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### Discrete Moment Problems with Logconcave and Logconvex Distributions

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DISCRETE MOMENT PROBLEMS WITH  
LOGCONCAVE AND LOGCONVEX DISTRIBUTIONS

by

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in partial fulfillment of the requirements  
for the degree of

Doctor of Philosophy  
in  
Operations Research

Melbourne, Florida  
May, 2020

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DISCRETE MOMENT PROBLEMS WITH  
LOGCONCAVE AND LOGCONVEX DISTRIBUTIONS BY TALAL ALHARBI

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## ABSTRACT

Title:

DISCRETE MOMENT PROBLEMS WITH  
LOGCONCAVE AND LOGCONVEX DISTRIBUTIONS

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We introduce new shape constraints, logconcavity and logconvexity, to discrete moment problems for bounding the  $k$ -out-of- $n$  type probabilities and expectations of higher order convex functions of discrete random variables with non-negative and finite support. The bounds are obtained as the optimum values of non-convex and convex nonlinear optimization problems, where the non-convex problem is reformulated as a bilinear optimization problem. We present numerical experiments to show the improvement in the tightness of the bounds when the shape of underlying unknown probability distribution is prescribed into discrete moment problems. We apply our optimization based bounding methodology in an insurance problem to estimate the expected stop-loss of aggregated insurance claims within a fixed period. The proposed bounding methodology is expected to expand the scope of applications for both discrete moment problems and logconcavity and logconvexity.

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# Dedication

I dedicate this dissertation to

The souls of my mom Fatmh and my father Mosaedd.

My wife Amani.

My sons Mohammed, Yazan, Alwaleed and Faris.

My daughter Alwateen.

# Chapter 1

## Introduction

Discrete Moment Problem (DMP) was formulated by Prékopa (1988) [1] as a linear programming problem to approximate linear functions on the unknown discrete probability distributions with finite support, where some of the power or binomial moments are known or given. Depending on the type of moments used, the problem is called a *discrete power moment problem (PMP)* or *discrete binomial moment problem (BMP)*.

The discrete moment problems came to prominence by the discovery that the classical probability bounds and expectations of higher order convex functions of discrete random variables with finite support can be obtained based on the knowledge of some of the binomial moments or power moments. In particular, Prékopa (1988, 1990) [1–3] had shown in his earlier work that sharp bounds for the probability of the union and the probability that at least  $k$  or exactly  $k$ -out-of- $n$  events occur can be formulated as DMPs.

Let  $X$  be a random variable with finite support  $\Omega = \{z_0, z_1, \dots, z_n\}$ . Assume that the probability distribution  $\{x_i\}$  defined as

$$x_i = P(X = z_i), \quad i = 0, 1, \dots, n$$

is unknown, but the first  $m$  power moments,

$$\mu_j = E(X^j), \quad j = 1, \dots, m,$$

where  $m < n$ , are assumed to be known and  $\mu_0 = 1$ .

Let us consider a given function  $f$  and introduce the notations

$$f_i = f(z_i), \quad i = 0, 1, \dots, n.$$

Then the discrete power moment problem (PMP) is given by

$$\begin{aligned} \min(\max) \quad & \sum_{i=0}^n f_i x_i \\ \text{subject to} \quad & \sum_{i=0}^n z_i^j x_i = \mu_j, \quad j = 0, 1, \dots, m \end{aligned} \tag{1.1}$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n.$$

Similarly, if we assume that the probability distribution  $\{x_i\}$  is unknown but the first  $m$  binomial moments,

$$S_j = E \left[ \binom{X}{j} \right], \quad j = 1, \dots, m,$$

are known, then we can formulate the discrete binomial moment problem (BMP) as follows

$$\begin{aligned} & \min(\max) && \sum_{i=0}^n f_i x_i \\ & \text{subject to} && \sum_{i=0}^n \binom{z_i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \end{aligned} \tag{1.2}$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n,$$

where  $S_0 = 1$ . Problems (1.1) and (1.2) can be transformed into each other by the use of the Stirling numbers of the first and second kind (Prékopa, 1995 [4]).

We remark that the coefficient matrix of problem (1.1) is a Vandermonde matrix and the coefficient matrix of problem (1.2) is a Pascal matrix, both of which can be badly ill-conditioned when  $n$  is large (see, for example, Pan, 2016 [5], Alonso et al., 2013 [6] and the references therein). Prékopa (1988) [1] developed a linear programming based methodology to solve the discrete moment problems (1.1) and (1.2) for the following three cases.

- Case 1. The function  $f$  has positive divided differences of order  $m + 1$  on the support set  $\Omega$ . In this case the optimum values of problems (1.1) and (1.2) provide us with sharp lower and upper bounds for  $E[f(X)]$ , that is, the expected value of higher order convex function of random variable  $X$ , based on the knowledge of first  $m$  power and binomial moments, respectively.
- Case 2.  $f_k = 1, f_i = 0, i \neq k$  for some  $0 \leq k \leq n$ . Then the optimum values of problems (1.1) and (1.2) give sharp lower and upper bounds for  $P(X = z_k)$ , based on the knowledge of first  $m$  power and binomial moments, respectively.
- Case 3.  $f_i = 0, i = 1, \dots, k - 1$  and  $f_k = \dots f_n = 1$  for some  $1 \leq k \leq n$ . In this case, the optimum values of problems (1.1) and (1.2) are the sharp lower and upper bounds for  $P(X \geq z_k)$ , based on the knowledge of first  $m$  power and binomial moments, respectively.

Let us assume that

$$z_i = i, \quad i = 0, 1, \dots, n$$

and the random variable  $X$  denotes the number of events  $A_1, \dots, A_n$ , associated with the probability space  $\Omega$ , occur. Then Case 2 provides us with sharp bounds for the probability that exactly  $k$ -out-of- $n$  events occur for some  $0 \leq k \leq n$ . Similarly, Case 3 can be used to obtain sharp bounds for the probability that at least  $k$ -out-of- $n$  events occur for some  $1 \leq k \leq n$ .

Prékopa extensively studied problems (1.1) and (1.2) for Cases 1-3 (Prékopa, 1988 [1], Boros and Prékopa, 1989 [7], Prékopa, 1989 [8] Prékopa, 1990 [2], Prékopa, 1990 [3], Prékopa, 1998, 2001 [9,10], Gao and Prékopa, 2002 [11], Prékopa and Gao, 2005 [11]). The central results in this respect include the characterization of dual feasible basis in problems (1.1) and (1.2) and obtaining closed form bounds for the

expectation  $E[f(X)]$  and probability of the union of events and probabilities that exactly  $k$  or at least  $k$ -out-of- $n$  events occur, that is,  $P(X \geq 1)$ ,  $P(X = k)$  for some  $0 \leq k \leq n$ , and  $P(X \geq k)$  for some  $1 \leq k \leq n$ , respectively.

Discrete moment problems were used in wide range of applications including maximum satisfiability problem (Boros and Prékopa, 1989 [12]), communication network reliability (Prékopa, Boros, and Lih, 1991 [13]), stochastic transportation network (Boros and Prékopa, 1991 [14]), telecommunication networks (Gao and Prékopa, 2001 [15]), PERT (Prékopa, Long, and Szántai, 2004 [16], Subasi, Subasi, Prékopa, 2009 [17]), reliability (Subasi, 2007 [18], Prékopa, Subasi, Subasi, 2008 [19]), and finance (Subasi, 2007 [18]). Since the problem is of practical importance, the theory of discrete moment problems continues to expand its purview and a wider range of investigations and arguments is developed. For more recent results the reader is referred to Prékopa, Ninh, and Alexe (2016) [20], Prékopa and Yoda (2016) [21].

While several attractive applications and theoretical investigations involving the discrete moment problems are presented in literature, little has been done to take into account the shape of the underlying probability distribution. Prékopa, Subasi, Subasi (2008) [19] and Subasi, Subasi, Prékopa (2009) [17] are the first to reformulate the discrete moment problems, where the underlying distribution is assumed to be increasing, decreasing, or unimodal with known or given mode. The problem is then used to obtain bounds for the probability of the union of events based on the knowledge of the first two binomial moments and for the expectations of higher order convex functions of discrete random variables based on the knowledge of the first two power moments.

Subasi et al. (2017) [22] and Subasi et al. (2018) [23] further expanded their investigations to binomial moment problem, where the first  $m$  binomial moments are known and the underlying probability distribution  $\{x_i\}$  is assumed to be unimodal with a known mode  $M$ , i.e., the following conditions are satisfied:

$$x_0 \leq x_1 \leq \dots \leq x_M \quad \text{and} \quad x_M \geq x_{M+1} \geq \dots \geq x_n \quad (1.3)$$

The starting point of the Subasi et al.'s investigation is the following linear program (Subasi, Subasi, Prékopa, 2017 [22]):

$$\begin{aligned} & \min(\max) && \sum_{i=k}^n x_i \\ & \text{subject to} && \\ & && \sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \\ & && x_0 \leq x_1 \leq \dots \leq x_M \\ & && x_M \geq x_{M+1} \geq \dots \geq x_n \\ & && x_i \geq 0, \quad i = 0, 1, \dots, n, \end{aligned} \quad (1.4)$$

where  $S_0 = 1$  and  $M$  is the mode of the distribution. Note that the optimum values of problem (1.4) give the sharp bounds for the probability that at least  $k$ -out-of- $n$  events occur,  $P(X \geq k)$ ,  $k = 1, \dots, n$ , where the underlying distribution is assumed to be unimodal with mode  $M$ .

Similarly, sharp bounds for the probability that exactly  $k$ -out-of- $n$  events occur,  $P(X = k)$ ,  $k = 0, \dots, n$ , where the underlying distribution is unimodal with mode  $M$  can be formulated as follows (Subasi, Subasi, Prékopa, 2018 [23]):

$$\begin{aligned}
& \min(\max) && x_k \\
& \text{subject to} && \\
& \sum_{i=0}^n \binom{i}{j} x_i = S_j, && j = 0, 1, \dots, m \\
& x_0 \leq x_1 \leq \dots \leq x_M && (1.5)
\end{aligned}$$

$$x_M \geq x_{M+1} \geq \dots \geq x_n$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n,$$

where  $S_0 = 1$ .

Subasi et al. (2017, 2018) [22, 23] fully characterized the dual feasible basis structures in binomial moment problems (1.4) and (1.5) and presented closed form bounds for the probabilities that at least  $k$  or exactly  $k$ -out-of- $n$  events occur based on the knowledge of the first two binomial moments. The authors also presented a dual type algorithm to obtain the customized algorithmic solutions of the linear programs (1.4) and (1.5).



Numerical investigations of Prékopa, Subasi, Subasi (2008) [19], Subasi, Subasi, Prékopa (2009) [17], and Subasi et al. (2017, 2018) [22, 23] suggest that, with the knowledge of the shape of the distribution, discrete moment problems renders significantly better lower and upper bounds for the expectations of the higher order convex functions of discrete random variables and probabilities that at least  $k$  or exactly  $k$ -out-of- $n$  events occur. Inclusion of the unimodality constraint into binomial moment problems (1.4) and (1.5) has drawn attention in the literature. We refer the readers to Kumaran and Swarnalatha (2017) [24] and Swarnalatha and Kumaran (2017) [25].

Motivated by the results of Prékopa, Subasi, Subasi (2008) [19], Subasi, Subasi, Prékopa (2009) [17], and Subasi et al. (2017, 2018) [22, 23], this dissertation's goal is to introduce two new shape constraints, logconvexity and logconcavity, into the discrete moment problems. We remark that a typical assumption for the discrete moment problem with unimodality constraint is that the location of the mode is known or given. This restriction is no longer required under logconvexity and logconcavity constraints.

The organization of the dissertation is as follows. In Chapter 2, we reformulate discrete moment problems, where the underlying distribution is assumed to be logconcave and present numerical examples to investigate the contribution of the logconcavity of the distribution on the bounds for the probability that at least  $k$ -out-of- $n$  events occur, based on the knowledge of first two and first three binomial moments.

In Chapter 3, we prescribe the logconvexity constraint into the power and binomial discrete moment problems and give numerical examples to demonstrate the contribution of the logconvexity of the distribution on the bounds for the probability that at least  $k$ -out-of- $n$  events occur, based on the knowledge of first two and first three binomial moments.

Chapter 4 presents an application of the discrete moment problem with log-concavity constraint. Finally, Chapter 5 concludes the dissertation with future directions.

# Chapter 2

## Discrete Moment Problems with Logconcave Distributions

### 2.1 Logconcavity Constraints

Logconcavity lies at the very heart of optimization theory and is a desired property in many fields such as economics, supply chain management, statistics, and stochastic optimization (see, e.g., Prékopa, 1995 [4], Johnson and Goldschmidt, 2006 [26], Ninh and Prékopa, 2013 [27], Alharbi, Subasi, Subasi, 2018 [28], and the references therein). In particular, Ninh and Prékopa (2013) studied the logconcavity property for compound distributions that are widely used to model intermittent demands (Axsäter, 2015 [29]).

Despite the considerable attention given to the logconcavity of continuous distributions, the theory of logconcavity of discrete distributions is still limited. The classical results in this respect is the notion of *r-times positive sequence* introduced by Fekete (1912) [30].

The sequence of nonnegative elements  $\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$  is said to be  $r$ -times positive if the matrix

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & a_0 & a_1 & a_2 & \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ & a_{-2} & a_{-1} & a_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

has no negative minor of order smaller than or equal to  $r$ . Further, if  $a_k = 0$  for  $k < 0$  and  $A$  has no negative minor of order smaller than or equal to  $r$ , then the concept of  $r$ -times positive students is known as  $PF_r$  sequence (Karlin, 1968 [31]).

The twice-positive sequences satisfy the following property

$$\begin{vmatrix} a_i & a_j \\ a_{i-t} & a_{j-t} \end{vmatrix} = a_i a_{j-t} - a_j a_{i-t} \geq 0 \quad (2.1)$$

for every  $i < j$  and  $t \geq 1$ . This holds if and only if

$$a_i^2 \geq a_{i-1} a_{i+1} \quad (2.2)$$

Fekete (1912) [30] also proved that the convolution of two  $r$ -times positive sequences is  $r$ -times positive. Twice-positive sequences are also called *logconcave sequences*. Hence, Fekete's theorem states that the convolution of two logconcave sequences is logconcave. A discrete probability distribution, defined on the real line, is said to be logconcave if the corresponding probability mass function is logconcave.

Note that a logconcave sequence should have no internal zero. Otherwise, basic convolution theorem of logconcave sequences will be violated (Ninh and Prékopa, 2013 [27]). In other words, there are no indices  $0 \leq i < j < r \leq n$  such that  $a_i \neq 0, a_j = 0, a_r \neq 0$ . Further, if the preceding inequality is reversed, then the sequence is called logconvex.

Finally, a sequence  $\{a_i\}_{i=0}^{\infty}$  is said to be concave if

$$2a_{i+1} \geq a_i + a_{i+2}, \quad i = 0, 1, 2, \dots$$

If this sequence is strictly positive, then it is logconcave if and only if  $\{\ln a_i\}_{i=0}^{\infty}$  is a concave sequence. Clearly, a logconcave sequence of positive terms is unimodal (Dharmadhikari and oag-Dev, 1988 [32], Stanley, 1989 [33]). Most recent result in this direction is due to Alharbi, Subasi, Subasi (2018) [28], where sufficient conditions for the logconcavity of multivariate discrete distributions are presented.

Incorporating shape constraints into discrete moment problems has shown that, with the knowledge of the shape of the distribution, the bounds for the probabilities and expectations can be significantly improved (Prékopa, Subasi, Subasi, 2008 [19], Subasi, Subasi, Prékopa, 2009 [17], and Subasi et al., 2017, 2018 [22, 23]).

Below we formulate discrete moment problems by prescribing the logconcavity constraint and discuss its solution approach.

A typical example involving logconcave discrete distributions is inventory management, where most demand distributions are assumed to be binomial, negative binomial, or compound Poisson distribution.

In this section, we assume that the discrete random variable  $X$  has a logconcave probability distribution  $\{x_i\}$ , i.e., the following conditions are satisfied

$$x_i^2 \geq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1. \quad (2.3)$$

Then the power moment problem (1.1) with logconcavity constraints is given by

$$\begin{aligned} & \min(\max) && \sum_{i=k}^n f_i x_i \\ & \text{subject to} && \sum_{i=0}^n z_i^j x_i = \mu_j, \quad j = 0, 1, \dots, m \end{aligned} \tag{2.4}$$

$$x_i^2 \geq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n.$$

Similarly, incorporating the logconcavity constraints (2.3) into the binomial moment problem (1.2), we obtain

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n f_i x_i \\
& \text{subject to} && \sum_{i=0}^n \binom{z_i}{j} x_i = S_j, \quad j = 0, 1, \dots, m
\end{aligned} \tag{2.5}$$

$$x_i^2 \geq x_{i-1} x_{i+1}, \quad i = 1, \dots, n-1$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n.$$

We remark that problems (2.4) and (2.5) fail to model the subtlety in the definition of a logconcave sequence. Recall that a probability sequence  $\{x_i\}$  is called logconcave if it has no internal zeros. This feature is satisfied by most discrete probability distributions in both theory and practice. In other words, any points in the support set should have a positive probability.

Let us introduce new decision variables

$$y_0, y_1, \dots, y_{n-1} \geq 0,$$

where

$$x_1 = y_0 x_0, \dots, x_n = y_{n-1} x_{n-1}.$$

Note that variables  $y_i, i = 0, \dots, n-1$  are the ratios between consecutive probabilities of the underlying random variable.

Substituting variables  $y_i, i = 0, \dots, n-1$ , we obtain an improved formulation of problems (2.4) and (2.5) as follows

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n f_i x_i \\
& \text{subject to} && \sum_{i=0}^n z_i^k x_i = \mu_j, \quad j = 0, \dots, m \\
& && x_i = x_{i-1} y_{i-1}, \quad i = 1, \dots, n
\end{aligned} \tag{2.6}$$

$$y_i \geq y_{i-1}, \quad i = 1, \dots, n$$

$$x_i \geq 0 \quad i = 0, 1, \dots, n$$

and

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n f_i x_i \\
& \text{subject to} && \sum_{i=0}^n \binom{z_i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \\
& && x_i = x_{i-1} y_{i-1}, \quad i = 1, \dots, n
\end{aligned} \tag{2.7}$$

$$y_i \geq y_{i-1}, \quad i = 1, \dots, n$$

$$x_i \geq 0 \quad i = 0, 1, \dots, n.$$



Note that problems (2.6) and (2.7) are nonconvex nonlinear programs due to the logconcavity constraints. These problems belong to a class of optimization problems called bilinear programs that have numerous applications in location theory, economics, risk management, etc. (see, e.g., Sherali and Alameddine, 1990 [34]). Solution methods to these type problems can be found in Konno (1976) [35], Liberti and Pantelides (2006) [36], Migladas and Pardalos (2013) [37]. To date, the Spatial Branch-and-Bound algorithms remain one of the most effective methods available for the global solution of nonconvex nonlinear programs (Tuy and Ghannadan, 1998 [38]).

Below we present numerical examples to investigate the contribution of the use of logconcavity constraints in discrete moment problems.

## 2.2 Numerical Examples: Logconcavity Constraints

Since problems (1.1) and (1.2) can be obtained from each other, we shall consider the following special case of binomial moment problem (1.2), where we assume that the underlying distribution is logconcave, that is,

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n x_i \\
& \text{subject to} && \\
& && \sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \\
& && x_i = x_{i-1} y_{i-1}, \quad i = 1, \dots, n \\
& && y_i \geq y_{i-1}, \quad i = 1, \dots, n \\
& && x_i \geq 0 \quad i = 0, 1, \dots, n.
\end{aligned} \tag{2.8}$$

Note that the optimum values of problem (2.8) provide us with lower and upper bounds for the probability that at least  $k$ -out-of- $n$  events occur,  $P(X \geq k)$  for some  $1 \leq k \leq n$ , where the underlying distribution is logconcave and the first  $m$  binomial moments are known.

Let  $A_1, \dots, A_n$  be arbitrary events in an arbitrary probability space  $\Omega$  and let  $X$  denote the number of those events that occur. The well-known Jordan's formulas (Jordan, 1927) [39] are available to compute the probabilities that at

least  $k$  ( $1 \leq k \leq n$ ) out of  $n$  events occur:

$$P(X \geq k) = \sum_{j=k}^n (-1)^{j-k} \binom{k-1}{j-1} S_j \quad (2.9)$$

where  $S_j$  is the  $j$ th binomial moment of the random variable  $X$  defined by

$$S_j = \sum_{1 \leq i_1 < \dots < i_j \leq n} P(A_{i_1} \cap \dots \cap A_{i_j}), \quad j = 1, \dots, n \quad (2.10)$$

and  $S_0 = 1$ , by definition. However, if  $n$  is large, it may not be possible to find the exact values of the probabilities in (2.9) because the total number of terms in (2.10) becomes an exponential function of  $n$ . On the other hand, it may be possible to compute a few of the binomial moments from the historical data and give lower and upper bounds for the probability  $P(X \geq k)$ . The goal of the traditional probability bounding approaches is to obtain the best possible approximation of  $P(X \geq k)$  based on the knowledge of the first  $m$  moments,  $S_0, \dots, S_m$ , ( $m < n$ ), or any finite collection of binomial moments.

The typical examples of probability  $P(X \geq k)$  is the reliability evaluations of  $k$ -out-of- $n$  systems such as multistate networks, including oil and gas supply systems, communication networks, power generation and transmission systems, and fault tolerant systems, including multidisplay system in a cockpit, multiengine system in an airplane, and multipump system in a hydraulic control system (see, e.g., Subasi et al., 2017 [22] and the references therein).

The probability bounding problem has been extensively studied throughout the history of probability theory. Boole (1854) [40] was the first who discovered a basic inequality and a general approximation scheme for the probability of the union of events. The next well-known bounds were presented by Bonferroni (1937) [41].

However, Boole and Bonferroni bounds are weak in general. Hailperin (1965) [42] proved that Boole's method was equivalent to Fourier-Motzkin elimination. Fréchet (1940) [43] obtained the first bounds based on the knowledge of the first binomial moment. His results were followed by Dawson and Sankoff (1967) [44] where the sharp Bonferroni bounds based on the first two binomial moments were presented. Kwerel (1975) [45] reproduced and extended Dawson-Sankoff results to present lower and upper bounds for the probability that exactly  $k$ -out-of- $n$  occurs, using the first three binomial moments. Prékopa (1988-1991) [1, 8, 46] discovered that the sharp bounds for the  $k$ -out-of- $n$  type probabilities and expectations of higher order convex functions of discrete random variables can be obtained as the optimum values of discrete moment problems. Boros and Prékopa (1989) [7] used a linear programming approach to produce the closed form Bonferroni bounds and bounds for the probabilities that at least  $k$  and exactly  $k$ -out-of- $n$  events occur, based on the knowledge of first four binomial moments. Samuels and Studden (1989) [47] independently discovered the sharp Bonferroni inequalities and moment problems, however, their method is applicable only to small size problems. Other closed form probability bounds were presented by Sathe et al. (1980) [48], Galambos et al. (1980, 1996) [49, 50], Móri and Székely (1985) [51], Prékopa and Gao (2001) [15], Gao and Prékopa (2002) [11], Dohmen and Tittmann (2007) [52], Petrov (2007) [53], Hoppe and Nediak (2008) [54], and Radwan et al. (2011) [55]. Probability bounds based on the probabilities of the individual events and their intersections and graph structures were presented by Hunter (1976) [56], Bukszár and Prékopa (2001) [57], Bukszár (2003) [58], Bukszár, Mási-Nagy, and Szántai (2012) [59], and Veneziani (2008) [60]. The other linear programming based bounding methodologies include Veneziani (2009) [60], Boros, Scozzari, Tardella,

and Veneziani (2014) [61], Prékopa, Ninh, and Alexe (2016) [20] and Yoda and Prékopa [21].

In order to utilize the contribution of the shape of the underlying distribution we proceed as follows:

- (1) Given an  $n$ , randomly generate a logconcave distribution  $\{x_i\}$  satisfying

- $\sum_{i=0}^n x_i = 1$
- $x_i^2 \geq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1$
- $x_i \geq 0, \quad i = 0, 1, \dots, n$

- (2) Given an  $m$ , compute the first  $m$  binomial moments,  $S_1, \dots, S_m$ , by the use of

$$\sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 1, \dots, m.$$

- (3) Assume that the distribution  $\{x_i\}$  is unknown and solve problem

$$\begin{aligned} & \min(\max) \quad \sum_{i=k}^n x_i \\ & \text{subject to} \quad \sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \end{aligned} \tag{2.11}$$

$$x_i \geq 0 \quad i = 0, 1, \dots, n$$

to obtain lower and upper bounds for  $P(X \geq k)$  for some  $1 \leq k \leq n$ , where the shape of the distribution is not used and the binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).

- (4) Next, solve problem (2.8), where the shape of the underlying distribution is used and the binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).
- (5) Compute the optimum values of the optimization problems involved and let
  - $LB$  and  $UB$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step 3, respectively
  - Let  $LB_{logconcavity}$  and  $UB_{logconcavity}$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step (4), respectively.
- (6) Compare the lower and upper bounds reported in Step (5).

### 2.2.1 Example 1. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity assumption

We take  $n = 10$  and generate a probability distribution as described in Section 2.2. In this example, bounds for the probability that  $k$ -out-of-10 events occur are obtained for

- $k = 1, 3, 5, 8, 9$ ,
- based on the first two binomial moments  $S_1, S_2$ ,
- based on the first three binomial moments  $S_1, S_2, S_3$ .

Table 2.1 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

Recall that that problem (2.8) uses the shape (logconcavity constraints) of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. We observe that the bounds obtained by (2.8) are much tighter than those obtained by problem (2.11) as shown in Figure 2.1.

Table 2.2 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

As can be seen from Figure 2.2, the bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problem (2.11). We also remark that the bounds become tighter if we use the first three binomial moments (Figure 2.2) instead of the first two binomial moments (Figure 2.1).

Table 2.1: Example 1. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.7375	1	0.8909	1
$k = 3$	0.3316	0.9319	0.5835	0.6696
$k = 5$	0.0579	0.6509	0.2723	0.3714
$k = 8$	0	0.1711	3.82E-07	0.0617
$k = 9$	0	0.1222	0	0.0259



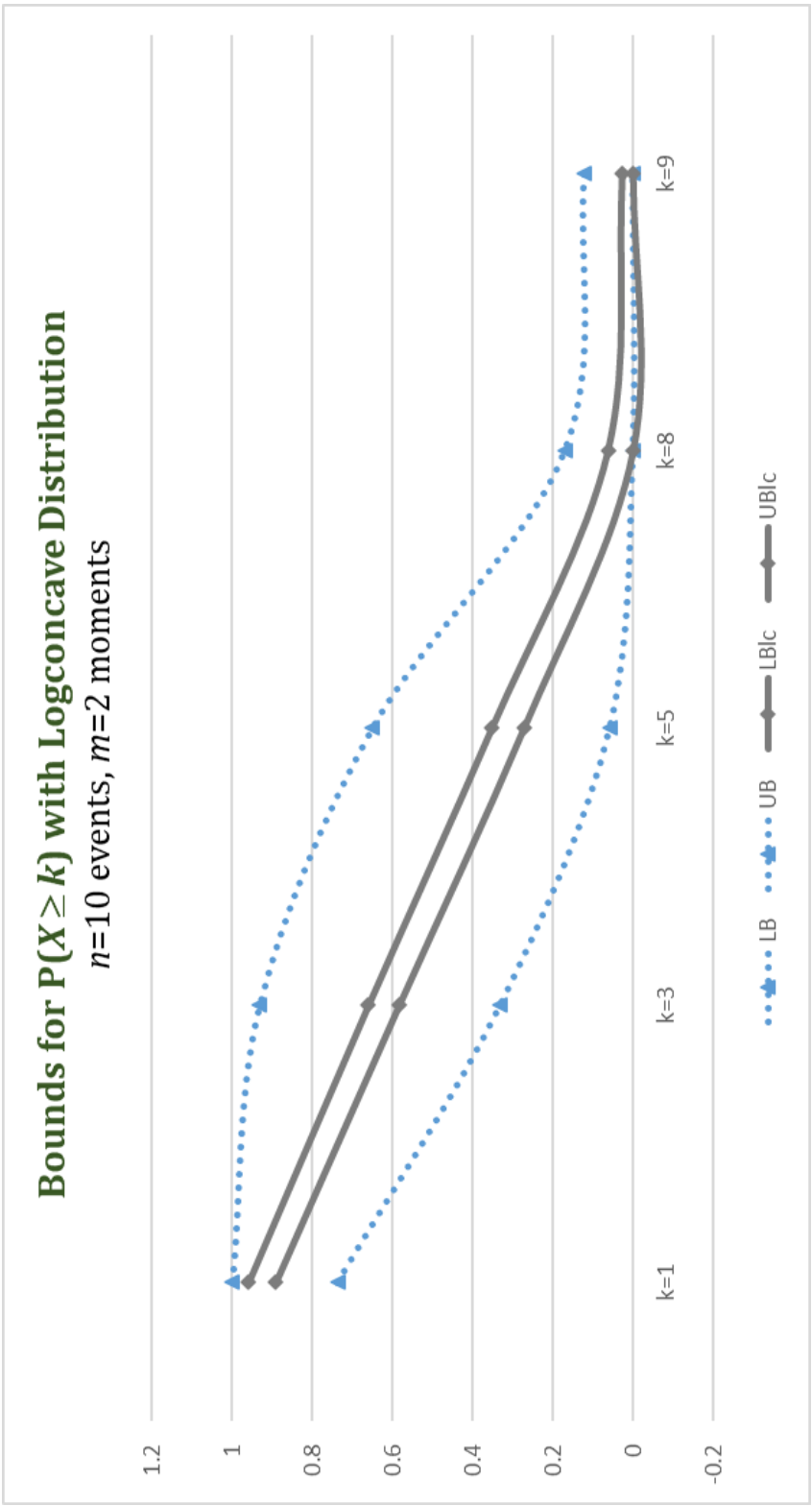


Figure 2.1: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Table 2.2: Example 1. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.8275	1	0.9299	0.9962
$k = 3$	0.3636	0.8729	0.5914	0.6692
$k = 5$	0.0675	0.5098	0.2744	0.3137
$k = 8$	0	0.1194	0.0329	0.051
$k = 9$	0	0.0707	0	0.0202

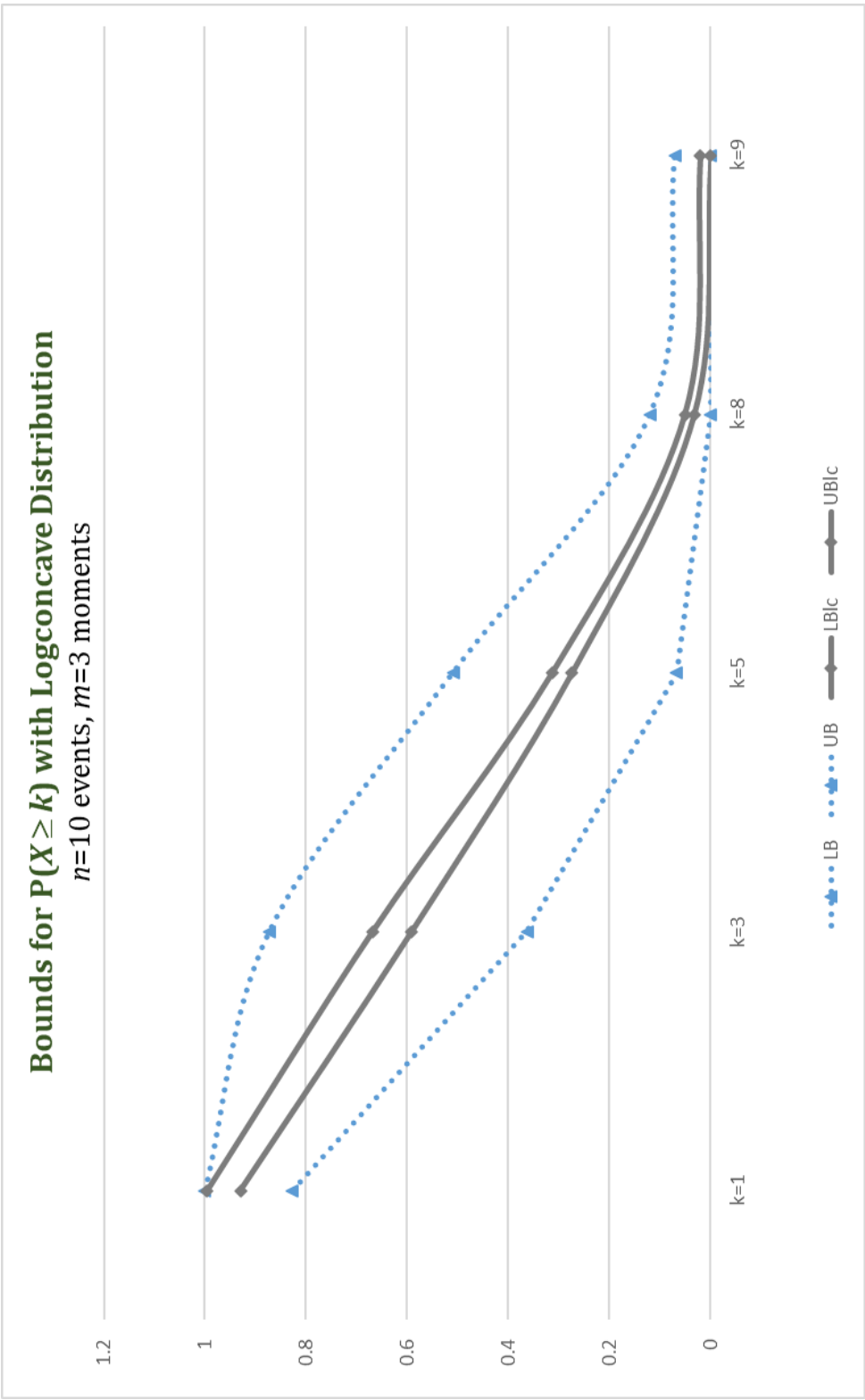


Figure 2.2: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

### 2.2.2 Example 2. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity assumption

We generate another probability distribution, where the size of the support set is  $n = 10$ . The distribution is generated as described in Section 2.2. As in Example 1, the bounds for the probability that  $k$ -out-of-10 events occur are obtained for

- $k = 1, 3, 5, 8, 9$ ,
- based on the first two binomial moments  $S_1, S_2$ ,
- based on the first three binomial moments  $S_1, S_2, S_3$ .

Table 2.3 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

Recall that that problem (2.8) uses the shape (logconcavity constraints) of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. We observe that the bounds obtained by (2.8) are much tighter than those obtained by problem (2.11) as shown in Figure 2.3.

Table 2.4 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

As can be seen from Figure 2.4, the bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problem (2.11). Note that the bounds become tighter if we use the first three binomial moments (Figure 2.4) instead of the first two binomial moments (Figure 2.3).

Table 2.3: Example 2. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.8451	1	0.9504	1
$k = 3$	0.571	1	0.7338	0.8532
$k = 5$	0.0661	0.7451	3.09E-01	0.4751
$k = 8$	0	0.1525	1.90E-07	0.0545
$k = 9$	0	0.1016	0	0.0214

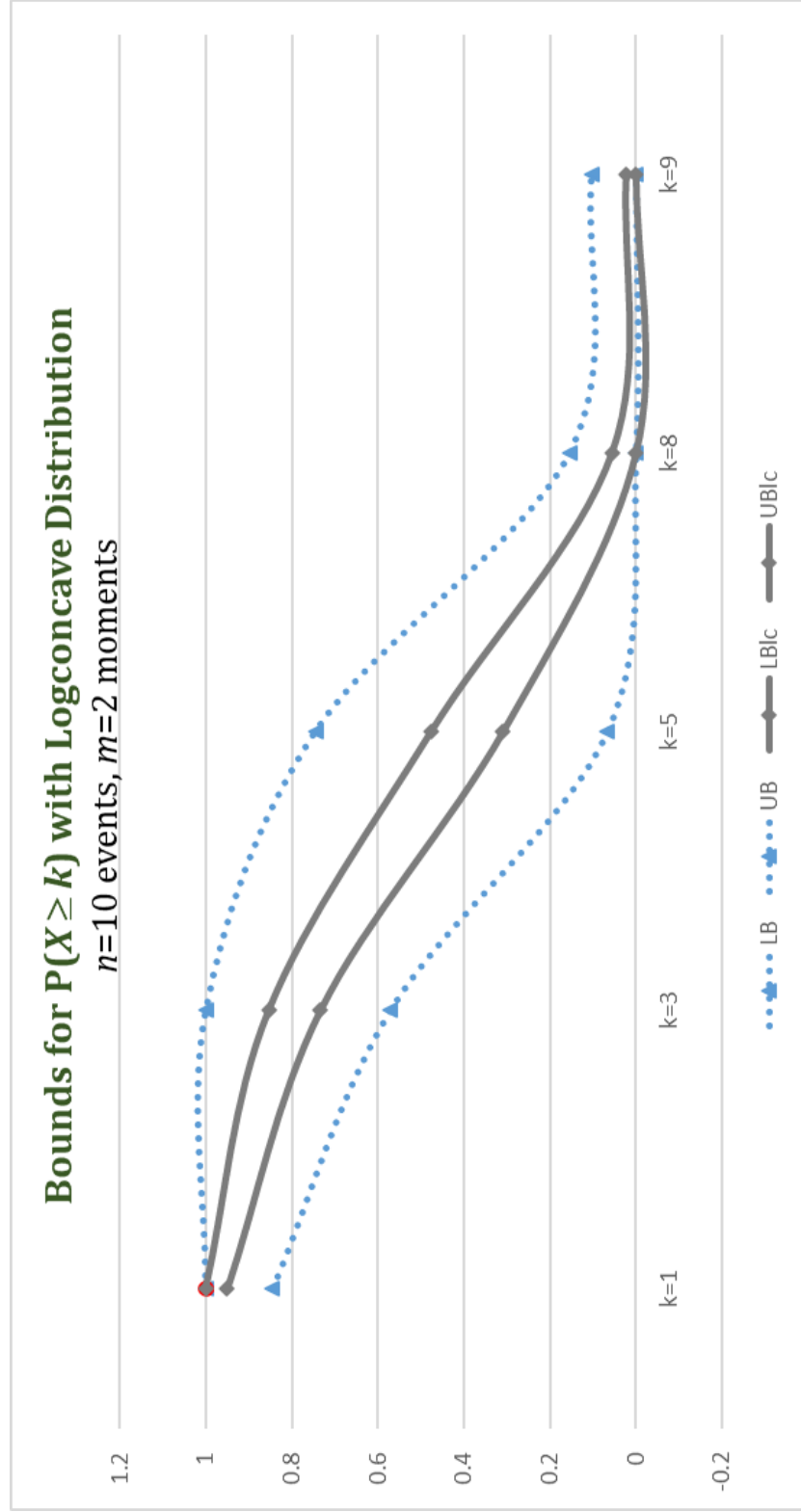


Figure 2.3: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Table 2.4: Example 2. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.9007	1	0.9706	1
$k = 3$	0.5915	0.9554	0.7432	0.821
$k = 5$	0.0831	0.6318	0.3298	0.4101
$k = 8$	0	0.0998	4.74E-07	0.0346
$k = 9$	0	0.0557	0	0.0117



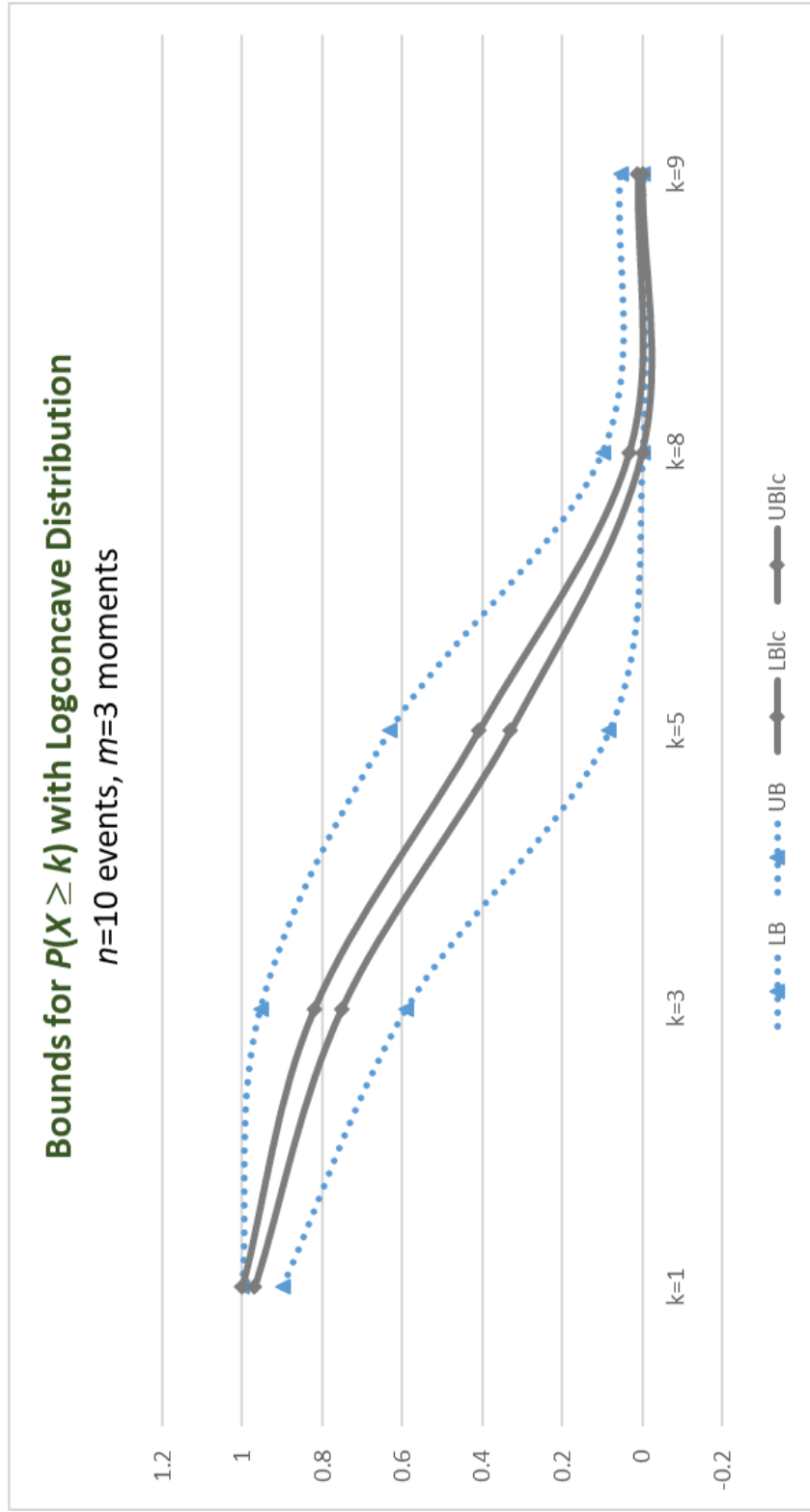


Figure 2.4: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

### 2.2.3 Example 3. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity assumption

As in Examples 1 and 2, the distribution with  $n = 10$  is generated as described in Section 2.2 and the bounds for the probability that  $k$ -out-of-10 events occur are obtained for

- $k = 1, 3, 5, 8, 9$ ,
- based on the first two binomial moments  $S_1, S_2$ ,
- based on the first three binomial moments  $S_1, S_2, S_3$ .

Table 2.5 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

Recall that that problem (2.8) uses the shape (logconcavity constraints) of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. We observe that the bounds obtained by (2.8) are much tighter than those obtained by problem (2.11) as shown in Figure 2.5.

Table 2.6 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8).

As can be seen from Figure 2.6, the bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problem (2.11). Here again we observe that the bounds become much tighter if we use the first three binomial moments (Figure 2.6) instead of the first two binomial moments (Figure 2.5).

Table 2.5: Example 3. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.8439	1	0.9553	1
$k = 3$	0.6315	1	0.8062	0.8899
$k = 5$	0.1618	0.8731	0.4593	0.5987
$k = 8$	0	0.2655	4.33E-08	0.1242
$k = 9$	0	0.1769	0	0.0549

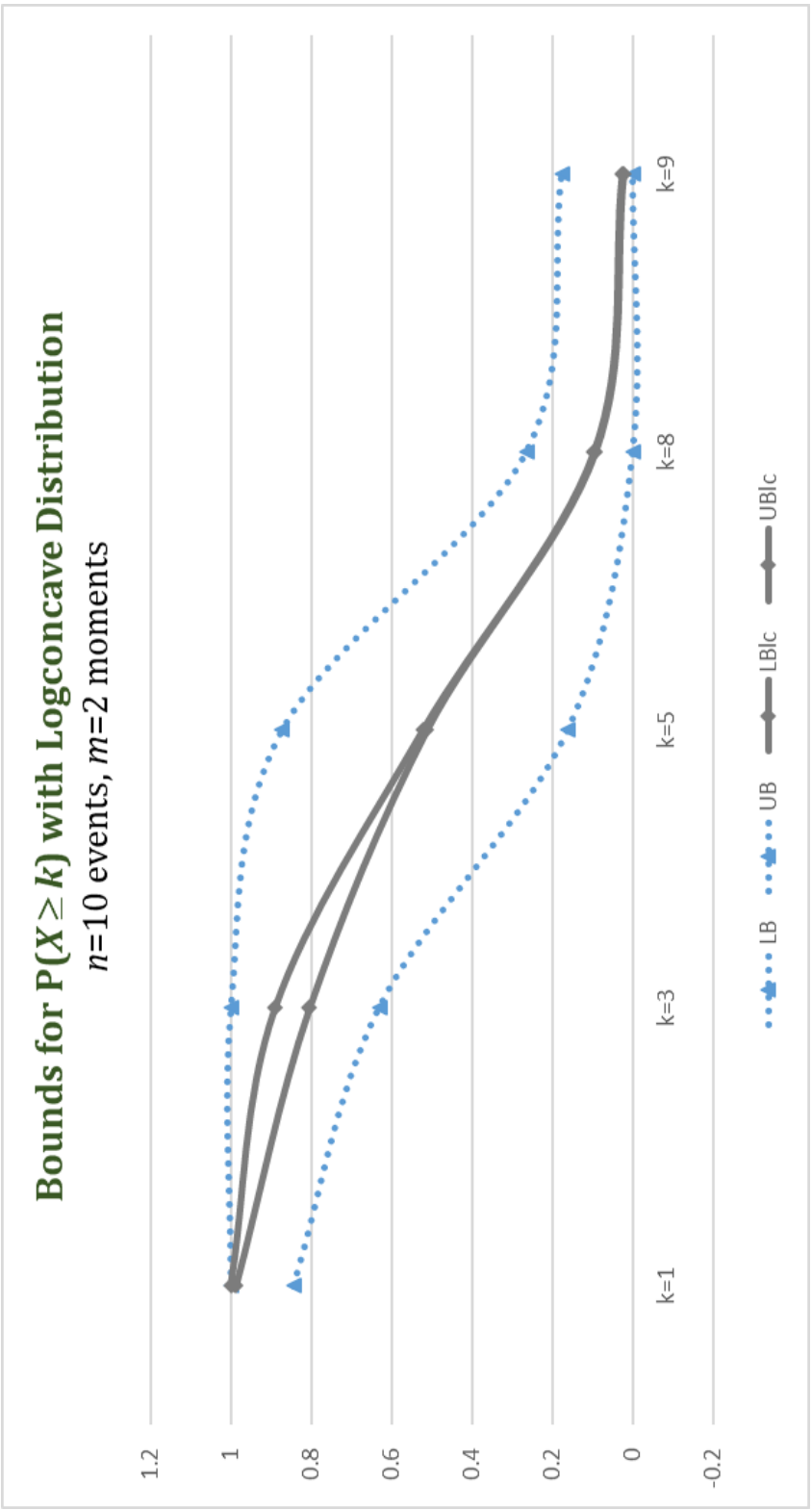


Figure 2.5: Example 3. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 2$  moments

Table 2.6: Example 3. Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{logconcavity}$	$UB_{logconcavity}$
$k = 1$	0.8899	1	0.9604	0.9821
$k = 3$	0.6332	0.9433	0.8121	0.83456
$k = 5$	0.3207	0.8457	0.5608	0.5928
$k = 8$	0	0.1682	0.0211	0.0702
$k = 9$	0	0.0897	0	0.0164

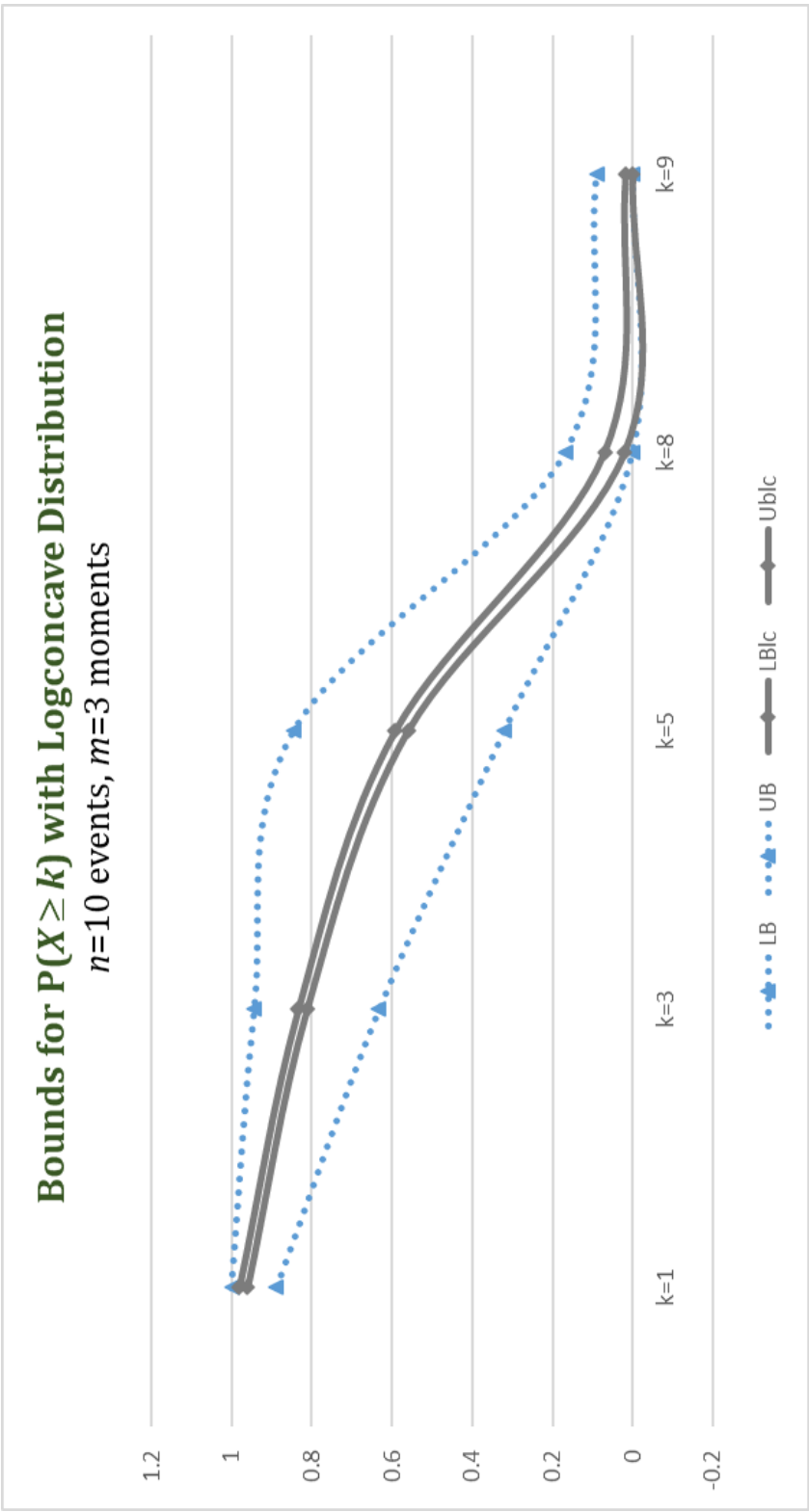


Figure 2.6: Example 3. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity:  $n = 10$  events,  $m = 3$  moments

## 2.3 Numerical Examples: Logconcavity vs. Unimodality

Recall that any logconcave discrete distribution is also unimodal (Fekete, 1912 [30], Prékopa, 1995 [4]). The goal of this section is to we make use of this information to compare the bounds for the probability that at least  $k$ -out-of- $n$  events occur for the following three cases:

- the shape of the underlying probability distribution is not taken into account, i.e., the bounds are obtained as the optimum values of problem (2.11),
- unimodality constraints are prescribed into the problem, i.e., the bounds are obtained as the optimum values of problem (1.4),
- logconcavity constraints are used, i.e., the bounds are the optimum values of problem (2.8).

In order to provide numerical examples, where we compare the contribution of the unimodality constraints (1.3) and (2.3), we follow the steps given below.

- (1) Given an  $n$ , randomly generate a logconcave distribution  $\{x_i\}$  satisfying

- $\sum_{i=0}^n x_i = 1$
- $x_i^2 \geq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1$
- $x_i \geq 0, \quad i = 0, 1, \dots, n$

- (2) Given an  $m$ , compute the first  $m$  binomial moments,  $S_1, \dots, S_m$ , by the use of

$$\sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 1, \dots, m.$$



- (3) Assume that the distribution  $\{x_i\}$  is unknown and solve problem (2.11) to obtain lower and upper bounds for  $P(X \geq k)$  for some  $1 \leq k \leq n$ , where the shape of the distribution is not used and the binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).
- (4) Solve problem (2.8), where the logconcavity of the underlying distribution is assumed and binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).
- (5) Next, find mode  $M$  of the distribution generated in Step (1) and solve problem (1.4) that uses the unimodality constraints and binomial moments,  $S_1, \dots, S_m$  found in Step (2).
- (6) Compute the optimum values of the optimization problems involved and let
  - $LB$  and  $UB$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step (3), respectively
  - Let  $LB_{logconcavity}$  and  $UB_{logconcavity}$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step (4), respectively
  - Let  $LB_{unimodality}$  and  $UB_{unimodality}$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step (5), respectively.
- (7) Compare the lower and upper bounds reported in Step (6).

### 2.3.1 Example 1. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity and unimodality assumption

We observe that the logconcave distribution in Example 1 in Section 2.2.1 is unimodal with mode  $M = 2$ . In this example, we apply the procedure described in Section 2.3 to find bounds for the probability that  $k$ -out-of-10 events occur, where

- $k = 1, 3, 5, 8, 9$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.7 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where

- $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively,
- $LB_{logconcavity}$  and  $UB_{logconcavity}$  are the lower and upper bounds obtained from problem (2.8), respectively,
- $LB_{unimodality}$  and  $UB_{unimodality}$  are the lower and upper bounds obtained from problem (1.4), respectively,
- $\Delta_{lc} = UB_{logconcavity} - LB_{logconcavity}$ ,
- $\Delta_u = UB_{unimodal} - LB_{unimodal}$ .

Recall that that problem (2.8) uses the logconcavity constraints and problem (1.4) uses the unimodality constraints as the shape of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. As can be observed in Figure 2.7, the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.8 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

Similar to the two moment case, we can see from Figure 2.8 that the bounds for the probability that at least  $k$ -out-of-10 events occur, for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). We also remark that the bounds become tighter if we use the first three binomial moments (Figure 2.8) instead of the first two binomial moments (Figure 2.7).

Table 2.7: Example 1. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 2$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.7375	1	0.8941	1	0.8909	1	0.1059	0.1091
$k = 3$	0.3316	0.9319	0.4079	0.6651	0.5835	0.6596	0.2472	0.0861
$k = 5$	0.0579	0.6509	0.1911	0.3578	0.2723	0.3714	0.1667	0.0991
$k = 8$	0	0.1711	0.00E+00	0.0917	3.82E-07	0.0617	0.0917	6.17E-02
$k = 9$	0	0.1222	0.00E+00	0.0458	0	0.0259	0.0458	0.0259

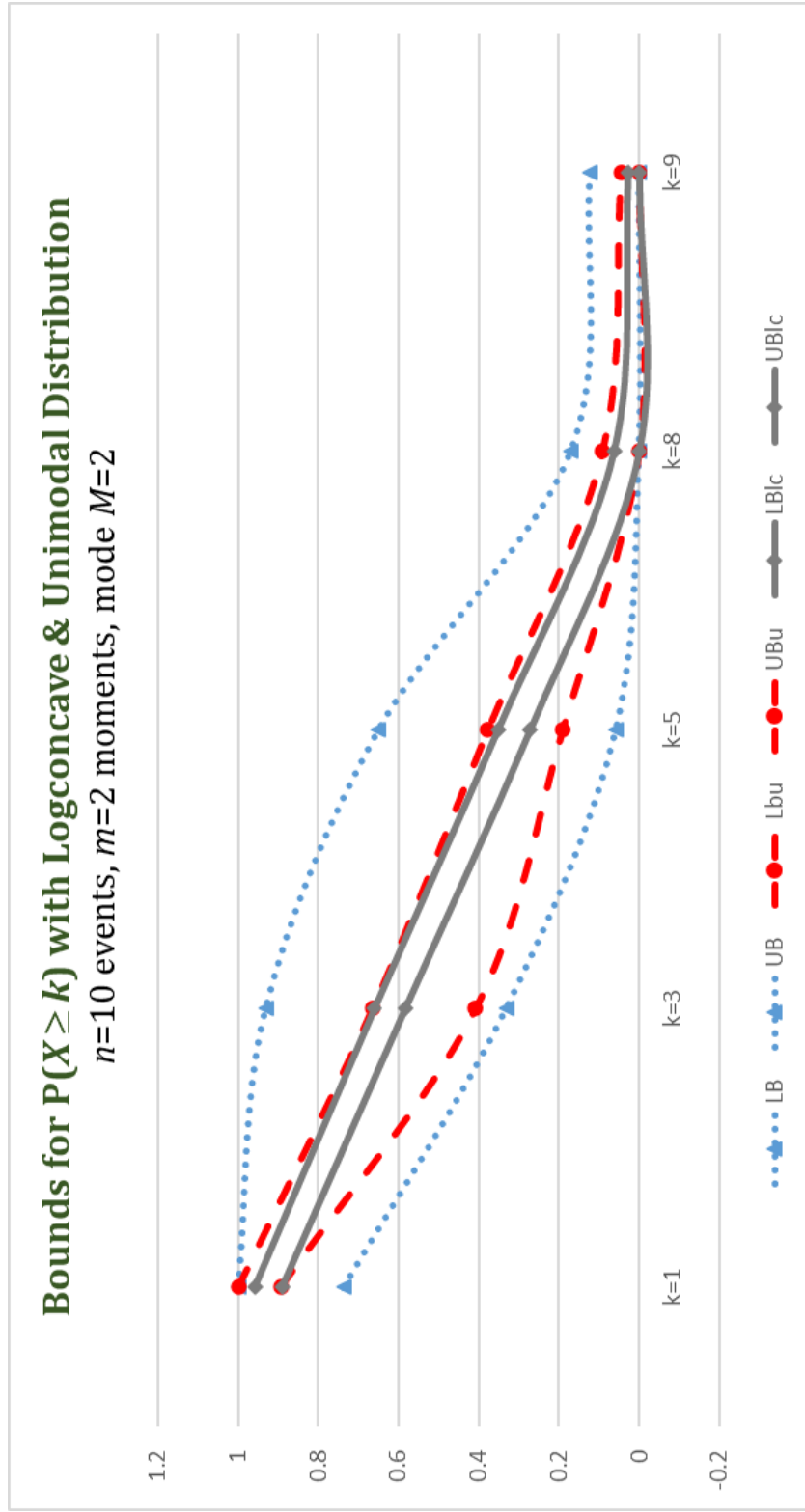


Figure 2.7: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 2$

Table 2.8: Example 1. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 2$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8275	1	0.9175	1	0.9299	0.9962	0.0825	0.0663
$k = 3$	0.3636	0.8729	0.5105	0.6651	0.5914	0.6592	0.1446	0.0778
$k = 5$	0.0675	0.5098	0.2393	0.3306	0.2744	0.3137	0.0913	0.0393
$k = 8$	0	0.1194	0.014991	0.0617	0.0329	0.051	0.046709	0.0181
$k = 9$	0	0.0707	0	0.0293	0	0.0202	0.0293	0.0202

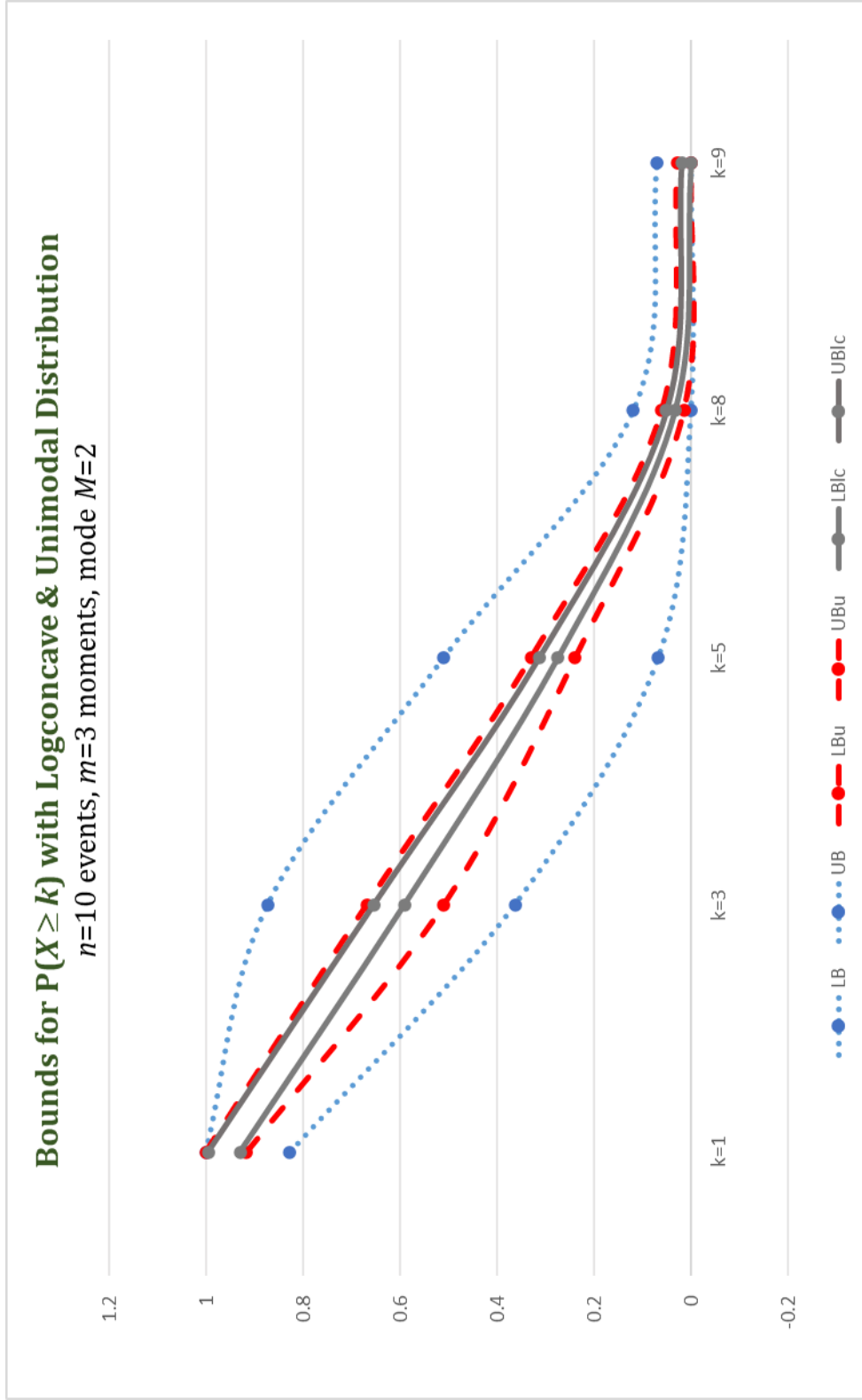


Figure 2.8: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 2$

### 2.3.2 Example 2. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity and unimodality assumption

We consider the logconcave distribution in Example 2 in Section 2.2.2. This distribution is unimodal with mode  $M = 4$ . In this example, we apply the procedure described in Section 2.3 to find bounds for the probability that  $k$ -out-of-10 events occur, where

- $k = 1, 3, 5, 8, 9$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.9 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

As can be seen in Figure 2.9, the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.10 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

Similar to the two moment case, Figure 2.10 shows that the bounds for the probability that at least  $k$ -out-of-10 events occur, for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Note also that the bounds become tighter if we use the first three binomial moments (Figure 2.10) instead of the first two binomial moments (Figure 2.9).



Table 2.9: Example 2. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 4$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8451	1	0.9322	1	0.9504	1	0.0678	0.0496
$k = 3$	0.571	1	0.7499	0.9311	0.7338	0.8532	0.1812	0.1194
$k = 5$	0.0661	0.7451	0.1499	0.4608	3.09E-01	0.4751	0.3109	0.1661
$k = 8$	0	0.1525	0.00E+00	0.0789	1.90E-07	0.0545	0.0789	5.45E-02
$k = 9$	0	0.1016	0.00E+00	0.0395	0	0.0214	0.0395	0.0214

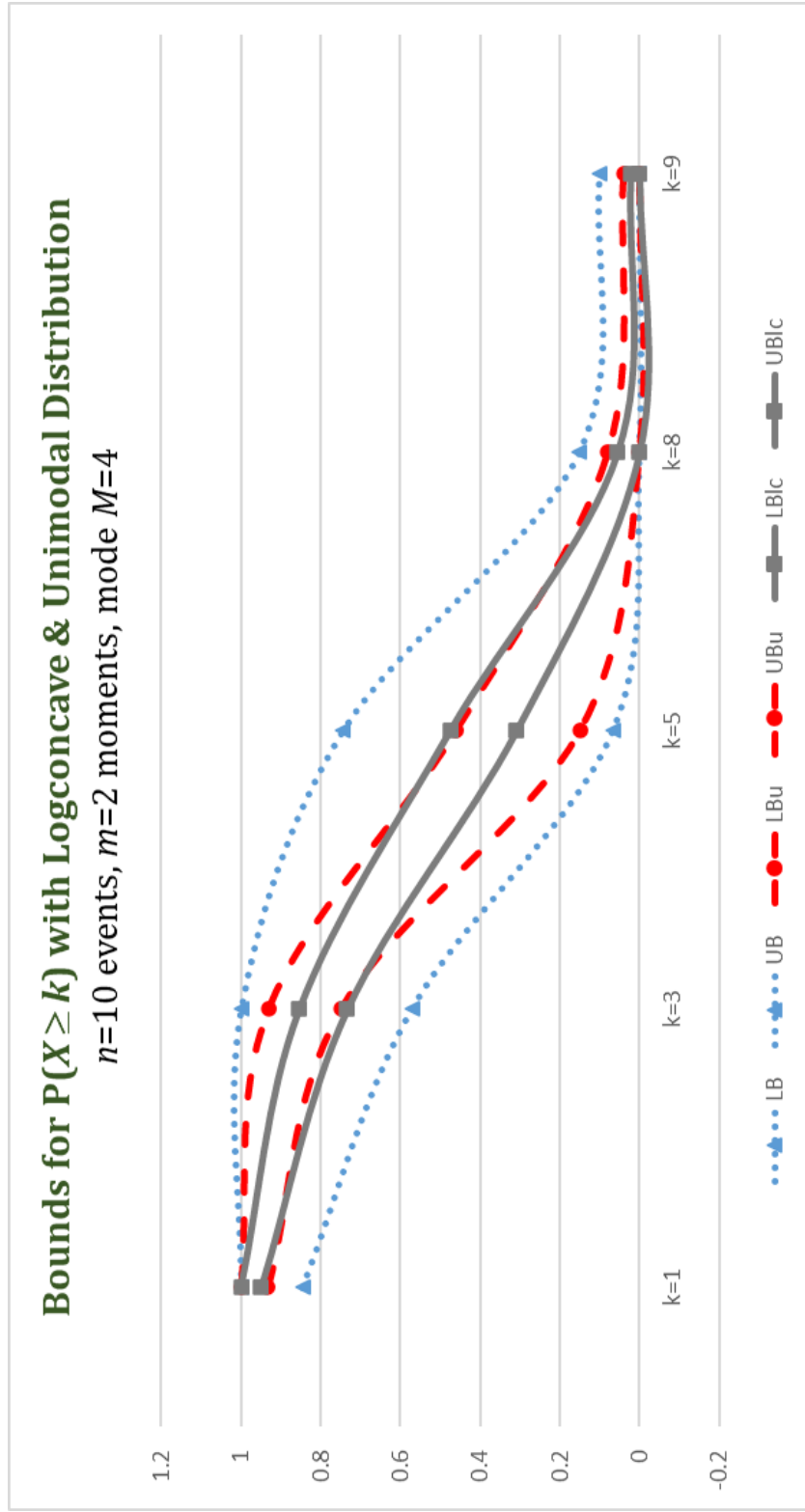


Figure 2.9: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 4$

Table 2.10: Example 2. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 4$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.9007	1	0.9482	1	0.9706	1	0.0518	0.0294
$k = 3$	0.5915	0.9554	0.7528	0.8829	0.7432	0.821	0.1301	0.0778
$k = 5$	0.0831	0.6318	0.1789	0.4081	0.3298	0.4101	0.2292	0.0803
$k = 8$	0	0.0998	0	0.0542	4.74E-07	0.0346	0.0542	3.46E-02
$k = 9$	0	0.0557	0	0.0256	0	0.0117	0.0256	0.0117

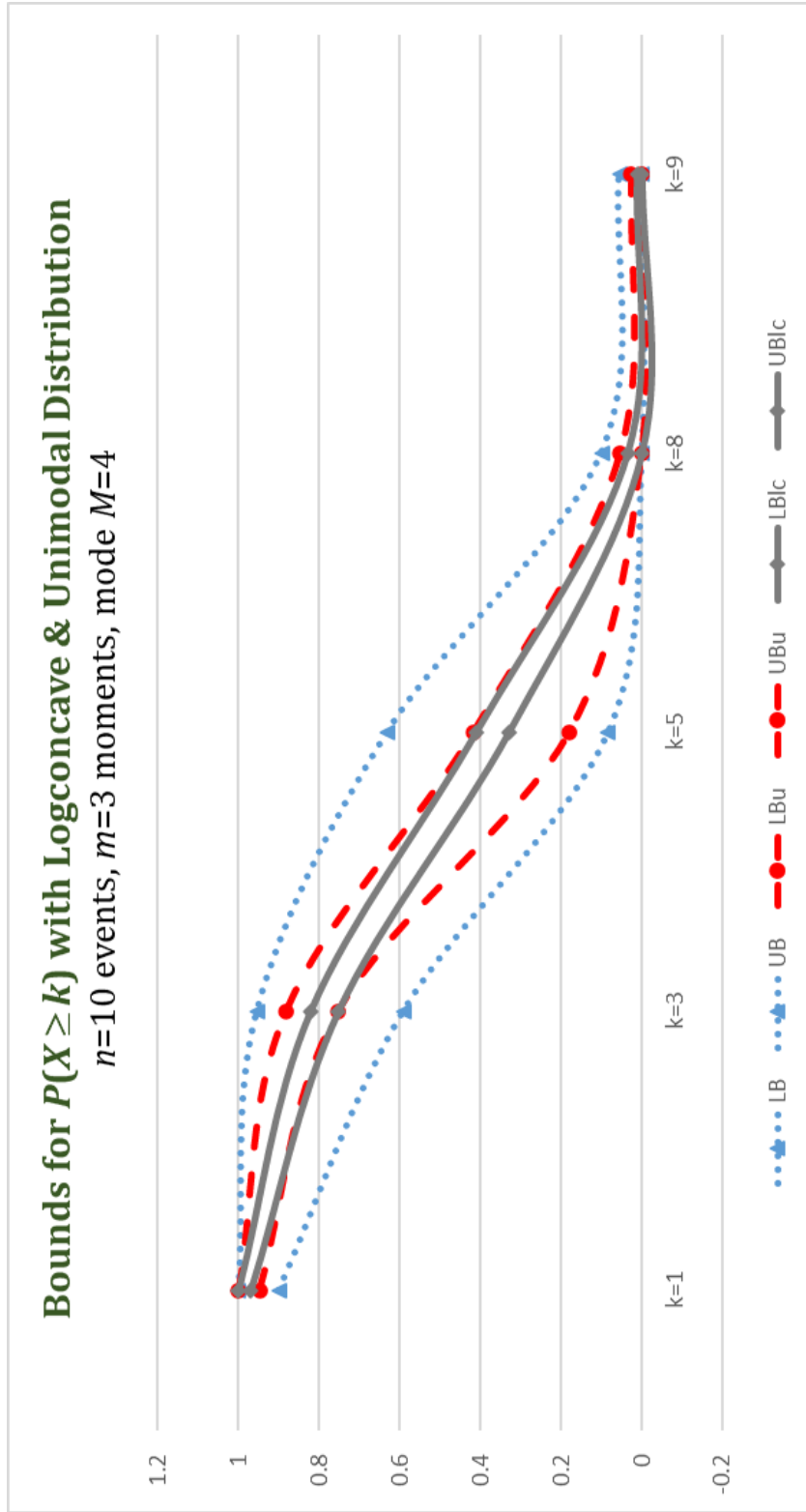


Figure 2.10: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 4$

### 2.3.3 Example 3. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconcavity and unimodality assumption

Next, we consider the logconcave distribution in Example 3 in Section 2.2.3. This distribution is unimodal with mode  $M = 6$ . In this example, we apply the procedure described in Section 2.3 to find bounds for the probability that  $k$ -out-of-10 events occur, where

- $k = 1, 3, 5, 8, 9$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.11 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

As can be seen in Figure 2.11, the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.12 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

Similar to the two moment case, Figure 2.12 shows that the bounds for the probability that at least  $k$ -out-of-10 events occur, for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Figures 2.11 and 2.12 show that the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.11: Example 3. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 6$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8439	1	0.9342	1	0.9553	1	0.0658	0.0447
$k = 3$	0.6315	1	0.7895	0.8939	0.8062	0.8899	0.1044	0.0837
$k = 5$	0.1618	0.8731	0.4992	0.7029	0.4593	0.5987	0.2037	0.1394
$k = 8$	0	0.2655	0.00E+00	0.1163	4.33E-08	0.1242	0.1163	1.24E-01
$k = 9$	0	0.1769	0.00E+00	0.0581	0	0.0549	0.0581	0.0549

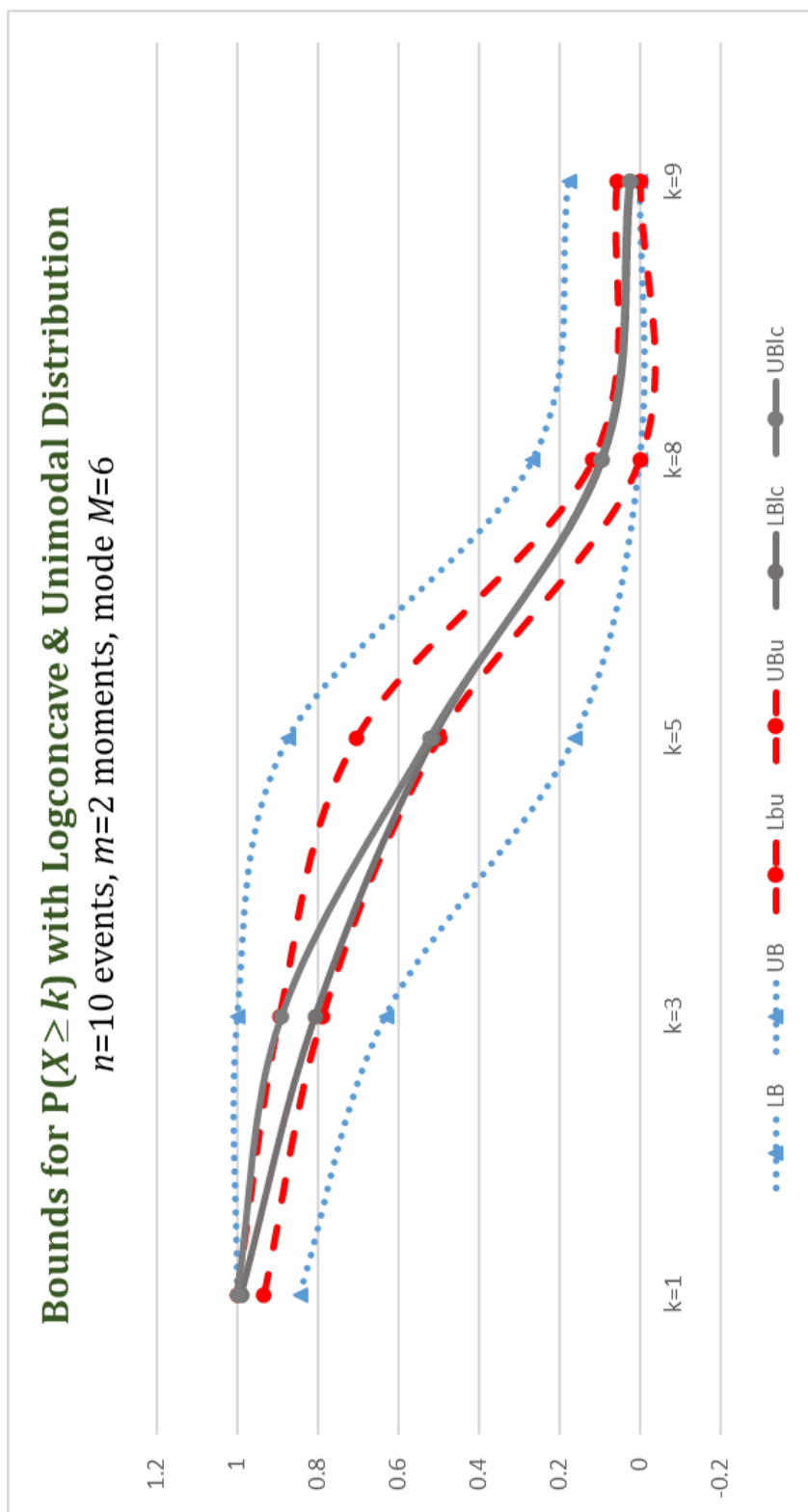


Figure 2.11: Example 3. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 2$  moments, mode  $M = 6$

Table 2.12: Example 3. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 6$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8899	1	0.9471	1	0.9604	0.9821	0.0529	0.0217
$k = 3$	0.6332	0.9433	0.7904	0.8657	0.8121	0.83456	0.0753	0.02246
$k = 5$	0.3207	0.8457	0.55081	0.6642	0.5608	0.5928	0.11339	0.032
$k = 8$	0	0.1682	0.0029	0.0874	0.0211	0.0702	0.0845	0.0491
$k = 9$	0	0.0897	0	0.0364	0	0.0164	0.0364	0.0164



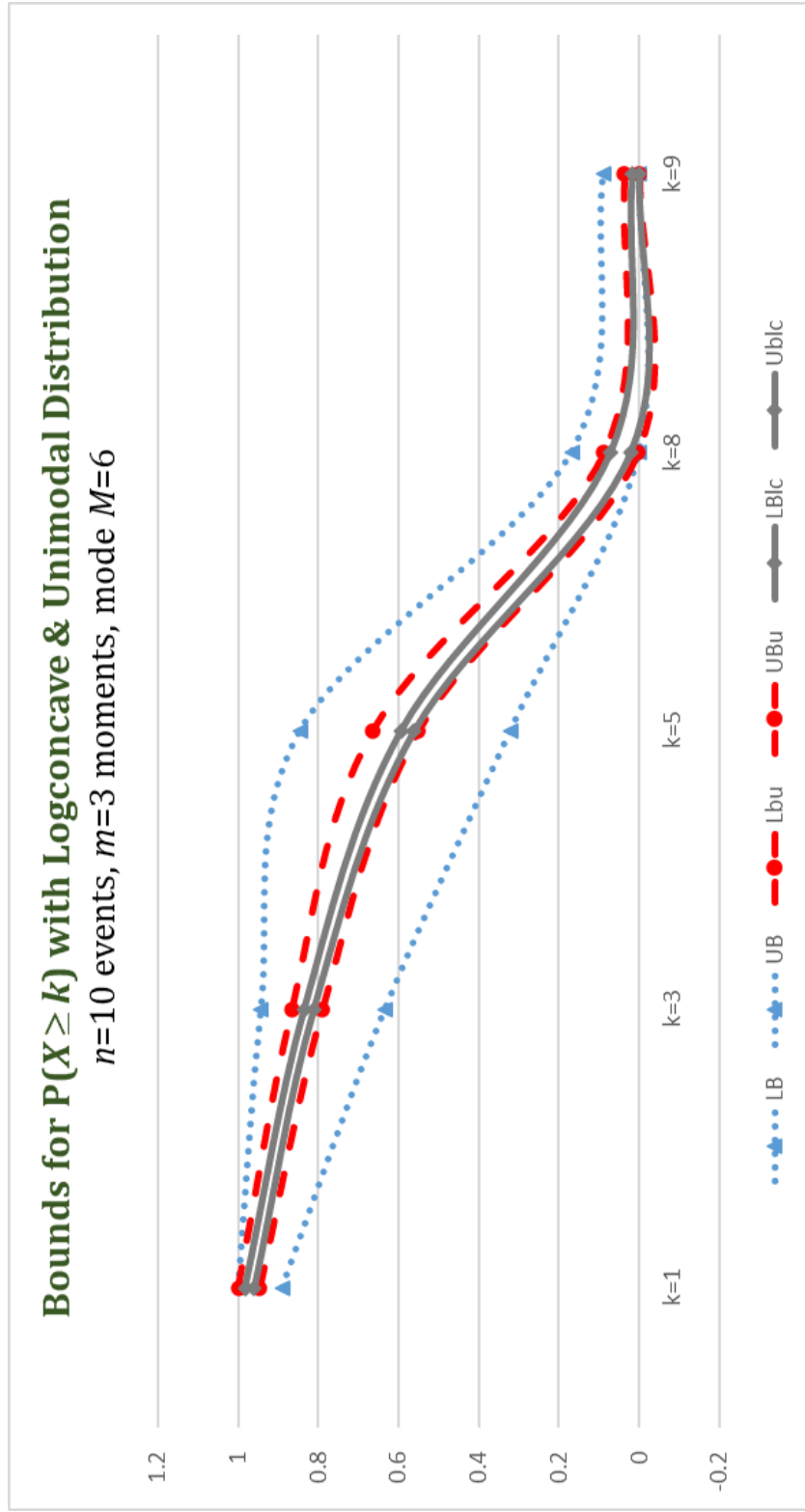


Figure 2.12: Example 3. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 10$  events,  $m = 3$  moments, mode  $M = 6$

#### 2.3.4 Example 4. Bounds for the probability that at least $k$ -out-of-20 events occur under the logconcavity and unimodality assumption

In this example we take  $n = 20$  and apply the procedure described in Section 2.3 to generate a new logconcave probability distribution. We observe that this distribution is unimodal with mode  $M = 3$ . We then obtain bounds for the probability that  $k$ -out-of-20 events occur, where

- $k = 1, 3, 5, 8, 9, 15, 18, 19$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.13 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

Figure 2.13 shows that the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.14 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events for  $k = 1, 3, 5, 8, 9, 15, 18, 19$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

As can be seen from Figure 2.14, the bounds for the probability that at least  $k$ -out-of-20 events occur, for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Figures 2.13 and 2.14 show that the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.13: Example 4. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 2$  moments, mode  $M = 3$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.7171	1	0.9329	1	0.9331	1	0.0671	0.0669
$k = 3$	0.5709	1	0.7963	1	0.8008	0.8902	0.2037	0.0894
$k = 5$	0.3377	1	0.5055	0.698	0.0.62091	0.6809	0.1925	0.0609
$k = 8$	0.0993	0.8052	0.2835	0.4816	3.98E-01	0.4789	0.1981	0.0809
$k = 9$	0.0736	0.7581	0.2468	0.419	0.33621	0.4156	0.1722	0.07939
$k = 15$	0	0.2557	2.62E-02	0.1224	0.0393	0.0964	0.0962	0.0571
$k = 18$	0	0.1515	1.20E-06	0.0489	9.63E-08	0.0312	0.048899	0.031199
$k = 19$	0	0.1298	4.49E-07	0.0245	0	0.0146	0.024499	0.0146

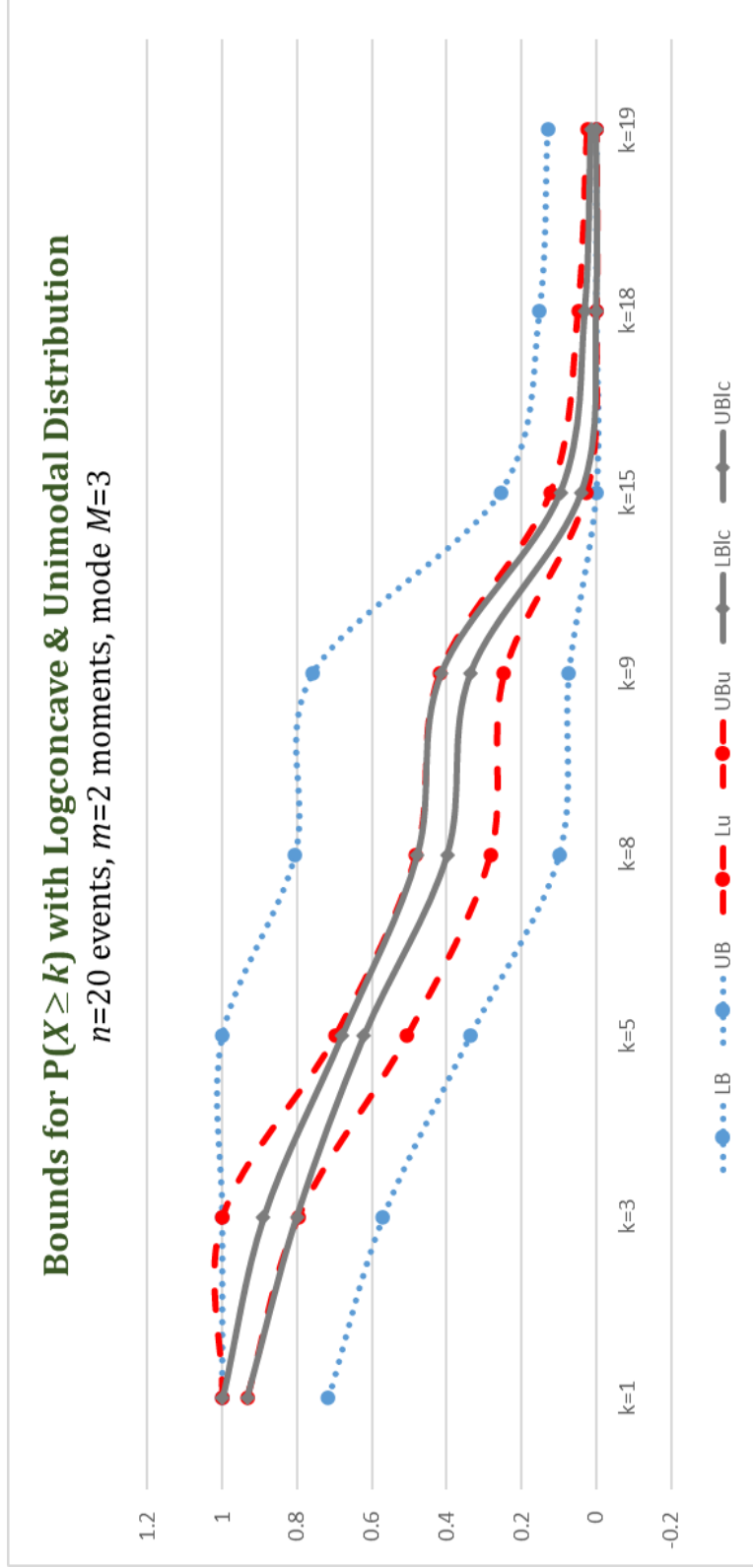


Figure 2.13: Example 4. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 2$  moments, mode  $M = 3$

Table 2.14: Example 4. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 3$  moments, mode  $M = 3$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8018	1	0.9448	1	0.9472	1	0.0552	0.0528
$k = 3$	0.6283	1	0.8109	0.9413	0.8177	0.8503	0.1304	0.0326
$k = 5$	0.3735	0.9065	0.5734	0.6849	0.6491	0.6962	0.1128	0.0458
$k = 8$	0.1587	0.8035	0.3898	0.4637	0.4279	0.4569	0.0739	0.029
$k = 9$	0.0771	0.6859	0.3293	0.3959	0.3579	0.3837	0.0666	0.0258
$k = 15$	0	0.2085	0.046	0.0975	0.0709	0.0892	0.0515	0.0183
$k = 18$	0	0.0908	0	0.0295	4.30E-07	0.0225	0.0295	0.022499
$k = 19$	0	0.0714	0	0.0147	0	0.0101	0.0147	0.0101

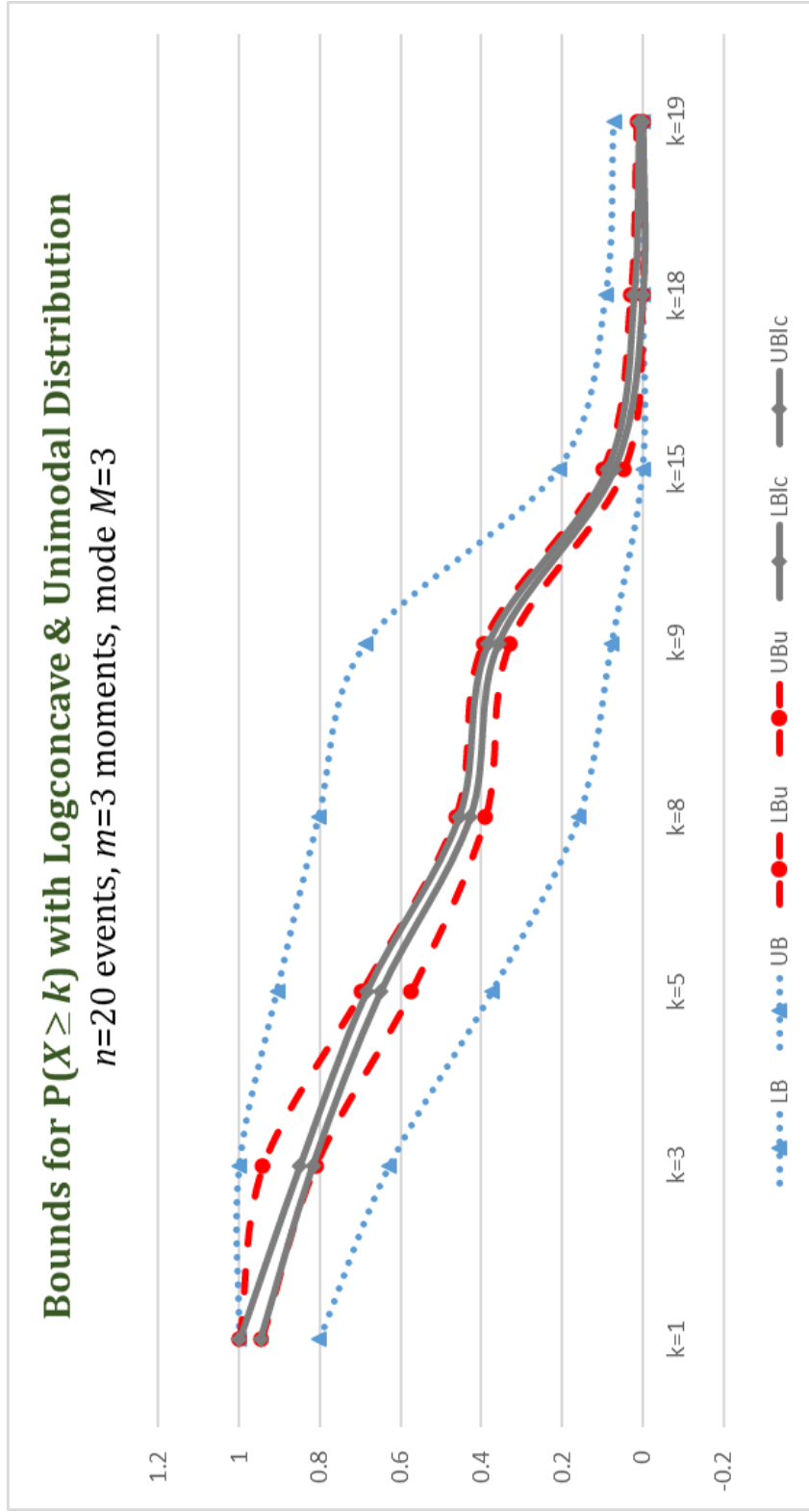


Figure 2.14: Example 4. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 3$  moments, mode  $M = 3$

### 2.3.5 Example 5. Bounds for the probability that at least $k$ -out-of-20 events occur under the logconcavity and unimodality assumption

In this example we take  $n = 20$  and apply the procedure described in Section 2.3 to generate another logconcave probability distribution. This new distribution is unimodal with mode  $M = 9$ . As in Example 4 in Section 2.3.4, we obtain the bounds for the probability that  $k$ -out-of-20 events occur, where

- $k = 1, 3, 5, 8, 9, 15, 18, 19$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.15 shows the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

Figure 2.15 gives the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Lower and upper bounds for the probability that at least  $k$ -out-of-20 events for  $k = 1, 3, 5, 8, 9, 15, 18, 19$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$  are presented in Table 2.16.

We observe from Figure 2.16 that the bounds for the probability that at least  $k$ -out-of-20 events occur, for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). As in previous examples, the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.15: Example 5. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 2$  moments, mode  $M = 9$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$UB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8595	1	0.9723	1	0.9809	1	0.0277	0.0191
$k = 3$	0.7945	1	0.9168	1	0.9364	0.9999	0.0832	0.0635
$k = 5$	0.686	1	0.8584	1	0.8682	0.9929	0.1416	0.1247
$k = 8$	0.3503	1	0.6503	0.8862	0.6545	0.7501	0.2359	0.0956
$k = 9$	0.1913	0.955	0.5159	0.8409	0.5535	0.6819	0.325	0.1284
$k = 15$	0	0.3925	0.0498	0.1919	0.0775	0.1799	0.1421	0.1024
$k = 18$	0	0.2029	0.00E+00	0.075	1.99E-08	0.0566	0.075	5.66E-02
$k = 19$	0	0.167	0.00E+00	0.0375	0.00E+00	0.0249	0.0375	0.0249



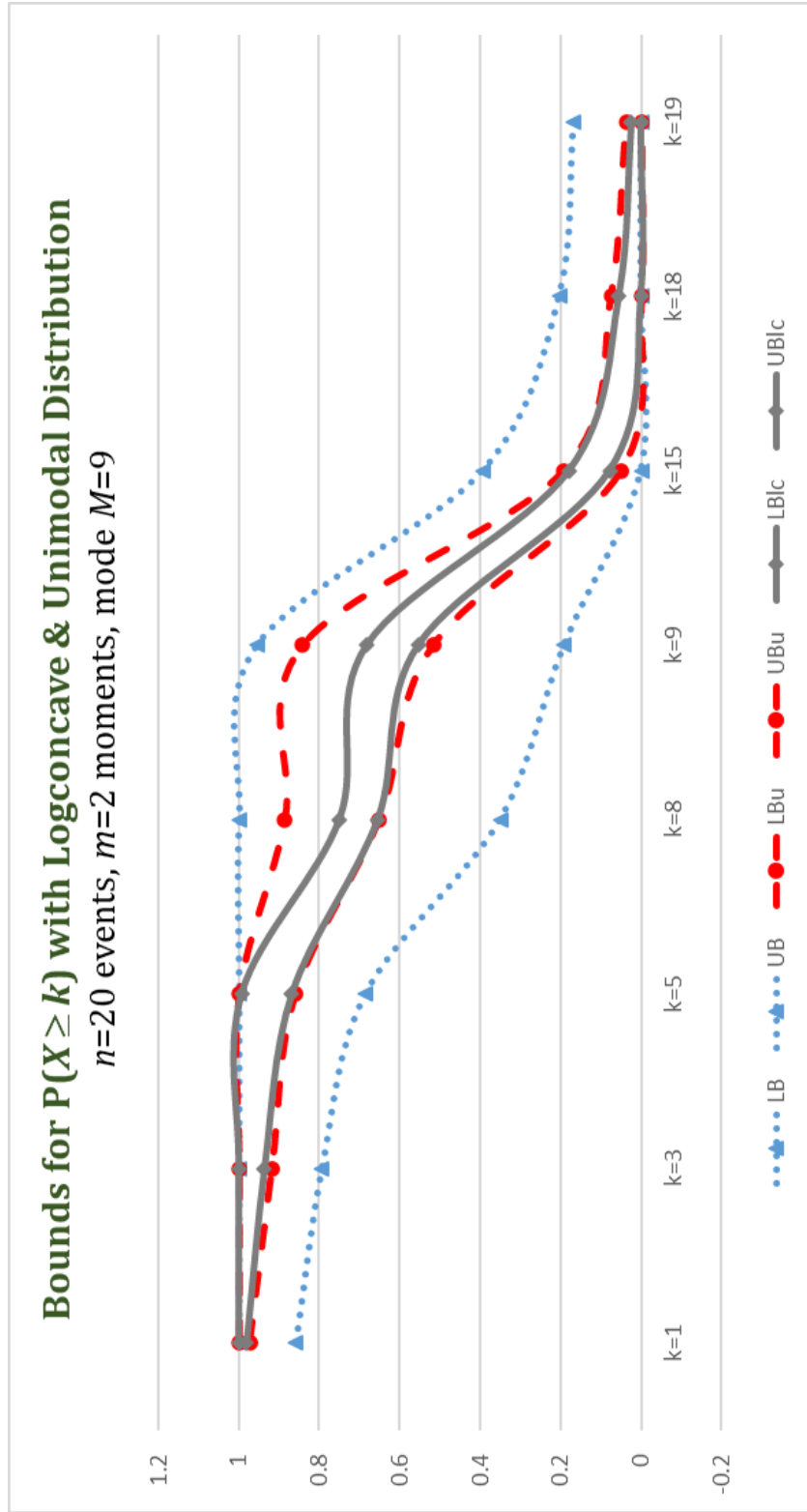


Figure 2.15: Example 5. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 2$  moments, mode  $M = 9$

Table 2.16: Example 5. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 3$  moments, mode  $M = 9$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.9267	1	0.9813	1	0.992	1	0.0187	0.008
$k = 3$	0.8702	1	0.9439	1	0.9655	0.9999	0.0561	0.0344
$k = 5$	0.7496	1	0.8817	0.9614	0.8909	0.9312	0.0797	0.0403
$k = 8$	0.4051	0.9571	0.6689	0.8678	0.6678	0.7412	0.1989	0.0734
$k = 9$	0.3055	0.9384	0.5754	0.8356	0.5909	0.6444	0.2602	0.0535
$k = 15$	0.0101	0.3707	0.1101	0.1845	0.1418	0.1798	0.0744	0.038
$k = 18$	0	0.1442	0	0.0622	7.92E-08	0.0444	0.0622	0.04439992
$k = 19$	0	0.1093	0	0.0311	0	0.0202	0.0311	0.0202

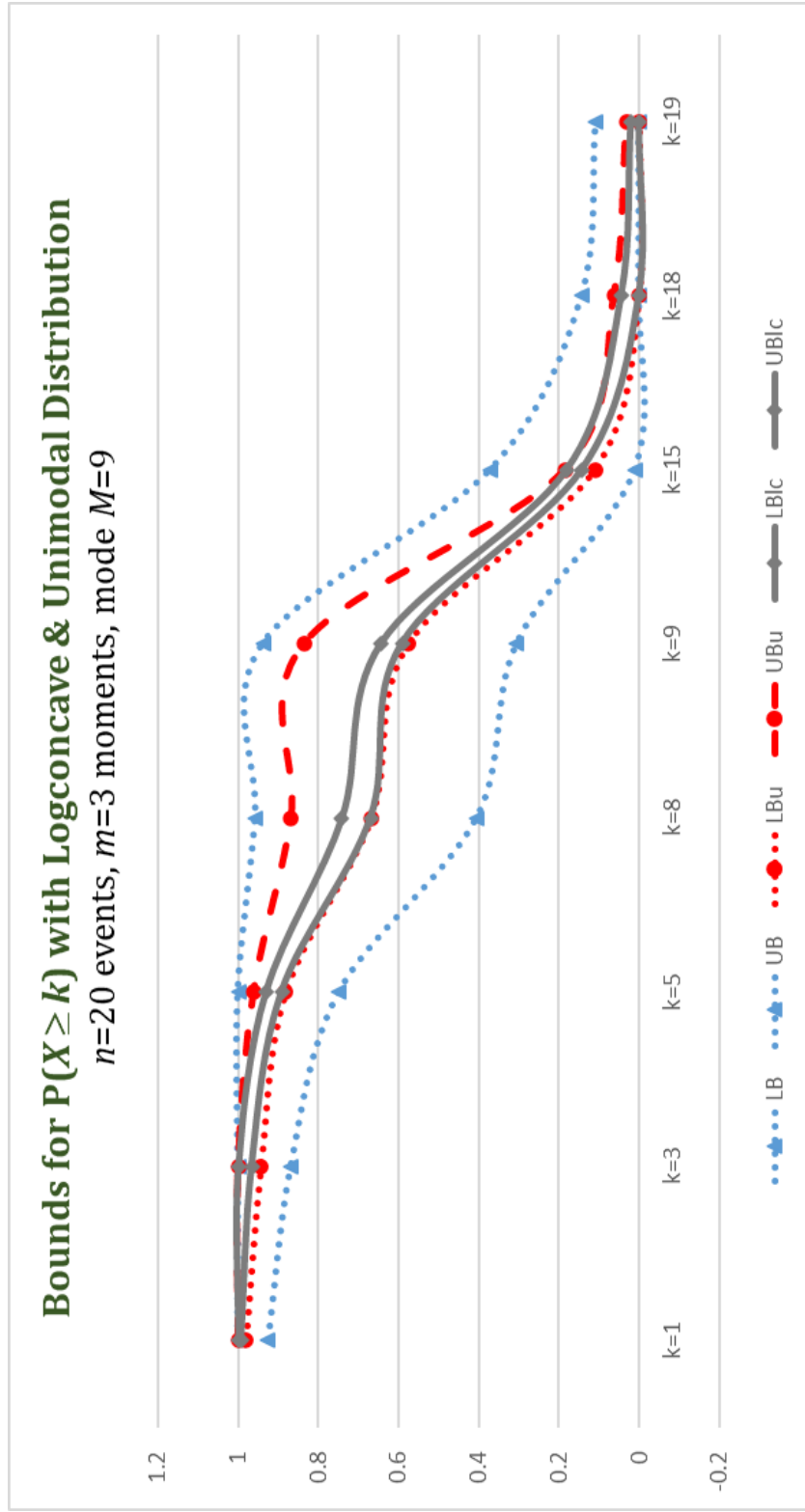


Figure 2.16: Example 5. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 20$  events,  $m = 3$  moments, mode  $M = 9$

### 2.3.6 Example 6. Bounds for the probability that at least $k$ -out-of-30 events occur under the logconcavity and unimodality assumption

In this example we take  $n = 30$  and apply the procedure described in Section 2.3 to generate a new logconcave probability distribution. We observe that this distribution is unimodal with mode  $M = 8$ . We obtain bounds for the probability that  $k$ -out-of-20 events occur, where

- $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.17 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

Figure 2.17 shows that the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.18 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

As can be seen from Figure 2.18, the bounds for the probability that at least  $k$ -out-of-20 events occur, for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Figures 2.17 and 2.18 show that the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.17: Example 6. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 8$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8294	1	0.9794	1	0.9867	1	0.0206	0.0133
$k = 3$	0.779	1	0.9382	1	0.9412	0.9999	0.0618	0.0587
$k = 5$	0.7042	1	0.8969	1	0.8926	0.9877	0.1031	0.0951
$k = 8$	0.5209	1	0.7606	1	0.7889	0.8616	0.2394	0.0727
$k = 9$	0.4348	1	0.5642	0.8476	0.7373	0.8024	0.2834	0.0651
$k = 15$	0.0614	0.8544	0.36793	0.38524	0.4234	0.4815	0.2058	0.0581
$k = 18$	0	0.6189	0.16086	0.2975	0.25392	0.25787	0.13664	0.0883
$k = 19$	0	0.5256	0.1312	0.2531	0.1763	0.2384	0.1219	0.0621
$k = 22$	0	0.3225	0.0423	0.1544	0.0278	0.121	0.1121	0.0932
$k = 25$	0	0.2083	0	0.093	0.0001	0.0634	0.093	0.0633
$k = 27$	0	0.1609	0	0.0558	0.0002	0.0294	0.0558	0.0292

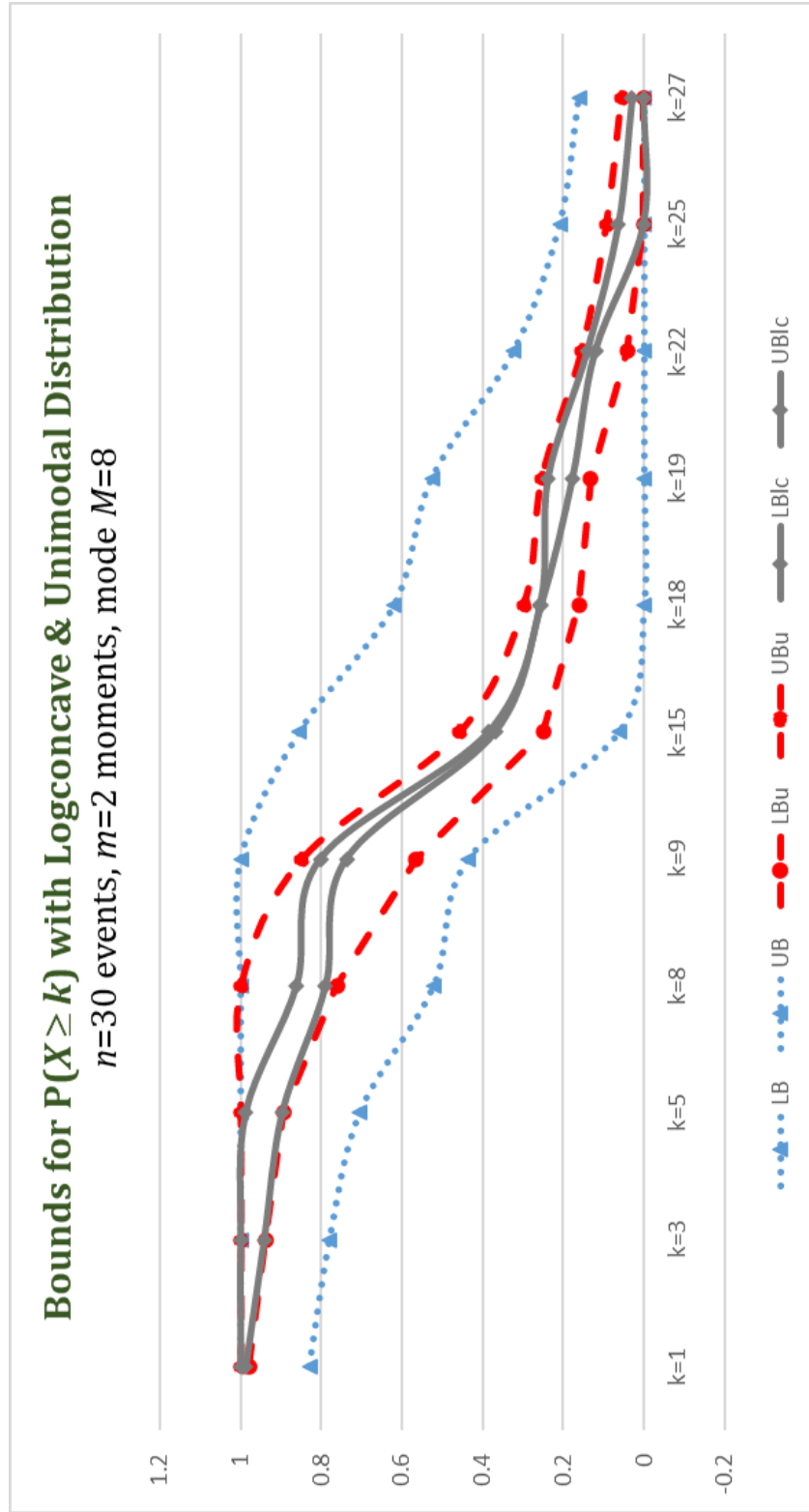


Figure 2.17: Example 6. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 8$

Table 2.18: Example 6. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  $n = 30$  events,  $m = 3$  moments, mode  $M = 8$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.9043	1	0.9854	1	0.9919	1	0.0146	0.0081
$k = 3$	0.8583	1	0.9561	1	0.9696	0.999	0.0439	0.0294
$k = 5$	0.7818	1	0.9218	1	0.9289	0.988	0.0782	0.0591
$k = 8$	0.556	1	0.7639	0.9501	0.8014	0.8474	0.1862	0.046
$k = 9$	0.4416	0.9819	0.6247	0.8166	0.7737	0.8006	0.1919	0.0269
$k = 15$	0.0713	0.7521	0.3153	0.4217	0.3697	0.4125	0.1064	0.0428
$k = 18$	0.0437	0.5242	0.17	0.2926	0.2256	0.2704	0.1226	0.0448
$k = 19$	0.0334	0.4716	0.1342	0.2526	0.1893	0.2231	0.1184	0.0338
$k = 22$	0	0.2965	0.0687	0.1385	0.10396	0.1163	0.0698	0.01234
$k = 25$	0	0.1593	0.007	0.0651	0.0312	0.0528	0.0581	0.0216
$k = 27$	0	0.1096	0	0.0377	2.10E-05	0.0271	0.0377	2.71E-02
$k = 29$	0	0.0778	0	0.0126	0	0.0078	0.0126	0.0078

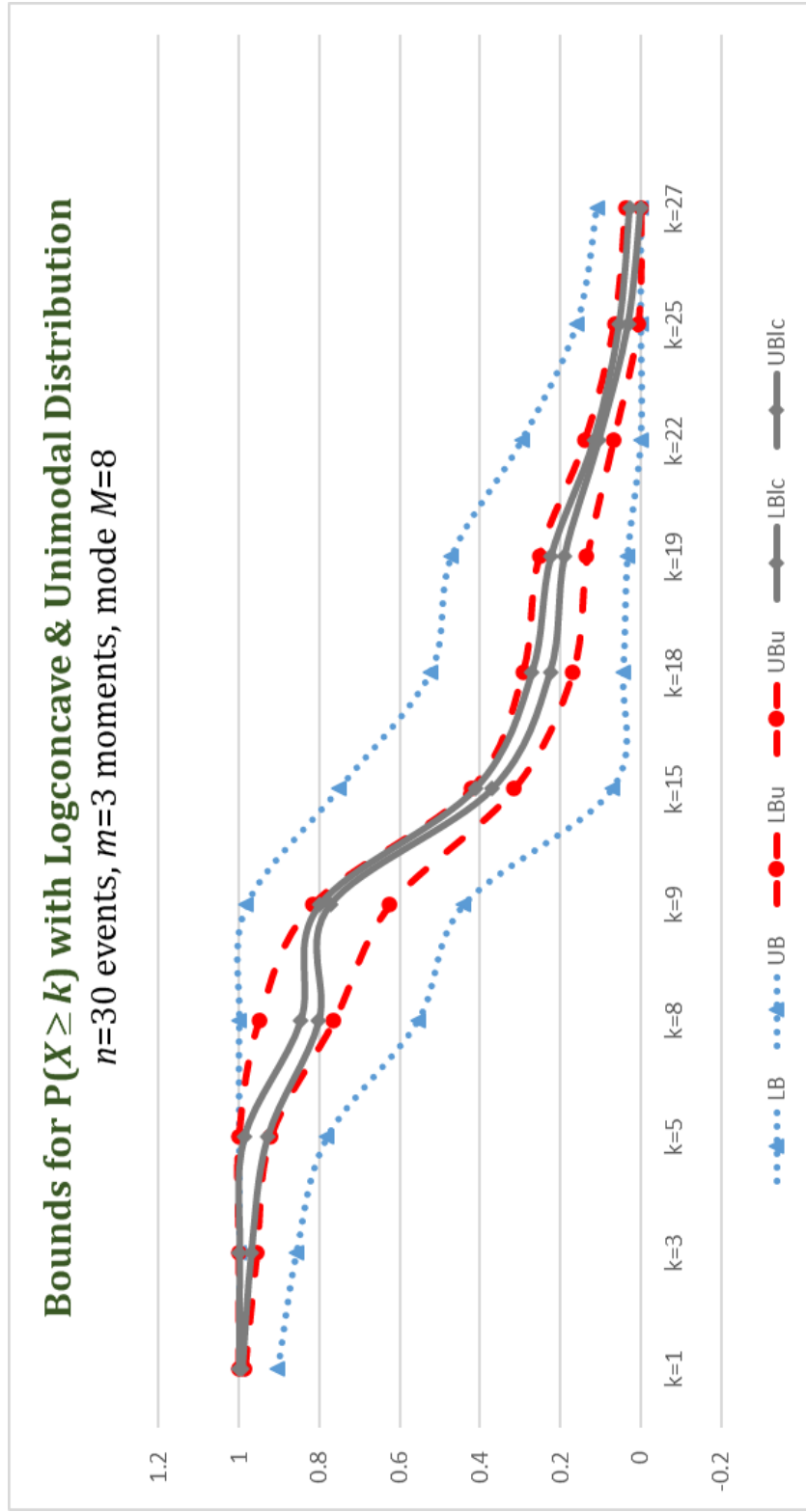


Figure 2.18: Example 6. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 3$  moments, mode  $M = 8$



### 2.3.7 Example 7. Bounds for the probability that at least $k$ -out-of-30 events occur under the logconcavity and unimodality assumption

We apply the procedure described in Section 2.3 to generate another logconcave probability distribution with  $n = 30$ . This distribution is unimodal with mode  $M = 14$ . We obtain bounds for the probability that  $k$ -out-of-20 events occur, where

- $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.19 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

Figure 2.19 gives that the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.20 presents the lower and upper bounds for the probability that at least  $k$ -out-of-20 events for  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

As can be observed from Figure 2.20, the bounds for the probability that at least  $k$ -out-of-20 events occur, for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Figures 2.19 and 2.20 show that the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.19: Example 7. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 14$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8122	1	0.969	1	0.9799	1	0.031	0.0201
$k = 3$	0.7568	1	0.9072	1	0.9279	0.9999	0.0928	0.072
$k = 5$	0.6786	1	0.8454	1	0.8775	0.9459	0.1546	0.0684
$k = 8$	0.4913	1	0.7502	0.9065	0.7655	0.8321	0.1563	0.0666
$k = 9$	0.4054	1	0.7093	0.8678	0.7119	0.7778	0.1585	0.0659
$k = 15$	0.0724	0.8438	0.2162	0.5304	0.3685	0.4823	0.3142	0.1138
$k = 18$	0	0.6479	0.1314	0.3679	0.2414	0.3132	0.2365	0.0718
$k = 19$	0	0.5563	0.1031	0.3182	0.1988	0.2676	0.2151	0.0688
$k = 22$	0	0.3497	0.0183	0.1688	0.0594	0.1432	0.1505	0.0838
$k = 25$	0	0.2295	0	0.0953	0.0086	0.0749	0.0953	0.0663
$k = 27$	0	0.1783	0	0.0572	6.57E-06	0.0382	0.0572	0.03819343

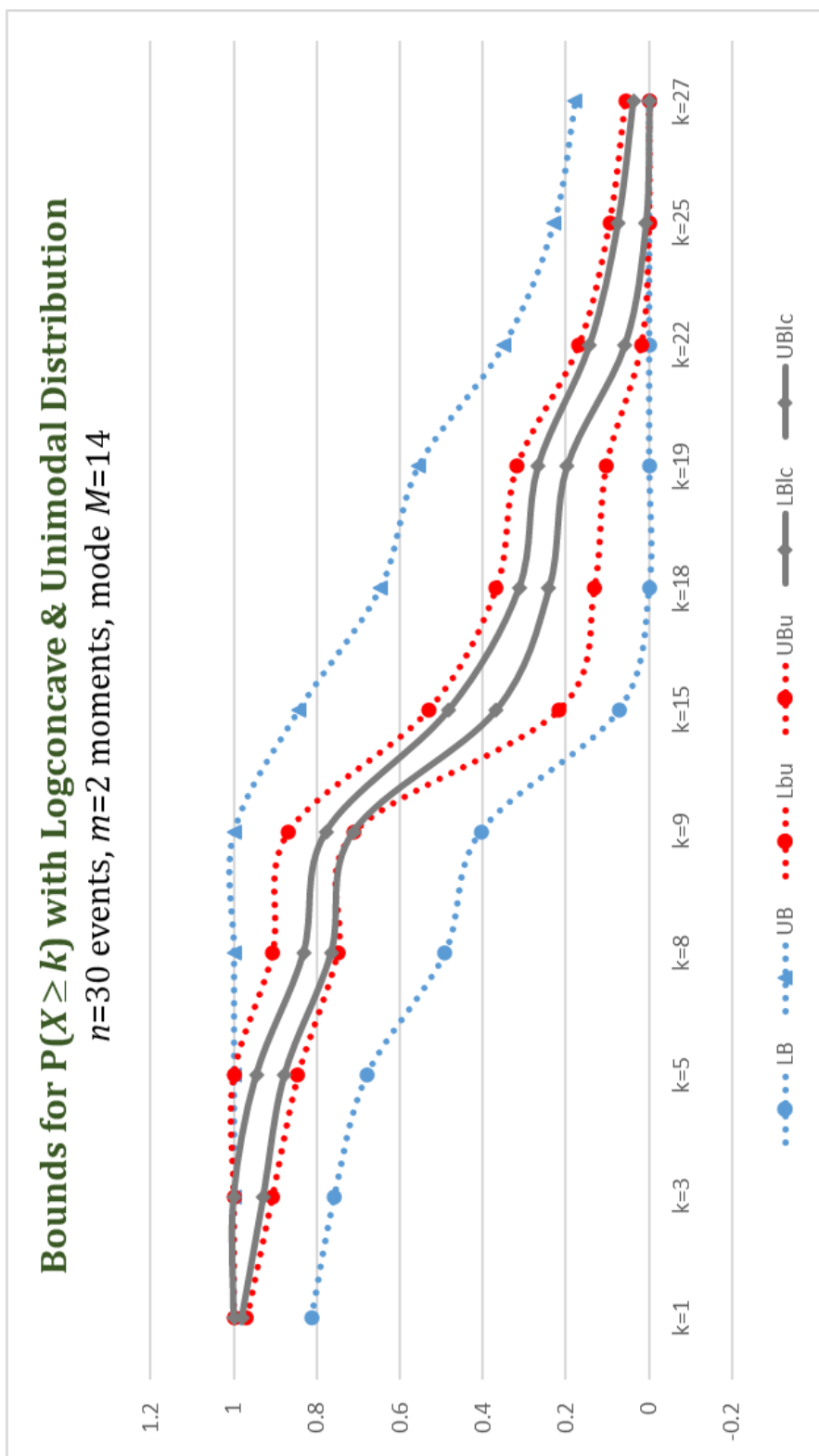


Figure 2.19: Example 7. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 14$

Table 2.20: Example 7. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 3$  moments, mode  $M = 14$

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8866	1	0.9777	1	0.987	1	0.0223	0.013
$k = 3$	0.8331	1	0.9332	1	0.9546	0.9999	0.0668	0.0453
$k = 5$	0.7458	1	0.8857	0.9557	0.9035	0.9448	0.07	0.0413
$k = 8$	0.5059	0.9878	0.7675	0.8655	0.7597	0.8181	0.098	0.0584
$k = 9$	0.4258	0.964	0.7154	0.8352	0.7065	0.7746	0.1198	0.0681
$k = 15$	0.0792	0.7631	0.2222	0.4633	0.3897	0.4323	0.2411	0.0426
$k = 18$	0.0486	0.5519	0.1599	0.2928	0.2419	0.2925	0.1329	0.0506
$k = 19$	0.0372	0.5025	0.1395	0.2559	0.2041	0.2478	0.1164	0.0437
$k = 22$	0	0.3211	0.0787	0.1473	0.1097	0.1412	0.0686	0.0315
$k = 25$	0	0.1728	0.0201	0.0755	0.0311	0.0589	0.0554	0.0278
$k = 27$	0	0.1189	0	0.0453	1.76E-05	0.0301	0.0453	0.0301
$k = 29$	0	0.0839	0	0.0151	0	0.0087	0.0151	0.0087

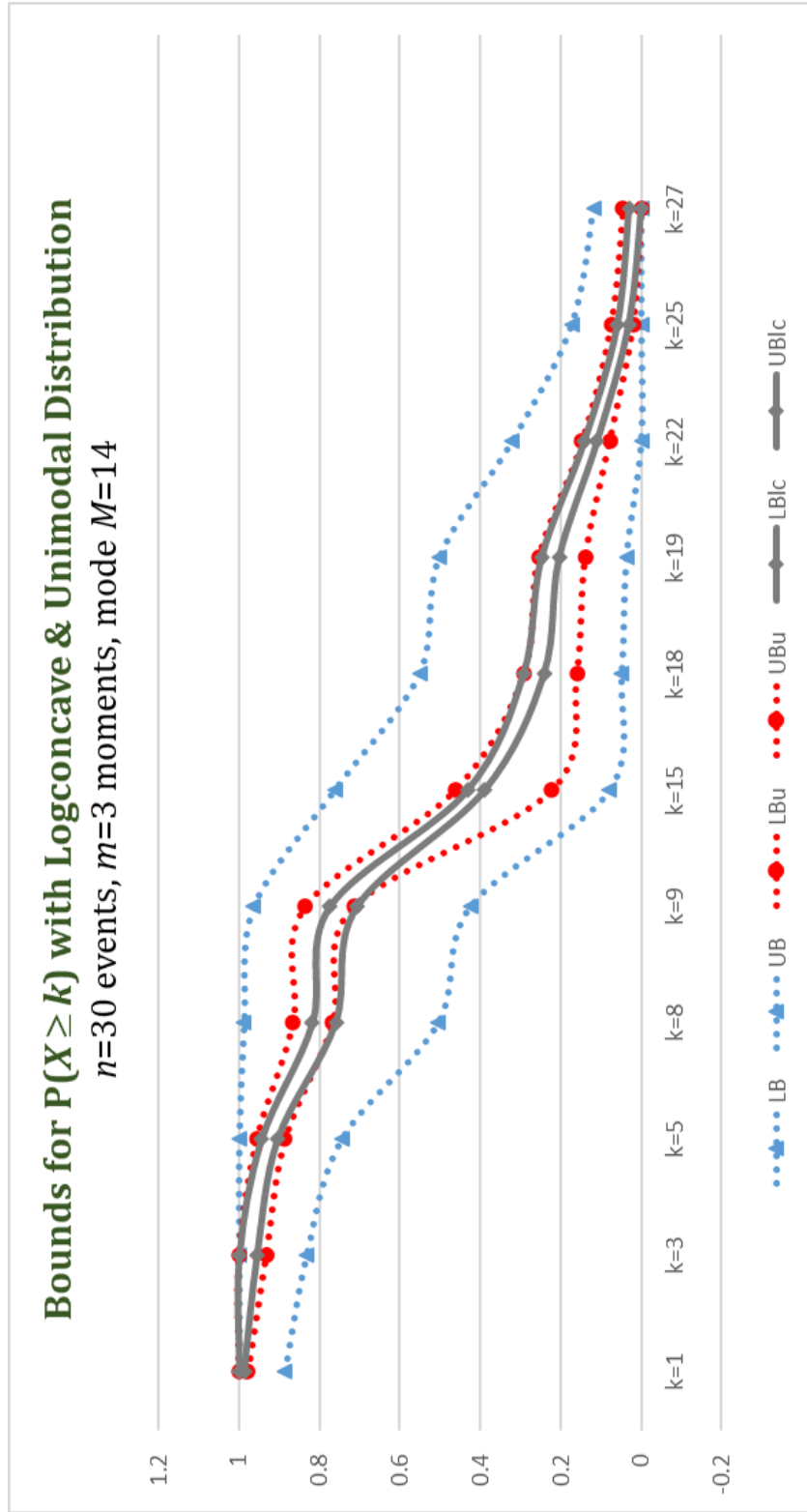


Figure 2.20: Example 7. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 3$  moments, mode  $M = 14$

### 2.3.8 Example 8. Bounds for the probability that at least $k$ -out-of-30 events occur under the logconcavity and unimodality assumption

We apply the procedure described in Section 2.3 to generate another logconcave probability distribution with  $n = 30$ . This distribution is unimodal with mode  $M = 21$ . We obtain bounds for the probability that  $k$ -out-of-20 events occur, where

- $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ ,
- $m = 2$ , i.e., the first two binomial moments  $S_1, S_2$  are used,
- $m = 3$ , i.e., the first three binomial moments  $S_1, S_2, S_3$  are used.

Table 2.21 gives the lower and upper bounds for the probability that at least  $k$ -out-of-20 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ .

Figure 2.21 gives that the bounds obtained by (2.8) are much tighter than those obtained by problems (1.4) and (2.11).

Table 2.22 presents the lower and upper bounds for the probability that at least  $k$ -out-of-20 events for  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ .

As can be observed from Figure 2.22, the bounds for the probability that at least  $k$ -out-of-20 events occur, for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (2.8) are much tighter than those obtained by problems (1.4) and (2.11). Figures 2.21 and 2.22 show that the probability bounds become tighter if we use the first three binomial moments instead of the first two binomial moments.

Table 2.21: Example 8. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 21$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.8674	1	0.9789	1	0.9895	1	0.0211	0.0105
$k = 3$	0.8347	1	0.9437	1	0.9639	0.9994	0.0563	0.0355
$k = 5$	0.7893	1	0.8942	1	0.9338	0.9929	0.1058	0.0591
$k = 8$	0.6824	1	0.8479	0.9718	0.8765	0.9606	0.1239	0.0841
$k = 9$	0.6316	1	0.8236	0.9566	0.8472	0.912	0.133	0.0648
$k = 15$	0.1842	0.9456	0.5745	0.8862	0.6216	0.6659	0.3117	0.0443
$k = 18$	0.089	0.8822	0.4252	0.7685	0.4109	0.5194	0.3433	0.1085
$k = 19$	0.0457	0.857	0.0372	0.6976	0.3632	0.4571	0.6604	0.0939
$k = 22$	0	0.5655	0	0.3096	0.2056	0.2779	0.3096	0.0723
$k = 25$	0	0.3554	0	0.1609	0.0041	0.1337	0.1609	0.1296
$k = 27$	0	0.2663	0	0.0966	2.23E-05	0.0815	0.0966	0.0815

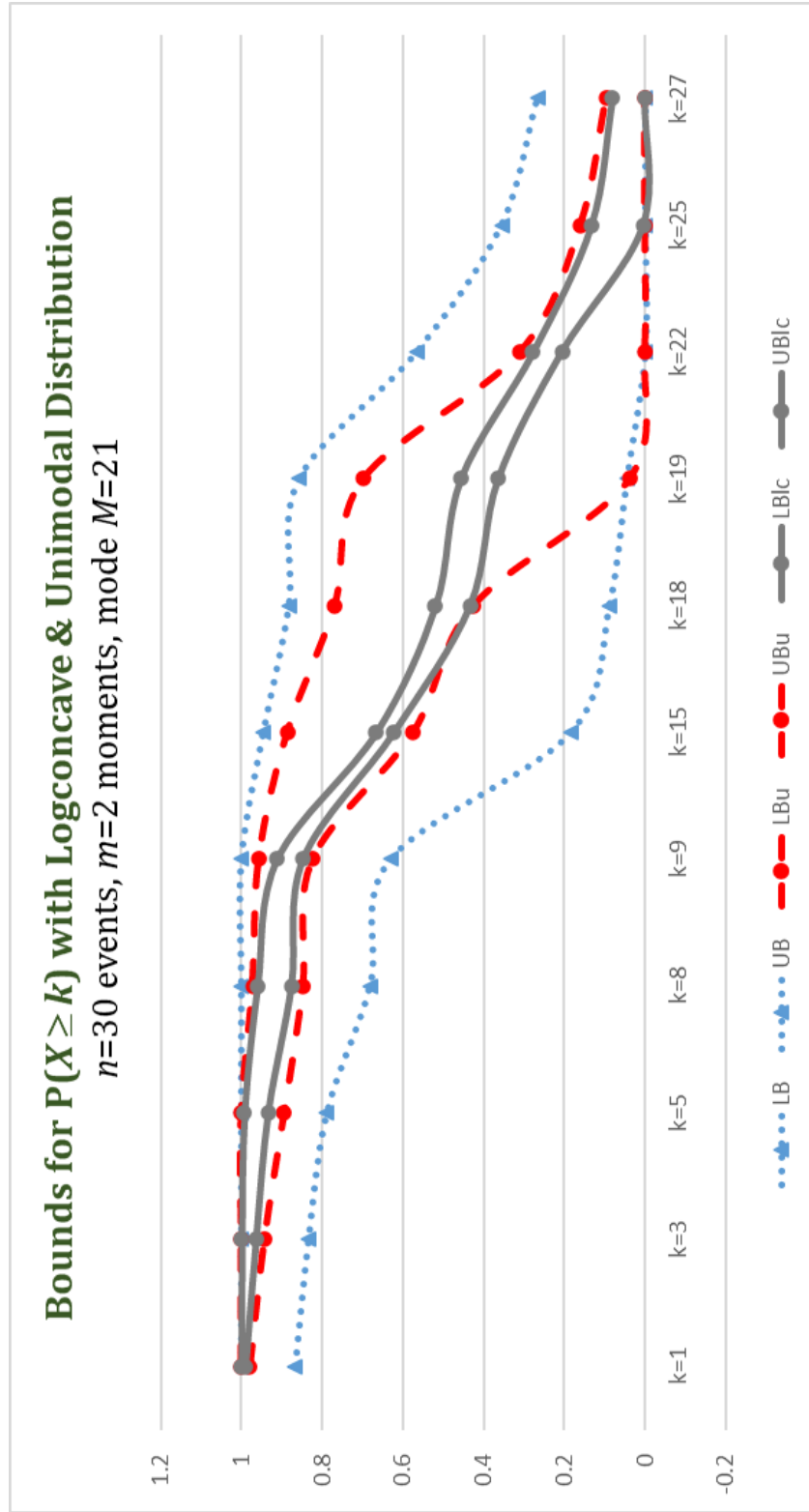


Figure 2.21: Example 8. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 2$  moments, mode  $M = 21$



Table 2.22: Example 8. Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 3$  moments, mode  $M = 21$

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{unimodality}$	$UB_{unimodality}$	$LB_{logconcavity}$	$LB_{logconcavity}$	$\Delta_u$	$\Delta_{lc}$
$k = 1$	0.9243	1	0.9652	1	0.9927	1	0.0348	0.0073
$k = 3$	0.8942	1	0.9417	1	0.9747	0.9999	0.0583	0.0252
$k = 5$	0.8467	1	0.9164	0.9961	0.9503	0.9774	0.0797	0.0271
$k = 8$	0.7177	1	0.85	0.9358	0.8849	0.9072	0.0858	0.0223
$k = 9$	0.6513	0.9999	0.8187	0.9151	0.8463	0.8806	0.0964	0.0343
$k = 15$	0.2886	0.9327	0.5929	0.6805	0.6063	0.6489	0.0876	0.0426
$k = 18$	0.1099	0.7858	0.4146	0.5438	0.4376	0.4857	0.1292	0.0481
$k = 19$	0.0926	0.7214	0.3513	0.4972	0.3739	0.4335	0.1459	0.0596
$k = 22$	0.0292	0.5643	0.1091	0.2998	0.2156	0.2635	0.1907	0.0479
$k = 25$	0	0.2933	0.0281	0.1223	0.0802	0.1196	0.0942	0.0394
$k = 27$	0	0.1908	0	0.0607	3.37E-05	0.0568	0.0607	0.0568
$k = 29$	0	0.1289	0	0.0231	0	0.0166	0.0231	0.0166

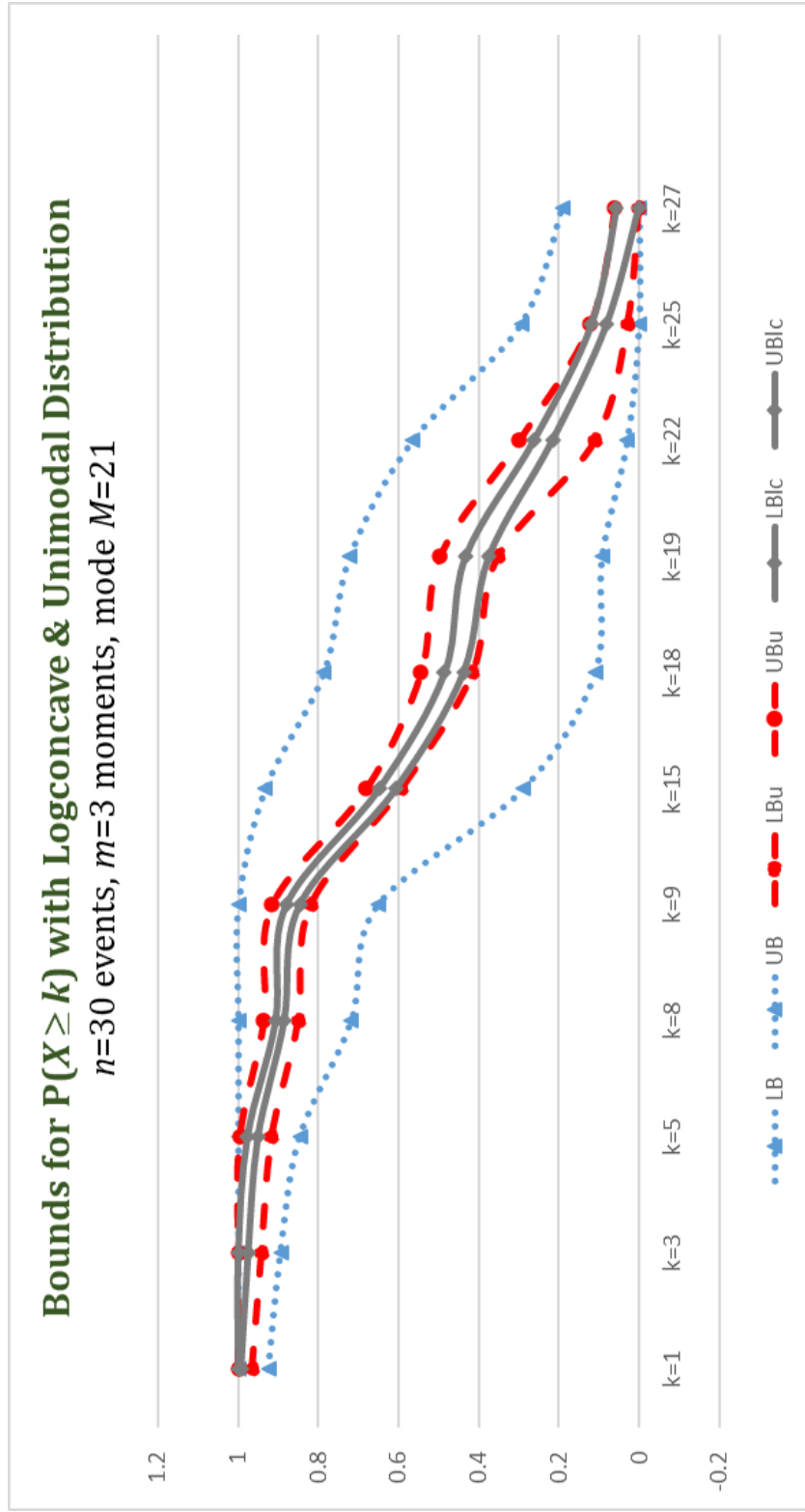


Figure 2.22: Example 8. Improvement on Bounds for  $P(X \geq k)$  with Logconcavity & Unimodality:  
 $n = 30$  events,  $m = 3$  moments, mode  $M = 21$

# Chapter 3

## Discrete Moment Problems with Logconvexity Constraints

### 3.1 Introduction

Though less popular than its counterpart, logconvexity is still very important in many application areas (Prékopa, 1995 [4], Lu, Simchi-Levi, 2013 [62]).

Recall that a probability distribution  $\{x_i\}$  defined on a finite support set

$$\Omega = \{z_0, z_1, \dots, z_n\}$$

is logconcave if conditions (2.3) are satisfied. Therefore, a probability distribution  $\{x_i\}$  is said to be logconvex if

$$x_i^2 \leq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1. \quad (3.1)$$

Under the assumption that the distribution  $\{x_i\}$  is logconvex, the power moment problem can be formulated as follows:

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n f_i x_i \\
& \text{subject to} && \sum_{i=0}^n z_i^j x_i = \mu_j, \quad j = 0, 1, \dots, m \\
& && (3.2)
\end{aligned}$$

$$x_i^2 \leq x_{i-1} x_{i+1}, \quad i = 1, \dots, n-1$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n.$$

Similarly, incorporating the logconvexity constraints (3.1) into the binomial moment problem (1.2), we obtain

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n f_i x_i \\
& \text{subject to} && \sum_{i=0}^n \binom{z_i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \\
& && (3.3)
\end{aligned}$$

$$x_i^2 \leq x_{i-1} x_{i+1}, \quad i = 1, \dots, n-1$$

$$x_i \geq 0, \quad i = 0, 1, \dots, n.$$

Note that the optimum values of problems (3.2) and (3.3) respectively provide us with sharp lower and upper bounds for

$$\sum_{i=k}^n f_i x_i,$$

where  $f_i = f(z_i)$ , based on the first  $m$  power and first  $m$  binomial moments of the logconvex probability distribution  $\{x_i\}$  defined on  $\Omega$ .

We remark that with logconvexity constraints being prescribed into problems (3.2) and (3.3), the convexity of the feasibility region is still preserved, and hence, both problems are convex nonlinear optimization problems (Boyd and Vandenberghe, 2004 [63], Nesterov, 2013 [64]). In particular,  $x_i^2 \leq x_{i-1}x_{i+1}$  is a second-order cone (see, e.g., Alizadeh, Goldfarb, 2003 [65]), and hence, both problems (3.2) and (3.3) can be solved efficiently by primal-dual interior point methods.

### 3.2 Numerical Examples: Logconvexity Constraint

As discussed in Chapter 1, problems (3.2) and (3.3) can be obtained from each other. For the sake of simplicity, we shall consider the following special case of binomial moment problem (3.3), where we assume that the underlying distribution is logconvex, that is,

$$\begin{aligned}
& \min(\max) && \sum_{i=k}^n x_i \\
& \text{subject to} && \\
& && \sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 0, 1, \dots, m \\
& && x_i^2 \leq x_{i-1} x_{i+1}, \quad i = 1, \dots, n-1 \\
& && x_i \geq 0 \quad i = 0, 1, \dots, n.
\end{aligned} \tag{3.4}$$

Note that the optimum values of problem (3.4) provide us with lower and upper bounds for the probability that at least  $k$ -out-of- $n$  events occur,  $P(X \geq k)$  for some  $1 \leq k \leq n$ , where the underlying distribution is logconvex and the first  $m$  binomial moments are known.

In order to illustrate the contribution of the logconvexity information of the underlying distribution we proceed as follows:

(1) Given an  $n$ , randomly generate a logconvex distribution  $\{x_i\}$  satisfying

- $\sum_{i=0}^n x_i = 1$
- $x_i^2 \leq x_{i-1}x_{i+1}, \quad i = 1, \dots, n-1$
- $x_i \geq 0, \quad i = 0, 1, \dots, n$

(2) Given an  $m$ , compute the first  $m$  binomial moments,  $S_1, \dots, S_m$ , by the use of

$$\sum_{i=0}^n \binom{i}{j} x_i = S_j, \quad j = 1, \dots, m.$$

(3) Assume that the distribution  $\{x_i\}$  is unknown and solve problem (2.11) to obtain lower and upper bounds for  $P(X \geq k)$  for some  $1 \leq k \leq n$ , where the shape of the distribution is not used and the binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).

(4) Next, solve problem (3.4), where the shape of the underlying distribution is used and the binomial moments,  $S_1, \dots, S_m$ , are those obtained in Step (2).

(5) Compute the optimum values of the optimization problems involved and let

- $LB$  and  $UB$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step 3, respectively
- Let  $LB_{logconvexity}$  and  $UB_{logconvexity}$  denote the lower and upper bounds for  $P(X \geq k)$  obtained in Step (4), respectively.

(6) Compare the lower and upper bounds reported in Step (5).

### 3.2.1 Example 1. Bounds for the probability that at least $k$ -out-of-10 events occur under the logconvexity assumption

We take  $n = 10$  and generate a probability distribution as described in Section 3.2. In this example, bounds for the probability that  $k$ -out-of-10 events occur are obtained for

- $k = 1, 3, 5, 8, 9$ ,
- based on the first two binomial moments  $S_1, S_2$ ,
- based on the first three binomial moments  $S_1, S_2, S_3$ .

Table 3.1 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11) and  $LB_{logconvexity}$  and  $UB_{logconvexity}$  are the lower and upper bounds obtained from problem (3.4), respectively.

Recall that that problem (3.4) uses the shape (logconvexity constraints) of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. We observe that the bounds obtained by (3.4) are much tighter than those obtained by problem (2.11) as shown in Figure 3.1.

Table 3.2 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11) and  $LB_{logconvexity}$  and  $UB_{logconvex}$  are the lower and upper bounds obtained from problem (3.4), respectively.



As can be seen from Figure 3.2, the bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (3.4) are much tighter than those obtained by problem (2.11). We also remark that the bounds become tighter if we use the first three binomial moments (Figure 3.2) instead of the first two binomial moments (Figure 3.1).

Table 3.1: Example 1. Bounds for  $P(X \geq k)$  with Logconvexity:  
 $n = 10$  events,  $m = 2$  moments

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{\logconvexity}$	$UB_{\logconvexity}$
$k = 1$	0.64153	1	0.78752	0.85683
$k = 3$	0.32518	0.84319	0.60862	0.63953
$k = 5$	0.2369	0.69162	0.43023	0.4773
$k = 8$	0	0.40092	0.20418	0.23577
$k = 9$	0	0.29719	0.10443	0.16779

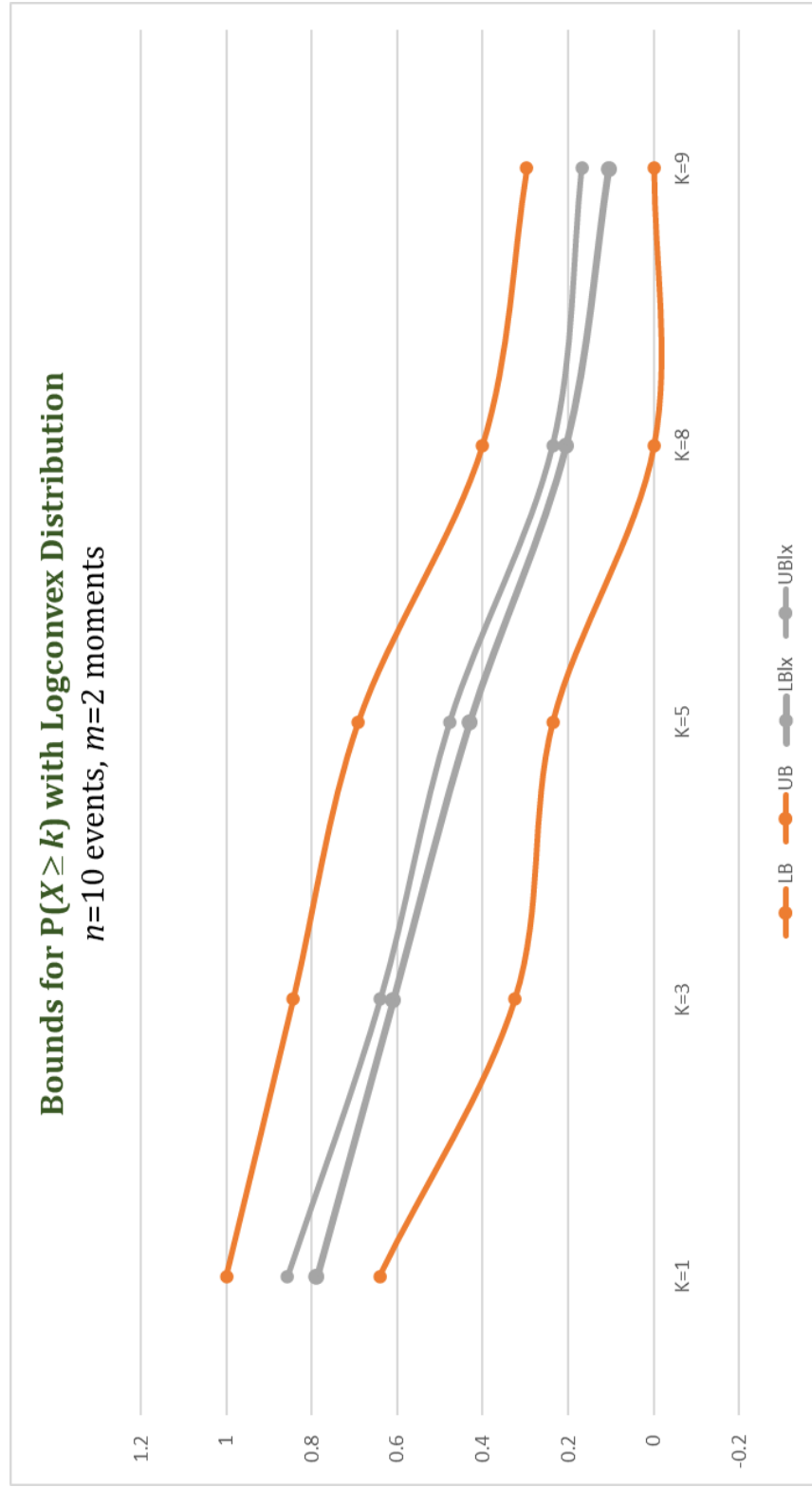


Figure 3.1: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 10$  events,  $m = 2$  moments

Table 3.2: Example 1. Bounds for  $P(X \geq k)$ :  $n = 10$  events,  $m = 3$  moments

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{logconvexity}$	$UB_{logconvexity}$
$k = 1$	0.67738	0.89367	0.69744	0.70109
$k = 3$	0.51628	0.71814	0.6809	0.68381
$k = 5$	0.38902	0.68094	0.54415	0.55014
$k = 8$	0.011265	0.29757	1.66E-01	0.16603
$k = 9$	0	0.14447	0.070504	0.070805

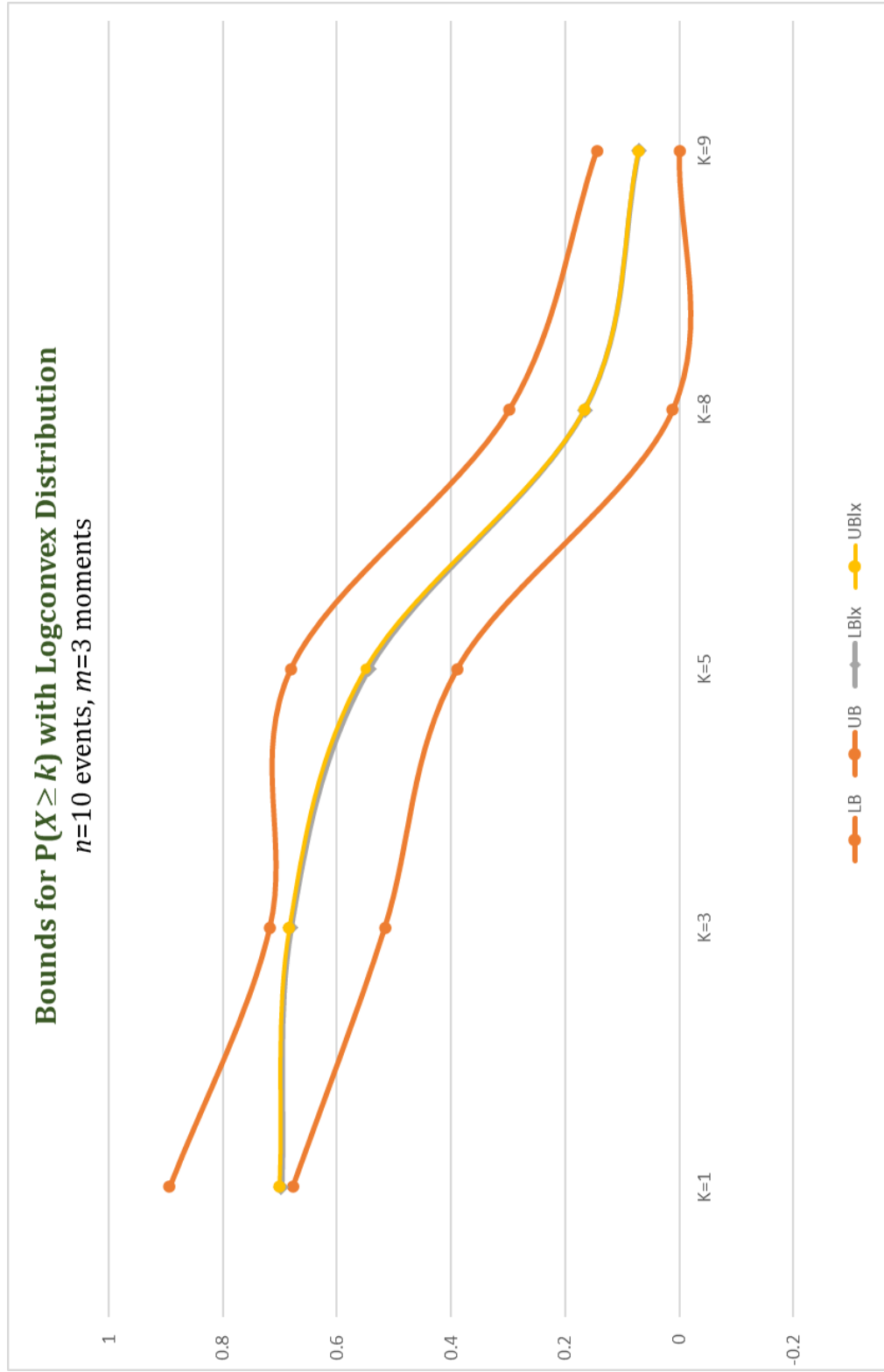


Figure 3.2: Example 1. Improvement on Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 10$  events,  $m = 3$  moments

### 3.2.2 Example 2. Bounds for the probability that at least $k$ -out-of-20 events occur under the logconvexity assumption

In this example we generate a new probability distribution with  $n = 10$ . The distribution is generated as described in Section 3.2. The bounds for the probability that  $k$ -out-of-20 events occur are obtained for

- $k = 1, 3, 5, 8, 9, 15, 18, 19$ ,
- based on the first two binomial moments  $S_1, S_2$ ,
- based on the first three binomial moments  $S_1, S_2, S_3$ .

Table 3.3 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9, 15, 18, 19$  and based on the knowledge of the first two binomial moments  $S_1, S_2$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconvexity}$  and  $UB_{logconvexity}$  are the lower and upper bounds obtained from problem (3.4).

Recall that that problem (3.4) uses the shape (logconvexity constraints) of the underlying probability distribution, whereas problem (2.11) does not take into account the shape of the distribution. We observe that the bounds obtained by (3.4) are much tighter than those obtained by problem (2.11) as shown in Figure 3.3.

Table 3.4 gives the lower and upper bounds for the probability that at least  $k$ -out-of-10 events for  $k = 1, 3, 5, 8, 9$ , based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , where  $LB$  and  $UB$  are the lower and upper bounds obtained from problem (2.11), respectively and  $LB_{logconvexity}$  and  $UB_{logconvexity}$  are the lower and upper bounds obtained from problem (3.4).

As can be seen from Figure 3.4, the bounds for the probability that at least  $k$ -out-of-10 events occur for the case of  $k = 1, 3, 5, 8, 9$  and based on the knowledge of the first three binomial moments  $S_1, S_2, S_3$ , obtained by problem (3.4) are much tighter than those obtained by problem (2.11). Note that the bounds become tighter if we use the first three binomial moments (Figure 3.4) instead of the first two binomial moments (Figure 3.3).

Table 3.3: Example 2. Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 20$  events,  $m = 2$  moments

Bounds based on $S_1, S_2$	$LB$	$UB$	$LB_{logconvexity}$	$UB_{logconvexity}$
$k = 1$	0.66048	1	0.88528	0.92461
$k = 3$	0.52673	1	0.76485	0.78517
$k = 5$	0.33979	0.99778	0.65136	0.66563
$k = 8$	0.19913	0.78637	0.49377	0.51362
$k = 9$	0.17777	0.74722	0.44465	0.46523
$k = 15$	0	0.43486	0.19592	0.20682
$k = 18$	0	0.26873	0.076118	0.10079
$k = 19$	0	0.2329	0.037626	0.068566



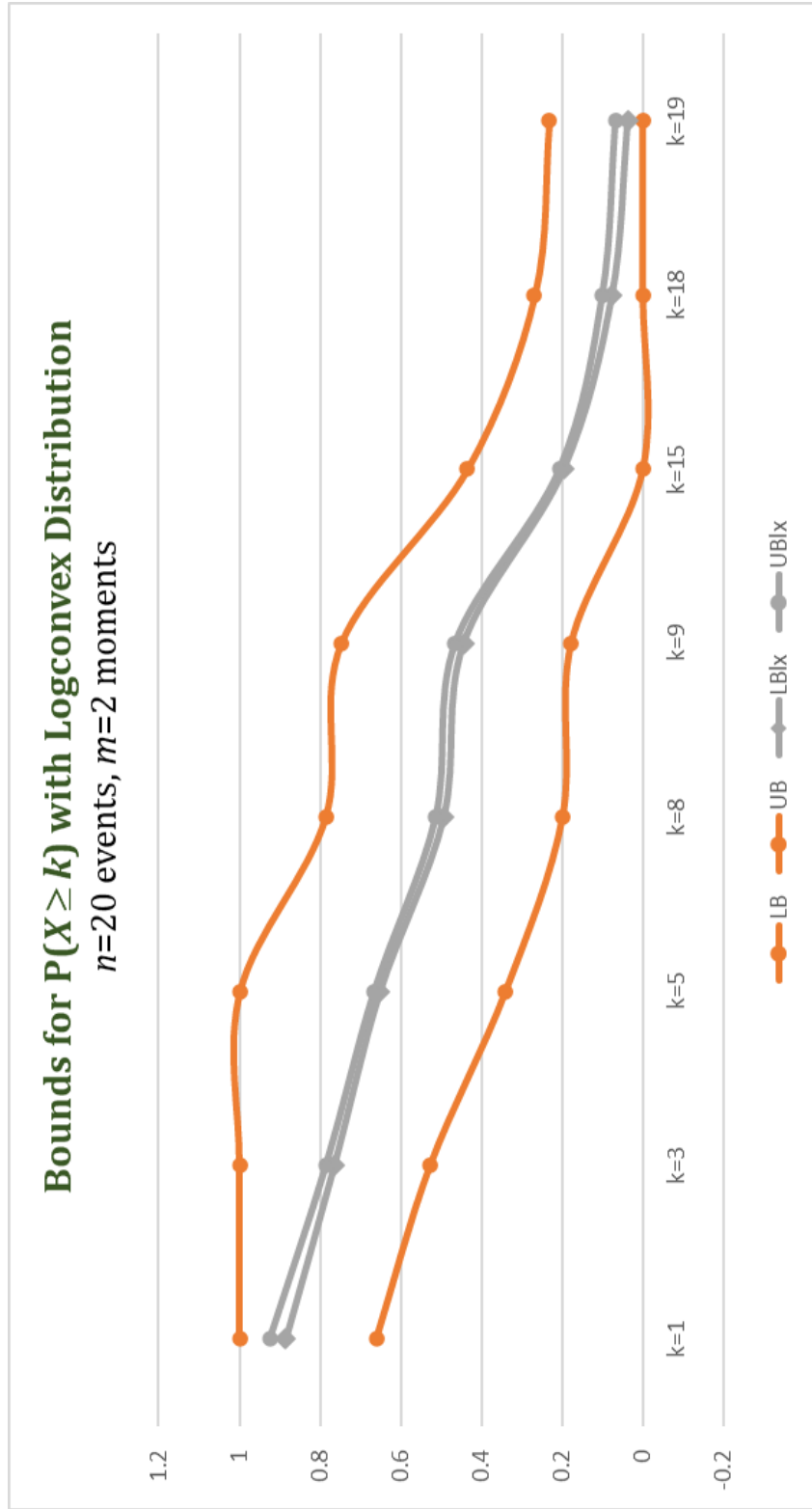


Figure 3.3: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 20$  events,  $m = 2$  moments

Table 3.4: Example 2. Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 20$  events,  $m = 3$  moments

Bounds based on $S_1, S_2, S_3$	$LB$	$UB$	$LB_{logconvexity}$	$UB_{logconvexity}$
$k = 1$	0.75838	1	0.91342	0.92293
$k = 3$	0.57228	1	0.77858	0.78202
$k = 5$	0.4012	0.86455	0.65589	0.66147
$k = 8$	0.26818	0.77233	0.49385	0.49991
$k = 9$	0.20754	0.73049	0.44576	0.45126
$k = 15$	0.055871	0.42744	0.19605	0.20676
$k = 18$	0	0.20571	0.087422	0.095223
$k = 19$	0	0.15945	0.044559	0.060346

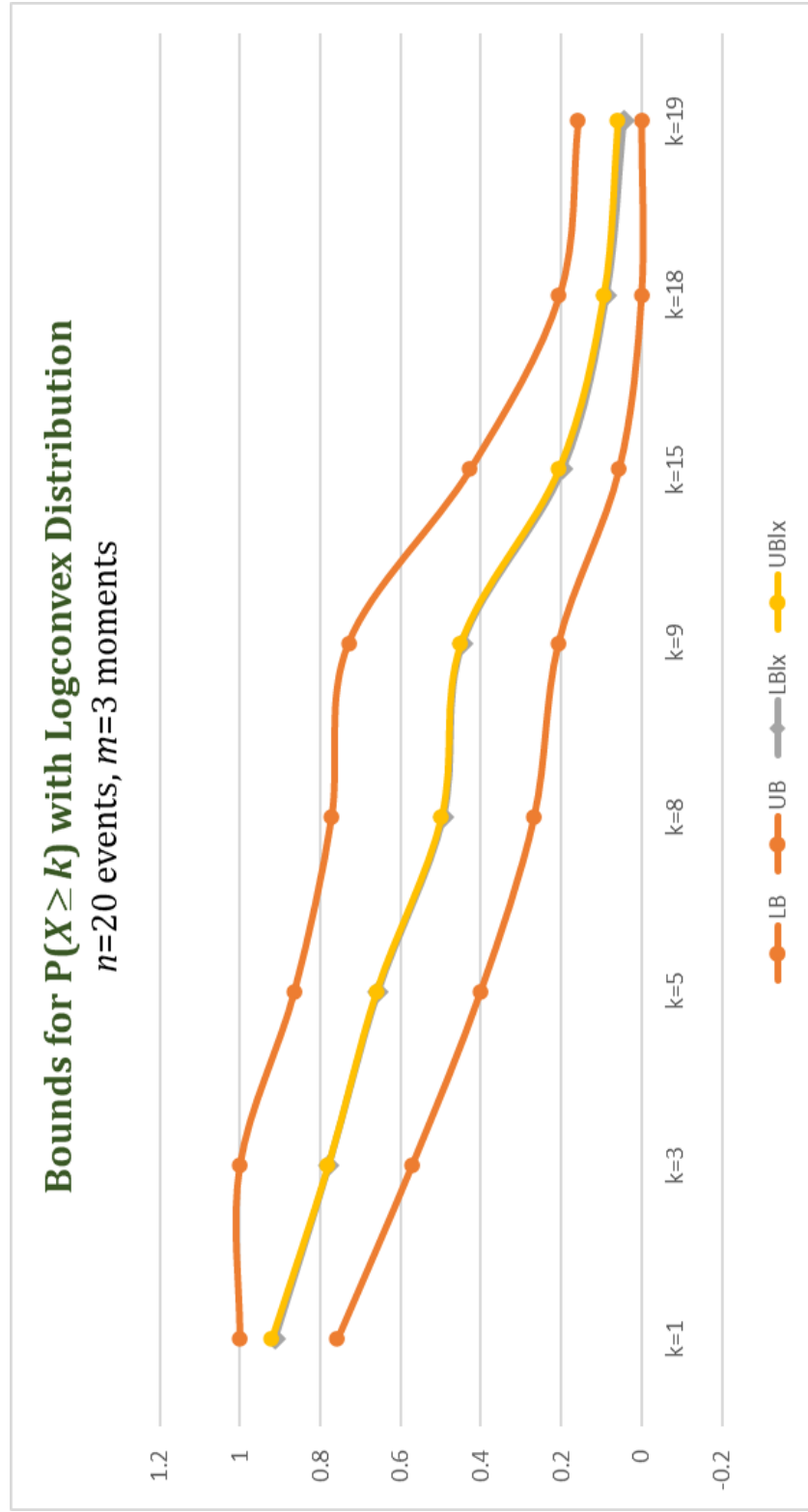


Figure 3.4: Example 2. Improvement on Bounds for  $P(X \geq k)$  with Logconvexity:  $n = 20$  events,  $m = 3$  moments

# Chapter 4

## Application

In this chapter we present an application of discrete moment problems in insurance problem to estimate the expected stop loss, where the underlying distribution is logconcave and the first three moments are known or can be obtained from historical data.

We consider the total claim amount in a fixed period in a portfolio of insurance contracts. Let  $X_i, i = 1, 2, \dots$  denote the amount of the  $i$ th claim arising from the policies in a given time period. Then the convolution (random sum)

$$X = X_1 + X_2 + \dots + X_N$$

represents the aggregated claims generated by the portfolio for the period under consideration, where the number of claims,  $N$ , payable by the insurer is a random variable and is associated with the frequency of claims.

The individual claim amounts,  $X_1, X_2, \dots$ , are independent identically distributed random variables and measure the severity of claims. Number of claims  $N$  and the

individual claims,  $X_1, X_2, \dots$ , are assumed to be independent random variables.

Let us consider the discrete expected stop loss defined as

$$E[(X - q)_+], \tag{4.1}$$

where  $q$  is a given constant in the support set of random variable  $X$  interpreted as aggregated loss from insurance claims. Then  $(X - q)_+$  is then considered as the excess of loss over the retention level  $q$ . Expected stop loss given in (4.1) is a risk measure that is widely used in finance and insurance (see, e.g., Courtois and Denuit, 2009 [66]).

In this context, if, for example, we assume that  $N$  has Poisson distribution with the probability function

$$p_n = P(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \dots,$$

where  $\lambda > 0$  is the expected number of claims, then total claim  $X$  has a compound Poisson distribution. We can utilize compound distributions in our application since their logconcavity property can be conveniently characterized (Ninh and Prékopa, 2013 [27]).

In particular, under the assumption that the individual claim amount  $X_i$ 's has a Bernoulli distribution with parameter  $0 < p < 1$ , the logconcavity, logconvexity, or unimodality of  $X$  depends on the logconcavity, logconvexity, or unimodality  $N$ .

In order to investigate the impact of the shape constraints, logconcavity, logconvexity, and unimodality, to the quality of the lower and upper bounds for the expected stop loss (4.1), we proceed as follows:

- (1) Let the support set of number of claim,  $N$ , be on  $\{1, \dots, 10\}$ .
- (2) Assume that the probability mass function of  $N$  is chosen to be logconcave to render the logconcavity for the aggregated claim  $X$ .
- (3) Compute the first three power moment of  $X$  and its mode  $M$  before assuming that the full distribution is unknown.
- (4) Given  $n$  (real value of  $N$ ), solve the following problem:

$$\begin{aligned}
& \min(\max) && E[(X - q)_+] \\
& \text{subject to} && \\
& && x_0 + x_1 + \dots + x_n = 1 \\
& && x_0 z_0 + \dots + x_n z_n = \mu_1 \\
& && x_0 z_0^2 + \dots + x_n z_n^2 = \mu_2 \\
& && x_0 z_0^3 + \dots + x_n z_n^3 = \mu_3 \\
& && x_0, \dots, x_n \geq 0.
\end{aligned} \tag{4.2}$$

The above problem serves as the benchmark for the performance of discrete moment problems with shape constraints (unimodality, logconcavity, or logconvexity).

Let

- $LB$ : optimum value of the minimization problem (4.2)
- $UB$ : optimum value of the maximization problem (4.2).

(5) Prescribe the logconcavity constraints into problem (4.2) and solve the power moment problem given below:

$$\begin{aligned}
& \min(\max) && E[(X - q)_+] \\
& \text{subject to} && \\
& && x_0 + x_1 + \cdots + x_n = 1 \\
& && x_0 z_0 + \cdots + x_n z_n = \mu_1 \\
& && x_0 z_0^2 + \cdots + x_n z_n^2 = \mu_2 \\
& && x_0 z_0^3 + \cdots + x_n z_n^3 = \mu_3 \\
& && x_i^2 \geq x_{i-1} x_{i+1}, \quad i = 1, \dots, n-1 \\
& && x_0, \dots, x_n \geq 0.
\end{aligned} \tag{4.3}$$

Let

- $LB_{lc}$ : optimum value of the minimization problem (4.3)
- $UB_{lc}$ : optimum value of the maximization problem (4.3).

(6) Prescribe the unimodality constraints into problem (4.2) and solve the following power moment problem:

$$\begin{aligned}
& \min(\max) && E[(X - q)_+] \\
& \text{subject to} && \\
& && x_0 + x_1 + \cdots + x_n = 1 \\
& && x_0 z_0 + \cdots + x_n z_n = \mu_1 \\
& && x_0 z_0^2 + \cdots + x_n z_n^2 = \mu_2 \\
& && x_0 z_0^3 + \cdots + x_n z_n^3 = \mu_3 \\
& && x_0 \leq \dots \leq x_M \\
& && x_M \geq \dots \geq x_n \\
& && x_0, \dots, x_n \geq 0.
\end{aligned} \tag{4.4}$$

Let

- $LB_u$ : optimum value of the minimization problem (4.4)
- $UB_u$ : optimum value of the maximization problem (4.4).

(7) Calculate

$$\Delta/\Delta_* = (UB - LB)/(UB_* - LB_*),$$

where \* represents *logconcavity* or *unimodality*.

Below, we present numerical results to compare the contribution of the shape constraints on the improvement of bounds for the expected stop-loss  $E[(X - q)_+]$ . All computations are done in MATLAB with BARON to solve discrete moment problems involved.



Table 4.1 show the lower and upper bounds for the expected stop-loss  $E[(X - q)_+]$  for the values of  $q = 2, \dots, 9$ . The bounds  $LB$  and  $UB$  are obtained from problem (4.2) that does not take into account the shape of the underlying distribution. The bounds  $LB_{lc}$  and  $UB_{lc}$  are the optimum values of problem (4.3), where the log-concavity constraints are prescribed. Finally, the  $LB_u$  and  $UB_u$  are the optimum values of problem (4.4) that uses the information that the underlying probability distribution is unimodal with mode  $M = 2$ . All bounds presented in Table 4.1 are the bounds for the expected stop-loss  $E[(X - q)_+]$  based on the knowledge of the first two binomial moments  $S_1$  and  $S_2$ . Figures 4.1, 4.2, and 4.3 show the comparison of, change in and improvement on the bounds for the expected stop-loss  $E[(X - q)_+]$  based on  $S_1$  and  $S_2$ , respectively.

Table 4.2 presents the lower and upper bounds for the expected stop-loss  $E[(X - q)_+]$  obtained from problems (4.2), (4.3), and (4.4) based on the knowledge of the first three binomial moments  $S_1$ ,  $S_2$ , and  $S_3$ . Similarly, Figures 4.4, 4.5, and 4.6 show the comparison of, change in and improvement on the bounds for the expected stop-loss  $E[(X - q)_+]$  based on  $S_1$ ,  $S_2$ , and  $S_3$ , respectively.

As in numerical experiments presented in Chapter 2, we observe that the use of the shape constraints greatly improve the bounds for the expected stop-loss, where the logconcavity constraints provide tighter bounds than those obtained by the use of the unimodality constraints. We remark that the bounds are very tight even for the case of first two binomial moments and as expected they are improved further when the first three binomial moments are used.

Table 4.1: Bounds for the Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 2$  moments, mode  $M = 2$

Bounds based on $S_1, S_2$	Without Unimodality		With Unimodality		Logconcavity		$\Delta = UB - LB$			Improvement	
	$LB$	$UB$	$LB_u$	$UB_u$	$LB_{lc}$	$UB_{lc}$	$\Delta$	$\Delta_u$	$\Delta_{lc}$	$\Delta / \Delta_u$	$\Delta / \Delta_{lc}$
1											
2	2.47	2.73	2.471	2.577	2.5084	2.5799	0.26	0.106	0.0715	2.453	3.636
3	1.47	1.99	1.471	1.789	1.7064	1.8059	0.52	0.318	0.0995	1.635	5.226
4	0.68	1.3	0.928	1.147	1.0432	1.157	0.62	0.219	0.1138	2.831	5.448
5	0.289	0.81	0.573	0.721	0.58934	0.66824	0.521	0.148	0.0789	3.520	6.603
6	0	0.51	0.279	0.458	0.28514	0.3719	0.51	0.179	0.10676	2.849	4.777
7	0	0.367	0.051	0.275	0.05330	0.15175	0.367	0.204	0.09845	1.799	3.728
8	0	0.244	0	0.138	3.36E-03	0.06993	0.244	0.138	0.06657	1.768	3.665
9	0	0.122	0	0.046	3.20E-07	0.01375	0.122	0.046	0.01375	2.652	8.873

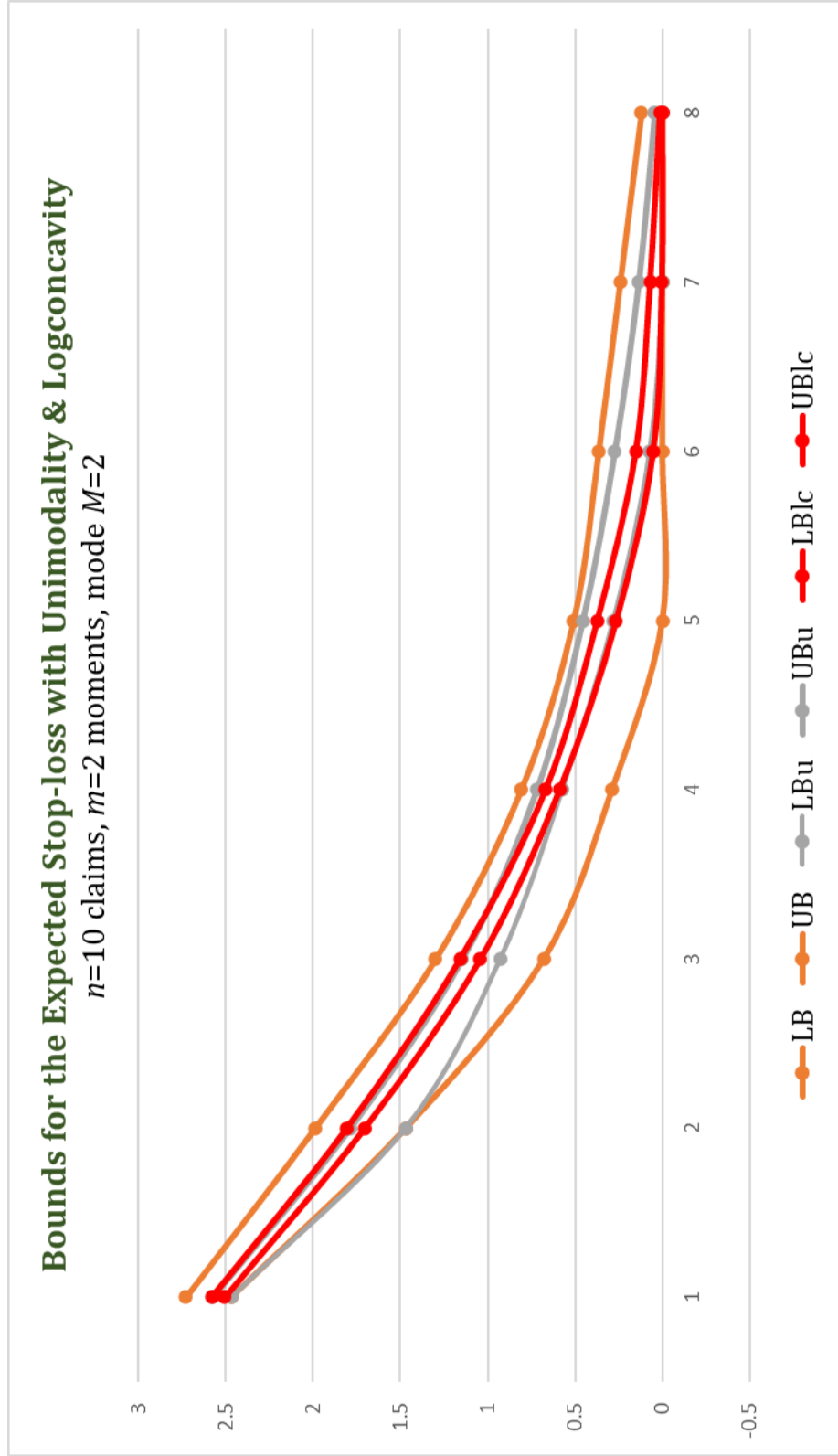


Figure 4.1: Comparison of the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 2$  moments, mode  $M = 2$

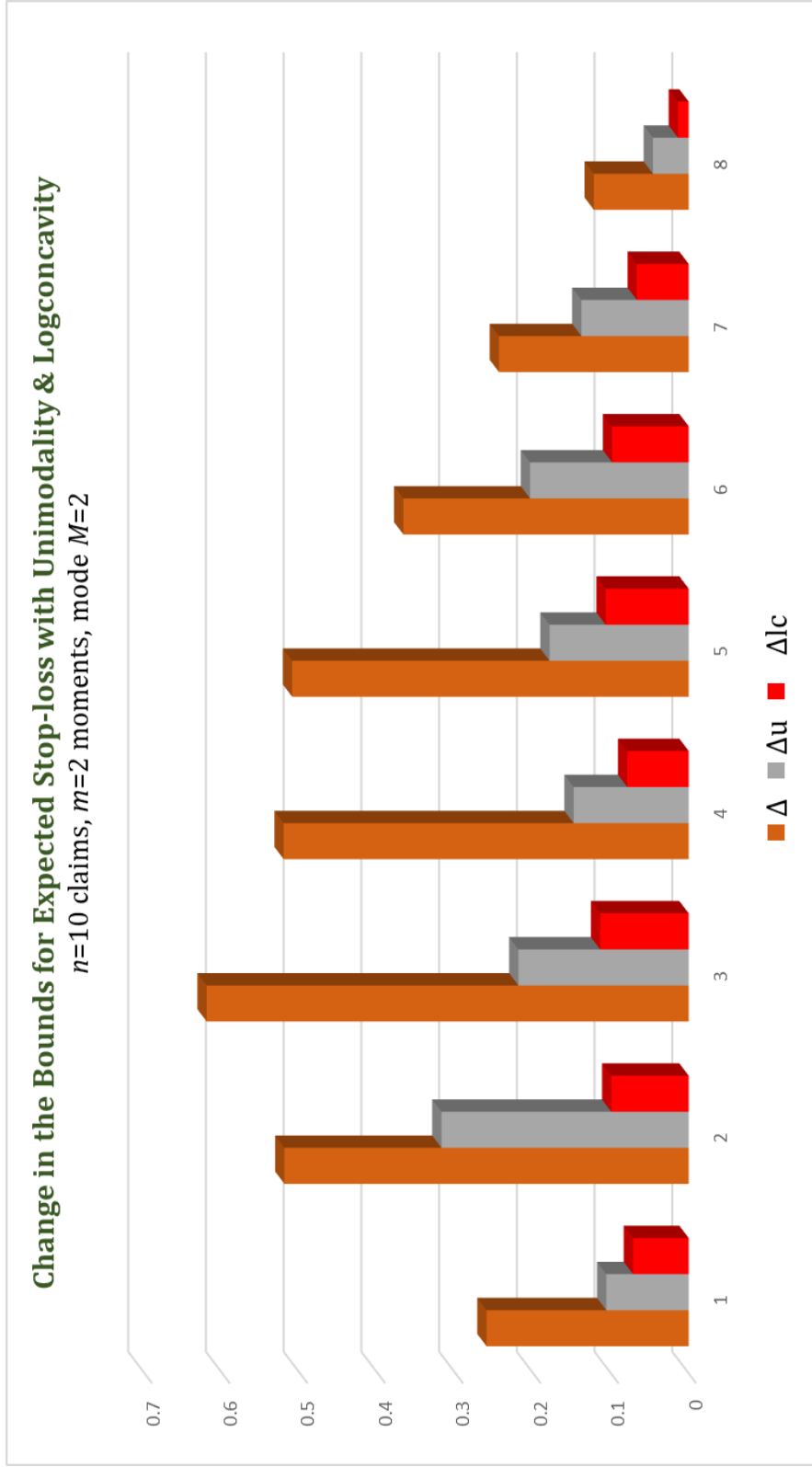


Figure 4.2: Change in the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 2$  moments, mode  $M = 2$

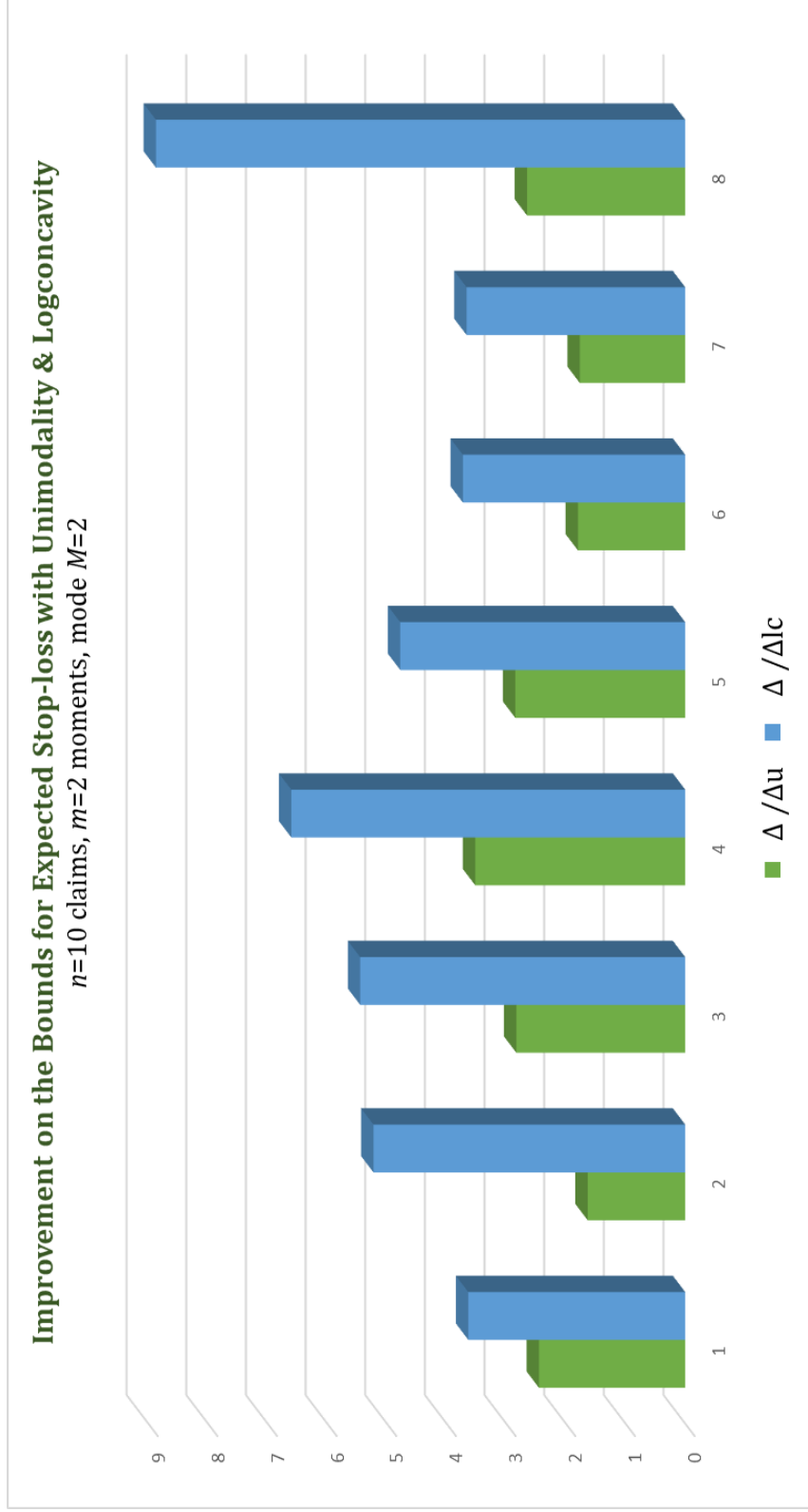


Figure 4.3: Improvement on the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 2$  moments, mode  $M = 2$

Table 4.2: Bounds for the Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 3$  moments, mode  $M = 2$

Bounds based on $S_1, S_2, S_3$	Without Unimodality		With Unimodality		Logconcavity		$\Delta = UB - LB$			Improvement	
	$LB$	$UB$	$LB_u$	$UB_u$	$LB_{lc}$	$UB_{lc}$	$\Delta$	$\Delta_u$	$\Delta_{lc}$	$\Delta / \Delta_u$	$\Delta / \Delta_{lc}$
q											
2	2.471	2.643	2.471	2.553	2.475	2.54	0.172	0.0821	0.065	2.095	2.646
3	1.529	1.832	1.62	1.723	1.682	1.718	0.303	0.103	0.036	2.942	8.417
4	0.852	1.198	1.04	1.131	1.046	1.108	0.346	0.091	0.062	3.802	5.581
5	0.338	0.807	0.588	0.701	0.61	0.669	0.469	0.113	0.059	4.15	7.949
6	0.178	0.446	0.292	0.379	0.328	0.359	0.268	0.087	0.031	3.08	8.645
7	0.034	0.239	0.129	0.185	0.146	0.168	0.205	0.056	0.022	3.661	9.318
8	0	0.141	0.022	0.088	0.041	0.071	0.141	0.066	0.03	2.1364	4.7
9	0	0.071	0	0.029	0	0.02	0.071	0.029	0.02	2.448	3.55

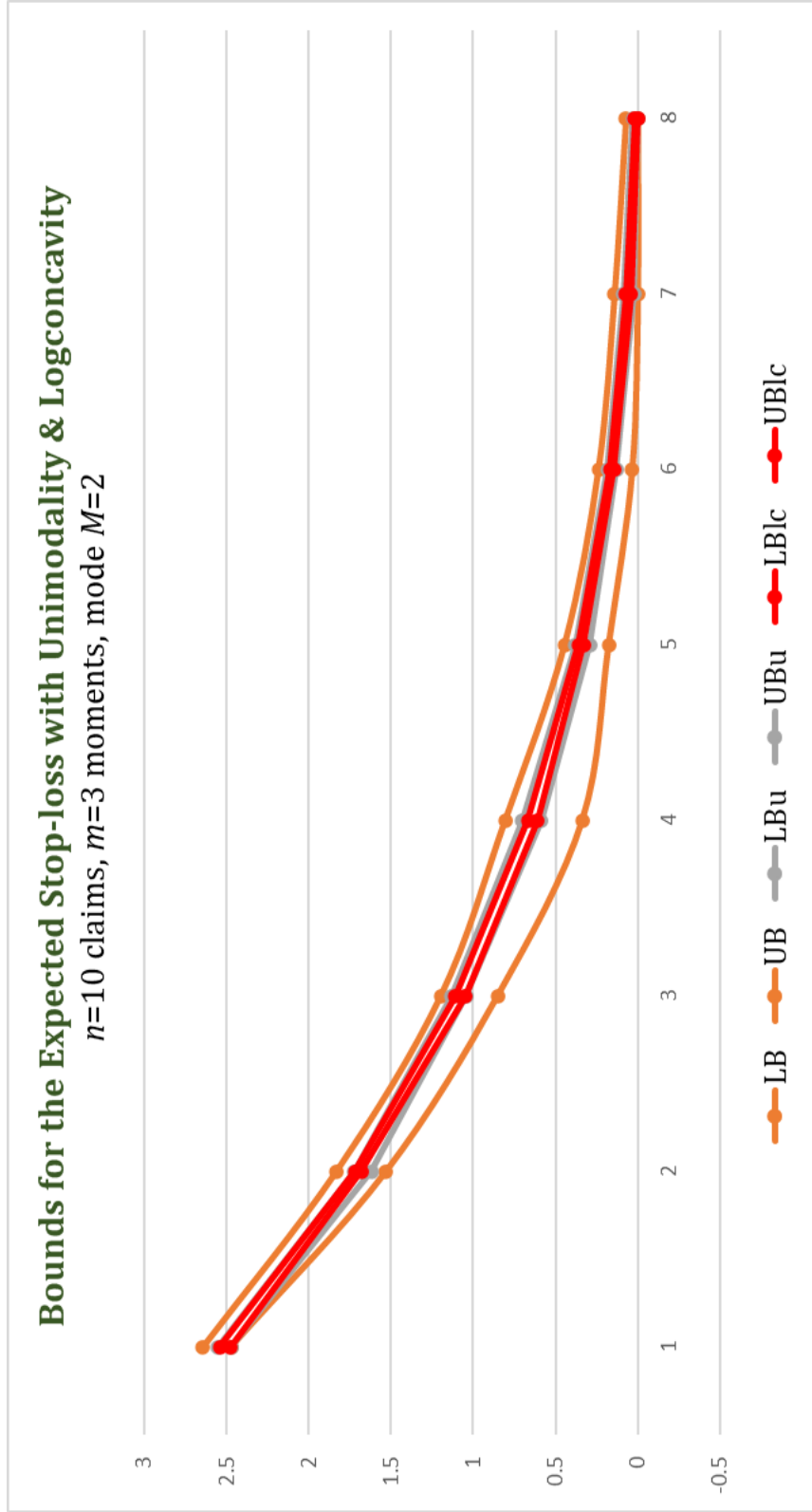


Figure 4.4: Comparison of the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 3$  moments, mode  $M = 2$

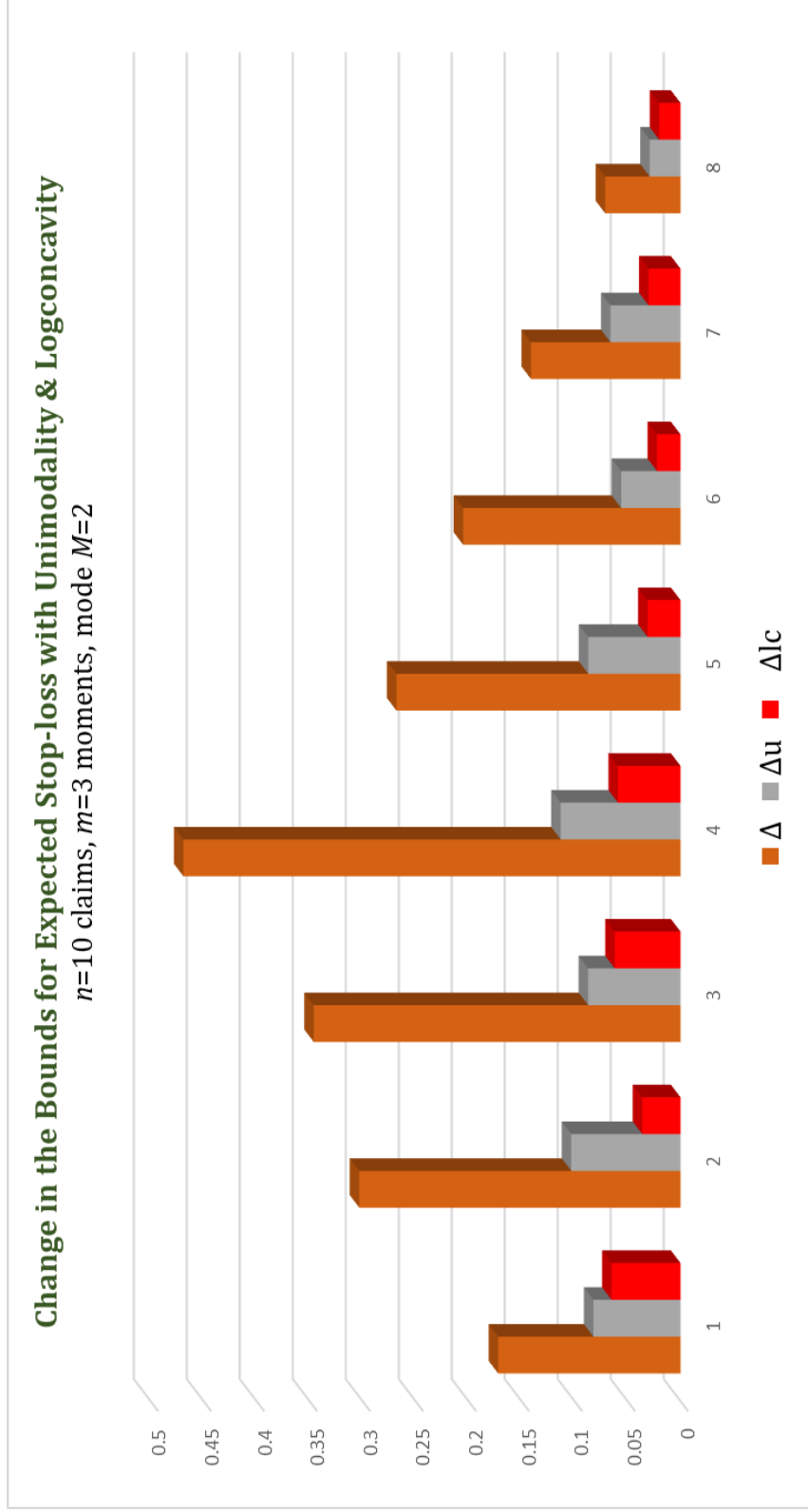


Figure 4.5: Change in the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 3$  moments, mode  $M = 2$



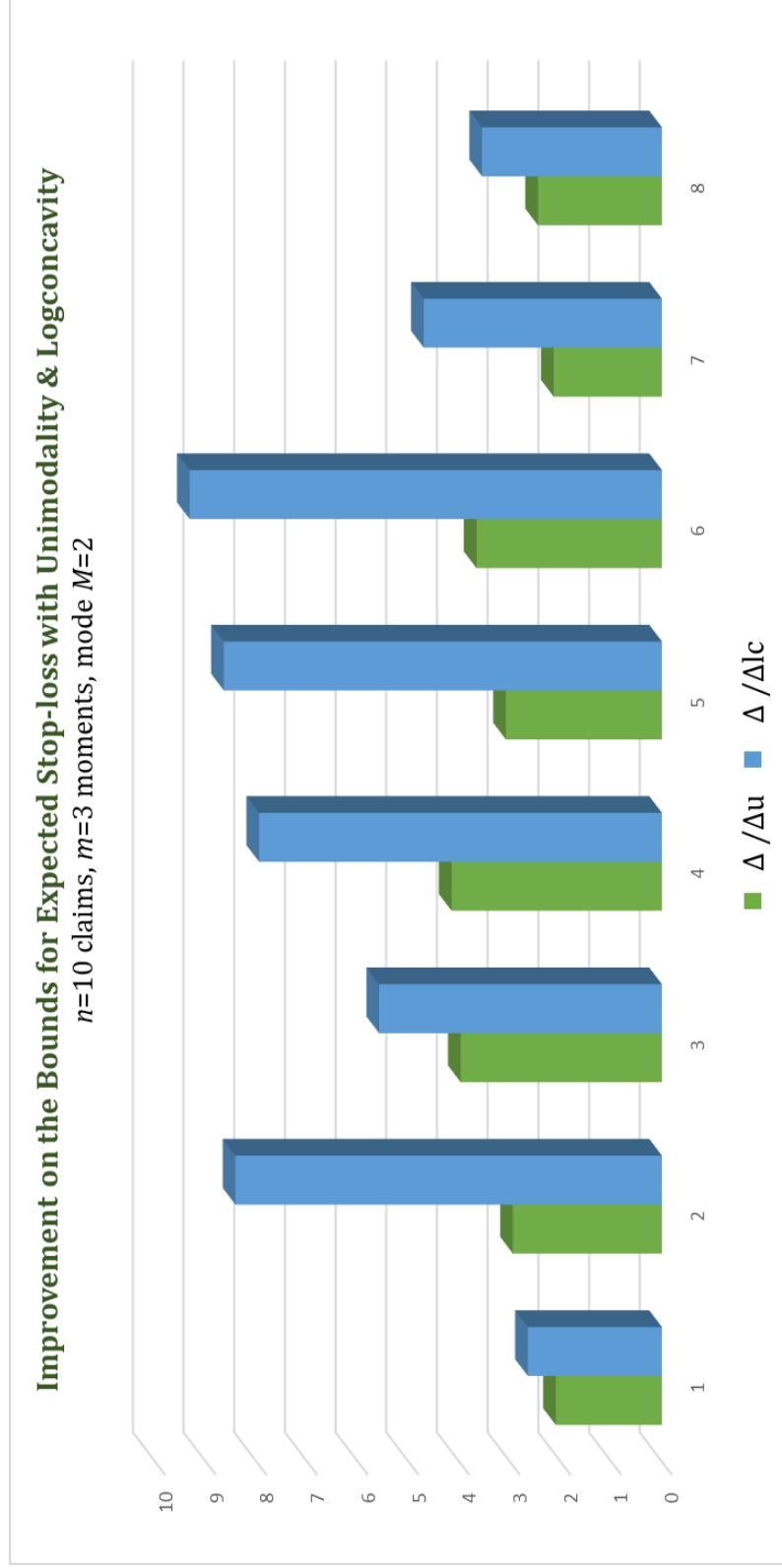


Figure 4.6: Improvement on the Bounds for Expected Stop-loss with Unimodality & Logconcavity:  
 $n = 10$  claims,  $m = 3$  moments, mode  $M = 2$

# Chapter 5

## Conclusion

In this research, we investigate the contribution of the shape constraints in discrete moment problems that was originally formulated as a linear programming problem to approximate the linear functions on the unknown discrete probability distributions non-negative and finite support, where some of the moments of the underlying distribution are known or obtained from historical data. The moments can be power, binomial or more general type. These problems came to prominence by the discovery that the classical probability bounds and expectations of discrete random variables can be obtained based on the knowledge of some of the binomial moments or power moments.

We introduce new shape constraints, logconcavity and logconvexity, to discrete moment problems for bounding the  $k$ -out-of- $n$  type probabilities and expectations of higher order convex functions of discrete random variables with non-negative and finite support, based on the knowledge of first  $m$  power or binomial moments where  $m$  is much smaller than the size of the support set of the underlying probability distribution. Discrete moment problem with logconcavity constraint is a

non-convex nonlinear optimization problem. We transform this problem into a bilinear optimization problem to solve it more efficiently. In case of logconvexity constraints, while the problem turns into a nonlinear program, the convexity of the problem is preserved. We perform several computational experiments, where we demonstrate the utility of the logconcavity and logconvexity property within the concept of probability bounding methodology. Numerical experiments show the improvement in the tightness of the bounds when the shape of underlying unknown probability distribution is prescribed into discrete moment problems even for the case of first two power or binomial moments. As expected from the theory of optimization, these results are further improved when the first three power or binomials are used. What makes it interesting and exciting is the improvement on the tightness of the bounds both in case of logconcave and logconvex distributions. We apply our optimization based bounding methodology in an insurance problem to estimate the expected stop-loss of aggregated insurance claims within a fixed period, where we assume the underlying claim distribution is normally distributed and therefore the distribution of the aggregated claims is the convolution of normally distributed random variables. We expect our proposed bounding methodology to be extended to various fruitful applications, including reliability, finance, and stochastic networks, where the underlying probability distribution is unknown, but the shape and the first two or three moments can be obtained from the historical data.

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## Publications

- Alharbi, T., Ninh, A., Subasi, E., Subasi, M.M. (2020). **Log-concavity, Log-convexity, and Discrete Moment Problems**. In preparation.

## Conference Presentations

- Alharbi, T., Ninh, A., Subasi, E., Subasi, M.M. **Log-concavity, Log-convexity, and Discrete Moment Problems**, 2019 INFORMS Annual Conference, October 20-23, 2019, Seattle, WA, USA.
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