g_1-g_2+g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma + \frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_{12}-g_2-g_{12}(\theta)+g_2(\theta)-g_1+g+g_1(\theta)-g(\theta)\right] + \frac{g_{12}-g_1-g_2+g}{1-\overset{\wedge}{\gamma}}\Gamma\gamma_0 + \frac{1}{\theta}\left[g_{012}-g_{02}-g_{012}(\theta)+g_{02}(\theta)-g_{01}+g_0+g_{01}(\theta)-g_0(\theta)\right]. \quad (4.2.73)$$

Finally, the transformation of the main functional Φ_ν reads,

$$\begin{aligned}
\overset{\wedge}{\Phi}{}^*(\theta,x,y) &= \int_{t=0}^{\infty} e^{-\theta t} D_p \mathcal{L}\mathcal{C}_q \Big[E z^{A(t)} e^{-i\eta P(t)} v^{A_{\nu-1}} e^{-i\varphi P_{\nu-1}} e^{-\psi t_{\nu-1}} \\
&\quad \times u^{A_\nu} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \Big](x,y)\,dt \\
&= \overset{\wedge}{\Phi}{}^*{}_{\nu_1<\nu_2}(\theta,x,y) + \overset{\wedge}{\Phi}{}^*{}_{\nu_1>\nu_2}(\theta,x,y) + \overset{\wedge}{\Phi}{}^*{}_{\nu_1=\nu_2}(\theta,x,y) \\
&= \gamma_0 \frac{g_1-g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma + \gamma_0 \frac{\Gamma g_1 - \Gamma g_1(\theta)}{1-\gamma} + \frac{g_1-g}{1-\overset{\wedge}{\gamma}}\Gamma\gamma_0 + \Gamma g_{01} - \Gamma g_{01}(\theta) \\
&\quad + \gamma_0 \frac{g_2-g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma + \gamma_0 \frac{\Gamma g_2 - \Gamma g_2(\theta)}{1-\gamma} \\
&\quad + \frac{g_2-g}{1-\overset{\wedge}{\gamma}}\Gamma\gamma_0 + \Gamma g_{02} - \Gamma g_{02}(\theta) + \gamma_0 \frac{g_{12}-g_1-g_2+g}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)}\Gamma\gamma \\
&\quad + \frac{\gamma_0}{1-\gamma}\frac{1}{\theta}\left[g_{12}-g_2-g_{12}(\theta)+g_2(\theta)-g_1+g+g_1(\theta)-g(\theta)\right] \\
&\quad + \frac{g_{12}-g_1-g_2+g}{1-\overset{\wedge}{\gamma}}\Gamma\gamma_0 \\
&\quad + \frac{1}{\theta}\left[g_{012}-g_{02}-g_{012}(\theta)+g_{02}(\theta)-g_{01}+g_0+g_{01}(\theta)-g_0(\theta)\right]
\end{aligned}$$

and after straightforward algebra is being reduced to

$$= \frac{\gamma_0}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)} \mathit{\Gamma}\gamma \mathit{\Gamma} g_{12}\theta + \frac{\gamma_0}{1-\gamma}\left[\mathit{\Gamma} g_{12} - \mathit{\Gamma} g_{12}(\theta)\right] + \frac{\mathit{\Gamma}\gamma_0}{1-\overset{\wedge}{\gamma}} \mathit{\Gamma} g_{12}\theta + \mathit{\Gamma} g_{012} - \mathit{\Gamma} g_{012}(\theta).$$

The above results can be summarized in the following theorem.

Theorem 4.2.1. *In a stochastic game of two active hostile players A and T (where T is the point process $\{t_0, t_1, \ldots\}$) and one passive player P, the transform*

$$\overset{\wedge}{\mathit{\Phi}}{}^{*}(\theta, x, y) = \int_{t=0}^{\infty} e^{-\theta t} D_p \mathcal{L}\mathcal{C}_q \Big[E z^{A(t)} e^{-i\eta P(t)} v^{A_{\nu-1}} e^{-i\varphi P_{\nu-1}} \times e^{-\psi t_{\nu-1}} u^{A_\nu} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \Big](x,y)\, dt \quad (4.2.74)$$

of the joint functional $\mathit{\Phi}_\nu$ (4.1.8) describing the evolution of the game at any time t from its inception to the end at t_ν (the first passage time) satisfies the following formulas:

$$\overset{\wedge}{\mathit{\Phi}}{}^{*}(\theta, x, y) = \frac{\gamma_0}{(1-\gamma)\left(1-\overset{\wedge}{\gamma}\right)} \mathit{\Gamma}\gamma \mathit{\Gamma} g_{12}\theta + \frac{\gamma_0}{1-\gamma}\left[\mathit{\Gamma} g_{12} - \mathit{\Gamma} g_{12}(\theta)\right] + \frac{\mathit{\Gamma}\gamma_0}{1-\overset{\wedge}{\gamma}} \mathit{\Gamma} g_{12}\theta + \mathit{\Gamma} g_{012} - \mathit{\Gamma} g_{012}(\theta). \quad (4.2.75)$$

where

$$\mathit{\Gamma}\gamma = \frac{1}{\theta}\left[\gamma\left(uvx, \varphi+\phi, \vartheta+\psi+y\right) - \gamma\left(uvx, \varphi+\phi, \theta+\vartheta+\psi+y\right)\right], \qquad (4.2.76)$$

$$\mathit{\Gamma}g_{12}\left(\theta\right) = \frac{1}{\theta}\left[g_{12}\left(\theta\right) - g\left(\theta\right)\right] = \frac{1}{\theta}\left[\gamma\left(u, \phi, \theta+\vartheta+y\right) - \gamma\left(ux, \phi, \theta+\vartheta+y\right)\right], \qquad (4.2.77)$$

$$\mathit{\Gamma}g_{12} = \frac{1}{\theta}\left[g_{12} - g\right] = \frac{1}{\theta}\left[\gamma\left(u, \phi, \vartheta\right) - \gamma\left(ux, \phi, \vartheta+y\right)\right], \qquad (4.2.78)$$

$$\mathit{\Gamma}g_{012}\left(\theta\right) = \frac{1}{\theta}\left[g_{012}\left(\theta\right) - g\left(\theta\right)\right] = \frac{1}{\theta}\left[\gamma_0\left(u, \phi, \theta+\vartheta+y\right) - \gamma_0\left(ux, \phi, \theta+\vartheta+y\right)\right], \qquad ((4.2.79))$$

$$\mathit{\Gamma}g_{012} = \frac{1}{\theta}\left[g_{012} - g\right] = \frac{1}{\theta}\left[\gamma_0\left(u, \phi, \vartheta+y\right) - \gamma_0\left(ux, \phi, \vartheta+y\right)\right], \qquad (4.2.80)$$

$$\overset{\wedge}{\gamma} = \gamma\left(uvx, \varphi+\phi, \vartheta+\psi+y\right), \qquad (4.2.81)$$

$$\gamma = \gamma\left(uvzx, \varphi+\phi+\eta, \theta+\vartheta+\psi+y\right), \qquad (4.2.82)$$

$$\gamma_0 = \gamma_0\left(uvzx, \varphi+\phi+\eta, \theta+\vartheta+\psi+y\right). \qquad (4.2.83)$$

□

Theorem 4.2.1 provides us with a closed-form expression in terms of the triple transform $\overset{\wedge}{\mathit{\Phi}}{}^{*}$ of functional $\mathit{\Phi}_{\nu}$. The inversion of $\overset{\wedge}{\mathit{\Phi}}{}^{*}$ is our main objective of the next section that we combine with applications.

4.3 Applications To Option Trading

For an illustration, consider the following special cases. Suppose that we observe a constantly fluctuating stock price of some company over the times $t_0 = 0, t_1, t_2, \ldots$ that starts off at time zero with a price of π_0 which is assumed to be a.s. a constant r.v. equal to p_0. Without making any further mention, we always refer to Theorem 4.2.1 under special assumptions.

Case 1. Observation of process $\boldsymbol{P(t)}$ when Player A's threshold is $\boldsymbol{M = 1}$.

1a. Suppose we are interested in the characteristics of the process around the period when the stock price drops for the first time or when maturity T is reached, whichever comes first. Because the stock prices cannot be modeled by a monotone process, we have the observed prices upon t's as the *passive component*, and introduce the *active component*

$$X_n = \begin{cases} 0, & \pi_n \geq 0 \\ 1, & \pi_n < 0 \end{cases}. \tag{4.3.1}$$

Our next assumption is that the marginal distribution of π_1 is Laplace with parameter μ and zero shift[1]. Thus the pdf of π_1 is

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x|}, x \in \mathbb{R}. \tag{4.3.2}$$

Also, suppose the $\Delta_n \in [\mathrm{Exp}(\lambda)]$, $n = 1, 2, 3 \ldots$. Then, with $\gamma_0(u, \phi, \vartheta) = Eu^{A_0}e^{-i\phi P_0}e^{-\vartheta t_0} = e^{-i\phi p_0}$, when $P_0 = \pi_0 = p_0$ a.s. constant, and $A_0 = t_0 = 0$ a.s. we have,

$$\gamma_0 = \gamma_0(uvzx, \varphi + \phi + \eta, \theta + \vartheta + \psi + y) = e^{-i(\varphi+\phi+\eta)p_0} \tag{4.3.3}$$

$$\mathit{\Gamma}\gamma_0 = \mathit{\Gamma} g_{012} = \mathit{\Gamma} g_{012}(\theta) = 0. \tag{4.3.4}$$

Hence

$$\mathit{\Phi}^*(\theta, x, y) = \frac{e^{-i(\varphi+\phi+\eta)p_0}}{(1-\gamma)\left(1-\hat{\gamma}\right)}\mathit{\Gamma}\gamma\mathit{\Gamma} g_{12}\theta + \frac{e^{-i(\varphi+\phi+\eta)p_0}}{1-\gamma}\left[\mathit{\Gamma} g_{12} - \mathit{\Gamma} g_{12}(\theta)\right] \tag{4.3.5}$$

implying that

$$e^{i(\varphi+\phi+\eta)p_0}\theta\mathit{\Phi}^*(\theta, x, y) = \int_{t=0}^{\infty} e^{-\theta t} D_p \mathcal{L}\mathcal{C}_q \Big[E z^{A(t)}$$

$$\times e^{-i\eta P(t)} v^{A_{\nu-1}} e^{-i\varphi P_{\nu-1}} e^{-\psi t_{\nu-1}} u^{A_\nu} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \Big]_{\nu_1(p),\nu_2(q)} (x, y)\, dt$$

---

[1]We note that this assumption is impractical, because it can send a stock to a negative territory. However, we do it for illustration purposes only. A more reasonable assumption would be a Laplace distribution with a non-zero shift.

$$= \left[[1-\gamma(uvzx, \varphi+\phi+\eta, \theta+\vartheta+\psi+y)]\,[1-\gamma(uvx, \varphi+\phi, \vartheta+\psi+y)] \right]^{-1}$$

$$\times [\gamma(uvx, \varphi+\phi, \vartheta+\psi+y) - \gamma(uvx, \varphi+\phi, \theta+\vartheta+\psi+y)]$$

$$[\gamma(u, \phi, \vartheta) - \gamma(ux, \phi, \vartheta+y)] + [1-\gamma(uvzx, \varphi+\phi+\eta, \theta+\vartheta+\psi+y)]^{-1}$$

$$\times \left[\gamma(u, \phi, \vartheta) - \gamma(ux, \phi, \vartheta+y) - \gamma(u, \phi, \theta+\vartheta) + \gamma(ux, \phi, \theta+\vartheta+y) \right]. \quad (4.3.6)$$

Next, with $z = v = u = 1$,

$$e^{i(\varphi+\phi+\eta)p_0}\theta\Phi^*(\theta, x, y)$$

$$= e^{i(\varphi+\phi+\eta)p_0}\theta \int_{t=0}^{\infty} e^{-\theta t} D_p \mathcal{L}\mathcal{C}_q \left[E e^{-i\eta P(t)} e^{-i\varphi P_{\nu-1}} e^{-\psi t_{\nu-1}} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \right](x, y)\, dt$$

$$= \left[[1-\gamma(x, \varphi+\phi+\eta, \theta+\vartheta+\psi+y)]\,[1-\gamma(x, \varphi+\phi, \vartheta+\psi+y)] \right]^{-1}$$

$$\times [\gamma(x, \varphi+\phi, \vartheta+\psi+y) - \gamma(x, \varphi+\phi, \theta+\vartheta+\psi+y)]\,[\gamma(1, \phi, \vartheta) - \gamma(x, \phi, \vartheta+y)]$$

$$+ [1-\gamma(x, \varphi+\phi+\eta, \theta+\vartheta+\psi+y)]^{-1}$$

$$\times \left[\gamma(1, \phi, \vartheta) - \gamma(x, \phi, \vartheta+y) - \gamma(1, \phi, \theta+\vartheta) + \gamma(x, \phi, \theta+\vartheta+y) \right]. \quad (4.3.7))$$

Furthermore with $\varphi = \psi = \phi = \vartheta = 0$,

$$e^{i\eta p_0}\theta\Phi^*(\theta, x, y) = e^{i\eta p_0}\theta \int_{t=0}^{\infty} e^{-\theta t} D_p \mathcal{L}\mathcal{C}_q \left[E e^{-i\eta P(t)} \mathbf{1}_{[0,t_\nu)}(t) \right](x, y)\, dt$$

$$= \left[[1-\gamma(x, \eta, \theta+y)]\,[1-\gamma(x, 0, y)] \right]^{-1}$$

$$\times \frac{1}{\theta} [\gamma(x, 0, y) - \gamma(x, 0, \theta+y)]\,[\gamma(1, 0, 0) - \gamma(x, 0, y)]$$

$$+[1-\gamma(x,\eta,\theta+y)]^{-1}\frac{1}{\theta}\Bigg[\gamma(1,0,0)-\gamma(x,0,y)-\gamma(1,0,\theta)+\gamma(x,0,\theta+y)\Bigg]. \quad (4.3.8)$$

Assuming position independent marking,

$$\gamma(u,\phi,\vartheta)=Eu^{x_1}e^{-i\phi\pi_1}e^{-\vartheta\Delta_1}=Eu^{x_1}e^{-i\phi\pi_1}\cdot Ee^{-\vartheta\Delta_1},$$

where $Ee^{-\vartheta\Delta_1}=\frac{\lambda}{\lambda+\vartheta}$, since $\Delta_1\in[\mathrm{Exp}(\lambda)]$. Then, under $M=1$,

$$\mathcal{D}_x^0 e^{i\eta p_0}\theta\varPhi^*(\theta,x,y)=e^{i\eta p_0}\theta\varPhi^*(\theta,0,y)=\Bigg[[1-\gamma(0,\eta,\theta+y)][1-\gamma(0,0,y)]\Bigg]^{-1}$$

$$\times[\gamma(0,0,y)-\gamma(0,0,\theta+y)][\gamma(1,0,0)-\gamma(0,0,y)]$$

$$+[1-\gamma(0,\eta,\theta+y)]^{-1}\Bigg[\gamma(1,0,0)-\gamma(0,0,y)-\gamma(1,0,\theta)+\gamma(0,0,\theta+y)\Bigg]. \quad (4.3.9)$$

Now

$$\gamma(0,\phi,\vartheta)=Eu^{x_1}e^{-i\phi\pi_1}\big|_{u=0}Ee^{-\Delta_1\vartheta}=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}\frac{\lambda}{\lambda+\vartheta}. \quad (4.3.10)$$

Due to our assumption on the pdf of π_1,

$$\gamma(0,\phi)=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}=\int_{x=0}^{\infty}e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx=\frac{1}{2}\frac{\mu}{\mu+i\phi};\ \gamma(0,0)=\frac{1}{2}. \quad (4.3.11)$$

Because $Ee^{-i\phi\pi_1}=Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq 0\}}+\mathbf{1}_{\{\pi_1<0\}}\right)$,we have

$$Ee^{-i\phi\pi_1}=Eu^{x_1}e^{-i\phi\pi_1}\big|_{u=1}Ee^{-\Delta_1\vartheta}_{\vartheta=0}=\gamma(1,\phi,0)$$

$$=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\int_{x=-\infty}^{0}e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\frac{1}{2}\frac{\mu}{\mu-i\phi}=\frac{1}{2}\mu\frac{2\mu}{\mu^2+\phi^2}=\frac{\mu^2}{\mu^2+\phi^2}. \quad (4.3.12)$$

Thus,

$$\gamma(1,\phi,\vartheta)=Ee^{-i\phi\pi_1}Ee^{-\Delta_1\vartheta}=Ee^{-i\phi\pi_1}\frac{\lambda}{\lambda+\vartheta}=\frac{\mu^2}{\mu^2+\phi^2}\frac{\lambda}{\lambda+\vartheta} \tag{4.3.13}$$

and

$$\gamma\left(1,0,\vartheta\right)=\frac{\lambda}{\lambda+\vartheta}. \tag{4.3.14}$$

implying that

$$e^{i\eta p_0}\varPhi^*\left(\theta,0,y\right)=e^{i\eta p_0}\int_{t=0}^{\infty}e^{-\theta t}D_p\mathcal{LC}_q\left[Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}\left(t\right)\right]\left(x,y\right)dt$$

$$=\left[1-\frac{1}{2}\frac{\mu}{\mu+i\eta}\frac{\lambda}{\lambda+\theta+y}\right]^{-1}\frac{1}{\lambda+\theta}. \tag{4.3.15}$$

with notation

$$A=\frac{1}{2}\frac{\lambda\mu}{\mu+i\eta}\ \text{ and } B=\lambda-A, \tag{4.3.16}$$

we have (4.3.15) in the form

$$e^{i\eta p_0}\varPhi^*\left(\theta,0,y\right)=\frac{1}{\lambda+\theta}+A\frac{1}{\lambda+\theta}\frac{1}{\theta+y+B}. \tag{4.3.17}$$

Due to the relation between the Laplace-Carson and Laplace transforms we will seek the Laplace inverse of the function

$$F\left(y,\theta\right)=\frac{1}{\lambda+\theta}\frac{1}{y}+A\frac{1}{\lambda+\theta}\frac{1}{\theta+y+B}\frac{1}{y} \qquad (4.3.18))$$

and then its Laplace inverse with respect to θ or the other way around which is about the same complexity. With

$$\mathcal{L}_y^{-1}\left[F\left(y,\theta\right)\right]\left(T\right)=\frac{1}{\lambda+\theta}+A\frac{1}{\lambda+\theta}\frac{1}{B+\theta}-A\frac{1}{\lambda+\theta}\frac{1}{B+\theta}e^{-BT}e^{-\theta T}\qquad(4.3.19)$$

and

$$\mathcal{L}_\theta^{-1}\left\{\mathcal{L}_y^{-1}\left[F\left(y,\theta\right)\right]\left(T\right)\right\}\left(t\right)$$

$$=\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\right)\left(t\right)+A\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\frac{1}{B+\theta}\right)\left(t\right)-Ae^{-BT}\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\frac{1}{B+\theta}e^{-\theta T}\right)\left(t\right)$$

$$=e^{-\lambda t}+\frac{A}{B-\lambda}e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{(\lambda-B)T}\right]-\frac{A}{B-\lambda}e^{-Bt}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)\right],\quad(4.3.20)$$

we arrive at

$$e^{i\eta p_0}\mathcal{L}_\theta^{-1}\mathcal{L}_y^{-1}\frac{1}{y}\Phi^*\left(\theta,0,y\right)\left(t,T\right)$$
$$=e^{-\lambda t}+\frac{A}{B-\lambda}e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{(\lambda-B)T}\right]-\frac{A}{B-\lambda}e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right)\quad(4.3.21)$$

and

$$Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}\left(t\right)$$
$$=e^{-i\eta p_0}\left[e^{-\lambda t}+\frac{A}{B-\lambda}e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{(\lambda-B)T}\right]-\frac{A}{B-\lambda}e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right)\right],$$
$$(4.3.22)$$

where (4.3.16) implies that $\frac{A}{B-\lambda}=-1$. So

$$Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}\left(t\right)=e^{-i\eta p_0}\left[e^{-\lambda t}-e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{AT}\right]+e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right)\right]$$

$$=e^{-\lambda t}e^{-i\eta p_0}\left[e^{At}\mathbf{1}_{[0,T]}\left(t\right)+e^{AT}\mathbf{1}_{(T,\infty)}\left(t\right)\right].\qquad(4.3.23)$$

In particular,

$$E\mathbf{1}_{[0,t_\nu)}(t) = P\{t_\nu > t\} = e^{-\lambda t}\left[e^{\lambda t/2}\mathbf{1}_{[0,T]}(t) + e^{\lambda T/2}\mathbf{1}_{(T,\infty)}(t)\right]. \tag{4.3.24}$$

The analytical result agrees with that obtained by simulation for the cumulative probability distribution function $P\{t_\nu \leq t\}$,

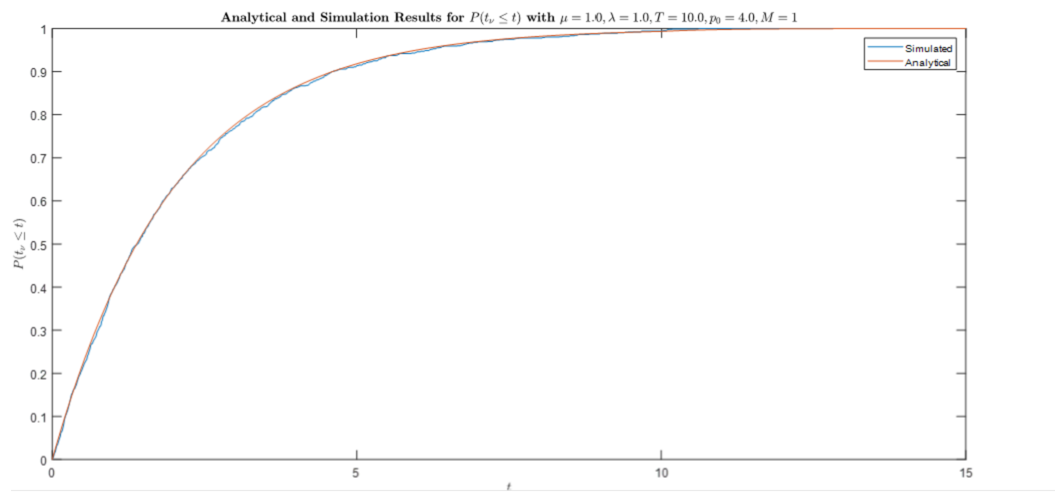

Figure 4.1: Analytical and simulated $P(t_\nu \leq t), \mu = 1, \lambda = 1, T = 10, p_0 = 4, M = 1$.

in particular, for $\mu = 1$, $T = 10$, and $p_0 = 4$ as seen from the above plots.

Returning to (4.3.23),

$$Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}(t) = e^{-\lambda t}\left[g(t)\,\mathbf{1}_{[0,T]}(t) + g(T)\,\mathbf{1}_{(T,\infty)}(t)\right], \tag{4.3.25}$$

where

$$g(\eta, t) = e^{\frac{1}{2}\frac{\lambda\mu}{\mu+i\eta}t - i\eta p_0}. \tag{4.3.26}$$

Hence we have that,

$$\frac{\partial}{\partial \eta} g(\eta, t) = g(\eta, t)\left[-i\frac{1}{2}\lambda\mu t\frac{1}{(\mu+i\eta)^2} - ip_0\right]. \tag{4.3.27}$$

and

$$i\frac{\partial}{\partial \eta} g(\eta, t)_{\eta=0} = e^{\frac{1}{2}\lambda t}\left(\frac{1}{2\mu}\lambda t + p_0\right) \tag{4.3.28}$$

implying that the expected stock price at any time t confined on the interval $[0, t_\nu)$ is

$$EP(t)\mathbf{1}_{[0,t_\nu)}(t) = i\frac{\partial}{\partial \eta} Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}(t)_{\eta=0}$$

$$= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t}\left(\frac{1}{2\mu}\lambda t + p_0\right)\mathbf{1}_{[0,T]}(t) + e^{\frac{1}{2}\lambda T}\left(\frac{1}{2\mu}\lambda T + p_0\right)\mathbf{1}_{(T,\infty)}(t)\right]. \tag{4.3.29}$$

The figures below compare analytical and simulated values of $EP(t)\mathbf{1}_{[0,t_\nu)}(t)$.

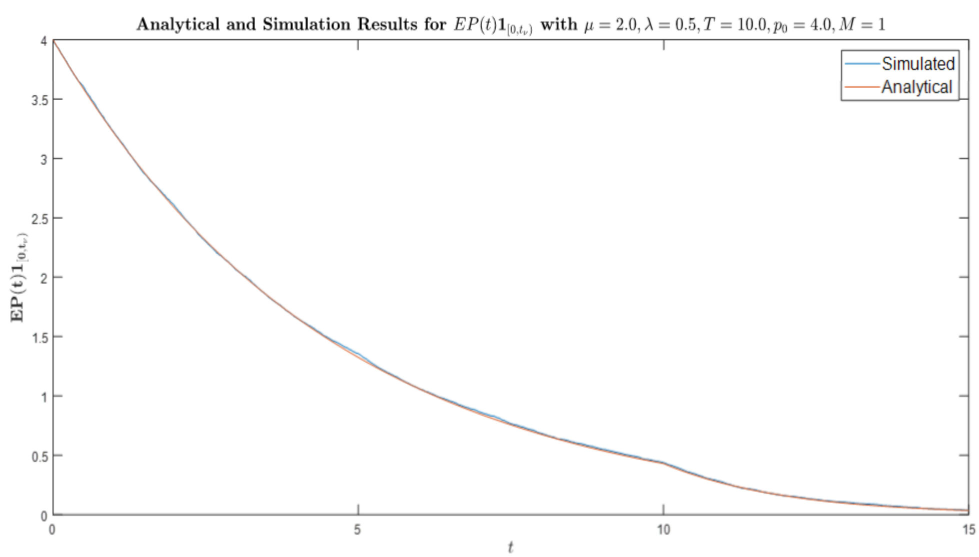

Figure 4.2: Plot 1: $\mu = 2$, $\lambda = 0.5$, $T = 10$, and $p_0 = 4$.

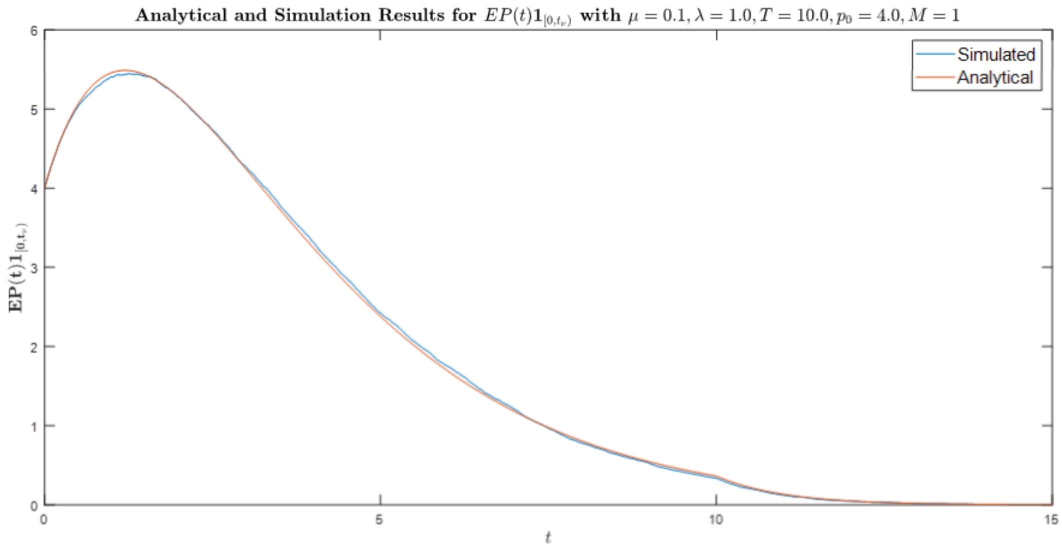

Figure 4.3: Plot 2: $\mu = 0.1$, $\lambda = 1$, $T = 10$, and $p_0 = 4$.

The following gives the joint density of the stock value at time $t \in [0, t_\nu)$.

Proposition 4.3.1. *It holds true that*

$$\frac{\partial}{\partial \vartheta} P\left\{P(t) \leq \vartheta, t_\nu > t\right\} = e^{-\lambda t}\Big\{\mathcal{F}^{-1}\left[e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}t - i\eta p_0}\right](\vartheta)\,\mathbf{1}_{[0,T]}(t)$$

$$+\mathcal{F}^{-1}\left[e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}t - i\eta p_0}\right](\vartheta)_{t=T}\,\mathbf{1}_{(T,\infty)}(t)\Big\},$$

where

$$\mathcal{F}^{-1}\left[e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}t - i\eta p_0}\right](\vartheta) = \delta(\vartheta - p_0)$$

$$+e^{-\mu(\vartheta - p_0)}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\left(\frac{1}{2}\lambda\mu t\right)^n(\vartheta - p_0)^{n-1}\,\mathbf{1}_{\mathbb{R}_+}(\vartheta - p_0),$$

where δ is the Dirac Delta function.

The figures below depict the obtained density of Proposition 4.3.1 for different values of parameter μ (of the assumed Laplace distribution) and λ (for the exponential distribution). Akin to rate parameter effects on Laplace and exponential density amplitudes, μ appears to affect the amplitude of the density curve of Proposition 4.3.1 and to a lesser extent its shift to the right, whereas λ only affects its amplitude.

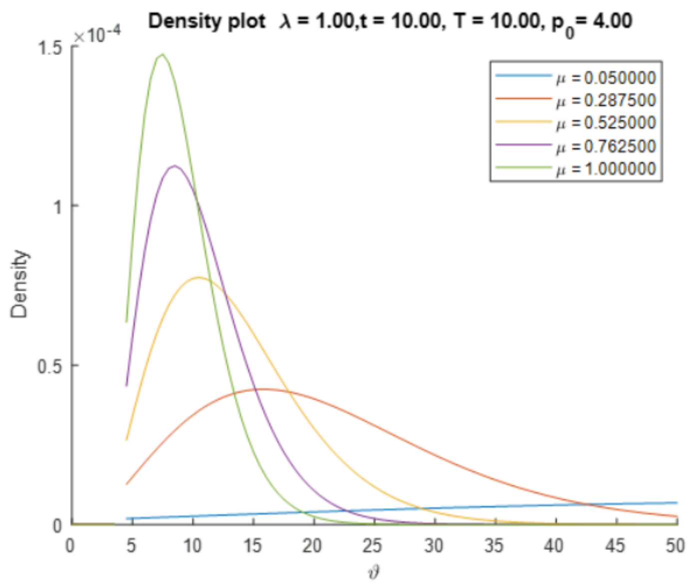

Figure 4.4: Density plots for varying μ

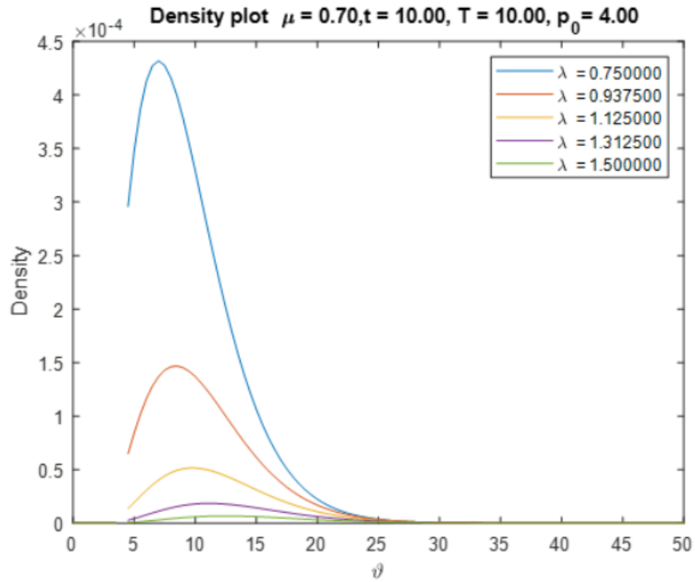

Figure 4.5: Density plots for varying λ

Proof. Recall that

$$\mathcal{F}^{-1}\left[g\left(\eta,t\right)\right]=\frac{1}{2\pi}\mathcal{F}\left[g\left(-\eta,t\right)\right] \tag{4.3.30}$$

where $g\left(\eta,t\right)=e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}t-i\eta p_0}$. From the expansion of

$$e^{\frac{1}{2}\lambda t\mu\frac{1}{\mu-i\eta}}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\lambda\mu t\right)^n\frac{1}{\left(\mu-i\eta\right)^n} \tag{4.3.31}$$

and due to

$$\mathcal{F}\left(\frac{1}{\left(\mu-i\eta\right)^n}\right)(\vartheta)=\begin{cases}0, & \vartheta<0\\ \frac{1}{(n-1)!}2\pi\vartheta^{n-1}e^{-\mu\vartheta}, & \vartheta>0\end{cases}, n=1,2,\ldots, \tag{4.3.32}$$

and

$$\mathcal{F}\left(\mathbf{1}\right)(\vartheta)=2\pi\delta\left(\vartheta\right) \tag{4.3.33}$$

we have that

$$\mathcal{F}\left(e^{\frac{1}{2}\lambda t\mu\frac{1}{\mu-i\eta}}\right)(\vartheta)=2\pi\delta(\vartheta)+2\pi e^{-\mu\vartheta}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\left(\frac{1}{2}\lambda\mu t\right)^{n}\vartheta^{n-1}\mathbf{1}_{\mathbb{R}_{+}}(\vartheta). \tag{4.3.34}$$

From

$$\mathcal{F}(g(\eta)\,e^{ib\eta})(\vartheta)=\mathcal{F}(g(\eta))(\vartheta-b),$$

and

$$\mathcal{F}g(-\eta,t)(\vartheta)=\mathcal{F}\left(e^{\frac{1}{2}\lambda\frac{\mu}{\mu-i\eta}t+i\eta p_0}\right)=\mathcal{F}\left(e^{\frac{1}{2}\lambda t\mu\frac{1}{\mu-i\eta}}\right)(\vartheta-p_0)$$

$$=2\pi\delta(\vartheta-p_0)+2\pi e^{-\mu(\vartheta-p_0)}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\left(\frac{1}{2}\lambda\mu t\right)^{n}(\vartheta-p_0)^{n-1}\mathbf{1}_{\mathbb{R}_{+}}(\vartheta-p_0), \tag{4.3.35}$$

we have that

$$\mathcal{F}^{-1}(g(\eta,t))(\vartheta)=\delta(\vartheta-p_0)$$

$$+e^{-\mu(\vartheta-p_0)}\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{(n-1)!}\left(\frac{1}{2}\lambda\mu t\right)^{n}(\vartheta-p_0)^{n-1}\mathbf{1}_{\mathbb{R}_{+}}(\vartheta-p_0) \tag{4.3.36}$$

and the assertion follows thereafter. □

1b. Next with player A's sustainability threshold $M=2$ we end the game when the stock price drops twice. Under equation (4.3.8),

$$e^{i\eta p_0}\theta\Phi^{*}(\theta,x,y)=\Big[[1-\gamma(x,\eta,\theta+y)][1-\gamma(x,0,y)]\Big]^{-1}$$

$$\times\frac{1}{\theta}[\gamma(x,0,y)-\gamma(x,0,\theta+y)][\gamma(1,0,0)-\gamma(x,0,y)]$$

$$+[1-\gamma(x,\eta,\theta+y)]^{-1}\frac{1}{\theta}\Big[\gamma(1,0,0)-\gamma(x,0,y)-\gamma(1,0,\theta)+\gamma(x,0,\theta+y)\Big]. \tag{4.3.37}$$

Now

$$\gamma(0,\phi,\vartheta) = Eu^{x_1}e^{-i\phi\pi_1}\big|_{u=0}Ee^{-\Delta_1\vartheta} = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}\frac{\lambda}{\lambda+\vartheta}. \tag{4.3.38}$$

With the pdf of π_1 as defined in (4.3.2),

$$\gamma(x,\eta,y) = Ex^{X_1}e^{-i\eta\pi_1}e^{-y\Delta_1}$$

due to position independent marking

$$= Ex^{X_1}e^{-i\eta\pi_1}\frac{\lambda}{\lambda+y}\left[\mathbf{1}_{\{X_1=0\}} + \mathbf{1}_{\{X_1=1\}}\right]$$

$$= \frac{\lambda}{\lambda+y}\left[\beta(0,\pi_1) + \beta(1,\pi_1)\right] \tag{4.3.39}$$

where

$$\beta(0,\pi_1) = Ex^{X_1}e^{-i\eta\pi_1}\mathbf{1}_{\{X_1=0\}}$$

$$= Ee^{-i\eta\pi_1}\mathbf{1}_{\{\pi_1\geq 0\}} = \int_{s=0}^{\infty}\frac{1}{2}\mu e^{-(\mu+i\eta)s}ds = \frac{1}{2}\frac{\mu}{\mu+i\eta} \tag{4.3.40}$$

and

$$\beta(1,\pi_1) = Ex^{X_1}e^{-i\eta\pi_1}\mathbf{1}_{\{X_1=1\}} = Ex^{X_1}e^{-i\eta\pi_1}\mathbf{1}_{\{\pi_1<0\}}$$

$$= x\int_{s=-\infty}^{0}e^{-i\eta s}\frac{1}{2}\mu e^{\mu s}ds = \frac{1}{2}\frac{\mu x}{\mu-i\eta}. \tag{4.3.41}$$

So

$$\gamma(x,\eta,y) = \frac{\lambda}{\lambda+y}\left[\beta(0,\pi_1) + \beta(1,\pi_1)\right] = \frac{\lambda}{\lambda+y}\frac{1}{2}\frac{\mu^2(1+x) - i\eta\mu(1-x)}{\mu^2+\eta^2} \tag{4.3.42}$$

and

$$\gamma(x,0,y) = \gamma(x,\eta,y)\Big|_{\eta=0} = \frac{\lambda}{\lambda+y}\frac{1+x}{2} \tag{4.3.43}$$

$$\gamma(1,0,y) = \frac{\lambda}{\lambda+y}. \tag{4.3.44}$$

Hence,

$$e^{i\eta p_0}\theta\varPhi^*(\theta,x,y) = \left[[1-\frac{\lambda}{\lambda+\theta+y}\frac{1}{2}\frac{\mu^2(1+x)-i\eta\mu(1-x)}{\mu^2+\eta^2}]\right]^{-1}\frac{\theta}{\lambda+\theta} \tag{4.3.45}$$

implying that

$$e^{i\eta p_0}\varPhi^*(\theta,x,y) = e^{i\eta p_0}\int_{t=0}^{\infty} e^{-\theta t} D_p\mathcal{L}\mathcal{C}_q\left[Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}(t)\right](x,y)\,dt$$

$$= \left[[1-\frac{\lambda}{\lambda+\theta+y}\frac{1}{2}\frac{\mu^2(1+x)-i\eta\mu(1-x)}{\mu^2+\eta^2}]\right]^{-1}\frac{1}{\lambda+\theta}. \tag{4.3.46}$$

Next, recalling (4.3.16) we have

$$\left[[1-\frac{\lambda}{\lambda+\theta+y}\frac{1}{2}\frac{\mu^2(1+x)-i\eta\mu(1-x)}{\mu^2+\eta^2}]\right]^{-1} = \frac{1}{1-A\frac{1}{\lambda+\theta+y}} = 1+A\frac{1}{\theta+y+B}$$

and

$$e^{i\eta p_0}\varPhi^*(\theta,x,y) = \frac{1}{\lambda+\theta}+A\frac{1}{\lambda+\theta}\frac{1}{\theta+y+B}. \tag{4.3.47}$$

The Laplace inverse with respect to y and θ of

$$F(y,\theta) = \frac{1}{\lambda+\theta}\frac{1}{y}+A\frac{1}{\lambda+\theta}\frac{1}{\theta+y+B}\frac{1}{y}. \tag{4.3.48}$$

because of

$$\mathcal{L}_y^{-1}\left[F\left(y,\theta\right)\right]\left(T\right)=\frac{1}{\lambda+\theta}+A\frac{1}{\lambda+\theta}\frac{1-e^{-(\theta+B)T}}{\theta+B}$$

$$=\frac{1}{\lambda+\theta}+A\frac{1}{\lambda+\theta}\frac{1}{B+\theta}-A\frac{1}{\lambda+\theta}\frac{1}{B+\theta}e^{-BT}e^{-\theta T} \qquad (4.3.49)$$

is

$$\mathcal{L}_\theta^{-1}\left\{\mathcal{L}_y^{-1}\left[F\left(y,\theta\right)\right]\left(T\right)\right\}\left(t\right)$$
$$=\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\right)(t)+A\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\frac{1}{B+\theta}\right)(t)-Ae^{-BT}\mathcal{L}_\theta^{-1}\left(\frac{1}{\lambda+\theta}\frac{1}{B+\theta}e^{-\theta T}\right)(t)$$

$$=e^{-\lambda t}+\frac{A}{B-\lambda}e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{(\lambda-B)T}\right]-\frac{A}{B-\lambda}e^{-Bt}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)\right]. \quad (4.3.50)$$

So,

$$e^{i\eta p_0}\mathcal{L}_\theta^{-1}\mathcal{L}_y^{-1}\frac{1}{y}\Phi^*\left(\theta,0,y\right)\left(t,T\right)$$
$$=e^{-\lambda t}+\frac{A}{B-\lambda}e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{(\lambda-B)T}\right]-\frac{A}{B-\lambda}e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right) \quad (4.3.51)$$

where

$$A=\frac{1}{2}\lambda\frac{\mu^2\left(1+x\right)-i\eta\mu\left(1-x\right)}{\mu^2+\eta^2}$$

and $B=\lambda-A$ and $\frac{A}{B-\lambda}=-1$.

Hence

$$e^{i\eta p_0}\mathcal{L}_\theta^{-1}\mathcal{L}_y^{-1}\frac{1}{y}\Phi^*\left(\theta,0,y\right)\left(t,T\right)=e^{-\lambda t}-e^{-\lambda t}\left[1-\mathbf{1}_{(T,\infty)}\left(t\right)e^{AT}\right]+e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right)$$

$$=\left[e^{-Bt}\mathbf{1}_{[0,T]}\left(t\right)+e^{-\lambda t}e^{AT}\mathbf{1}_{(T,\infty)}\left(t\right)\right]=e^{-\lambda t}\left[e^{At}\mathbf{1}_{[0,T]}\left(t\right)+e^{AT}\mathbf{1}_{(T,\infty)}\left(t\right)\right].$$

$$= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}e^{\frac{1}{2}\lambda t\frac{\mu x}{\mu-i\eta}}\mathbf{1}_{[0,T]}(t)+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}e^{\frac{1}{2}\lambda\frac{\mu x}{\mu-i\eta}T}\mathbf{1}_{(T,\infty)}(t)\right]. \tag{4.3.52}$$

Next we apply the $\mathcal{D}$-operator to

$$G(x) = e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}e^{\frac{1}{2}\lambda t\frac{\mu x}{\mu-i\eta}}\mathbf{1}_{[0,T]}(t)+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}e^{\frac{1}{2}\lambda\frac{\mu x}{\mu-i\eta}T}\mathbf{1}_{(T,\infty)}(t)\right]. \tag{4.3.53}$$

to get

$$\begin{aligned}
\mathcal{D}_x^1 G(x) &= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}\mathcal{D}_x^1\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\lambda t\frac{\mu}{\mu-i\eta}\right)^n x^n\mathbf{1}_{[0,T]}(t)\right.\\
&\left.+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}\mathcal{D}_x^1\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\lambda T\frac{\mu}{\mu-i\eta}\right)^n x^n\mathbf{1}_{(T,\infty)}(t)\right]\\
&= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{1}{2}\lambda t\frac{\mu}{\mu-i\eta}\right)^n\mathbf{1}_{[0,T]}(t)\right.\\
&\left.+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{1}{2}\lambda T\frac{\mu}{\mu-i\eta}\right)^n\mathbf{1}_{(T,\infty)}(t)\right]\\
&= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}\left[1+\frac{1}{2}\lambda t\frac{\mu}{\mu-i\eta}\right]\mathbf{1}_{[0,T]}(t)\right.\\
&\left.+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}\left[1+\frac{1}{2}\lambda T\frac{\mu}{\mu-i\eta}\right]\mathbf{1}_{(T,\infty)}(t)\right].
\end{aligned} \tag{4.3.54}$$

(4.3.54) implies that

$$\begin{aligned}
Ee^{-i\eta P(t)}\mathbf{1}_{[0,t_\nu)}(t) &= e^{-(\lambda t+i\eta p_0)}\left[e^{\frac{1}{2}\lambda t\frac{\mu}{\mu+i\eta}}\left[1+\frac{1}{2}\lambda t\frac{\mu}{\mu-i\eta}\right]\mathbf{1}_{[0,T]}(t)\right.\\
&\left.+e^{\frac{1}{2}\lambda\frac{\mu}{\mu+i\eta}T}\left[1+\frac{1}{2}\lambda T\frac{\mu}{\mu-i\eta}\right]\mathbf{1}_{(T,\infty)}(t)\right].
\end{aligned} \tag{4.3.55}$$

In particular,

$$
\begin{aligned}
&E\mathbf{1}_{[0,t_\nu)}(t) = P\{t_\nu > t\} \\
&= e^{-\lambda t}\left[e^{\frac{1}{2}\lambda t}\left[1+\frac{1}{2}\lambda t\right]\mathbf{1}_{[0,T]}(t) + e^{\frac{1}{2}\lambda T}\left[1+\frac{1}{2}\lambda T\right]\mathbf{1}_{(T,\infty)}(t)\right].
\end{aligned} \tag{4.3.56}
$$

The latter agrees with simulation as depicted in the figure below.

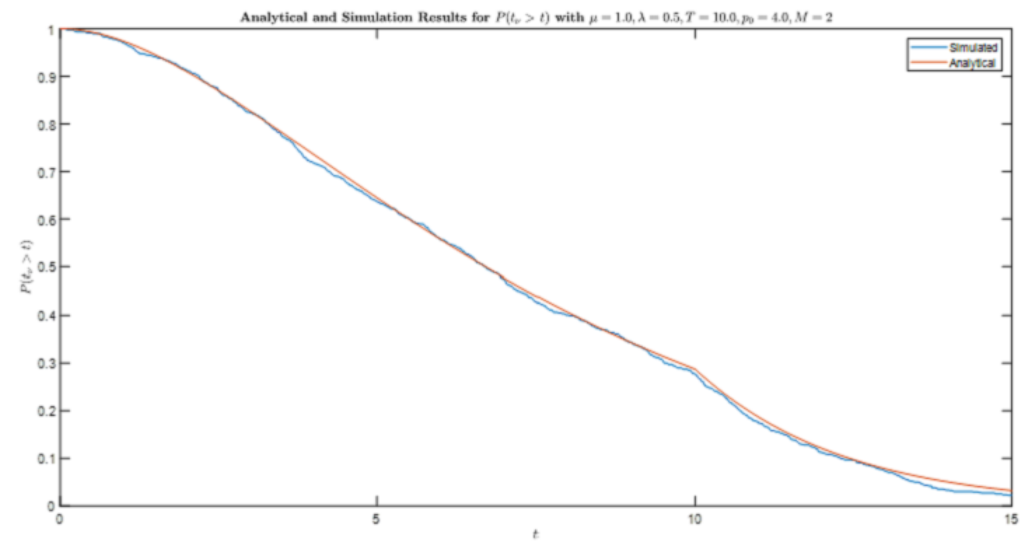

Figure 4.6: Analytical and simulated $P(t_\nu > t), \mu = 1, \lambda = 1, T = 10, p_0 = 4, M = 2$.

Chapter 5

Summary, Future Work, and Conclusion

5.1 Summary

We began our work by studying a real time observation of a delayed renewal point process marked by a monotone process A that evolves until it hits a targeted fixed level. We acquired the explicit formulas for the time sensitive joint probability functional

$$Ez^{N_t}u^{A_{\nu-1}}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\,, \tag{5.1.1}$$

$$\|z\| \leq 1,\ \|u\| \leq 1,\ \|v\| \leq 1,\ \mathrm{Re}\vartheta_0 \geq 0,\ \mathrm{Re}\vartheta \geq 0,$$

of the time dependent counting process N_t, first passage time t_ν, pre-first passage time t_ν, first excess level A_ν and the pre-first excess level $A_{\nu-1}$. Here $(t_{\nu-1}, t_\nu]$ is a random interval after the pre-first time and up to the first passage time. From the joint functional we also obtained closed form marginals of these entities via examples of a marked Poission process with position independent marking where the marks $x_1, x_2, \ldots \in [\mathrm{Geo}_1(p)]$ are independent and identically distributed (iid). We validated our results via probabilistic arguments.

We continued with Chapter 3 in which we studied a class of signed marked random measures with position dependent marking driven by a non-monotone process being watched by a monotone process. We targeted the critical behavior of the underlying stochastic process about a fixed threshold in the context of time sensitivity. The latter means that all related characteristic, such as first passage time and the location of the process upon crossing the threshold relate to deterministic time $t \geq 0$. The major benefit of this study was to utilize stochastic control over the process that must traditionally be considered on time interval $[0, t], t \geq 0$. Using and further embellishing fluctuation theory, we found explicitly the functionals

$$\Phi_\nu(t) = Ez^{N_t} e^{-i\eta \Pi_t} e^{-i\varphi P_{\nu-1}} u^{A_{\nu-1}} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t) \; 5.1.2$$

and

$$\overset{\wedge}{\Phi}_\nu(t) = Ez^{N_t} e^{-i\eta \Pi_t} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0, t_\nu)}(t) \tag{5.1.3}$$

with respect to time $t \in [t_{\nu-1}, t_\nu)$ and $t \in [0, t_\nu)$, respectively. These functionals describe the status of underlying processes $N_t = \sum_{n=0}^{\infty} x_n \varepsilon_{t_n}[0, t]$ and $\Pi_t = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}[0, t]$, along with other characteristics like the values of these processes upon the crossing as well as just prior to crossing the threshold.

We discussed various applications to the finance (stock option trading) and risk theory. A number of special cases and examples demonstrated analytic tractability of the results obtained.

Next, in Chapter 4 we studied a class of signed marked random measures with position dependent marking that is driven by a stochastic game of three players A,T, and P, of whom A and T are active antagonistic players exerting damages of random magnitudes upon each other at random epochs of them $\{t_0, t_1, \ldots\}$, whereas player P receives hits but does not respond to the attacks. The position of the passive player P is described by a non-monotone process $P(t)$ often related to stock price variations. We developed new techniques (in the framework of fluctuation theory), to find a closed-form expression for the functional

$$\Phi_\nu(t) = Ez^{A(t)} e^{-i\eta P(t)} v^{A_{\nu-1}} e^{-i\varphi P_{\nu-1}} e^{-\psi t_{\nu-1}} u^{A_\nu} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t) \tag{5.1.4}$$

in terms of multiple transforms organized in Theorem 4.2.1. With the inversion of the transform of Φ_ν, we provided ample evidence to the claim for Φ_ν being analytically tractable also illustrating them in the analysis of stock option trading. The numerical values of analytical results agreed with simulation for a number of special cases.

We obtained analytical solutions for the expectation functional

$$\overset{\wedge}{\Phi}_\nu(t) = Ee^{-i\eta P(t)} \mathbf{1}_{[0,t_\nu)}(t)\,. \tag{5.1.5}$$

This functional gives the status of underlying stock price process $P(t) = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}[0,t]$ at any time in interval $[0,t_\nu)$. Specifically, using player's A threshold levels $M=1$ and $M=2$ which represent the first and second drop of the stock price process $P(t)$, respectively, we derived formulas for $EP(t)\,1_{[0,t_\nu)}(t)$ and $P(t_\nu > t)$ and validated the analytical results by simulation. Furthermore, for $M = 1$, we established an explicit formula for density $\frac{\partial}{\partial\vartheta}P\{P(t) \le \vartheta, t_\nu > t\}$.

5.2 Future Work

An interesting direction is to modify our time interval in chapter 4 from $[0, t_\nu)$ to $[0, t_{\nu-1} + \delta)$, where $\delta \ge 0$ is not necessarily distributed as the $(\nu+1)$ th time increment Δ_ν. This modification is suitable for scenarios where we need to scrutinize the status of our game from the start to random times after the pre-first passage time $t_{\nu-1}$ but not necessarily up to the first passage time t_ν. Suppose $A(t)$, $P(t)$, $A_{\nu-1}$, $P_{\nu-1}$, $t_{\nu-1}, A_\nu, P_\nu$, t_ν, are independent and stationary increment processes which are valued in $\mathbb{N}_0$, $\mathbb{R}, \mathbb{N}_0, \mathbb{R}$, $\mathbb{R}^+, \mathbb{N}_0, \mathbb{R}$, $\mathbb{R}^+$ respectively and for which we require probabilistic information as players A and T fluctuate about their thresholds at time $t \in [0, t_{\nu-1} + \delta)$. Hence the new time sensitive joint functional is

$$\Phi_\nu(t) = Ez^{A(t)}e^{-i\eta P(t)}v^{A_{\nu-1}}e^{-i\varphi P_{\nu-1}}e^{-\psi t_{\nu-1}}u^{A_\nu}e^{-i\phi P_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_{\nu-1}+\delta)}(t) \quad (5.2.1.1)$$

For example, say you would like to sell some of your stock shares before the stock price drops for the Mth time and keep some on hand in case the stock price rises sharply thereafter. We can find conditional marginal LSTs and pgfs of the passage times, stock price process, and the drop monitoring process upon the time before the Mth drop or after that time. Recall that $t_\nu = t_{\nu-1} + \Delta_\nu$. Thus, for $\delta = \Delta_\nu$, the interval $[0, t_{\nu-1} + \delta)$ coincides with $[0, t_\nu)$ and (5.2.1.1) reduces to functional

(4.1.8). Recall that for the stock example in chapter 4, we observe the prices and sell the shares before the Mth drop. It will be interesting to see how (5.2.1.1) generalizes the results in Chapter 4.

5.3 Conclusion

In conclusion, this research has contributed to the field of fluctuation analysis of stochastic processes by laying out a systematic approach to finding closed form solutions of time sensitive joint functionals that describe real life phenomena. For example, one application of our first result (the one-dimensional monotone result) is in computer network security. Suppose we have a network that is under attack and repairs are not being done fast enough. We would like to predict when the entire network will fail so we watch the failed computers' accumulation for when the total number of failed computers will surpass a critical level. We can use our results to predict the exact time that the critical level is crossed, the time right before the crossing happens and the number of downed computers at those times. Based on our results, certain measures such as shutting down the entire network can be undertaken to prevent total system failure. Yet another example is in cancer treatment. Our results can predict the cancer progression (or regression) and the exact time the cancer will become untreatable by radiation therapy ahead of the time so that another course of action is undertaken before the patient's condition deteriorates. Another example is in queuing theory where a server continues a service and goes on a vacation when the queue drops to a certain level and only returns under certain conditions. For example, in the N-policy queuing system, the server returns to the queue only after the queue has accumulated to N. Using our results and the distributions of the inter-arrival times and input process, we can predict the time when the queue length will cross N and thus when a server should return and resume working, thereby reducing service delay costs.

Our second result (the one-dimensional non-monotone result) has applications in stock price trading. Here we can use our results to monitor the stock price for the number of drops and by how much. This informs a trader when to buy or sell stock shares based on their decision threshold. Finally, our two-dimensional result finds applications in game theory. This result differs from the second result by adding a second monotone principal player with their own pre-defined threshold. For example, in the stock option application that we investigated the two principle players are the time and a player monitoring the stock price for drops. Here the two thresholds are the number of times the stock price is allowed to drop and its maturity date. Yet another example is the N-T policy in queuing. Here, the server goes on vacations of T length and resumes service when the queue accumulates to N units. With our results we can obtain the distribution of the queueing process which is limited by both the total length of the server's vacations and the total number of customers in the queue.

In all our results we focused on the real time observation of the first crossing of the specified thresholds at any time t within either of the time intervals $[t_{\nu-1}, t_\nu)$ and $[0, t_\nu)$, where t_ν was the first passage time and $t_{\nu-1}$ the pre-first passage time. This is a very significant milestone because unlike the previous work in this field, now we can refine and condition the stochastic quantities in our joint functional, making our results useful for stochastic control.

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Appendix A

Proposition A.1. *The series*

$$\sum_{n=1}^{\infty} F_n^* (\theta) = \sum_{n=1}^{\infty} \int_{t=0}^{\infty} e^{-\theta t} E z^{N_t} u^{A_{n-1}} v^{A_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \leq t < t_n\}} dt$$

converges to

$$\sum_{n=1}^{\infty} F_n^* (\theta) = \frac{1}{\theta} \gamma_0 (uvz, \vartheta_0 + \vartheta + \theta) [\gamma (v, \vartheta) - \gamma (v, \vartheta + \theta)] \frac{1}{1 - \gamma (uvz, \vartheta_0 + \vartheta + \theta)}$$

and $\|\gamma (uvz, \vartheta_0 + \vartheta + \theta)\| < 1.$ (A.1.1)

Proof. The first part of the proposition is due to the above steps that ended in formula (3.6). The inequality (A.1.1) is due to the following arguments.

$$\left\|\gamma(uvz, \vartheta_0 + \vartheta + \theta)\right\| = \left\|E (uvz)^{X_1} e^{-(\vartheta_0 + \vartheta + \theta)\Delta_1}\right\|$$

$$= \| \sum_{k=0}^{\infty} (uvz)^k \int_{t=0}^{\infty} e^{-(\vartheta_0 + \vartheta + \theta)t} P_{X_1 \otimes \Delta_1} (k, dt) \|$$

$$\leq \sum_{k=0}^{\infty} \| (uvz) \|^k \int_{t=0}^{\infty} \|e^{-(\vartheta_0 + \vartheta + \theta)t}\| P_{X_1 \otimes \Delta_1} (k, dt)$$

$$= \sum_{k=0}^{\infty} \| (uvz) \|^k \int_{t=0}^{\infty} e^{-Re(\vartheta_0 + \vartheta + \theta)t} P_{X_1 \otimes \Delta_1} (k, dt)$$

$$= \int_{t=0}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(0,dt) + \sum_{k=1}^{\infty} \left\| (uvz) \right\|^k \int_{t=0}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(k,dt)$$

$$< \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(0,dt) + e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(0,dt)$$
$$+ \left\| (uvz) \right\| \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(k,dt) + \left\| (uvz) \right\| \sum_{k=1}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(k,dt) ,$$

since $\left\| (uvz) \right\| > \left\| (uvz) \right\|^k$ for $\left\| (uvz) \right\| < 1$ and $k > 1$.

Let

$$\begin{aligned}
a &:= \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(0,dt)\\
b &:= \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(0,dt)\\
c &:= \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(k,dt)\\
d &:= \sum_{k=1}^{\infty} e^{-Re(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(k,dt) .
\end{aligned}$$

Then clearly, $a + b + c + d = 1$ and thus,

$$a + e^{-Re(\vartheta_0+\vartheta+\theta)} b + \left\| (uvz) \right\| c + \left\| (uvz) \right\| e^{-Re(\vartheta_0+\vartheta+\theta)} d < 1$$

since $\left\| (uvz) \right\| < 1$ and because $Re(\vartheta_0 + \vartheta + \theta) \geq 0$ is a requirement for the existence of the LST, $e^{-Re(\vartheta_0+\vartheta+\theta)} \leq 1$.

$\square$

Appendix B

B.1 Matlab Code

```
%--------------------------------------------------------------------%
%Chapter 4 Time vs P Game Simulation and Comparison with
%Analytical Results
%---------------------------------------------------------------------

%Parameter Initialization
T=10;
t=transpose(0:0.01:15);
mu=1;
L=1; %lambda
n=10000; %number of iterations
M=1;
p0=4;
t_nu=zeros(1,n);
p_tI=zeros(length(t),n);
p_t=zeros(length(t),n);
p_t(1,:)=repmat(p0,1,n);
p_tI(1,:)=repmat(p0,1,n);

for i=1:n

```

```
    %time has exponential increments with parameter lambda
    time= cumsum([0,exprnd(1/L,1,1000)]);

    %passive process has Laplace Increments...
    %shift parameter 0 and scale parameter 1/mu
    P=cumsum([p0,randlap(1,1000,mu,0)]);

    %active process increments each time passive process
    %drops
    A=cumsum([0,diff(P)<0]);

    %t_nu occurs the first time that (time>=T) or A>=M
    t_nu(i)=time(find((time>=T)|(A>=M),1));

    for j=2:length(t)

        %For P(t) in [t_(j-1),t_(j)), P(t)=t_(j-1);
        p_t(j,i)=P(find(time<t(j),1,'last'));

        %with indicator, P(t)1(t<t_nu)
        p_tI(j,i)=p_t(j,i)*(t(j)<t_nu(i));

    end
end
%simulated probability P(t_nu>t) for each t value
```

```
prob_sim=mean(t_nu>t,2);

%analytical probability P(t_nu>t) for each t value
%when M==1
prob1=exp(-L.*t).*(exp(0.5.*L.*t).*(((0<=t)&(t<=T)))...
      +exp(0.5.*L.*T).*(t>T));

%when M==2
prob2=exp(-L.*t).*(exp(0.5.*L.*t).*(1+0.5*L*t).*...
      (((0<=t)&(t<=T)))+exp(0.5.*L.*T).*(1+0.5*L*t).*(t>T));

%simulated expected value of P(t)1(t<t_nu) for each t value
Exp_p_tI_sim=mean(p_tI,2);

%analytical expected value of P(t)1(t<t_nu) for each t value
%for M==1
Exp_p_tI=...
      exp(-L.*t).*((exp(0.5.*L.*t).*(p0+0.5.*L./mu.*t)...
      .*(((0<=t)&(t<=T)))...
      +exp(0.5.*L.*T).*(p0+0.5.*L./mu.*T).*(t>T)));

% cdf P(t_nu<=t) plots for M==1
figure()
plot(t,1-prob_sim,t,1-prob2)
```

```
title(sprintf(['\\bf{Analytical and Simulation Results'...
    'for $P(t_{\\nu} > t)$ with $\\mu = %.1f, \\lambda'...
    '= %.1f, T = %.1f, p_0= %.1f,M=%d$}'],mu,L,T,p0,M)...
    ,'Interpreter','latex')
xlabel('\bf{$t$}','Interpreter','latex')
ylabel('\bf{$P(t_{\nu}> t)$}','Interpreter','latex')
legend(Simulated, Analytical)


%expected value of P(t)1(t<t_nu) plots for M=1
figure()
plot(t,Exp_p_tI_sim,t,Exp_p_tI)
title(sprintf(['\\bf{Analytical and Simulation Results'...
    'for $EP(t){\\bf 1}_{[0,t_{\\nu})}$ with'...
    '$\\mu = %.1f,\\lambda = %.1f, T = %.1f,p_0= %.1f,M=1}']...
    ,mu,L,T,p0) ,'Interpreter','latex')
xlabel('\bf{$t$}','Interpreter','latex')
ylabel('$\bf{EP(t){\bf 1}_{[0,t_{\nu})}}$','Interpreter','latex')
legend(Simulated, Analytical)



%our density function for M=1
```

```
syms n p L mu T t v

f = @(n,p,L,mu,T,t,v)...
      exp(-L.*t).*(((0<=t)&&(t<=T)).*(symsum(1./(gamma(n+1)...
      *gamma(n)).*(0.5.*L.*t.*mu).^n.*(v-p).^(n-1)...
      .*exp(-mu*(v-p)).*((v-p)>0),n,1,Inf)+dirac(v-p))...
      + (t>T).*(symsum(1./(gamma(n+1)*gamma(n)).*...
      (0.5.*L.*T.*mu).^n.*(v-p).^(n-1).*exp(-mu*(v-p))...
      .*((v-p)>0),n,1,Inf)+dirac(v-p))...
      );

%some sample parameter values
L=1; %lambda
T=10;
t=10;
p0=4;
w=0:0.5:30;
mu=linspace(0.05,1,5);
y=zeros(length(mu),length(w));
figure();
Legend=cell(1,length(mu));

% results for sample parameter values
for i=1:length(mu)
      y(i,:)=vpa(f(n,p0,L,mu(i),T,t,w));
```

```
    Legend{i}=sprintf('\\mu = %f',mu(i));
    hold on
end

% density plots for different mus
plot(w,y);
title(sprintf(['    Density plot   \\lambda = %.2f,t = %.2f,' ...
    ' T = %.2f,  p_0= %.2f'],L,t,T,p0))
xlabel('\vartheta')
ylabel('Density')
legend(Legend)
hold off;


% Function randlap to generate an n by m matrix of random
%numbers fromthe Laplace distribution. Here we assume the
%Laplace pdf is:
%         f(x) = 1/2 *mu*exp(-mu*abs(x-b))
% and cdf
%        F(x))=  1/2*exp(mu*(x-b)),x<=b
%        F(x) = 1-1/2*exp(-mu*(x-b)), x>=b/
%  and so inverse cdf
% invF(y) = b - (1/mu) * sign(y-0.5).* ln(1- 2* abs(y-0.5))
    .
```

```
% We use the fact that the cdf of a random variable has a
%uniform distibution on [0,1]. if we draw y1 from Unif(0,1)
%, then x1=invF(y1)) is from the Laplace distribution.

function x1 = randlap(n,m,mu,b)
y1 = rand(n,m);
x1 = b - (1/mu) * sign(y1-0.5).* log(1- 2* abs(y1-0.5));
end
```