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Global Upper Bounds for The Landau Equation of Plasma Physics in The Very Soft Potentials Case

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Global upper bounds for the Landau equation of plasma physics in the very soft potentials case

By

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"Global upper bounds for the Landau equation of plasma physics in the very soft potentials case"
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Abstract

Title: *Global upper bounds for the Landau equation of plasma physics in the very soft potentials case*

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This paper explores global upper bounds for solutions of the Landau equation in the soft potentials case ($\gamma < -2$). In particular, this paper explores the case of $\gamma \in [-3, -2)$. Working with a classical solution to the Landau equation weighted by a cut-off function χ and using the Moser iteration, an upper bound for the L^∞_v norm of the solution to the Landau equation f is obtained proportianally to the L^2_v norm of f with the assumptions of positive, essentially bounded coefficients. The supremum of f for $t \in [0, T], x \in \mathbb{R}^3, v \in B_R$ for some large radius R is shown to be bounded polynomially in R.

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List of Notations

- $g = \chi f^q$

  where χ is a cutoff function, f is a solution to the Landau equation, and q is a constant.

- $Q(\rho) = \big\{(x, v, t) : |x| < \rho^3, |v| < \rho, t \in (-\rho^2, 0]\big\}$

  where $0 < \rho \leq 1$.

- χ is defined as a smooth cutoff function in $\mathbb{R}^7$ such that

$$\chi = \begin{cases} 1 \text{ in} & Q(\rho_{\text{int}}) \\ 0 \text{ outside} & Q(\rho_{\text{ext}}) \end{cases}$$

- $r = \rho_{\text{ext}} - \rho_{\text{int}}$

- $|f|_{L^p(\Omega)} = \left(\int_\Omega f^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$

- $\Lambda = |A|_{L^\infty}$

  where A is a coefficient in the Landau equation

- $A \lesssim B \Rightarrow A \leq CB$ for some constant C depending only on γ.

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Chapter 1

Introduction

1.1 General overview

The Landau equation reads

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + Cf$$

with nonlocal coefficients $A(t,x,v) \in \mathbb{R}^{3\times 3}$, $B(t,x,v) \in \mathbb{R}^3$, and $C(t,x,v) \in \mathbb{R}$ defined by

$$\begin{aligned} A(t,x,v) &= A_\gamma \int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(t,x,v-w)\,\mathrm{d}w \\ B(t,x,v) &= B_\gamma \int_{\mathbb{R}^3} |w|^\gamma w f(t,x,v-w)\,\mathrm{d}w \\ C(t,x,v) &= C_\gamma \int_{\mathbb{R}^3} |w|^\gamma f(t,x,v-w)\,\mathrm{d}w \end{aligned}$$

where A_γ, B_γ, and C_γ are positive constants.

The potentials are considered "hard" when $\gamma > 0$, "Maxwellian" when $\gamma = 0$, "moderately soft" when $\gamma \in [-2, 0)$, "very soft" when $\gamma \in [-3, -2)$, and "Coulomb" when $\gamma = -3$.

Above is the definition of the Landau equation which models the particle density of a plasma. It was derived in 1936 by Landau. The equation combines nonlocal diffusion in v with transport in t and x. Here, we look at the $\gamma \in [-3, -2)$ case which is the most physically relevant as well as the most mathematically difficult since it contains $\gamma = -3$.

It is important to note the solution f as it relates to the macroscopic properties of mass, energy, and entropy stated below respectively.

$$
\begin{aligned}
M_f(t) &= \int_{\mathbb{R}^6} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\
E_f(t) &= \int_{\mathbb{R}^6} |v|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\
H_f(t) &= \int_{\mathbb{R}^6} f(t,x,v)\log f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x
\end{aligned}
$$

It is known that mass and energy are conserved and that entropy is nonincreasing. It is also known that any solution of the form $c_1 e^{c_2|v-v_0|^2}$ is an equilibrium solution. [16]

This paper is motivated by the problem of solving the global existance of the Landau equation.

Global existence is known when the initial data is close to equilibrium, which was proven by Y. Guo [12]. Global existence is also known in the space homogeneous case, but only when $\gamma \geq -2$ [2] [7] [18].

Global existence for general initial data (spatially varying and not close to equilibrium) is a very difficult open problem. A stepping stone to this difficult problem is conditional regularity: if the mass, energy, and entropy densities of f are uniformly under control in t and x, then f is smooth and can be continued past a given time. (Note that mass, energy, and entropy densities are integrals on only v, whereas mass, energy, and entropy are integrals in x and v) This was done in the following paper, but only when $\gamma > -2$ [13]. When $\gamma \leq -2$, one needs more conditional assumptions on f to get conditional regularity. A uniform bound on the L^p_v norm, where $p > 3/(3+\gamma)$, is sufficient [14]. The goal of this paper is to show an L^∞ bound requiring the L^p_v norm be bounded with $p > 3/(5+\gamma)$ at the cost of growing polynomial weight in velocity.

We first will prove the bound of the L^∞ norm of f as Theorem 1.1.1

Theorem 1.1.1. *Let f be any solution of the Landau equation on $(-1,0]\times B_1\times\mathbb{R}^3$. Then the following local estimate holds:*

$$|f|_{L^\infty(Q(\frac{1}{2}))}\leq C\left(\frac{|A|_{L^\infty}}{\lambda}\left(1+|A|_{L^\infty}+|B|_{L^\infty}+|C|_{L^\infty}\right)\right)^{\frac{19}{2}}|f|_{L^2(Q(1))}$$

where the constant $C>0$ depends only on γ, and $\lambda>0$ is the lower ellipticity constant of A. Note that this theorem could be applied to cylinders centered at points other than $(0,0,0)$, by translation.

We will then prove the bound of the supremum of f depending on $t\in[0,T]$, $x\in\mathbb{R}^3$ and $v\in B_R$ for a large radius R as Theorem 1.1.2

Theorem 1.1.2. *Let f be a solution of the Landau equation on $[0,T]\times\mathbb{R}^3\times\mathbb{R}^3$. Then, for any sufficiently large radius $R>0$, one has*

$$\sup_{[0,T]\times\mathbb{R}^3\times B_R} f\leq CR^{-19\gamma}$$

for a constant $C>0$ depending only on the initial data and the $L^\infty_{t,x}L^1_v$ and $L^\infty_{t.x}L^p_v$ norms of f. The important aspect of this theorem is that the constant C is independent of the L^∞ norm of f.

A goal for future work is to remove the dependence of R, i.e. proving a global unweighted L^∞*estimate.*

1.1.1 Proof Methods

We use a Moser iteration method to prove our local L^∞ bound. A related method was applied in [10] . Compared to [10], which applies to general linear kinetic equations, this proof utilizes the coupling among A, B, C and the solution f to get a better constant in terms of L^∞ norms of A, B and C. We will also utilize the definitions of A, B and C as well as the scaling symmetries of the Landau equation.

1.2 Related work

In this section, we highlight papers related to this paper's research of the Landau equation in addition to those that have been referenced above. The related works in separated into two categories: research of the homogeneous Landau equation and research of the inhomogeneous Landau equation. The homogeneous Landau equation results when f is assumed to be independent of x.

1.2.1 Homogeneous Landau:

This section highlights some of the most important works on the homogeneous Landau equation.

For the spatially homogeneous Landau equation with moderately soft potentials, any weak solution instantaneously becomes smooth and remains smooth over time. For very soft potentials, a conditional regularity result assumed to hold uniformly over time exists dependent on a nonlinear Morrey space bound. Such a bound holds true for very soft potentials, and nearly holds for the general potential case. [11] For Coulomb interactions, when bounding the entropy dissipation by an H^1 norm of $f^{\frac{1}{2}}$, an $L^1_t(L^3_v)$ estimate is obtained as well as the propogation of L^1. These estimates are then applied to the Landau equation with (moderately) soft potentials. [6] This next paper looks at the spatially homogeneous Boltzmann equation with grazing collisions. For a very broad class of potentials, when studying the Boltzmann equation without a cut-off, the Fokker-Planck-Landau equation, and the asymptotics of grazing collisions, the Landau equation for Coulomb potentials can be rigorously derived by introducing a new form of weak solutions using the entropy production. [17] The set of singular times for weak solutions of the spatially homogeneous Landau equation with Coulomb potentials found in [17] has at most a Hausdorff dimension of $\frac{1}{2}$. [9]

1.2.2 Inhomogeneous Landau:

This section highlights some of the most important works on the inhomogeneous Landau equation.

When approximating the Boltzmann equation by the Landau equation for grazing collisions, for the space-dependent equation with only assuming finite mass, energy, entropy and entropy production combined with entropy production smoothing effects, realistic singularities of Coulomb type and approximations of the Debye cut can be treated. Unfortunately, while the Landau equation describes long-time corrections to the Vlasov-Poisson equation, this method only works for finite time. Physically relevant situations may be delt with after time rescaling if the mean-field interaction is neglected. [2] For very soft and Coulomb potentials, Type 1 self-similar blow-up solutions are non-existent for the homogeneous and inhomogeneous Landau equation when assuming mild decay. [3] For the inhomogeneous Landau equation with moderately soft potential with arbitrary initial data, assuming that mass, energy, and entropy densities stay under control, pointwise estimates decay polynomially in the velocity variable. Further, if the initial data is satisfied by a Gaussian upper bound, this upper bound applies to all positive times. [5] For the spatially inhomogeneous Landau equation with moderately soft potentials on $\mathbb{R}^3$, if the initial data is close to the vacuum solution, then the solution remains globally regular in time. Additionally, near-vacuum solutions approach solutions to the linear transport equation for $t \to +\infty$. Further, generally, solutions do not approach a traveling global Maxwellian as $t \to +\infty$. [15]

The following paper primarily talks about the Boltzmann equation, but the proof also applies to the Landau equation. For some strong estimates of smoothness, decay at large velocities and strict positivity, estimates on the convergence rate of solutions of the Boltzmann equation such as $O(t^{-\infty})$ may be obtained. A key point of this paper is that Maxwellian equilibrium states are global attractors, i.e. under suitable conditions, solutions converge to equilibrium even if they do not start close to equilibrium. [8]

Chapter 2

Estimation of the L^∞-norm

We begin by presenting the proof of Theorem 1.1.1 using a number of supporting lemmas. Afterward, we give the proofs of the lemmas, which makes up the bulk of our work.

Let χ be chosen such that in $Q(\rho_{\text{ext}}) \setminus Q(\rho_{\text{int}})$

$$\begin{cases} \partial_{v_i}\chi \lesssim r^{-1} \\ \partial_{v_i v_j}\chi \lesssim r^{-2} \\ (\partial_t + v\cdot\nabla_x)\chi \lesssim r^{-2} \end{cases}$$

Here, we apply the constraints that A, B and C are measurable and satisfy

$$\begin{cases} 0 < \lambda I \le A \le \Lambda I \\ |B| \text{ essentially bounded} \\ C \text{ essentially bounded} \end{cases}$$

From Lemma 2.4, we have

$$|f|_{L^{pq}(Q(\rho_{\text{int}}))}^{2q} \le |g|_{L^p}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\left(r^{-4}+q^2\right)|f|_{L^{2q}(Q(\rho_{\text{ext}}))}^{2q}$$

with $p = 42/19$, for any $q > 1$, $r = \rho_{\text{ext}} - \rho_{\text{int}}$ and any choice of concentric cylinders $Q(\rho_{\text{int}})$ and $Q(\rho_{\text{ext}})$.

$$|f|_{L^{pq}(Q(\rho_{\text{int}}))} \lesssim \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{1}{2q}} \left(r^{-4}+q^2\right)^{\frac{1}{2q}}|f|_{L^{2q}(Q(\rho_{\text{ext}}))}$$

Let us define $\rho_i = \frac{1}{2} + (\frac{1}{2})^i$ and $q_i = (\frac{21}{19})^i$ and $r_i = \rho_i - \rho_{i+1} = (\frac{1}{2})^{i+1}$.

$$|f|_{L^{2q_{i+1}}(Q(\rho_{i+1}))} \lesssim \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{1}{2q_i}} \left(r_i^{-4}+q_i^2\right)^{\frac{1}{2q_i}}|f|_{L^{2q_i}(Q(\rho_i))}$$

Since $q_i^2 = (\frac{21}{19})^{2i} \lesssim 16(16)^i = r_i^{-4}$,

$$|f|_{L^{2q_{i+1}}(Q(\rho_{i+1}))} \lesssim 2^{(2i+\frac{5}{2})(\frac{19}{21})^i} \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{1}{2}(\frac{19}{21})^i} |f|_{L^{2q}(Q(\rho_i))}$$

After iterating, we get

$$|f|_{L^\infty(Q(\frac{1}{2}))} \lesssim 2^{\sum_{i=1}^\infty (2i+\frac{5}{2})(\frac{19}{21})^i} \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{1}{2}\sum_{i=1}^\infty(\frac{19}{21})^i} |f|_{L^2(Q(1))}$$

Evaluating the summations,

$$|f|_{L^\infty(Q(\frac{1}{2}))} \lesssim 2^{\frac{893}{4}} \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{19}{4}} |f|_{L^2(Q(1))}$$

$$|f|_{L^\infty(Q(\frac{1}{2}))} \lesssim \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left(1+|A|_{L^\infty}^2+|B|_{L^\infty}^2+|C|_{L^\infty}^2\right)\right)^{\frac{19}{4}} |f|_{L^2(Q(1))}$$

$$|f|_{L^\infty(Q(\frac{1}{2}))} \lesssim \left(\left(\frac{|A|_{L^\infty}}{\lambda}\right)^2 \left[(1+|A|^2_{L^\infty}+|B|^2_{L^\infty}+|C|^2_{L^\infty})^{\frac{1}{2}}\right]^2\right)^{\frac{19}{4}} |f|_{L^2(Q(1))}$$

By assuming any terms is the maximum term it can be shown that

$$\left(1+|A|^2_{L^\infty}+|B|^2_{L^\infty}+|C|^2_{L^\infty}\right)^{\frac{1}{2}} \lesssim 1+|A|_{L^\infty}+|B|_{L^\infty}+|C|_{L^\infty}$$

$$|f|_{L^\infty(Q(\frac{1}{2}))} \lesssim \left(\frac{|A|_{L^\infty}}{\lambda}\left(1+|A|_{L^\infty}+|B|_{L^\infty}+|C|_{L^\infty}\right)\right)^{\frac{19}{2}} |f|_{L^2(Q(1))}$$

End of Proof

Lemma 2.1. *If f solves the Landau equation, then g satisfies this inequality with H_0 and H_1 defined as follows:*

$$\partial_t g + v\cdot\nabla_x g \leq H_0 + \nabla_v\cdot H_1 + \nabla_v\cdot(A\nabla_v g)$$

$$\begin{aligned} H_0 =& f^q[(\partial_t + v\cdot\nabla_x)\chi + \nabla\cdot(A\nabla_v\chi) - \nabla_v\cdot(\chi B) + q\chi C] \\ H_1 =& f^q[-2A\nabla_v\chi + \chi B] \end{aligned}$$

Proof. Let us start with the following equation.

$$\begin{aligned} \partial_t g + v\cdot\nabla_x g =& (\partial_t + v\cdot\nabla_x)g \\ =& f^q(\partial_t + v\cdot\nabla_x)\chi + q\chi f^{q-1}[\nabla_v\cdot(A\nabla_v f) + B\cdot\nabla_v f + Cf] \\ =& f^q(\partial_t + v\cdot\nabla_x)\chi + q\chi f^{q-1}\nabla_v\cdot(A\nabla_v f) + q\chi f^{q-1}B\cdot\nabla_v f + q\chi C f^q \\ =& f^q(\partial_t + v\cdot\nabla_x)\chi + q\chi f^{q-1}\nabla_v\cdot(A\nabla_v f) + q\chi B\cdot\nabla_v f^q + q\chi C f^q \end{aligned}$$

Let us first analyze the A term.

$$\begin{aligned}
q\chi f^{q-1}\nabla_v\cdot(A\nabla_v f) =&\nabla_v\cdot(q\chi f^{q-1}A\nabla_v f) - \nabla_v(q\chi f^{q-1})\cdot A\nabla_v f\\
=&\nabla_v\cdot(\chi A\nabla_v f^q) - q\chi\nabla_v f^{q-1}\cdot A\nabla_v f - qf^{q-1}\nabla_v\chi\cdot A\nabla_v f\\
=&\nabla_v\cdot(A\nabla_v g) - \nabla_v\cdot(f^q A\nabla_v\chi)\\
&-\frac{q-1}{q}\chi\nabla_v f^{\frac{q}{2}}\cdot A\nabla_v f^{\frac{q}{2}} - \nabla_v f^q\cdot A\nabla_v\chi
\end{aligned}$$

Since A is a positive definite matrix,

$$q\chi f^{q-1}\nabla_v\cdot(A\nabla_v f) \leq \nabla_v\cdot(A\nabla_v g) - \nabla_v\cdot(f^q A\nabla_v\chi) - \nabla_v f^q\cdot A\nabla_v\chi$$

Applying product rule on the last term,

$$q\chi f^{q-1}\nabla_v\cdot(A\nabla_v f) \leq \nabla_v\cdot(A\nabla_v g) - \nabla_v\cdot(f^q[2A\nabla_v\chi]) + f^q\nabla\cdot(A\nabla_v\chi) \tag{2.1.1}$$

Next, we analyze the B term.

$$q\chi B\cdot\nabla_v f^q = \nabla_v\cdot(\chi f^q B) - f^q\nabla_v\cdot(\chi B)$$

Thus, recombining the terms, we get

$$\partial_t g + v\cdot\nabla_x g \leq H_0 + \nabla_v\cdot H_1 + \nabla_v\cdot(A\nabla_v g)$$

$$\begin{aligned}
H_0 =& f^q[(\partial_t + v\cdot\nabla_x)\chi + \nabla\cdot(A\nabla_v\chi) - \nabla_v\cdot(\chi B) + q\chi C]\\
H_1 =& f^q[-2A\nabla_v\chi + \chi B]
\end{aligned}$$

□

Lemma 2.2. *With f, H_0 and H_1 as above, H_0 and H_1 satisfy the following inequalities:*

$$|H_0|^2_{L^2} \lesssim \left(1 + |A|^2_{L^\infty} + |B|^2_{L^\infty} + |C|^2_{L^\infty}\right)\left(r^{-4} + q^2\right)|f|^{2q}_{L^{2q}(Q(\rho_{ext}))}$$
$$|H_1|^2_{L^2} \lesssim \left(1 + |A|^2_{L^\infty} + |B|^2_{L^\infty} + |C|^2_{L^\infty}\right)\left(r^{-4} + q^2\right)|f|^{2q}_{L^{2q}(Q(\rho_{ext}))}$$

Furthermore, with $\tilde{g}$ satisfying this equation

$$\partial_t \tilde{g} + v \cdot \nabla_x \tilde{g} = H_0 + \nabla_v \cdot H_1 + \nabla_v \cdot (A \nabla_v \tilde{g})$$

the following estimate is true:

$$|\nabla_v \tilde{g}|^2_{L^2} \lesssim \frac{1}{\lambda^2}(|H_0|^2_{L^2} + |H_1|^2_{L^2})$$

Proof. Since g is now a subsolution due to the inequality in (2.1.1), let us define $\tilde{g}$ as the solution.

$$\partial_t \tilde{g} + v \cdot \nabla_x \tilde{g} = H_0 + \nabla_v \cdot H_1 + \nabla_v \cdot (A \nabla_v \tilde{g})$$

We start by integrating the above equation with respect to $\tilde{g}$.

$$\int_{\mathbb{R}^7} \tilde{g}\partial_t \tilde{g} + \int_{\mathbb{R}^7} \tilde{g} v \cdot \nabla_x \tilde{g} = \int_{\mathbb{R}^7} \tilde{g} H_0 + \int_{\mathbb{R}^7} \tilde{g} \nabla_v \cdot H_1 + \int_{\mathbb{R}^7} \tilde{g} \nabla_v \cdot (A \nabla_v \tilde{g})$$

Applying integration by parts,

$$\int_{\mathbb{R}^7} \tilde{g}\partial_t \tilde{g} - \int_{\mathbb{R}^7} \frac{1}{2}\tilde{g}^2 \nabla_x \cdot v = \int_{\mathbb{R}^7} \tilde{g} H_0 - \int_{\mathbb{R}^7} \nabla_v \tilde{g} \cdot H_1 - \int_{\mathbb{R}^7} \nabla_v \tilde{g} \cdot (A \nabla_v \tilde{g})$$

It is important to note that the boundary terms disappear due to the compact support of $\tilde{g}$ inherited from χ. Using the lower ellipticity bound,

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 - \int_{\mathbb{R}^7} \frac{1}{2}\tilde{g}^2 \nabla_x \cdot v \leq \int_{\mathbb{R}^7} \tilde{g} H_0 - \int_{\mathbb{R}^7} \nabla_v \tilde{g} \cdot H_1 - \lambda |\nabla_v \tilde{g}|^2_{L^2}$$

Since v & x are independent,

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 \leq \int_{\mathbb{R}^7} \tilde{g} H_0 - \int_{\mathbb{R}^7} \nabla_v \tilde{g} \cdot H_1 - \lambda |\nabla_v \tilde{g}|_{L^2}^2$$

Using Young's inequality,

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 \leq \int_{\mathbb{R}^7} \tilde{g} H_0 + \frac{\lambda}{2}\int_{\mathbb{R}^7} |\nabla_v \tilde{g}|^2 + \frac{1}{2\lambda}\int_{\mathbb{R}^7} |H_1|^2 - \lambda |\nabla_v \tilde{g}|_{L^2}^2$$

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 \leq \int_{\mathbb{R}^7} \tilde{g} H_0 + \frac{1}{2\lambda}|H_1|_{L^2}^2 - \frac{\lambda}{2}|\nabla_v \tilde{g}|_{L^2}^2$$

Now let us look at the time derivative.

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 = \int_{x,v\in\mathbb{R}^6} \int_{-\rho_{\mathrm{ext}}^2}^{0} \frac{1}{2}\partial_t \tilde{g}^2$$

Using the Fundamental Theorem of Calculus, we get

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 = \int_{x,v\in\mathbb{R}^6} \frac{1}{2}[\tilde{g}^2(t=0) - \tilde{g}^2(t=-\rho_{\mathrm{ext}}^2)]$$

The term at $t = -\rho_{\mathrm{ext}}^2$ will be zero due to the cutoff that $\tilde{g}$ inherited from g.

The term at $t = 0$ will be positive, thus we get

$$\int_{\mathbb{R}^7} \frac{1}{2}\partial_t \tilde{g}^2 \geq 0$$

We can now move the $\nabla_v \tilde{g}$ term to the left-hand side.

$$\frac{\lambda}{2}|\nabla_v \tilde{g}|_{L^2}^2 \leq \int_{\mathbb{R}^7} \tilde{g} H_0 + \frac{1}{2\lambda}|H_1|_{L^2}^2$$

Using Young's inequality again and Poincare's inequality,

$$\frac{\lambda}{2}|\nabla_v \tilde{g}|_{L^2}^2 \lesssim \frac{\lambda}{4}\int_{\mathbb{R}^7} |\nabla_v \tilde{g}|^2 + \frac{1}{\lambda}\int_{\mathbb{R}^7} H_0^2 + \frac{1}{2\lambda}|H_1|_{L^2}^2$$

$$\frac{\lambda}{4}|\nabla_v \tilde{g}|_{L^2}^2 \leq \frac{1}{\lambda}|H_0|_{L^2}^2 + \frac{1}{2\lambda}|H_1|_{L^2}^2$$

$$|\nabla_v \tilde{g}|_{L^2}^2 \lesssim \frac{1}{\lambda^2}(|H_0|_{L^2}^2 + |H_1|_{L^2}^2)$$

Let us now look at the L^2 norms of H_0 and H_1.

$$\begin{aligned}
|H_0|_{L^2}^2 =& |f^q[(\partial_t + v\cdot\nabla_x)\chi + \nabla\cdot(A\nabla_v\chi) - \nabla_v\cdot(\chi B) + q\chi C]|_{L^2}^2 \\
|H_1|_{L^2}^2 =& |f^q[-2A\nabla_v\chi + \chi B]|_{L^2}^2
\end{aligned}$$

Applying product rule,

$$\begin{aligned}
|H_0|_{L^2}^2 =& |f^q[(\partial_t + v\cdot\nabla_x)\chi + \sum_{ij} A_{ij}\partial_{v_i v_j}\chi - 2\nabla_v\chi\cdot B + (q+1)\chi C]|_{L^2}^2 \\
|H_1|_{L^2}^2 =& |f^q[-2A\nabla_v\chi + \chi B]|_{L^2}^2
\end{aligned}$$

$$\begin{aligned}
|H_0|_{L^2}^2 \leq& |(\partial_t + v\cdot\nabla_x)\chi + \sum_{ij} A_{ij}\partial_{v_i v_j}\chi - 2\nabla_v\chi\cdot B + (q+1)\chi C|_{L^\infty}^2 |f^q|_{L^2(Q_{\text{ext}})}^2 \\
|H_1|_{L^2}^2 \leq& |-2A\nabla_v\chi + \chi B|_{L^\infty}^2 |f^q|_{L^2(Q_{\text{ext}})}^2
\end{aligned}$$

Applying the χ bounds,

$$\begin{aligned}
|H_0|_{L^2}^2 &\lesssim \left(r^{-2} + r^{-2}|A|_{L^\infty} + r^{-1}|B|_{L^\infty} + (q+1)|C|_{L^\infty}\right)^2 |f^q|_{L^2(Q_{\text{ext}})}^2 \\
|H_1|_{L^2}^2 &\lesssim \left(r^{-1}|A|_{L^\infty} + |B|_{L^\infty}\right)^2 |f^q|_{L^2(Q_{\text{ext}})}^2
\end{aligned}$$

Keeping the worst exponents,

$$\begin{aligned}
|H_0|_{L^2}^2 &\lesssim \left(1 + |A|_{L^\infty}^2 + |B|_{L^\infty}^2 + |C|_{L^\infty}^2\right)\left(r^{-4} + q^2\right)|f^q|_{L^2(Q_{\text{ext}})}^2 \\
|H_1|_{L^2}^2 &\lesssim \left(1 + |A|_{L^\infty}^2 + |B|_{L^\infty}^2 + |C|_{L^\infty}^2\right)\left(r^{-4} + q^2\right)|f^q|_{L^2(Q_{\text{ext}})}^2
\end{aligned} \tag{2.2.1}$$

□

Lemma 2.3. *Let $\tilde{g}$ satisfy the equation:*

$$\partial_t \tilde{g} + v \cdot \nabla_x \tilde{g} = H_0 + \nabla_v \cdot H_1 + \nabla_v \cdot (A \nabla_v \tilde{g})$$

the following L^p estimate is true for $p = \frac{42}{19}$

$$|\tilde{g}|^2_{L^p} \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 (|H_0|^2_{L^2} + |H_1|^2_{L^2})$$

where $|A|_{L^\infty}/\lambda$ is dependent on A.

Proof. We want to use the following estimate to find a lower bound. [4]

$$\begin{aligned} |D_t^{1/3} \tilde{g}|^2_{L^2} + |D_x^{1/3} \tilde{g}|^2_{L^2} \lesssim & |\tilde{g}|^2_{L^2} + |\nabla_v \tilde{g}|_{L^2} |(1 + |v|^2)^{1/2} H_0|_{L^2} \\ & + |\nabla_v \tilde{g}|^{4/3}_{L^2} |(1 + |v|^2)^{1/2} (H_1 - A \nabla_v \tilde{g})|^{2/3}_{L^2} \\ & + |\nabla_v \tilde{g}|_{L^2} |(1 + |v|^2)^{1/2} (H_1 - A \nabla_v \tilde{g})|_{L^2} \end{aligned}$$

Note that H_0 is constant and H_1 is a vector.

Since $|v| \leq \rho \leq 1$, we get

$$\begin{aligned} |D_t^{1/3} \tilde{g}|^2_{L^2} + |D_x^{1/3} \tilde{g}|^2_{L^2} \lesssim & |\tilde{g}|^2_{L^2} + |\nabla_v \tilde{g}|_{L^2} |H_0|_{L^2} \\ & + |\nabla_v \tilde{g}|^{4/3}_{L^2} |(H_1 - A \nabla_v \tilde{g})|^{2/3}_{L^2} + |\nabla_v \tilde{g}|_{L^2} |(H_1 - A \nabla_v \tilde{g})|_{L^2} \end{aligned}$$

Applying Young's inequality to each term and adding $|\nabla_v \tilde{g}|^2_{L^2}$ to both sides, we get

$$|\nabla_v \tilde{g}|^2_{L^2} + |D_t^{1/3} \tilde{g}|^2_{L^2} + |D_x^{1/3} \tilde{g}|^2_{L^2} \lesssim |\tilde{g}|^2_{L^2} + |\nabla_v \tilde{g}|^2_{L^2} + |H_0|^2_{L^2} + |H_1 - A \nabla_v \tilde{g}|^2_{L^2}$$

Applying the AM-QM inequality,

$$|\nabla_v \tilde{g}|^2_{L^2} + |D_t^{1/3} \tilde{g}|^2_{L^2} + |D_x^{1/3} \tilde{g}|^2_{L^2} \lesssim |\tilde{g}|^2_{L^2} + |\nabla_v \tilde{g}|^2_{L^2} + |H_0|^2_{L^2} + |H_1|^2_{L^2} + |A \nabla_v \tilde{g}|^2_{L^2}$$

Using the upper ellipticity bound ($|A|_{L^\infty} \geq 1$) and applying Poincare's inequality,

$$|\nabla_v \tilde{g}|_{L^2}^2 + |D_t^{1/3} \tilde{g}|_{L^2}^2 + |D_x^{1/3} \tilde{g}|_{L^2}^2 \lesssim |A|_{L^\infty}^2 |\nabla_v \tilde{g}|_{L^2}^2 + |H_0|_{L^2}^2 + |H_1|_{L^2}^2$$

Applying Lemma 2.2 and using the fact that $|A|_{L^\infty}/\lambda \geq 1$,

$$|\nabla_v \tilde{g}|_{L^2}^2 + |D_t^{1/3} \tilde{g}|_{L^2}^2 + |D_x^{1/3} \tilde{g}|_{L^2}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 (|H_0|_{L^2}^2 + |H_1|_{L^2}^2)$$

Applying Sobolev Embedding to the left-hand side with $p = \frac{42}{19}$, we get

$$|\tilde{g}|_{L^p}^2 \lesssim |\nabla_v \tilde{g}|_{L^2}^2 + |D_t^{1/3} \tilde{g}|_{L^2}^2 + |D_x^{1/3} \tilde{g}|_{L^2}^2$$

$$|\tilde{g}|_{L^p}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 (|H_0|_{L^2}^2 + |H_1|_{L^2}^2)$$

□

Lemma 2.4. *Let $\tilde{g}$ and g satisfy:*

$$\partial_t \tilde{g} + v \cdot \nabla_x \tilde{g} = H_0 + \nabla_v \cdot H_1 + \nabla_v \cdot (A \nabla_v \tilde{g})$$

and

$$\partial_t g + v \cdot \nabla_x g \leq H_0 + \nabla_v \cdot H_1 + \nabla_v \cdot (A \nabla_v g)$$

Any subsolution g will have the following L^p estimate

$$|g|_{L^p}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 \left(1 + |A|_{L^\infty}^2 + |B|_{L^\infty}^2 + |C|_{L^\infty}^2\right) \left(r^{-4} + q^2\right) |f|_{L^{2q}(Q(\rho_{ext}))}^{2q}$$

where $|A|_{L^\infty}/\lambda$ is dependent on A.

Proof. Subtracting the g & $\tilde{g}$ equations,

$$\partial_t (g - \tilde{g}) + v \cdot \nabla_x (g - \tilde{g}) - \nabla_v \cdot (A \nabla_v (g - \tilde{g})) \leq 0$$

Using the Maximum Principle, since $g - \tilde{g} = 0$ on the boundary,

$$g - \tilde{g} \leq 0$$

Thus we get

$$g \leq \tilde{g}$$

and

$$|g|_{L^p} \leq |\tilde{g}|_{L^p}$$

From Lemma 2.3, we have

$$|g|_{L^p}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 (|H_0|_{L^2}^2 + |H_1|_{L^2}^2)$$

Now using (2.2.1),

$$|g|_{L^p}^2 \lesssim \left(\frac{|A|_{L^\infty}}{\lambda} \right)^2 \left(1 + |A|_{L^\infty}^2 + |B|_{L^\infty}^2 + |C|_{L^\infty}^2\right) \left(r^{-4} + q^2\right) |f|_{L^{2q}(Q(\rho_{\text{ext}}))}^{2q}$$

$\square$

Chapter 3

Asymptotic upper bounds for large velocity.

We will now prove Theorem 1.1.2

3.1 Coefficient bounds

The following lemma gives upper bounds for the coefficients A, B, C, under the assumption that L^1_v and $L^{p+\delta}_v$ norms of f are bounded, where $p = 3/(5+\gamma)$. The dependence on L^∞ norms needs to be tracked precisely in the argument below.

Lemma 3.2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ belong to the space $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Let $p = \dfrac{3}{5+\gamma}$, and let $\delta > 0$ be an arbitrary small number. Assume further that $|f|_{L^p(\mathbb{R}^3)} \leq |f|_{L^\infty(\mathbb{R}^3)}$.*

Then the coefficients A, B, and C defined above satisfy, for all $v \in \mathbb{R}^3$,

$$|A(v)| \leq C,$$

$$|B(v)| \leq C|f|^{1-p(\gamma+4)/3}_{L^\infty(B_1(v))},$$

$$|C(v)| \leq C|f|^{1-p(\gamma+3)/3}_{L^\infty(B_1(v))},$$

for a constant $C > 0$ depending only on the $L^{p+\delta}(\mathbb{R}^3)$ and $L^1(\mathbb{R}^3)$ norms of f.

Note that this lemma will be applied for each fixed value of t and x, where $f(t, x, v)$ solves the Landau equation.

Also, the assumption $|f|_{L^p(\mathbb{R}^3)} \le |f|_{L^\infty(\mathbb{R}^3)}$ is reasonable in our context since if this assumption is false, our main theorem is automatically true.

Proof. First, we see that the components of A and B can be bounded above (up to a constant) by simpler convolutions: since $|w_i w_j |w|^{-2}| \le 1$,

$$\begin{aligned}
|A_{ij}(v)| &\le A_\gamma \int_{\mathbb{R}^3} \left|\delta_{ij} - \frac{w_i w_j}{|w|^2}\right| |w|^{\gamma+2} f(v-w)\,\mathrm{d}w \le 2a_\gamma \int_{\mathbb{R}^3} |w|^{\gamma+2} f(v-w)\,\mathrm{d}w, \\
|B_i(v)| &\le |B_\gamma| \int_{\mathbb{R}^3} |w|^{\gamma+1} f(v-w)\,\mathrm{d}w.
\end{aligned}$$

Since $\bar{c}^f$ is already of the same form (with the exponent γ), we see that bounding convolutions of the form $|v|^\sigma * f$ with $-3 < \sigma < 0$ will be sufficient to bound our coefficients.

We break the integral over $\mathbb{R}^3$ into integrals over B_r and $\mathbb{R}^3 \setminus B_r$, with $r > 0$ to be chosen later, and use Hölder's inequality on both pieces:

$$\begin{aligned}
(|v|^\sigma * f)(v) &= \int_{B_r} |w|^\sigma f(v-w)\,\mathrm{d}w + \int_{\mathbb{R}^3 \setminus B_r} |w|^\sigma f(v-w)\,\mathrm{d}w \\
&\le |f|_{L^\infty(B_r(v))} \int_{B_r} |w|^\sigma\,\mathrm{d}w + \left(\int_{\mathbb{R}^3 \setminus B_r} |w|^{q\sigma}\,\mathrm{d}w\right)^{1/q} \left(\int_{\mathbb{R}^3 \setminus B_r} f^p(v-w)\,\mathrm{d}w\right)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$ and all norms are over $\mathbb{R}^3_v$. We need $q\sigma < -3$ to have a convergent integral, or equivalently, $p < 3/(3+\sigma)$. For such a σ, we have

$$(|v|^\sigma * f)(v) \le |f|_{L^\infty(B_r(v))} C r^{\sigma+3} + C r^{\sigma + 3/q} |f|_{L^p(\mathbb{R}^3)}.$$

To get the optimal upper bound, we choose $r > 0$ so that the two terms balance, or more precisely, so that the dependence on f is the same in both terms. We guess that the right r is of the form

$$r = |f|^a_{L^\infty(B_1(v))} |f|^b_{L^p(\mathbb{R}^3)}$$

for some exponents a, b, and plug into the above to obtain

$$(|v|^\sigma * f)(v) \le C|f|_{L^\infty}^{1+a(\sigma+3)}|f|_{L^p}^{b(\sigma+3)} + C|f|_{L^\infty}^{a(\sigma+3/q)}|f|_{L^p}^{1+b(\sigma+3/q)}.$$

Setting the exponents of $|f|_{L^\infty}$ equal gives $1 + a(\sigma+3) = a(\sigma+3/q)$, or $a = -q/(3(q-1))$. Setting the exponents of $|f|_{L^p}$ equal gives $b(\sigma+3) = 1 + b(\sigma+3/q)$, or $b = q/(3(q-1))$. Since we would like our final estimate in terms of $p = q/(q-1)$ instead of q, we write $a = -p/3$ and $b = p/3$. This determines our r, and since $|f|_{L^p} \le |f|_{L^\infty}$, we in fact have $r \le 1$. With this choice, the estimate becomes

$$(|v|^\sigma * f)(v) \le C|f|_{L^\infty(B_1(v))}^{1-p(\sigma+3)/3}|f|_{L^p}^{p(\sigma+3)/3}, \quad \text{if } p < \frac{3}{3+\sigma}.$$

This estimate (applied for $\sigma = \gamma+1$ and $\sigma = \gamma$) implies the desired estimates for $B(v)$ and (when $\gamma \in (-3,-2]$) $C(v)$. In the case $\gamma = -3$, we have $C(v) = f(v)$, and we simply have $|C(v)|_{L^\infty(\mathbb{R}^3)} \le |f|_{L^\infty(B_1(v))}$, as claimed.

For A, we proceed slightly differently: with $\delta > 0$ small,

$$\begin{aligned}
|A(v)| &\lesssim \left(\int_{B_r} + \int_{B_r^c}\right) |w|^{\gamma+2} f(v-w)\,\mathrm{d}w \\
&\le |f|_{L^{p+\delta}(\mathbb{R}^3)} r^{\gamma+2+3/(p+\delta)'} + r^{\gamma+2}|f|_{L^1(\mathbb{R}^3)} \\
&\le |f|_{L^{p+\delta}(\mathbb{R}^3)}^{-(p+\delta)'(\gamma+2)/3} |f|_{L^1(\mathbb{R}^3)}^{1+(p+\delta)'(\gamma+2)/3},
\end{aligned}$$

where $(p+\delta)' = (p+\delta)/(p+\delta-1)$. We need to take $p+\delta$ rather than p here because $|w|^{p'(\gamma+2)}$ is not integrable near $w = 0$.

Note that in the statement of the lemma, we have absorbed the dependence on $L^1(\mathbb{R}^3)$ and $L^{p+\delta}(\mathbb{R}^3)$ norms into the constant C. Also, note that the $L^p(\mathbb{R}^3)$ norm of f is bounded in terms of the L^1 and $L^{p+\delta}$ norms of f, via a standard interpolation. $\square$

The following known estimate for the lower ellipticity bound λ is also needed. It applies when the initial data satisfies good lower bounds, and f is uniformly bounded in L^1_v and $L^{p+\delta}_v$.

Lemma 3.3. *[14, Lemma 2.5] Let $f : [0,T] \times \mathbb{R}^6 \to [0,\infty)$ be a solution of the Landau equation. Assume that $f(t,x,\cdot)$ is bounded in $L^1(\mathbb{R}^3) \cap L^{p+\delta}(\mathbb{R}^3)$, uniformly in (t,x), where $p = \dfrac{3}{5+\gamma}$, and*

$$f(0,x,v) \geq c_f, \quad v \in B_\rho(v_0),$$

uniformly in x, for some $\rho > 0$ and $v_0 \in \mathbb{R}^3$.

Then the coefficient A satifies the lower ellipticity bound

$$A(t,x,v) \geq c_A(1+|v|)^\gamma,$$

where $c_A > 0$ depends only on c_f, ρ, $|v_0|$, and $|f|_{L^\infty_{t,x}L^1_v} + |f|_{L^\infty_{t,x}L^{p+\delta}_v}$.

3.4 Improving the estimate through scaling

By direct calculation, if f is a solution of the Landau equation, then the rescaled function

$$f_r(t,x,v) = r^{5+\gamma} f(t_0 + r^2 t, x_0 + r^3 x + r^2 t v_0, v_0 + rv)$$

is also a solution to the Landau equation. The new coefficients corresponding to f_r are related to the f coefficients by

$$\begin{aligned} A_r(t,x,v) &= A(t,x,v), \\ B_r(t,x,v) &= rB(t,x,v), \\ C_r(t,x,v) &= r^2 C(t,x,v). \end{aligned}$$

Applying the above L^∞ estimate to f_r, we have

$$\begin{aligned} f(t_0,x_0,v_0) &\leq |f|_{L^\infty(Q_{r/2}(z_0))} \\ &= \frac{|f_r|_{L^\infty(Q_{1/2}(0))}}{r^{5+\gamma}} \\ &\lesssim \left(\frac{|A|_{L^\infty}}{\lambda}(1 + |A|_{L^\infty} + r|B|_{L^\infty} + r^2|C|_{L^\infty}) \right)^{19/2} \frac{|f_r|_{L^2(Q_1)}}{r^{5+\gamma}}. \end{aligned}$$

We analyze the L^2 norm on the right as follows:

$$\frac{|f_r|_{L^2(Q_1)}}{r^{5+\gamma}} \leq \frac{|f_r|_{L^\infty_{t,x}L^2_v(Q_1)}}{r^{5+\gamma}} = r^{-3/2}|f|_{L^\infty_{t,x}L^2_v(Q_r)}.$$

For brevity, let $L_0 = |f|_{L^\infty(Q_1(z_0))}$. We can assume $L_0 \geq 1$ since we need to study this inequality at points z_0 where f is large. With this notation, and incorporating the coefficient estimates from Lemma 3.2, we have

$$f(t_0, x_0, v_0) \lesssim \frac{1}{\lambda^{19/2}} \left(1 + rL_0^{1-p(\gamma+4)/3} + r^2 L_0^{1-p(\gamma+3)/3}\right)^{19/2} r^{-3/2}|f|_{L^\infty_{t,x}L^2_v(Q_r)}$$

The optimal scale r is chosen so that the terms inside the parentheses balance:

$$r = L_0^{-p/3},$$

and the estimate becomes

$$f(t_0, x_0, v_0) \lesssim \frac{1}{\lambda^{19/2}} \left(1 + L_0^{1-p(\gamma+5)/3}\right)^{19/2} L_0^{p/2}|f|_{L^\infty_{t,x}L^2_v(Q_r)}.$$

Recalling our choice $p = \dfrac{3}{5+\gamma}$, we have

$$f(t_0, x_0, v_0) \lesssim \frac{1}{\lambda^{19/2}} L_0^{p/2}|f|_{L^\infty_{t,x}L^2_v(Q_r)}.$$

For the $L^\infty_{t,x}L^2_v$ norm, we interpolate between L^∞ and $L^{p+\delta}$ norms, since we can choose δ small enough that $p + \delta < 2$:

$$|f|_{L^\infty_{t,x}L^2_v(Q_r(z_0))} \leq L_0^{(1-p-\delta)/2}|f|^{(p+\delta)/2}_{L^\infty_{t,x}L^{p+\delta}_v(Q_r(z_0))} \lesssim L_0^{(1-p-\delta)/2}.$$

This gives

$$f(t_0, x_0, v_0) \lesssim \frac{1}{\lambda^{19/2}} L_0^{p/2} L_0^{(1-p-\delta)/2} \lesssim \frac{1}{\lambda^{19/2}} L_0^{(1-\delta)/2}.$$

Now we use the lower bound of Lemma 3.3 to write $\lambda \gtrsim (1+|v_0|)^\gamma$, which gives

$$f(t_0,x_0,v_0) \lesssim (1+|v_0|)^{-19\gamma/2} L_0^{(1-\delta)/2}.$$

Taking the supremum over $(t_0,x_0,v_0) \in [0,T]\times\mathbb{R}^3\times B_R$, and recalling the definition of L_0, we obtain

$$\sup_{[0,T]\times\mathbb{R}^3\times B_R} f \leq CR^{-19\gamma/2}\left(\sup_{[0,T]\times\mathbb{R}^3\times B_{R+1}} f\right)^{(1-\delta)/2}$$

If R is large enough, we must have $\sup_{[0,T]\times\mathbb{R}^3\times B_R} f \geq \frac{1}{2}\sup_{[0,T]\times\mathbb{R}^6} f \geq \frac{1}{2}\sup_{[0,T]\times\mathbb{R}^3\times B_{R+1}}$. Rearranging terms then gives

$$\left(\sup_{[0,T]\times\mathbb{R}^3\times B_R} f\right)^{(1+\delta)/2} \leq CR^{-19\gamma/2},$$

or

$$\sup_{[0,T]\times\mathbb{R}^3\times B_R} f \leq CR^{-19\gamma/(1+\delta)}.$$

Discarding the small constant $\delta > 0$ gives the statement of the main theorem:

$$\sup_{[0,T]\times\mathbb{R}^3\times B_R} f \leq CR^{-19\gamma},$$

for a constant $C > 0$ depending only on the initial data and the $L^\infty_{t,x}L^1_v$ and $L^\infty_{t.x}L^{p+\delta}_v$ norms of f.

End of Proof

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