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Existence of Smooth Solutions for the Landau Equation with Hard Potentials

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"Existence of Smooth Solutions for the Landau Equation with Hard Potentials"
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Abstract

Title: *Existence of Smooth Solutions for the Landau Equation with Hard Potentials*

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This dissertation is concerned with the Landau equation, an integro-differential equation that models the particle density of a plasma as it evolves in phase space. The main topic is the (large-data) local existence of classical solutions to the Landau equation in the case of hard potentials ($\gamma \in (0, 1]$). Solutions have previously been constructed by Chaturvedi [SIAM J. Math. Anal., 55(5), 5345–5385, 2023] for initial data in an exponentially-weighted Sobolev space of order 10, but it is not a priori clear whether these solutions have more regularity than the initial data. We improve Chaturvedi's existence result in two ways: our solutions are infinitely differentiable for positive times, and we allow initial data that is more general in terms of regularity and decay, at the cost of requiring a mild positivity condition at time zero. We also prove uniqueness, under the additional assumption that the initial data is Hölder continuous.

Along the way, we establish some useful results that were previously only known in the case of soft potentials, including spreading of positivity and propagation of Hölder continuity. Many of the proof strategies from the soft potentials case do not apply here because of the more severe loss of velocity moments.

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Dedication

This thesis is dedicated to all pursuers of knowledge. Those who endeavour to discover truth in this world, that will shed light on the world to come.

"Our ideas of education take too narrow and too low a range. There is need of a broader scope, a higher aim. True education means more than the pursual of a certain course of study. It means more than a preparation for the life that now is. It has to do with the whole being, and with the whole period of existence possible to man. It is the harmonious development of the physical, the mental, and the spiritual powers. It prepares the student for the joy of service in this world and for the higher joy of wider service in the world to come." — Ellen G. White, Education

Chapter 1

Introduction

We are interested in the Landau equation, a kinetic integro-differential model from plasma physics. Let us refer to [36] for Landau's original 1936 paper that introduced the equation, and [9, 37, 1, 38, 46] for some general references about the equation, including physical modeling issues. For time $t \geq 0$, location $x \in \mathbb{R}^3$, and velocity $v \in \mathbb{R}^3$, the initial-boundary value problem for the particle density $f(t, x, v) \geq 0$ reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f, f), \\ f(0, x, v) = f_0(x, v), \\ f(t, \cdot, v) \text{ periodic in } x \text{ with period } 1, \end{cases} \tag{1.0.1}$$

where f_0 is some given initial condition. Letting $\mathbb{T}^3$ denote the three-dimensional torus of side length 1, since $f(t, \cdot, v)$ is periodic, we may equivalently consider it as a function on $\mathbb{T}^3$ with periodic boundary conditions on $\partial\mathbb{T}^3$. We will often use this equivalence between periodic functions on $\mathbb{R}^3$ and functions on $\mathbb{T}^3$ throughout this work.

Next, $Q(f, g)$ is Landau's bilinear collision operator, which acts only in the velocity variable, and is defined by

$$Q(f,g) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v-w)[f(w)\nabla_v g(v) - f(v)\nabla_w g(w)] \, \mathrm{d}w \right),$$

for any functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, and the matrix $a(z)$ is defined by

$$a(z) = a_\gamma \left(I - \frac{z \otimes z}{|z|^2} \right) |z|^{\gamma+2},$$

where, in general, $\gamma \in [-3, 1]$, $a_\gamma > 0$ is a constant depending on γ, and I is the 3×3 identity matrix.

The following regimes for the parameter γ are considered in the literature:

- $\gamma > 0$: hard potentials.
- $\gamma = 0$: Maxwellian molecules.
- $-2 \leq \gamma < 0$: moderately soft potentials.
- $-3 \leq \gamma < -2$: very soft potentials.
- $\gamma = -3$: Coulomb potentials.

We are concerned with the case $\gamma > 0$.

Physically, the Landau equation is a kinetic model that seeks to understand the evolution of a plasma, which is made up of many small particles, by studying the particle distribution function $f(t, x, v)$ rather than tracking individual particles. While f itself is not physically observable, any average (integral) of f in (x, v) space over a set $\Omega \subset \mathbb{R}^6$ gives the expected number of particles with position and velocity in Ω at time t.

The left-hand side of (1.0.1) represents transport, and $Q(f, f)$ describes how the particle distribution changes as a result of binary collisions.

The model assumes that the plasma is diffuse enough that only binary collisions are frequent enough to contribute, but not so diffuse that the approximation of discrete particles with a continuous density is invalid. The model also assumes that the boundary is far away enough to be neglected—this is the justification for the periodic boundary conditions.

The standard kinetic equation for studying many-particle systems with binary collisions is the Boltzmann equation [44]. However, the Boltzmann equation is not well-defined for Coulomb interactions [1], and in the case of plasmas, the particles are highly charged ions, so Coulomb interactions are the most relevant case. This was Landau's original motivation for deriving equation (1.0.1) in [36]. Briefly, the derivation proceeds as follows: to obtain a well-defined collision operator, one modifies the Boltzmann collision operator by screening the Coulomb interaction potential at some length scale λ. The Landau collision operator defined above, is then obtained as the leading order term as $\lambda \to \infty$. This limiting process is highly involved, so we refer to [1, 12] for more details.

Let us recall some basic properties of the Landau equation. Define the following quantities:

$$\begin{aligned}
\text{Mass} &= \iint_{\mathbb{R}^6} f(t,x,v)\,dv\,dx,\\
\text{Momentum} &= \iint_{\mathbb{R}^6} v f(t,x,v)\,dv\,dx,\\
\text{Energy} &= \iint_{\mathbb{R}^6} |v|^2 f(t,x,v)\,dv\,dx,\\
\text{Entropy} &= \iint_{\mathbb{R}^6} f(t,x,v)\log f(t,x,v) dv\,dx.
\end{aligned}$$

For f a solution of (1.0.1), the mass, momentum, and energy are conserved, and the entropy is decreasing. Furthermore, functions of the form $c_1 e^{-c_2|v|^2}$ (called *Maxwellians*) are equilibrium solutions.

Global existence with general initial data poses a very challenging open problem. Therefore, the scope of this dissertation focuses on the nontrivial question of local existence, which is less understood in the case $\gamma > 0$. More specifically, our goal is to prove existence of a classical solution on some time interval $[0, T]$, given some "large" (i.e. not necessarily close to equilibrium) initial data f_0. Other aspects of the well-posedness question (global solutions near equilibrium, renormalized solutions, space homogeneous solutions, etc.) will be briefly surveyed in Section 1.2 below.

Local existence for large initial data began with [27], which addressed the case $\gamma = -3$, and [30], which addressed $\gamma \in [-3, 0)$. The hard potentials case was established last by Chaturvedi in [10]. This is the most difficult case for local existence because the growth of $a(z)$ for large z leads to a loss of velocity moments in various estimates of the collision operator. In particular, the analysis in [10], which involves a heirarchy of weighted Sobolev norms, is noticeably more intricate than the existence proof for the soft potentials case in [30].

Regarding the allowable spaces of initial data, all three of the mentioned works [27, 30, 10] worked with initial data in Sobolev spaces of high degree (at least 4) with either exponential or high-degree polynomial decay in velocity. The next step in large-data well-posedness was to enlarge the allowable space of initial data f_0 to include functions with no, or minimal, regularity hypotheses, while still recovering smoothness of f for positive times. This is a natural goal because the Landau equation is known to have a hypoelliptic smoothing effect [21, 29]. This was accomplished for the case $\gamma \in [-3, 0)$ in [31], which took initial data f_0 with $(1 + |v|^5)f_0 \in L^\infty$. Again, the proof does not extend naturally to hard potentials because of velocity moment loss in several steps of the argument.

The only prior existence result for hard potentials is still [10], which does not imply any smoothing for the solution. This leaves open two questions:

1. Do the solutions constructed in [10] regularize? There is an *a priori* smoothing theorem for the hard potentials case [41], but it is not straightforward to apply this result to the solutions contstructed in [10] because [41] assumes a uniform lower bound on the mass density.

2. Can the allowable space of initial data be enlarged beyond the space used in [10], which is essentially $\{f_0 : e^{\rho|v|} f_0 \in H^{10}(\mathbb{R}^6)\}$?

The current work answers these questions affirmatively by constructing a solution to the Landau equation (1.0.1) with $\gamma \in (0, 1]$, given initial data f_0 with $e^{\rho|v|^\beta} f_0 \in L^\infty(\mathbb{R}^6)$ for some $\beta \in [\gamma, 1]$. We also need to assume f_0 is uniformly positive in some small ball in (x, v) space. Therefore, our results are comparable to those in [31], except that we assume sub-exponential decay in v instead of polynomial. As in [31], we require additional hypotheses on f_0 (Hölder continuity and a stronger lower bound assumption) to prove uniqueness of solutions.

Parts of this dissertation appeared, in a somewhat modified form, in the article [43].

1.1 Main results

Recall the notation $\langle v\rangle = \sqrt{1+|v|^2}$.

Theorem 1.1.1. *Let $\gamma \in (0,1]$, and let $f_0 : \mathbb{R}^3_x \times \mathbb{R}^3_v \to [0,\infty)$ be periodic in the x variable and satisfy*

$$\|e^{\rho\langle v\rangle^\beta} f_0\|_{L^\infty(\mathbb{R}^6)} \le K_0,$$

for some $\rho, K_0 > 0$ and $\beta \in [\gamma, 1]$. Furthermore, if $\gamma \in (0,1)$, assume there exist $(x_m, v_m) \in \mathbb{R}^6$ and $\delta, r > 0$ such that

$$f_0(x,v) \ge \delta, \quad |x - x_m| < r, |v - v_m| < r.$$

If $\gamma = 1$, assume there exist $\delta, r, R > 0$ such that for every $x_m \in \mathbb{R}^3$, there exists a $v_m \in B_R(0)$ with

$$f_0(x,v) \ge \delta, \quad |x - x_m| < r, |v - v_m| < r.$$

Then there exist $T, \sigma > 0$ depending on γ, β, and K_0 (but not on δ or r) and a classical solution f to the Landau equation (1.0.1) on $[0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ such that $e^{(\rho-\sigma t)\langle v\rangle^\beta} f(t) \in L^\infty(\mathbb{R}^6)$ for each $t \in [0,T]$. This solution is periodic in the x variable with the same period as f_0, and infinitely differentiable in $(0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$, with any partial derivative ∂f in (t,x,v) variables bounded uniformly on any compact subset of $(0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$.

The solution f agrees with the initial data in the following sense: for any test function $\phi \in C^1_{t,x} C^2_v$ with compact support in $[0,T) \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$\int_{\mathbb{R}^6} f_0(x,v)\phi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^6} [f(\partial_t + v\cdot\nabla_x)\phi + Q(f,f)\phi]\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.$$

Several comments on the statement of Theorem 1.1.1 are in order:

- The assumption of periodicity in x is purely a technical condition, and could be removed at the cost of more technical arguments. The period does not affect our estimates quantitatively. If the spatial domain were not periodic, one would need to assume that lower bounds for f_0 are "well-distributed" in the sense of [31], i.e. that no x location is too far away from a location where f satisfies positive lower bounds.

- The extra lower bound condition when $\gamma = 1$ may be an artifact of our proof. However, it is interesting to note that the existence result [10] also contains minor differences in assumptions between the $\gamma \in (0,1)$ and $\gamma = 1$ cases, suggesting $\gamma = 1$ may be a genuine borderline.

- If the initial data is continuous, then it can be shown that $f(t,x,v) \to f_0(x,v)$ pointwise as $t \to 0$, as expected. The proof is identical to [31, Proposition 3.1], so we omit it.

Next, we state our main uniqueness result. As in the corresponding study of the soft potentials case [31] and the related work [33] on the Boltzmann equation, stronger assumptions are required to prove uniqueness than existence.

Theorem 1.1.2. *Let $f_0 : \mathbb{R}^3_x \times \mathbb{R}^3_v \to [0,\infty)$ be periodic in the x variable and satisfy*

$$\|e^{\rho_0 \langle v \rangle^\gamma} f_0\|_{C^\alpha_{k,x,v}(\mathbb{T}^3 \times \mathbb{R}^3)} \leq K_0,$$

for some $\rho_0, K_0 > 0$ and $\alpha \in (0,1)$. Furthermore, assume there are $\delta, r, R > 0$ such that for every $x_m \in \mathbb{R}^3$, there exists $v_m \in B_R(0)$ with

$$f_0(x,v) \geq \delta, \quad |x - x_m| < r, |v - v_m| < r.$$

Let $f : [0,T] \times \mathbb{T}^3 \times \mathbb{R}^3 \to [0,\infty)$ be the solution to the Landau equation (1.0.1) guaranteed by Theorem 1.1.1.

There exists $T_U \in (0,T]$, depending on K_0, ρ_0, and α, such that for any classical solution $g \geq 0$ to (1.0.1) on $[0,T_U] \times \mathbb{T}^3 \times \mathbb{R}^3$ with $g(0,x,v) = f_0(x,v)$, there must hold $f = g$.

The uniqueness or non-uniqueness of the solutions constructed in Theorem 1.1.1, when the extra hypotheses of Theorem 1.1.2 are not satisfied, remains an open question.

1.2 Related work

1.2.1 Overview of prior work on the Landau equation

In this subsection, we present some important standard references on the Landau equation.

The Landau equation was introduced in 1936 in order to deal with the mathematical failure of the Boltzmann equation in the case of Coulomb interactions. For the physical background, see the treatise of Lifshitz and Pitaevskii [37].

In [1], Alexandre-Villani presented an argument that directly addresses the problem of justifying the Landau approximation, i.e. realizing the Landau equation as an appropriate scaling limit of the Boltzmann equation.

The article [13] presents an estimate that will bound below the entropy dissipation of the Landau operator with Coulomb interaction by a weighted H^1 norm of $\sqrt{f}$. In addition, the article presents applications to existence theory.

The work [21] presents the Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and applications to the Landau equation. Specifically, this paper presents the study of the Hölder regularity and establish a Harnack inequality for solutions to a general linear equation of Fokker-Planck type whose coefficients are merely measurable and essentially bounded. These general results are then applied to the non-negative essentially bounded weak solutions of the Landau equation with inverse-power law $\gamma \in [-d, 1]$ whose mass, energy and entropy density are bounded, and mass is bounded away from 0, implying the Hölder regularity of these solutions.

In the article [41], the author considers Gaussian bounds for the inhomogeneous Landau equation with hard potential. The goal of the paper is to prove that solutions of the Landau equation with hard potentials are bounded above and below by Maxwellians.

The article [3] rules out the existence of some self-similar blowup solutions to the Landau equation. However, the question of large-data global existence vs. breakdown remains open for the inhomogeneous equation.

1.2.2 Existence theory

In this subsection, let us discuss references on the existence theory for the Landau equation.

We begin with the article [15], where the Cauchy problem is examined for the homogeneous (i.e. x-independent) Landau equation, specifically for the case of hard potentials. For a large class of initial data, it is proven that there exists a unique weak solution to this problem, which becomes immediately smooth and rapidly decaying at infinity. This paper gives a detailed discussion of the Cauchy problem and the qualitative properties of the solutions, specifically, the smoothing effects.

In [47], the author establishes some global in time a priori estimates of the spatially homogeneous Landau equation for moderately soft potentials, meaning $\gamma \in [-2, 0)$. The global well-posedness results for $\gamma \in [-2, 0)$ is deduced as an application, where the estimates include the critical case of $\gamma = -2$.

Recently, it was shown in [24] that the space homogeneous Landau equation is globally well-posed for large initial data, even in the case of very soft and Coulomb potentials $\gamma \in [-3, 2)$.

Other works on existence and regularity theory for the spatially homogeneous Landau equation include [17, 40, 23, 6, 5, 20, 22, 14, 2, 19].

When the initial data f_0 is sufficiently close to a Maxwellian equilibrium state $M(v) = c_1 e^{-c_2|v|^2}$ with $c_1, c_2 > 0$, the Landau equation has a solution that is global in time. This has been known since the work of Guo [25]. Other works on the close-to-equilibrium case include [8, 7, 35, 16, 26, 18] and the references therein.

Global solutions close to the vacuum state $f \equiv 0$ were constructed in [39, 11].

A suitable notion of generalized solution, known as "renormalized solutions with defect measure," were shown to exist globally for the inhomogeneous equation by Villani [45]. However, the regularity and uniqueness of these solutions are not understood.

Furthermore, we discuss the spatially inhomogeneous Landau equation with soft potentials, inclusive of the case of Coulomb interactions. As discussed briefly above, [30] established the existence of solutions for a short time, with the assumption that the initial data is in a fourth-order Sobolev space and has Gaussian decay in the velocity variable (no decay assumptions are made in the spatial variable). Secondly, the evolution instantaneously spreads mass to every point in its domain. The article presents the optimal result of the pointwise lower bounds for a sub-Gaussian rate of decay.

For other works on the existence and regularity of the inhomogeneous Landau equation without a close-to-equilibrium assumption, see [42, 34].

Next, the article [31] presented a solution that is constructed for any bounded, measurable initial data with uniform polynomial decay in the velocity variable. This solution also satisfies a lower bound assumption. The assumption is made, for the uniqueness in this weak class, that the initial data is Hölder continuous.

Finally, let us discuss the article [10] in more detail. This work focused on the spatially inhomogeneous Landau equation with hard potentials on the whole space $\mathbb{R}^3_x$. With the assumption that the initial data is in a weighted tenth-order Sobolev space and decays exponentially in the velocity variable, the existence and uniqueness of the solutions for a small time is proven. The proof presented relies on a weighted hierarchy of norms that depends on the number of spatial and velocity derivatives in an asymmetric way, in order to overcome the moment loss issue. Thus, the hierarchy makes it possible to deal with the terms most affected by the moment loss. As mentioned above, our goal is to improve the result of [10].

1.3 Difficulties and proof strategy

As mentioned above, the results in this paper are in a similar spirit to [31], which considered the case of soft potentials ($\gamma \in [-3, 0)$) and used norms with polynomial velocity weights (as opposed to sub-exponential decay as in the current work). In general, the current study has to deal with the main difficulties encountered in [31] in addition to the new issues brought about by the stronger loss of moments when $\gamma > 0$.

Let us now discuss the proof strategies for three main areas of this work.

1.3.1 Spreading of positivity

A key tool in [31] is the positivity-spreading result of [30], which was proven via a probabilistic argument that requires $\gamma < 0$ in an apparently essential way. In the current work (see Theorem 5.0.1), we extend positivity-spreading to $\gamma > 0$ via a deterministic barrier argument inspired by a similar argument from the study of the Boltzmann equation [32]. The basic steps of this argument are as follows: (i) propagate local lower bounds forward for a short amount of time, (ii) spread lower bounds to high velocities, (iii) spread lower bounds in x, using the fact that f has positive lower bounds at a desired velocity. All three of these steps are proven using barriers. In the case $\gamma > 0$ considered here, the step of spreading to large v is technically more challenging than [32] and involves localizing in both x and v.

This argument shows f has sufficient lower bounds at all x locations to use the smoothing properties of the collision operator.

1.3.2 Existence

The fundamental lemma for local existence in [31] is an a priori estimate in the polynomially weighted space $L^\infty_q(\mathbb{R}^6)$ of the form $\|f(t)\|_{L^\infty_q(\mathbb{R}^6)} \leq C\|f_0\|_{L^\infty_q(\mathbb{R}^6)}$ for t less than some T depending only on $\|f_0\|_{L^\infty_q(\mathbb{R}^6)}$. This proof relies on the fact that $h(t,v) = e^{\beta t}\langle v\rangle^{-q}$ is a supersolution of the linear Landau equation for suitable β and q, but this fact is false when $\gamma > 0$, because the coefficient $\bar{c}^f$ grows too fast in velocity.

In this work, the polynomially-decaying function h is replaced with sub-exponentially decaying supersolutions $\phi(t,v) = e^{(\rho-\sigma t)\langle v\rangle^\beta}$, for some $\beta \in [\gamma, 1]$. The benefit of ϕ is that $\partial_t \phi$ produces a term $-\sigma\langle v\rangle^\beta \phi$, which has the right sign to absorb the extra moments produced by $\bar{c}^f$ when $\gamma > 0$. This provides an estimate of the form $\|e^{(\rho-\sigma t)\langle v\rangle^\beta} f(t)\|_{L^\infty(\mathbb{R}^6)} \leq C\|e^{\rho\langle v\rangle^\beta} f_0\|_{L^\infty(\mathbb{R}^6)}$ for t in some time interval $[0,T]$.

Once this estimate is available, existence follows by an approximation argument similar to [31, 33]. One smooths the initial data and cuts off large velocities, applies the prior existence result of [10], and uses the estimate described in the previous paragraph to apply regularity theory and take the limit by compactness, as the cutoff and smoothing vanish.

1.3.3 Uniqueness

There is a fundamental difficulty, seen already in [31, 33], with proving uniqueness in the low-regularity setting. Namely, to control the difference between two solutions f and g, one needs some velocity regularity for one of the solutions (say f). Naively applying regularity estimates yields a constant that blows up too fast as $t \to 0$ to be useful. Therefore, one must take initial data that is Hölder continuous and free of vacuum regions. By studying the evolution of a finite difference of the solution, one can propagate the Hölder modulus forward to positive times. This provides enough regularity to prove uniqueness.

In the hard potentials case, the same overall strategy works, but implementing the details requires, as usual, controlling velocity moments that are lost in the estimates. As in the proof of existence, sub-exponential weights like $e^{(\rho-\sigma t)\langle v\rangle^\beta}$ are needed because of the good term produced by the time derivative falling on the weight.

1.4 Notation

For $f, g : \mathbb{R}^3 \to \mathbb{R}$, it is well-known that Landau's collision operator $Q(f,g)$ can be written as a second-order diffusion operator, in either divergence form

$$Q(f,g) = \nabla_v \cdot (\bar{a}^f \nabla_v g) + \bar{b}^f \cdot \nabla_v g + \bar{c}^f g,$$

or non-divergence form

$$Q(f,g) = \mathrm{tr}(\bar{a}^f D_v^2 g) + \bar{c}^f g,$$

where the nonlocal coefficients are defined by

$$\bar{a}^f(v) := a_\gamma \int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(v-w)\,\mathrm{d}w, \tag{1.4.1}$$

$$\bar{b}^f(v) := b_\gamma \int_{\mathbb{R}^3} |w|^\gamma w f(v-w)\,\mathrm{d}w, \tag{1.4.2}$$

$$\bar{c}^f(v) := c_\gamma \int_{\mathbb{R}^3} |w|^\gamma f(v-w)\,\mathrm{d}w, \tag{1.4.3}$$

where I is the 3×3 identity matrix and a_γ, b_γ, c_γ are constants depending on γ. When f is a function of (t, x, v), then these coefficients naturally also depend on all three variables: t, x, and v.

We often use the notation $z = (t, x, v)$ to denote a point in $\mathbb{R}^7$.

Throughout this work, all constants may depend on the parameter γ, even when not explicitly noted.

We always assume the solution f and initial data f_0 are periodic in x with period 1. Usually, we write the x domain as $\mathbb{R}^3$, but sometimes we write $\mathbb{T}^3$ to emphasize this periodicity. These points of view are equivalent, since any function on $\mathbb{T}^3$ can be extended by periodicity to $\mathbb{R}^3$.

Chapter 2

Preliminaries and known results

2.1 Existence for regular initial data

The following existence result was proven by Chaturvedi [10]. We state a simplified version of his theorem with less sharp hypotheses, that is sufficient for our purposes:

Theorem 2.1.1. *Let* $M_0 > 0, \gamma \in [0,1], d_0 > 0, f_0$ *be such that*

$$\sum_{|\alpha|+|\beta|\leq 10} ||\partial_x^\alpha \partial_v^\beta (e^{d_0\langle v\rangle} \cdot f_0)||^2_{L^2_x L^2_v} \leq M_0.$$

Then for some $T > 0$*, depending on* γ*,* d_0 *and* M_0*, there is a non-negative solution* f *to the Landau equation with* $f(0,x,v) = f_0(x,v)$*.*

Moreover, $e^{(d_0-\kappa t)\langle v\rangle} f \in C([0,T], H^{10}_{x,v}(\mathbb{R}^6))$*.*

Below, we will construct a solution by smoothing our initial data and cutting off large velocities, applying Theorem 2.1.1, and deriving sufficient estimates on these approximate solutions to take the limit.

2.2 Coefficient bounds

There are two types of upper bounds for $\bar{a}^f$, $\bar{b}^f$, and $\bar{c}^f$ we will need. The first type is based on L^1-norms of f, and are available in the literature, as in the following lemma:

Lemma 2.2.1. *[41, Lemma 2.1] Let f satisfy*

$$\int_{\mathbb{R}^3} (1+|v|^{\gamma+2}) f(t,x,v)\,\mathrm{d}v \le K_0, \quad \text{for all } t\in[0,T], x\in\mathbb{R}^3.$$

Then there exist constants C_1, C_2 C_3, depending only on K_0, such that

$$\bar{a}^f_{ij}(t,x,v)e_ie_j \le C_1 \begin{cases} (1+|v|)^{\gamma+2}, & e\in\mathbb{S}^{d-1}, \\ (1+|v|)^{\gamma}, & e\cdot v = |v|, \end{cases}$$
$$|\bar{b}^f(t,x,v)| \le C_2(1+|v|)^{\gamma+1},$$
$$\bar{c}^f(t,x,v) \le C_3(1+|v|)^{\gamma}.$$

The next type of coefficient estimate is based on weighted L^∞-norms of f, and is essentially understood in the literature as well.

Lemma 2.2.2. *If $f\in L^\infty_q([0,T]\times\mathbb{R}^6)$ for some $q>\gamma+5$, then*

$$\bar{a}^f_{ij}(t,x,v)e_ie_j \le C\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)}(1+|v|)^{\gamma+2}$$
$$|\bar{b}^f(t,x,v)| \le C\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)}(1+|v|)^{\gamma+1},$$
$$\bar{c}^f(t,x,v) \le C\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)}(1+|v|)^{\gamma},$$

for a constant $C>0$ depending only on γ and q.

Lemma 2.2.1 can be proven using the convolution estimate $|(g*|\cdot|^r)(v)| \le C\|g\|_{L^\infty_q(\mathbb{R}^3)}\langle v\rangle^r$ where $r>0$ and $q>r+3$. We omit the details.

Furthermore, we have a lower ellipticity estimate for the matrix $\bar{a}^f$. The proof is the same as [30, Lemma 4.3].

Lemma 2.2.3. *If $f : \mathbb{R}^3 \to \mathbb{R}$ is nonnegative and satisfies the lower bound*

$$f(v) \geq \delta, \quad \text{for all } v \in B_r(v_0),$$

for some $r, \delta > 0$ and $v_0 \in \mathbb{R}^3$, then for all $v \in \mathbb{R}^3$, the matrix $\bar{a}^f(v)$ defined by (1.4.1) *satisfies*

$$\bar{a}^f_{ij}(v) e_i e_j \geq c_1 \begin{cases} (1+|v|)^\gamma, & e \in \mathbb{S}^2, \\ (1+|v|)^{\gamma+2}, & e \cdot v = 0. \end{cases} \tag{2.2.1}$$

with $c_1 > 0$ depending only on δ, r, and $|v_0|$.

2.3 Kinetic Hölder norms

The regularity of the inhomogeneous Landau equation is most naturally measured with respect to a metric which respects the invariance of kinetic equations with respect to rescalings of the form $(t, x, v) \mapsto (r^2 t, r^3 x, rv)$ and Galilean shifts. In more detail, define the kinetic distance

$$d_k(z, z') = |t - t'|^{1/2} + |x' - x - (t' - t)v|^{1/3} + |v' - v|.$$

Technically, d_k is not a metric on $\mathbb{R}^7$ because it does not satisfy the triangle inequality and is not symmetric. However, this fact causes no issues in our analysis. For any $\alpha \in (0, 1)$ and domain $\Omega \subset \mathbb{R}^7$, we define the kinetic Hölder seminorm

$$[u]_{C^\alpha_k(\Omega)} = \sup_{z, z' \in \Omega} \frac{|u(z) - u(z')|}{d_k(z, z')^\alpha},$$

as well as the norm $\|u\|_{C^\alpha_k(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{C^\alpha_k(\Omega)}$, and the Hölder space $C^\alpha_k(\Omega) = \{u : \Omega \to \mathbb{R}, \|u\|_{C^\alpha(\Omega)} < \infty\}$.

Let us also define the second-order space $C^{2,\alpha}_k(\Omega)$ using the norm

$$\|u\|_{C^{2,\alpha}_k(\Omega)} = \|u\|_{L^\infty(\Omega)} + \|\nabla_v u\|_{L^\infty(\Omega)} + \|D^2_v u\|_{C^\alpha_k(\Omega)} + \|(\partial_t + v \cdot \nabla_x) u\|_{C^\alpha_k(\Omega)}.$$

We sometimes apply the norm $\|\cdot\|_{C^{2,\alpha}}$ to functions u where $\partial_t u$ and $\nabla_x u$ may not be defined pointwise. In this case, the differential operator $(\partial_t + v \cdot \nabla_x)$ has been extended by density.

Next, we recall the standard kinetic cylinders, defined for some point $z_0 \in \mathbb{R}^7$ and radius $r > 0$ by

$$\begin{aligned} Q_r(z_0) &= \{z = (t,x,v) \in \mathbb{R}^7 : t < t_0 \text{ and } d_k(z,z_0) < r\} \\ &= \{(t,x) \in \mathbb{R}^4 : t_0 - r^2 < t < t_0 \text{ and } |x - x_0 - (t-t_0)v_0| < r^3\} \times B_r(v_0). \end{aligned}$$

We also use the notations $Q_r = Q_r(0)$ and

$$Q_r^{t,x}(z_0) = \{(t,x) \in \mathbb{R}^4 : t_0 - r^2 < t < t_0 \text{ and } |x - x_0 - (t-t_0)v_0| < r^3\}.$$

The following is a standard result about the Hölder seminorm of a product, which we state without proof:

Lemma 2.3.1. *For any subset $\Omega \subset \mathbb{R}^7$, and any $f, g \in C_k^\alpha(\Omega)$, the following inequality holds:*

$$[fg]_{C_k^\alpha(\Omega)} \le \|f\|_{L^\infty(\Omega)}[g]_{C^\alpha(\Omega)} + [f]_{C^\alpha(\Omega)}\|g\|_{L^\infty(\Omega)}.$$

2.4 Sub-exponential functions

Throughout the paper, we make use of functions of the form $\phi = e^{\rho\langle v\rangle^\beta}$ for some $\rho \in \mathbb{R}$ and $\beta > 0$. Sometimes, ρ will be replaced by a linear function of t. Let us collect a few useful properties:

$$\partial_{v_i}\phi = \rho\beta\phi\langle v\rangle^{\beta-2}v_i, \quad i = 1,2,3, \tag{2.4.1}$$

$$\partial_{v_i v_j}\phi = \rho\beta\langle v\rangle^{\beta-4}\phi\left[(\beta-2)v_iv_j + \langle v\rangle^2\delta_{ij} + \rho\beta\langle v\rangle^\beta v_iv_j\right], \quad i,j = 1,2,3, \tag{2.4.2}$$

$$\bar{a}^g_{ij}\partial_{v_iv_j}\phi = \rho\beta\langle v\rangle^{\beta-4}\phi\left[\left((\beta-2) + \rho\beta\langle v\rangle^\beta\right)\bar{a}^g_{ij}v_iv_j + \langle v\rangle^2\mathrm{tr}(\bar{a}^g)\right], \tag{2.4.3}$$

where $\bar{a}^g$ is defined by (1.4.1) for any function g, and the expression in (2.4.3) is summed over repeated indices.

We also have two interpolation lemmas with sub-exponential weights, that will be used in our proof of the propagation of a Hölder modulus:

Lemma 2.4.1. *For any $\theta, \alpha \in (0,1]$, any $z_0 \in \mathbb{R}^7$ any $\beta \in [0,1]$, any $\rho_1 \geq \rho_0 \geq 0$, and any $g : Q_\theta(z_0) \to \mathbb{R}$ such that the right-hand side is finite, there holds*

$$[e^{[(\rho_0+\rho_1)/2]\langle v\rangle^\beta} g]_{C_k^{\alpha/2}(Q_\theta(z_0))} \leq C[e^{\rho_0\langle v\rangle^\beta} g]^{1/2}_{C_k^\alpha(Q_\theta(z_0))} \|e^{\rho_1\langle v\rangle^\beta} g\|^{1/2}_{L^\infty(Q_\theta(z_0))}, \tag{2.4.4}$$

for a constant C depending on ρ, β, and α.

Furthermore, the same interpolation holds for functions defined on $[0,T] \times \mathbb{R}^6$:

$$[e^{[(\rho_0+\rho_1)/2]\langle v\rangle^\beta} g]_{C_k^{\alpha/2}([0,T]\times\mathbb{R}^6)} \leq C[e^{\rho_0\langle v\rangle^\beta} g]^{1/2}_{C_k^\alpha([0,T]\times\mathbb{R}^6)} \|e^{\rho_1\langle v\rangle^\beta} g\|^{1/2}_{L^\infty([0,T]\times\mathbb{R}^6)},$$

with C as above.

Proof. Define

$$R = \left(\frac{\|e^{\rho_1\langle v\rangle^\beta} g\|_{L^\infty(Q_\theta(z_0))}}{e^{[(\rho_1-\rho_0)/2]\langle v_0\rangle^\beta}[e^{\rho_0\langle v\rangle^\beta} g]_{C_k^\alpha(Q_\theta(z_0))}} \right)^{2/\alpha}.$$

Taking distinct $z_1, z_2 \in Q_\theta(z_0)$, there are two cases. If $d_k(z_1,z_2) \geq R$, then

$$\begin{aligned}\frac{|g(x_1,v_1) - g(x_2,v_2)|}{d_k(z_1,z_2)^{\alpha/2}} &\leq CR^{-\alpha/2} e^{-\rho_1\langle v_0\rangle^\beta} \|e^{\rho_1\langle v_0\rangle^\beta} g\|_{L^\infty(Q_\theta(z_0))} \\ &= Ce^{-[(\rho_0+\rho_1)/2]\langle v_0\rangle^\beta} [e^{\rho_0\langle v\rangle^\beta} g]^{1/2}_{C_k^\alpha(Q_\theta(z_0))} \|e^{\rho_1\langle v\rangle^\beta} g\|^{1/2}_{L^\infty(Q_\theta(z_0))}.\end{aligned}$$

If $d_k(z_1,z_2) < R$, then with Lemma 2.3.1, we have

$$\begin{aligned}\frac{|g(x_1,v_1) - g(x_2,v_2)|}{d_k(z_1,z_2)^{\alpha/2}} &\leq \frac{|g(x_1,v_1) - g(x_2,v_2)|}{d_k(z_1,z_2)^{\alpha}} d_k(z_1,z_2)^{\alpha/2} \\ &\leq Ce^{-\rho_0\langle v_0\rangle^\beta} [e^{\rho_0\langle v\rangle^\beta} g]_{C_k^\alpha(Q_\theta(z_0))} R^{\alpha/2} \\ &= Ce^{-[(\rho_0+\rho_1)/2]\langle v_0\rangle^\beta} [e^{\rho_0\langle v\rangle^\beta} g]^{1/2}_{C_k^\alpha(Q_\theta(z_0))} \|e^{\rho_1\langle v\rangle^\beta} g\|^{1/2}_{L^\infty(Q_\theta(z_0))}.\end{aligned}$$

In either case, we see that $e^{[(\rho_0+\rho_1)/2]\langle v_0\rangle^\beta}[g]_{C_k^{\alpha/2}(Q_\theta(z_0))}$ is bounded by the right-hand side of (2.4.4). Conclusion (2.4.4) then follows after applying Lemma 2.3.1 again.

To prove the interpolation inequality on the whole space, cover $[0,T]\times\mathbb{R}^6$ with a countable union of kinetic cylinders with radius 1 centered at z_i, and note that

$$\|e^{[(\rho_0+\rho_1)/2]\langle v\rangle^\beta} g\|_{C_k^{\alpha/2}([0,T]\times\mathbb{R}^6)} \approx \sum_{i=1}^\infty \|e^{[(\rho_0+\rho_1)/2]\langle v\rangle^\beta} g\|_{C_k^{\alpha/2}(Q_1(z_i)\cap([0,T]\times\mathbb{R}^6))}.$$

The inequality then follows from applying (2.4.4) for each z_i. □

Lemma 2.4.2. *For $g:\mathbb{R}^6\to\mathbb{R}$ such that the right-hand side is finite, there holds for any $z_0\in\mathbb{R}_+\times\mathbb{R}^6$ and any $\theta\in(0,\min\{1,\sqrt{t_0/2}\})$,*

$$\|e^{\rho\langle v\rangle^\beta} D_v^2 g\|_{L^\infty(Q_\theta(z_0))} \le C[D_v^2 g]_{C_k^{2\alpha/3}(Q_\theta(z_0))}^{1-\frac{2\alpha}{6-\alpha}} \|e^{\rho'\langle v\rangle^\beta} g\|_{C_k^\alpha(Q_\theta(z_0))}^{\frac{2\alpha}{6-\alpha}},$$

with $\rho' = \rho\left(\dfrac{6}{\alpha}-2\right)$, and $C>0$ a constant depending on ρ, α, and β.

Proof. First, we apply a standard unweighted interpolation between C^2, $C^{2,\alpha/3}$, and C^α norms, obtaining

$$\|e^{\rho\langle v\rangle^\beta} D_v^2 g\|_{L^\infty(Q_\theta(z_0))} \le [e^{\rho\langle v\rangle^\beta} D_v^2 g]_{C_k^{\alpha/3}(Q_\theta(z_0))}^{1-\frac{\alpha}{6-2\alpha}} [e^{\rho\langle v\rangle^\beta} g]_{C_k^\alpha(Q_\theta(z_0))}^{\frac{\alpha}{6-2\alpha}}.$$

Next, we apply Lemma 2.4.1 to $D_v^2 g$, with $2\alpha/3$ replacing α, and with $\rho_0=0$ and $\rho_1=2\rho$:

$$\begin{aligned}
&\|e^{\rho\langle v\rangle^\beta} D_v^2 g\|_{L^\infty(Q_\theta(z_0))}\\
&\le [D_v^2 g]_{C_k^{2\alpha/3}(Q_\theta(z_0))}^{\frac12-\frac{\alpha/2}{6-2\alpha}} \|e^{2\rho\langle v\rangle^\beta} D_v^2 g\|_{L^\infty(Q_\theta(z_0))}^{\frac12-\frac{\alpha/2}{6-2\alpha}} [e^{\rho\langle v\rangle^\beta} g]_{C_k^\alpha(Q_\theta(z_0))}^{\frac{\alpha}{6-2\alpha}} \qquad (2.4.5)\\
&\le Ce^{\rho\langle v_0\rangle^\beta(1-\frac{\alpha}{6-2\alpha})} [D_v^2 g]_{C_k^{2\alpha/3}(Q_\theta(z_0))}^{\frac12-\frac{\alpha/2}{6-2\alpha}} \|e^{\rho\langle v\rangle^\beta} D_v^2 g\|_{L^\infty(Q_\theta(z_0))}^{\frac12-\frac{\alpha/2}{6-2\alpha}} [e^{\rho\langle v\rangle^\beta} g]_{C_k^\alpha(Q_\theta(z_0))}^{\frac{\alpha}{6-2\alpha}}.
\end{aligned}$$

To absorb the exponential factor, we apply Lemma 2.3.1:

$$
\begin{aligned}
&e^{\rho\langle v_0\rangle^\beta(1-\frac{\alpha}{6-2\alpha})}[e^{\rho\langle v\rangle^\beta}g]^{\frac{\alpha}{6-2\alpha}}_{C^\alpha_k(Q_\theta(z_0))}\\
&= \left(e^{\rho\langle v_0\rangle^\beta(\frac{6}{\alpha}-3)}[e^{\rho\langle v\rangle^\beta(3-\frac{6}{\alpha})}e^{\rho\langle v\rangle^\beta(\frac{6}{\alpha}-2)}g]_{C^\alpha_k(Q_\theta(z_0))}\right)^{\frac{\alpha}{6-2\alpha}}\\
&\le C\left([e^{\rho\langle v\rangle^\beta(\frac{6}{\alpha}-2)}g]_{C^\alpha_k(Q_\theta(z_0))}\right.\\
&\quad\left.+e^{\rho\langle v_0\rangle^\beta(\frac{6}{\alpha}-3)}[e^{\rho\langle v\rangle^\beta(3-\frac{6}{\alpha})}]_{C^\alpha_k(Q_\theta(z_0))}\|e^{\rho\langle v\rangle^\beta(\frac{6}{\alpha}-2)}g\|_{L^\infty(Q_\theta(z_0))}\right)^{\frac{\alpha}{6-2\alpha}}\\
&\le C\|e^{\rho\langle v\rangle^\beta(\frac{6}{\alpha}-2)}g\|^{\frac{\alpha}{6-2\alpha}}_{C^\alpha_k(Q_\theta(z_0))},
\end{aligned}
$$

where we used the fact that $[e^{\rho\langle v\rangle^\beta(3-\frac{6}{\alpha})}]_{C^\alpha_k(Q_\theta(z_0))} \le C_{\beta,\alpha,\rho}e^{\rho\langle v_0\rangle^\beta(3-\frac{6}{\alpha})}$, since $\beta \le 1$. Returning to (2.4.5), we now have

$$
\|e^{\rho\langle v\rangle^\beta}D^2_v g\|_{L^\infty(Q_\theta(z_0))} \le C[D^2_v g]^{\frac{1}{2}-\frac{\alpha/2}{6-2\alpha}}_{C^{2\alpha/3}_k(Q_\theta(z_0))}\|e^{\rho\langle v\rangle^\beta}D^2_v g\|^{\frac{1}{2}-\frac{\alpha/2}{6-2\alpha}}_{L^\infty(Q_\theta(z_0))}\|e^{\rho\langle v\rangle^\beta(\frac{6}{\alpha}-2)}g\|^{\frac{\alpha}{6-2\alpha}}_{C^\alpha_k(Q_\theta(z_0))}.
$$

Absorbing the middle factor on the right into the left-hand side and simplifying, we obtain the conclusion of the lemma. □

Chapter 3

Regularity theory

3.1 Change of variables

When applying regularity estimates for the Landau equation, the ellipticity of the matrix $\bar{a}^f$ degenerate for large v. A change of variables was developed in [4] to precisely track this degeneration.

For a fixed $z_0 = (t_0, x_0, v_0) \in [\tau, T] \times \mathbb{R}^6$, if $|v_0| > 2$, let S be the linear transformation defined by

$$S\xi = \begin{cases} |v_0|^{1+\gamma/2}\xi, & \xi \perp v_0, \\ |v_0|^{\gamma/2}\xi, & \xi \parallel v_0. \end{cases}$$

If $|v_0| \leq 2$, then we define S as the identity matrix. Next, define

$$\mathcal{T}_{z_0}(t, x, v) = (t_0 + t, x_0 + Sx + tv_0, v_0 + Sv).$$

Given a solution to the Landau equation (1.0.1) on $[0, T] \times \mathbb{R}^6$, one then defines

$$r_1 = \begin{cases} |v_0|^{-1-\gamma/2} \min\left(1, \sqrt{t_0/2}\right), & |v_0| > 2, \\ \min\left(1, \sqrt{t_0/2}\right), & |v_0| \leq 2, \end{cases} \tag{3.1.1}$$

and

$$f_{z_0}(t,x,v) := f(\mathcal{T}_{z_0}(\delta_{r_1}(z))), \quad z \in Q_1(0),$$

where $\delta_{r_1}(z) = (r_1^2 t, r_1^3 x, r_1 v)$. By direct calculation, f_{z_0} satisfies both the divergence form equation

$$\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \nabla_v \cdot (A(z)\nabla_v f_{z_0}) + B(z) \cdot \nabla_v f_{z_0} + C(z) f_{z_0}, \tag{3.1.2}$$

and the nondivergence-form equation

$$\partial_t f_{z_0} + v \cdot \nabla_x f_{z_0} = \mathrm{tr}(A(z) D_v^2 f_{z_0}) + C(z) f_{z_0}, \tag{3.1.3}$$

in $Q_1(0)$, where the coefficients are defined by

$$\begin{aligned} A(z) &= S^{-1}\bar{a}^f(\mathcal{T}_{z_0}(\delta_{r_1}(z)))S^{-1}, \\ B(z) &= r_1 S^{-1}\bar{b}^f(\mathcal{T}_{z_0}(\delta_{r_1}(z))) \\ C(z) &= r_1^2 \bar{c}^f(\mathcal{T}_{z_0}(\delta_{r_1}(z))). \end{aligned} \tag{3.1.4}$$

The key properties of f_{z_0} and the transformed equation are contained in the following lemma, which first appeared in [4] and was originally derived for the case $\gamma \in (-2, 0)$. However, as pointed out in [41], the result extends to the case $\gamma > 0$ with essentially the same proof. The lemma is as follows:

Lemma 3.1.1 ([4]). *With $z_0 \in (0,T] \times \mathbb{R}^6$ given, let S and $\mathcal{T}_{z_0}$ be defined as above.*

(a) There exists a constant $C > 0$ independent of z_0, such that

$$C^{-1}|v_0| \le |v_0 + r_1 S v| \le C|v_0|, \quad v \in B_1(0).$$

(b) Let f be a solution of the Landau equation on $[0,T]\times\mathbb{R}^6$, satisfying

$$\bar{a}^f(t,x,v)\xi_i\xi_j \approx \begin{cases} \langle v\rangle^\gamma|\xi|^2, & \xi\perp v,\\ \langle v\rangle^{\gamma+2}|\xi|^2, & \xi \parallel v,\end{cases} \quad |\bar{b}^f(t,x,v)| \lesssim \langle v\rangle^{\gamma+1}, \quad \bar{c}^f(t,x,v)\lesssim \langle v\rangle^\gamma. \tag{3.1.5}$$

Then the coefficients A, B, and C defined in (3.1.4) satisfy

$$\begin{aligned} \lambda I \le A(z) &\le \Lambda I,\\ B(z) &\le \Lambda\langle v_0\rangle^{1+\gamma/2},\\ C(z) &\le \Lambda\langle v_0\rangle^\gamma, \end{aligned} \tag{3.1.6}$$

for all $z\in Q_1(0)$, where λ and Λ are constants depending only on the implied constants in (3.1.5).

We also note the following properties of our change of variables, which can be verified by a direct computation: for any $z_1, z_2 \in Q_1$,

$$d_k(\delta_{r_1}(z_1),\delta_{r_1}(z_2)) = r_1 d_k(z_1,z_2), \tag{3.1.7}$$

and

$$\min\{1,\sqrt{t_0/2}\}\langle v_0\rangle^{-1} d_k(z_1,z_2) \le d_k(\mathcal{T}_{z_0}(\delta_{r_1}(z_1)),\mathcal{T}_{z_0}(\delta_{r_1}(z_2))) \le \min\{1,\sqrt{t_0/2}\} d_k(z_1,z_2). \tag{3.1.8}$$

The following lemma relates the regularity of f_{z_0} to the regularity of f:

Lemma 3.1.2. *Let $f:[0,T]\times\mathbb{R}^6\to\mathbb{R}$ and $z_0\in(0,T]\times\mathbb{R}^6$ be given. Let r_1 be defined by (3.1.1), and let $r_0=\min\{\sqrt{t_0/2},1\}$*

(a) If $f\in L^\infty_q(Q^{t,x}_{r_0}(z_0)\times\mathbb{R}^3)$ for some $q>0$, then

$$\|f_{z_0}\|_{L^\infty(Q_1)} \le C\langle v_0\rangle^{-q}\|f\|_{L^\infty_q(Q^{t,x}_{r_0}(z_0)\times\mathbb{R}^3)}.$$

(b) If $f_{z_0} \in C^\alpha_k(Q_\theta)$ for some $\alpha, \theta \in (0,1]$, then

$$[f]_{C^\alpha_k(Q_{r_1\theta}(z_0))} \leq C \max\{1, t_0^{-\alpha/2}\} \langle v_0 \rangle^\alpha [f_{z_0}]_{C^\alpha_k(Q_\theta)}.$$

(c) If $f \in C^\alpha_k(Q_\theta(z_0))$, for some $\alpha, \theta \in (0,1]$, then

$$[f_{z_0}]_{C^\alpha_k(Q_\theta)} \leq C \min\{1, t_0^{\alpha/2}\} [f]_{C^\alpha_k(Q_\theta(z_0))}.$$

In all three estimates, the constant $C > 0$ is independent of f and z_0.

Proof. First, we note that note that $Q_{r_1\theta}(z_0) \subset \mathcal{T}_{z_0}(\delta_{r_1}(Q_\theta)) \subset Q_{r_0\theta}(z_0)$. Conclusion (a) then follows from Lemma 3.1.1(a), and conclusions (b) and (c) follow by applying (3.1.8). □

The purpose of the following lemma is to pass regularity of f to the coefficients A and C. Since A and C are nonlocal in the v variable, the assumption of Hölder continuity for f must be made on the entire velocity domain $\mathbb{R}^3$. The proof of this lemma is the same as [28, Lemma 3.3] or [31, Lemma 2.7].

Lemma 3.1.3. *Let f be defined in $\Omega \times \mathbb{R}^3$ for some (t,x) domain $\Omega \subset \mathbb{R}^4$, and let z_0 be such that $Q_{r_1}(z_0) \subset \Omega \times \mathbb{R}^3$. Assume that $\langle v \rangle^m f \in C^\alpha_k(\Omega \times \mathbb{R}^3)$ for some $\alpha \in (0,1)$ and $m > 5 + \gamma + \alpha/3$.*

Then the coefficients A and C defined in (3.1.4) are Hölder continuous in Q_1, and

$$\begin{aligned} [A]_{C^{2\alpha/3}_k(Q_1)} &\leq C \langle v_0 \rangle^{2+\alpha/3} [\langle v \rangle^m f]_{C^\alpha_k(\Omega\times\mathbb{R}^3)}, \\ [C]_{C^{2\alpha/3}_k(Q_1)} &\leq C \langle v_0 \rangle^{\alpha/3} [\langle v \rangle^m f]_{C^\alpha_k(\Omega\times\mathbb{R}^3)}. \end{aligned} \tag{3.1.9}$$

The constant C depends only on γ, α, and m.

3.2 Regularity estimates

3.2.1 Local $C^{2,\alpha}$ estimate

The following lemma is a Schauder estimate for the Landau equation. Because of the nonlocality of the coefficients, this estimate depends on the Hölder continuity of f over the entire velocity domain.

Lemma 3.2.1. *Let f be a solution to the Landau equation on $[0,T]\times\mathbb{R}^6$. Let $z_0 \in (0,T]\times\mathbb{R}^6$ be given, and define*

$$\Omega(z_0) = (Q_{r_0}^{t,x}(z_0)\times\mathbb{R}^3_v),$$

where $r_0 = \min\{1, \sqrt{t_0/2}\}$. For some q, m, and α with $\alpha\in(0,1)$ and

$$q > m > 5+\gamma+\alpha/3,$$

assume that

$$f(t,x,v) \le K_0\langle v\rangle^{-q}, \quad \text{in } \Omega(z_0), \tag{3.2.1}$$

and $\langle v\rangle^m f \in C^\alpha_k(\Omega)$. Assume further that

$$\bar a^f_{ij}(t,x,v)\xi_i\xi_j \ge \lambda_0 \begin{cases} \langle v\rangle^\gamma, & \xi\perp v, \\ \langle v\rangle^{\gamma+2}, & \xi \parallel v, \end{cases} \quad \text{for all } (t,x,v)\in\Omega(z_0) \text{ and } \xi\in\mathbb{R}^3. \tag{3.2.2}$$

Then

$$\begin{aligned}&[D_v^2 f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} + [(\partial_t + v\cdot\nabla_x)f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} \\ &\le C(1+t_0^{-1-\alpha/3})\langle v_0\rangle^{-(q+2m)/3+9+3\alpha+6/\alpha+2\alpha^2/9+\gamma}\times\left(1+[\langle v\rangle^m f]^{3+2\alpha/3+3/\alpha}_{C_k^\alpha(\Omega(z_0))}\right),\end{aligned}$$

where $C>0$ is a constant depending only on K_0 and λ_0, and r_1 is defined in (3.1.1).

Proof. Defining f_{z_0} as above, with base point z_0, we will work with the nondivergence-form equation (3.1.3). Our first step is to verify the hypotheses of Lemma 3.1.1. From Lemma 2.2.2, our assumption (3.2.1) provides suitable upper bounds on $\bar{c}^f$ as in (3.1.5). The lower bound on $\bar{a}^f$ in (3.1.5) follows from (3.2.2), and the upper bound follows from Lemma 2.2.1. (Note that the $L^\infty_q(\mathbb{R}^3_v)$ norm of f bounds the quantity $\int_{\mathbb{R}^3}(1+|v|^{\gamma+2})f(t,x,v)\,dv$.) Therefore, the bounds (3.1.6) are valid for the coefficients A and C defined in (3.1.4), with constants depending only on λ_0 and K_0.

The Schauder estimate of [28, Theorem 2.9], with $2\alpha/3$ replacing α, yields

$$\begin{aligned}[D_v^2 f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} &+ [(\partial_t + v\cdot\nabla_x) f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} \\ &\le C\left([C f_{z_0}]_{C_k^{2\alpha/3}(Q_1)} + \|A\|^{3+2\alpha/3+3/\alpha}_{C_k^{2\alpha/3}(Q_1)}\|f_{z_0}\|_{L^\infty(Q_1)}\right).\end{aligned}$$

Applying Lemma 2.3.1 for the product Cf_{z_0}, and using Lemma 3.1.3 and the upper bounds on A and C from Lemma 3.1.1, we have

$$\begin{aligned}&[D_v^2 f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} + [(\partial_t + v\cdot\nabla_x) f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} \\ &\qquad \le C\Big(\|f_{z_0}\|_{L^\infty(Q_1)}\langle v_0\rangle^{\alpha/3}[\langle v\rangle^m f]_{C_k^\alpha(\Omega(z_0))} + [f_{z_0}]_{C_k^{2\alpha/3}(Q_1)}\langle v_0\rangle^\gamma \\ &\qquad\quad + \left(\langle v_0\rangle^{2+\alpha/3}[\langle v\rangle^m f]_{C_k^\alpha(\Omega(z_0))}\right)^{3+2\alpha/3+3/\alpha}\|f_{z_0}\|_{L^\infty(Q_1)}\Big).\end{aligned} \tag{3.2.3}$$

Next, we use Lemma 3.1.2(c), the interpolation $[f]_{C_k^{2\alpha/3}(Q_{r_0}(z_0))} \le C[f]^{2/3}_{C_k^\alpha(Q_{r_0}(z_0))}\|f\|^{1/3}_{L^\infty(Q_{r_0}(z_0))}$, and Lemma 2.3.1 to write

$$\begin{aligned}[f_{z_0}]_{C_k^{2\alpha/3}(Q_1)} &\le C\min\{1, t_0^{\alpha/3}\}[f]_{C_k^{2\alpha/3}(Q_{r_0}(z_0))} \\ &\le C[f]^{2/3}_{C_k^\alpha(Q_{r_0}(z_0))}\|f\|^{1/3}_{L^\infty(Q_{r_0}(z_0))} \\ &\le C\langle v_0\rangle^{-(q+2m)/3}[\langle v\rangle^m f]^{2/3}_{C_k^\alpha(Q_{r_0}(z_0))}\|f\|^{1/3}_{L^\infty_q(\Omega(z_0))}.\end{aligned}$$

Returning to (3.2.3), using $\|f_{z_0}\|_{L^\infty(Q_1)} \lesssim \langle v_0\rangle^{-q}\|f\|_{L^\infty_q(\Omega(z_0))}$, absorbing the norm $\|f\|_{L^\infty_q}$ into the constant, and keeping only the largest powers of $\langle v_0\rangle$ and $[\langle v\rangle^m f]_{C_k^\alpha}$, we obtain

$$\begin{aligned}[D_v^2 f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} &+ [(\partial_t + v\cdot\nabla_x) f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} \\ &\le C\langle v_0\rangle^{-(q+2m)/3+7+7\alpha/3+6/\alpha+2\alpha^2/9}\left(1+[\langle v\rangle^m f]_{C_k^\alpha(\Omega(z_0))}^{3+2\alpha/3+3/\alpha}\right),\end{aligned} \tag{3.2.4}$$

Finally, we translate from f_{z_0} to f, using the chain rule and (3.1.8). In particular, with $\|S\|$ denoting the operator matrix norm of S, we have

$$\begin{aligned}[D_v^2 f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} &\le C r_1^{-2}\|S\|^{-2}(1+t_0^{-\alpha/3})\langle v_0\rangle^\alpha [D_v^2 f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} \\ &\le C(1+t_0^{-1-\alpha/3})\langle v_0\rangle^{2+2\alpha/3}[D_v^2 f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})},\end{aligned}$$

and

$$\begin{aligned}[(\partial_t + v\cdot\nabla_x) f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} &\le C r_1^{-2}(1+t_0^{-\alpha/3})\langle v_0\rangle^\alpha [(\partial_t + v\cdot\nabla_x) f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})} \\ &\le C(1+t_0^{-1-\alpha/3})\langle v_0\rangle^{2+\gamma+2\alpha/3}[(\partial_t + v\cdot\nabla_x) f_{z_0}]_{C_k^{2\alpha/3}(Q_{1/2})},\end{aligned}$$

which, combined with (3.2.4), imply the conclusion of the lemma. □

The following local estimate is used to take the limit in our approximation procedure when proving existence:

Proposition 3.2.1. *Let f be a solution to the Landau equation on $[0,T]\times\mathbb{R}^6$. Let $z_0 \in (0,T]\times\mathbb{R}^6$ be given, and define $\Omega(z_0)$ as in Lemma 3.2.1. For some $q > 5+2\gamma+4\alpha/3$, assume that*

$$f(t,x,v) \le K_0\langle v\rangle^{-q}, \quad \text{in } \Omega(z_0),$$

and

$$\bar a_{ij}^f(t,x,v)\xi_i\xi_j \ge \lambda_0 \begin{cases}\langle v\rangle^\gamma, & \xi\perp v,\\ \langle v\rangle^{\gamma+2}, & \xi \parallel v,\end{cases} \quad \text{for all } (t,x,v)\in\Omega(z_0) \text{ and } \xi\in\mathbb{R}^3. \tag{3.2.5}$$

Then there exist $\alpha\in(0,1)$ and $C>0$ depending only on K_0 and λ_0, and $p_1, p_2 > 0$ depending *on α and γ, such that*

$$[D_v^2 f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} + [(\partial_t + v\cdot\nabla_x)f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} \le C\left(1+t_0^{-p_1}\right)\langle v_0\rangle^{-q+p_2},$$

where r_1 is defined in (3.1.1).

Proof. Once again, we work with the transformed function f_{z_0}. As in the proof of Lemma 3.2.1, our hypotheses imply that Lemma 3.1.1 is satisfied, which in particular implies the bounds (3.1.6) for the coefficients A, B, and C defined in (3.1.4). This allows us to apply the C^α estimate of [21], which applies to the divergence-form equation (3.1.2) and yields

$$\|f_{z_0}\|_{C_k^\alpha(Q_{1/2}(0))} \le C\left(\|f_{z_0}\|_{L^2(Q_1)} + \|C(z)f_{z_0}\|_{L^\infty(Q_1)}\right),$$

with $C > 0$ and $\alpha \in (0,1)$ depending only on λ_0 and K_0. From our bounds on $C(z)$ and f_{z_0} (which come from Lemmas 3.1.1 and 3.1.2(a) respectively), this implies

$$\|f_{z_0}\|_{C_k^\alpha(Q_{1/2}(0))} \le C\langle v_0\rangle^{-q+\gamma}.$$

Undoing the change of variables via Lemma 3.1.2(b), we have

$$\begin{aligned}\|f\|_{C_k^\alpha(Q_{r_1/2}(z_0))} &\le \|f\|_{L^\infty(Q_{r_1/2}(z_0))} + C\left(1+t_0^{-\alpha/2}\right)\langle v_0\rangle^\alpha [f_{z_0}]_{C_k^\alpha(Q_{1/2})}\\ &\le C\left(1+t_0^{-\alpha/2}\right)\langle v_0\rangle^{-q+\gamma+\alpha},\end{aligned} \tag{3.2.6}$$

since $f \in L_q^\infty$. To get an estimate for the C_k^α norm of f on the larger cylinder $Q_{r_0}(z_0)$, we use the straightforward interpolation

$$\|f\|_{C_k^\alpha(Q_{r_0}(z_0))} \le C\left(r_1^{-\alpha}\|f\|_{L^\infty(Q_{r_0}(z_0))} + \sup_{z_1\in Q_{r_0}(z_0)}\|f\|_{C_k^\alpha(Q_{r_1}(z_1))}\right),$$

and replace z_0 with $z_1 \in Q_{r_0}(z_0)$ in (3.2.6). This yields

$$\begin{aligned}\|f\|_{C_k^\alpha(Q_{r_0}(z_0))} &\le C\left(1+t_0^{-\alpha/2}\right)\langle v_0\rangle^{\alpha(1+\gamma/2)}\|f\|_{L^\infty(Q_{r_0}(z_0))} \\ &\quad + \sup_{z_1\in Q_{r_0}(z_0)} \|f\|_{C_k^\alpha(Q_{r_1}(z_1))} \\ &\le C\left(1+t_0^{-\alpha/2}\right)\left(\langle v_0\rangle^{-q+\alpha(1+\gamma/2)} + \langle v_0\rangle^{-q+\gamma+\alpha}\right) \\ &\le C\left(1+t_0^{-\alpha/2}\right)\langle v_0\rangle^{-q+\gamma+\alpha}.\end{aligned}$$

Applying Lemma 2.3.1 with $g = \langle v\rangle^{q-\gamma-\alpha}$, we have

$$[\langle v\rangle^{q-\gamma-\alpha} f]_{C_k^\alpha(Q_{r_0}(z_0))} \le C\left(1+t_0^{-\alpha/2}\right).$$

The same estimate holds with arbitrary $z \in \Omega(z_0)$ replacing z_0, so we have $[\langle v\rangle^{q-\gamma-\alpha} f]_{C_k^\alpha(\Omega(z_0))} \le C\left(1+t_0^{-\alpha/2}\right)$. This allows us to apply the Schauder estimate of Lemma 3.2.1 with $m = q-\gamma-\alpha > 5+\gamma+\alpha/3$, to conclude

$$\begin{aligned}[D_v^2 f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} &+ [(\partial_t + v\cdot\nabla_x)f]_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))} \\ &\le C(1+t_0^{-5/2-11\alpha/6-\alpha^2/3})\langle v_0\rangle^{-q+5\gamma/3+9+11\alpha/3+6/\alpha+2\alpha^2/9},\end{aligned}$$

as claimed. □

3.2.2 Global C^2 estimate

Proposition 3.2.2. *Let f be a solution the Landau equation on $[0,T]\times\mathbb{R}^6$, and for some $\tau \in (0,T]$, $\rho_0 > 0$, $\beta \in [\gamma,1]$, and $\alpha \in (0,1)$, assume that*

$$e^{\rho_0\langle v\rangle^\beta} f \in C_k^\alpha([0,\tau]\times\mathbb{R}^6).$$

and

$$\bar{a}^f_{ij}(t,x,v)\xi_i\xi_j \geq \lambda_0 \begin{cases} \langle v\rangle^{\gamma}, & \xi \perp v, \\ \langle v\rangle^{\gamma+2}, & \xi \parallel v, \end{cases} \quad \text{for all } (t,x,v) \in [0,\tau]\times\mathbb{R}^6 \text{ and } \xi \in \mathbb{R}^3. \tag{3.2.7}$$

Then

$$\|e^{\rho\langle v\rangle^\beta} D_v^2 f\|_{L^\infty([\tau/2,\tau]\times\mathbb{R}^6)} \leq C\left(1+\tau^{-1+\frac{\alpha^2}{6-\alpha}}\right)\left(1+\|e^{\rho_0\langle v\rangle^\beta} f\|^{P(\alpha)}_{C^\alpha_k([0,\tau]\times\mathbb{R}^6)}\right),$$

where $\rho = \dfrac{\alpha}{6-2\alpha}\rho_0$*,* $P(\alpha) > 1$ *is an exponent depending only on* α*, and* $C > 0$ *is a constant depending only on* ρ_0*,* β*,* α*,* λ_0*, and* $\|e^{\rho_0\langle v\rangle^\beta} f\|_{L^\infty([0,\tau]\times\mathbb{R}^6)}$*.*

Proof. Let $z_0 \in [\tau/2,\tau]\times\mathbb{R}^6$ be fixed, and let r_1 be defined by (3.1.1) and $r_0 = \min\{1, \sqrt{t_0/2}\}$. As in Lemma 3.2.1, define $\Omega(z_0) = Q^{t,x}_{r_0}(z_0)\times\mathbb{R}^3_v$. Since our assumption on f implies polynomial decay of all orders, we choose $m > 5+\gamma+\alpha/3$ arbitrarily, and choose $q > m$ large enough that the exponent of $\langle v_0\rangle$ in Lemma 3.2.1 is negative, i.e.

$$-(q+2m)/3 + 9 + 3\alpha + 6/\alpha + 2\alpha^2/9 + \gamma \leq 0.$$

We apply the weighted interpolation of Lemma 2.4.2 (note that $\rho' = \rho_0$ with our choice of ρ), followed by Lemma 3.2.1 with our choices of m and q:

$$\begin{aligned}
&\|e^{\rho\langle v\rangle^\beta} D_v^2 f\|_{L^\infty(Q_{r_1/2}(z_0))} \\
&\quad\leq C[D_v^2 f]^{1-\frac{2\alpha}{6-\alpha}}_{C_k^{2\alpha/3}(Q_{r_1/2}(z_0))}\|e^{\rho_0\langle v\rangle^\beta} f\|^{\frac{2\alpha}{6-\alpha}}_{C^\alpha_k(Q_{r_1/2}(z_0))} \\
&\quad\leq C\left[\left(1+t_0^{-1-\alpha/3}\right)\left(1+[\langle v\rangle^m f]^{3+2\alpha/3+3/\alpha}_{C^\alpha_k(\Omega(z_0))}\right)\right]^{1-\frac{2\alpha}{6-\alpha}}\|e^{\rho_0\langle v\rangle^\beta} f\|^{\frac{2\alpha}{6-\alpha}}_{C^\alpha_k(Q_{r_1/2}(z_0))} \\
&\quad\leq C\left(1+t_0^{-1+\frac{\alpha^2}{6-\alpha}}\right)\left(1+\|e^{\rho_0\langle v\rangle^\beta} f\|^{(3+2\alpha/3+3/\alpha)(1-\frac{2\alpha}{6-\alpha})+\frac{2\alpha}{6-\alpha}}_{C^\alpha_k(\Omega(z_0))}\right)
\end{aligned}$$

where in the last line, we used $Q_{r_1/2}(z_0) \subset \Omega(z_0)$ and the crude upper bound $[\langle v\rangle^m f]_{C^\alpha_k(\Omega(z_0))} \leq \|e^{\rho_0\langle v\rangle^\beta} f\|_{C^\alpha_k(\Omega(z_0))}$. Since $z_0 \in [\tau/2,\tau]\times\mathbb{R}^6$ was arbitrary, and $t_0 \approx \tau$, the proof is complete. Note that the constant C depends on $\|e^{\rho_0\langle v\rangle^\beta} f\|_{L^\infty([0,\tau]\times\mathbb{R}^6)}$ due to the dependence on K_0 in

Lemma 3.2.1. □

3.3 Regularity in time

The following proposition says that (x, v) regularity implies t regularity, for solutions of a class of linear kinetic equations that include the (linear) Landau equation. A similar result was shown in [31, Proposition A.1], under a stronger assumption on the coefficients, namely that the zeroth-order coefficient c is uniformly bounded. To prove the more general form that we state here, one can modify the proof in [31] in a straightforward way to account for a $c(t, x, v)$ that grows polynomially in v. Therefore, we omit the proof.

Proposition 3.3.1. *Suppose that $f : [0, T] \times \mathbb{R}^6 \to [0, \infty)$ is a solution of the linear equation*

$$\partial_t f + v \cdot \nabla_x f = \operatorname{tr}(a D_v^2 f) + cf,$$

where the coefficients a and c satisfy

$$\|\langle v\rangle^{\gamma+2} a_{ij}\|_{L^\infty([0,T]\times\mathbb{R}^6)} + \|\langle v\rangle^{\gamma} c\|_{L^\infty([0,T]\times\mathbb{R}^6)} \le K_0.$$

Furthermore, assume f is locally Hölder continuous in (x, v) variables, and that $\langle v\rangle^\gamma f$ is bounded. Then f is locally Hölder continuous in all three variables, and the estimate

$$\|f\|_{C_k^\alpha(Q_1(z_0)\cap[0,T]\times\mathbb{R}^6)} \le C\langle v_0\rangle^{\alpha(1+\gamma/2)+\gamma} \left(\|f|_{L_\gamma^\infty([0,T]\times\mathbb{R}^6)} + \sup_{\substack{0\le t\le t_0 \\ t_0 - t \le 1}} [f(t,\cdot,\cdot)]_{C_{k,x,v}^\alpha(B_2(x_0,v_0))} \right)$$

holds, where $C > 0$ is a constant depending only on γ and K_0.

Chapter 4

Decay estimates for large velocity

Our first decay estimate shows that sub-exponential decay in velocity is propagated forward in time.

Lemma 4.0.1. *Let f be a classical solution to the Landau equation* (1.0.1) *on* $[0,T]\times\mathbb{R}^6$, *periodic in the x variable, such that*

$$f(0,x,v)\le K_0 e^{-\rho\langle v\rangle^\beta},\quad x\in\mathbb{R}^3, v\in\mathbb{R}^3,$$

for some $\rho, K_0>0$ and $\beta\in[\gamma,1]$, and such that

$$\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)}\le L_0,$$

for some $q>5+\gamma$ and $L_0>0$.

Then there exists $\sigma>0$, depending only on γ, ρ, q, and L_0, so that f satisfies

$$f(t,x,v)\le K_0 e^{-(\rho-\sigma t)\langle v\rangle^\beta},\quad 0\le t\le\min\left\{T,\frac{\rho}{2\sigma}\right\}, x\in\mathbb{R}^3, v\in\mathbb{R}^3.$$

In fact, the conclusion holds for any $\sigma>CL_0\rho(1+\rho)$, where $C>0$ is a constant depending only on γ and q.

Proof. With $\sigma, \kappa > 0$ to be chosen later, and $\varepsilon > 0$ an arbitrary small number, define the barrier

$$\begin{aligned}
\phi(t,v) &= \phi_1(t,v) + \varepsilon\phi_2(t,v),\\
\phi_1(t,v) &= K_0 e^{-(\rho-\sigma t)\langle v\rangle^\beta},\\
\phi_2(t,v) &= e^{\kappa t}\langle v\rangle^\gamma.
\end{aligned}$$

The purpose of the error term $\varepsilon\phi_2(t,v)$ is to ensure the existence of a first crossing point at a positive time. Later, we will send $\varepsilon \to 0$.

We want to show that

$$f(t,x,v) \le \phi(t,v), \quad 0 \le t \le \min\left\{T, \frac{\rho}{2\sigma}\right\}, x \in \mathbb{R}^3, v \in \mathbb{R}^3. \tag{4.0.1}$$

If this inequality is false, then we claim there is a point (t_0, x_0, v_0), with $t_0 > 0$, where f and ϕ touch for the first time. Indeed, $f(0,x,v) < \phi(0,v)$ by construction, and since f is bounded, there is some $M > 0$ (depending on ε) so that $f(t,x,v) < \phi(t,v)$ whenever $|v| > R$. Since f is periodic in x, the existence of $z_0 = (t_0, x_0, v_0)$ then follows from the compactness of the domain $[0,T] \times \mathbb{T}^3 \times \overline{B_R(0)}$ and continuity in t.

To keep the notation clean, for the remainder of this proof, evaluations of f, $\bar{a}^f$, and $\bar{c}^f$ are assumed to be at z_0 unless otherwise noted, and evaluations of ϕ, ϕ_1, and ϕ_2 are at (t_0, v_0).

At the first crossing point z_0, we have $\partial_t(\phi - f) \le 0$, $\nabla_x(\phi - f) = 0$, and $D_v^2(\phi - f) \ge 0$. These inequalities imply

$$\sigma\langle v_0\rangle^\beta\phi_1 + \varepsilon\kappa\phi_2 = \partial_t\phi \le \partial_t f = \mathrm{tr}(\bar{a}^f D_v^2 f) + \bar{c}^f f \le \mathrm{tr}(\bar{a}^f D_v^2\phi) + \bar{c}^f\phi, \tag{4.0.2}$$

since $\bar{a}^f$ is non-negative definite. By linearity, this right-hand side equals

$$\left[\mathrm{tr}(\bar{a}^f D_v^2\phi_1) + \bar{c}^f\phi_1\right] + \varepsilon\left[\mathrm{tr}(\bar{a}^f D_v^2\phi_2) + \bar{c}^f\phi_2\right].$$

Let us treat the ϕ_1 and ϕ_2 terms in this expression separately. First, with (2.4.3) and Lemma

2.2.2, we have

$$
\begin{aligned}
&\operatorname{tr}(\bar{a}^f D_v^2\phi_1) + \bar{c}^f\phi_1 \\
&\leq -(\rho - \sigma t_0)\beta\langle v_0\rangle^{\beta-4}\phi\left[\left((\beta-2) - (\rho-\sigma t_0)\beta\langle v_0\rangle^{\beta}\right)\bar{a}^f_{ij}(v_0)_i(v_0)_j + \langle v_0\rangle^2\operatorname{tr}(\bar{a}^f)\right] + \bar{c}^f\phi_1 \\
&\leq (\rho-\sigma t_0)\beta C L_0\langle v_0\rangle^{\gamma+\beta-2}\phi\left(2-\beta+(\rho-\sigma t_0)\beta\langle v_0\rangle^{\beta}\right) + CL_0\langle v_0\rangle^{\gamma}\phi_1 \\
&\leq (1+\rho^2)CL_0\left(\langle v_0\rangle^{\gamma+2\beta-2} + \langle v_0\rangle^{\gamma}\right)\phi_1 \\
&\leq CL_0\rho(1+\rho)\langle v_0\rangle^{\gamma}\phi_1,
\end{aligned}
$$

since $\beta \leq 1$. Here, C is a constant depending only on γ and q, as in Lemma 2.2.2. Next, a direct calculation shows that

$$
(D_v^2\phi_2(t,v))_{ij} = e^{\kappa t}\left[\gamma(\gamma-2)\langle v\rangle^{\gamma-4}v_iv_j + \gamma\langle v\rangle^{\gamma-2}\delta_{ij}\right],
$$

so that, after discarding negative terms,

$$
\begin{aligned}
\operatorname{tr}(\bar{a}^f D_v^2\phi_2) + \bar{c}^f\phi_2 &\leq e^{\kappa t_0}\left(\gamma\langle v_0\rangle^{\gamma-2}\operatorname{tr}(\bar{a}^f) + \bar{c}^f\langle v_0\rangle^{\gamma}\right) \\
&\leq CL_0\gamma e^{\kappa t_0}\langle v_0\rangle^{2\gamma} \\
&= CL_0\gamma\langle v_0\rangle^{\gamma}\phi_2.
\end{aligned}
$$

Returning to (4.0.2), we have, at the crossing point z_0,

$$
\sigma\langle v_0\rangle^{\beta}\phi_1 + \varepsilon\kappa\phi_2 \leq CL_0\rho(1+\rho)\langle v_0\rangle^{\gamma}\phi_1 + \varepsilon CL_0\gamma\langle v_0\rangle^{\gamma}\phi_2
$$

Since $\beta \geq \gamma$, this inequality is a contradiction if σ is chosen greater than $CL_0\rho(1+\rho)$ and κ is chosen greater than $CL_0\gamma$. We have established (4.0.1), and the proof is complete after sending $\varepsilon \to 0$. □

Next, we assume that the initial data f_0 satisfies uniform Gaussian upper bounds:

$$
f_0(x,v) \leq C_0 e^{-\rho|v|^2}, \quad (x,v) \in \mathbb{R}^6,
$$

for some $\rho, C_0 > 0$. As our next result shows, these upper bounds are also propagated forward in time, i.e. $f(t,x,v) \le Ce^{\beta t}e^{-\rho|v|^2}$ for $t > 0$. We prove it using a maximum principle argument similar to [41, Lemma 3.1(a)].

The following is an a priori estimate for solutions with qualitative decay that is sufficiently fast. Later, this a priori estimate will be combined with an approximation argument to extend the estimate to solutions that are not assumed to decay rapidly.

Lemma 4.0.2. *For $f \ge 0$ a smooth, rapidly decaying solution of the Landau equation on $[0,T] \times \mathbb{R}^6$, with $e^{\rho|v|^2} f_0(x,v) \in L^\infty(\mathbb{R}^6)$ for some $\rho > 0$, there holds*

$$f(t,x,v) \le Ne^{\beta t}e^{-\rho|v|^2},$$

with N and β depending only on $\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)}$ for some $q > \gamma + 5$ and the initial data.

Proof. Define the barrier

$$g(t,x,v) = Ne^{\beta t}e^{-\rho|v|^2},$$

with $N, \beta, \rho > 0$ to be chosen later. We want to show $f < g$ everywhere in the domain $[0,T] \times \mathbb{T}^3 \times \mathbb{R}^3$. We will proceed by contradiction.

We claim $f < g$ everywhere in $[0,T] \times \mathbb{R}^6$. If this is false, then f and g must cross for the first time at some location (t_0, x_0, v_0). From our assumption on f_0, and by choosing $N > \|e^{\rho|v|^2} f_0\|_{L^\infty}$, we have $f_0(x,v) < g(0,x,v)$. By continuity in t and the rapid decay of f, we must have $t_0 > 0$. At the first crossing location, $g - f$ is decreasing, so we have

$$\partial_t g(t_0,x_0,v_0) - \partial_t f(t_0,x_0,v_0) \le 0.$$

The crossing point is a minimum of $g - f$ in x, v variables, so we have

$$\nabla_v f(t_0,x_0,v_0) = \nabla_v g(t_0,x_0,v_0), \quad \nabla_x f(t_0,x_0,v_0) = \nabla_x g(t_0,x_0,v_0) = 0, \tag{4.0.3}$$

since g is constant in x. From this, we obtain, at the crossing point,

$$\partial_t g \leq \partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{4.0.4}$$

from the equation satisfied by f. We will use this inequality to derive a contradiction. For the left side, we have $\partial_t g = \beta N e^{\beta t} e^{-\rho|v|^2}$. For the right side, since $g - f$ has a local minimum at (t_0, x_0, v_0), we have $D_v^2 f \leq D_v^2 g$, and using $\bar{a}^f \geq 0$, we have

$$Q(f,f) = \mathrm{tr}(\bar{a}^f D_v^2 f) + \bar{c}^f f \leq \mathrm{tr}(\bar{a}^f D_v^2 g) + \bar{c}^f g = \sum_{i=1}^{3}\sum_{j=1}^{3} \bar{a}_{ij}^f \partial_{v_i}\partial_{v_j} g + \bar{c}^f g,$$

where we have used $f(t_0, x_0, v_0) = g(t_0, x_0, v_0)$ in the last term.

Overall, we have

$$\beta N e^{\beta t} e^{-\rho|v|^2} \leq \sum_{i=1}^{3}\sum_{j=1}^{3} \bar{a}_{ij}^f \partial_{v_i}\partial_{v_j} g + \bar{c}^f g, \tag{4.0.5}$$

at the point (t_0, x_0, v_0). By direct calculation,

$$\begin{aligned} D_v^2 g &= (\partial_{v_i}\partial_{v_j} g)_{i,j=1,2,3} \\ &= N e^{\beta t} \partial_{v_i} \left(\partial_{v_j} e^{-\rho(v_1^2+v_2^2+v_3^2)} \right) \\ &= N e^{\beta t} \partial_{v_i} \left(-2\rho v_j e^{-\rho|v|^2} \right) \\ &= N e^{\beta t} \left(-2\rho \delta_{ij} e^{-\rho|v|^2} + 4\rho^2 v_i v_j e^{-\rho|v|^2} \right), \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In order to use the anisotropic lower bounds for $\bar{a}^f$ given by (2.2.1), we need to write $D_v^2 g$ in terms of directions perpendicular to v and parallel to v. Namely, we have

$$\partial_{v_j}\partial_{v_j} g = N e^{\beta t} e^{-\rho|v|^2} \left[-2\rho(\delta_{ij} - |v|^{-2} v_i v_j) + (4\rho^2 - 2\rho|v|^{-2}) v_i v_j \right].$$

Returning to (4.0.5), we now have

$$\beta N e^{\beta t} e^{-\rho|v|^2} \leq \sum_{i=1}^{3}\sum_{j=1}^{3} \bar{a}_{ij}^{f} N e^{\beta t} \left[-2\rho(\delta_{ij} - |v|^{-2} v_i v_j) + (4\rho^2 - 2\rho|v|^{-2}) v_i v_j\right] e^{-\rho|v|^2}$$

$$+ (\bar{c}^f + v \cdot \nabla_x \varphi_R + C\|f\|_{L^\infty_q}) N e^{\beta t} e^{-\rho|v|^2}.$$

After cancelling like terms, we get

$$\beta \leq \sum_{i=1}^{3}\sum_{j=1}^{3} \bar{a}_{ij}^{f}(-2\rho(\delta_{ij} - |v|^{-2} v_i v_j) + (4\rho^2 - 2\rho|v|^{-2}) v_i v_j) + \bar{c}^f.$$

Applying the lower and upper bounds for $\bar{a}^f$ from (2.2.1) and Lemma 2.2.1, we obtain

$$\beta \leq -\rho c_1 (1 + |v|)^{\gamma+2} + |4\rho^2 - \rho|v|^{-2}| C_1 \|f\|_{L^\infty_q} (1 + |v|)^{\gamma+2} + C\|f\|_{L^\infty_q} (1 + |v|)^{\gamma},$$

for some constants $c_1, c_2, C_1 > 0$.

Now we consider the case of large $|v|$. Choosing $\rho < c_1/(2C_1\|f\|_{L^\infty_q})$, we see that the term proportional to $|v|^{\gamma+2}$ will predominate for large $|v|$, and this term is negative by our choice of ρ. More precisely, there is some $R_0 > 0$ such that the right-hand side of our inequality is negative whenever $|v| > R_0$, which contradicts $\beta > 0$, so we conclude a crossing point cannot happen when $|v| > R_0$. Next, to rule out a crossing point with $|v| \leq R$, we see that the right-hand side is bounded by $4C(1 + R_0)^{\gamma+2} + C_3(1 + R_0)^{\gamma}$. Choosing β larger than this number, we conclude the inequality must be false in this case as well, and we conclude there is no crossing point, and our desired upper bound holds. □

Chapter 5

Lower bounds

In this step, let f be a smooth solution of (1.0.1) on $[0,T]\times\mathbb{R}^6$. We assume the initial data has a core of mass:

$$f_0(x,v)\geq\delta,\quad x\in B_r(x_0), v\in B_r(v_0),$$

for some $r,\delta>0$ and $x_0,v_0\in\mathbb{R}^3$. Our goal is to show that this positive lower bound is propagated forward in time. The statement is similar to [30, Theorem 1.3], but we need to use a different proof.

The following lemma is inspired by [32, Lemma 3.1], which applied to the Boltzmann equation, but we need to adapt the argument for the Landau equation.

Lemma 5.0.1. *Let $f\geq 0$ solve (1.0.1) on $[0,T]\times\mathbb{R}^6$, and assume $f\in L^\infty_q([0,T]\times\mathbb{R}^6)$.*

If $f(0,x,v)\geq\delta 1_{|x-x_0|<r,|v-v_0|<r/\sigma}$ for some $(x_0,v_0)\in\mathbb{R}^6$ and $\delta,r,\sigma>0$, then the lower bound

$$f(t,x,v)\geq\frac{\delta}{2}$$

holds whenever $0\leq t\leq\min\{T,\sigma\}$ and, for a universal constant C,

$$\frac{|v-v_0|^2}{r^2/\sigma^2}+\frac{|x-x_0-tv|^2}{r^2}<\frac{1}{4},\qquad \text{and } t<\frac{C\|f\|^{-1}_{L^\infty_q}(r/\sigma)^2}{\langle|v_0|+r/\sigma\rangle^{\gamma+2}}.$$

Proof. Consider the function

$$\underline{f}(t,x,v) := -c_1 t + c_2 \left(1 - \frac{|v-v_0|^2}{r^2/\sigma^2} - \frac{|x-x_0-tv|^2}{r^2}\right) \tag{5.0.1}$$

with $c_1, c_2 > 0$ chosen later.

We wish to show that $\underline{f}$ is a subsolution to the linear Landau equation, at least at points where it is positive. Assume that (t,x,v) is such that $\underline{f} > 0$. We clearly have $\mathrm{tr}(\bar{a}^f D_v^2 \underline{f}) = \mathrm{tr}(\bar{a}^f D_v^2(\underline{f} + c_1 t))$, so that (5.0.1) implies

$$\begin{aligned} \partial_t \underline{f} &= -c_1 + c_2 \cdot \frac{2}{r^2}(x - x_0 - tv)\cdot v, \\ v \cdot \nabla_x \underline{f} &= c_2 \frac{2}{r^2}(x - x_0 - tv)\cdot v, \\ \partial_t \underline{f} + v \cdot \partial_x \underline{f} &= -c_1. \end{aligned} \tag{5.0.2}$$

Next, by direct calculation,

$$\begin{aligned} \left|\partial_{ij}\underline{f}(v)\right| &= |4C_2 r^{-4}(\sigma^2(v-v_0) - t(x-x_0-tv))_i(\sigma^2(v-v_0) - t(x-x_0-tv))_j \\ &\quad + 2C_2 r^{-2}\delta_{ij}(\sigma^2 + t^2)| \\ &\le Cc_2((t^2+\sigma^2)r^{-2} + r^{-2}\sigma^2) \\ &\le Cc_2\sigma^2 r^{-2}. \end{aligned} \tag{5.0.3}$$

We have used $t \le T$, and that $\underline{f} \le 0$ if $\frac{|v-v_0|^2}{r^2/\sigma^2} + \frac{|x-x_0-tv|^2}{r^2} > 1$. Using (5.0.3), Lemma 2.2.1, and $\bar{c}^f \underline{f} \ge 0$, we have, at points where $\underline{f}$ is positive,

$$\begin{aligned} Q(f,\underline{f}) &= \mathrm{tr}(\bar{a}^f D_v^2 \underline{f}) + \bar{c}^f \underline{f} \\ &\ge -\left|\mathrm{tr}(\bar{a}^f D_v^2 \underline{f})\right| \\ &\ge -C\left\|f\right\|_{L^\infty_q} \langle v_0\rangle^{\gamma+2}\sigma^2 r^{-2}. \end{aligned} \tag{5.0.4}$$

Combining this with (5.0.2) gives

$$\partial_t \underline{f} + v\cdot\nabla_x \underline{f} = -c_1 < -C\|f\|_{L^\infty_q}\langle v\rangle^{\gamma+2}\sigma^2 r^{-2} \le Q(f,\underline{f}), \tag{5.0.5}$$

if we make the choice

$$c_1 = 2\mathrm{C}\|f\|_{L^\infty_q}\langle|v_0| + r/\sigma\rangle^{\gamma+2}\sigma^2 r^{-2}. \tag{5.0.6}$$

Thus,

$$\partial_t \underline{f} + v \cdot \nabla_x \underline{f} = -c_1 < Q(f, \underline{f}) \text{ for all } v \in B_r(v_0) \tag{5.0.7}$$

Now, we claim $f > \underline{f}$ for all (t, x, v) such that $\underline{f}(t, x, v) > 0$. By choosing $c_2 = \frac{3\delta}{4}$, this claim is true for t=0. If the claims fails, then there is a first crossing point (t_c, x_c, v_c) with $\underline{f}(t_c, x_c, v_c) > 0$, such that $f(t_c, x_c, v_c) = \underline{f}(t_c, x_c, v_c)$ and $f(t, x, v) > \underline{f}(t, x, v)$ whenever $\underline{f}(t, x, v) > 0$ and $t < t_c$.

The strict positivity of t_c follows from the compact support of $\underline{f}(t, \cdot, \cdot)$ for each t. We also have $f(t_c, x, v) \geq \underline{f}(t_c, x, v)$ for all $(x, v) \in \mathbb{R}^6$.

Letting $g = f - \underline{f}$, we have $\partial_t g(t_c, x_c, v_c) \leq 0$ and $\nabla_x g(t_c, x_c, v_c) = 0$, so that (5.0.7) implies

$$0 \geq (\partial_t + v_c \cdot \nabla_x) g(t_c, x_c, v_c) > Q(f, g). \tag{5.0.8}$$

Next, since g has a local minimum in v at the crossing point, we have

$$\mathrm{tr}(\bar{a}^f D_v^2 g)(t_c, x_c, v_c) \geq 0.$$

Since we also have $g = 0$ at the crossing point, we in fact have $Q(f, g) \geq 0$, contradicting (5.0.8).

This contradiction implies $f \geq \underline{f}$ whenever $\underline{f}(t, x, v) > 0$. The conclusion then follows by choosing C according to the constant in (5.0.6) and using the definition of $\underline{f}$. □

The purpose of the next lemma is to spread lower bounds to large velocities.

Lemma 5.0.2. *Let f be a solution of the Landau equation* (1.0.1) *on $[0, T] \times \mathbb{R}^6$, such that*

$$\sup_{t,x} \int_{\mathbb{R}^3} f(t, x, v)(1 + |v|^{\gamma+2})\, \mathrm{d}v \leq K_0,$$

and for some $\delta, r > 0$, $\tau \in (0,1]$, and $x_0, v_0 \in \mathbb{R}^3$, assume that f satisfies the lower bound

$$f(t,x,v) \geq \delta, \quad t \in [0,\tau], x \in B_r(x_m), v \in B_r(v_m).$$

Then, for any $R > 1$, f also satisfies the lower bound

$$f(t,x,v) \geq \frac{\delta}{4} e^{-\kappa t^{-1}|v-v_m|^2}, \quad t \in [0,\tau'], x \in B_{r/4}(x_m + tv_m), v \in B_R(v_m),$$

where $\kappa > 0$ depends on δ, K_0, and r, and $\tau' = \min\{\tau, Cr/R\}$ for a constant $C > 0$ depending on K_0.

Proof. First, we recenter around the origin by defining

$$\widetilde{f}(t,x,v) = f(t, x_m + x + tv_m, v_m + v).$$

A direct calculation shows that $\widetilde{f}$ satisfies the Landau equation in $[0,\tau] \times \mathbb{R}^6$. Our assumptions for f imply

$$\widetilde{f}(t,x,v) \geq \delta, \quad t \in [0,\widetilde{\tau}], x \in B_{r/2}(0), v \in B_r(0), \tag{5.0.9}$$

where $\widetilde{\tau} = \min\{\tau, r/(2|v_m|)\}$. For the remainder of the proof, we write f instead of $\widetilde{f}$.

Define $\zeta(x) = 1 - |x|^2/(r/2)^2$, and note that $\zeta(x) \leq 1_{B_{r/2}}(x)$. Let $\xi_R : \mathbb{R}^3 \to [0,\infty)$ be a smooth, radially decreasing cutoff with $\xi_R = 1$ in B_R and $\xi_R = 0$ outside B_{2R}, with $|\nabla \xi_R| \leq CR^{-1}$ and $|D^2\xi_R| \leq CR^{-2}$ globally in $\mathbb{R}^3$, for some universal constant C. Next, define

$$\psi(t,x,v) = \delta(\zeta(x) - A_1 t)(\xi_R(v) - A_2 t) e^{-\kappa t^{-1}|v|^2},$$

where $\kappa, A_1, A_2 > 0$ are constants to be chosen later. For some small $\varepsilon > 0$, we claim that

$$f(t,x,v) > \psi(t,x,v) - \varepsilon, \quad \text{in } \Omega := [0,\widetilde{\tau}] \times \mathbb{R}^3_x \times \{|v| \geq r/2\}, \tag{5.0.10}$$

where we extend ψ smoothly by zero on $\{t = 0\} \times \mathbb{R}^3_x \times \{|v| \geq r/2\}$.

First, let us show that $f > \psi - \varepsilon$ on the (parabolic) boundary of Ω. When $t = 0$ and $|v| \geq r/2$, we have $\psi(0, x, v) = 0$ and $f(0, x, v) \geq 0 > \psi(0, x, v) - \varepsilon$.

When $|v| = r/2$ and $t \in [0, \widetilde{\tau}]$, since $\zeta(x) \leq 1_{B_{r/2}}(x)$ and $\xi_R(v) = 1_{B_r}(v) = 1$, we have

$$\psi(t, x, v) \leq \delta 1_{B_{r/2}}(x)\xi_R(v)e^{-\kappa t^{-1}r^2/4} < \delta 1_{B_{r/2}}(x) = \delta 1_{B_{r/2}}(x)1_{B_r}(v) \leq f(t, x, v) < f(t, x, v)+\varepsilon,$$

where we used (5.0.9) and the fact that $e^{-\kappa t^{-1}r^2/4} < 1$.

Next, we claim that if (5.0.10) is false, there is a point $z_0 = (t_0, x_0, v_0)$ where f and $\psi - \varepsilon$ cross for the first time, with $t_0 > 0$. This follows from the fact that $f \geq 0 > \psi - \varepsilon$ for all (x, v) outside of a compact domain. Naturally, the crossing point satisfies $|x_0| \leq r/2$ and $|v_0| \leq 2R$.

At the crossing point z_0, as above we have

$$\partial_t(f - \psi) \leq 0, \quad \nabla_x(f - \psi) = 0, \quad D_v^2(f - \psi) \geq 0,$$

which implies

$$\partial_t\psi + v_0 \cdot \nabla_x\psi \geq \partial_t f + v_0 \cdot \nabla_x f = \text{tr}(\bar{a}^f D_v^2 f) + \bar{c}^f f \geq \text{tr}(\bar{a}^f D_v^2\psi), \tag{5.0.11}$$

since $\bar{c}^f f \geq 0$ and $\bar{a}^f$ is nonnegative definite. To bound the right side of (5.0.11) from below, we first find via direct calculation

$$\begin{aligned}\partial_{v_iv_j}\psi(t, x, v) = \delta(\zeta(x) - A_1t)e^{-\kappa t^{-1}|v|^2} &\left[(4\kappa t^{-2}v_iv_j - 2\kappa t^{-1}\delta_{ij})(\xi_R - A_2t)\right. \\ &\left. -2\kappa t^{-1}\left(v_j\partial_{v_i}\xi_R + v_i\partial_{v_j}\xi_R\right) + \partial_{v_iv_j}\xi_R\right].\end{aligned}$$

Therefore, at z_0 we have

$$\begin{aligned}\text{tr}(\bar{a}^f D_v^2\psi) = \delta(\zeta(x_0) - A_1t_0)e^{-\kappa t_0^{-1}|v_0|^2} &\left[(4\kappa t_0^{-2}\bar{a}^f(v_0)_i(v_0)_j - 2\kappa t_0^{-1}\text{tr}(\bar{a}^f))(\xi_R(v_0) - A_2t_0)\right. \\ &\left. -4\kappa t_0^{-1}\bar{a}^f_{ij}(v_0)_j\partial_{v_i}\xi_R + \bar{a}^f_{ij}\partial_{v_iv_j}\xi_R\right].\end{aligned}$$

Since ξ_R is radially decreasing and $\bar{a}^f$ is positive-definite, we have $-\bar{a}^f_{ij}(v_0)_j\partial_{v_i}\xi_R(v_0) \geq 0$.

Next, using Lemmas 2.2.3 and 2.2.1 and $|D_v^2 \xi_R| \le CR^{-2}$, we have

$$\begin{aligned}\operatorname{tr}(\bar{a}^f D_v^2 \psi) \ge \delta(\zeta(x_0) - A_1 t_0) e^{-\kappa t_0^{-1}|v_0|^2} & \left[\kappa t_0^{-1}\left(c_1 \langle v_0\rangle^{\gamma} \kappa t_0^{-1} |v_0|^2 - C_2 \langle v_0\rangle^{\gamma+2}\right)(\xi_R(v_0) - A_2 t_0)\right. \\ & \left. - C_2 \langle v_0\rangle^{\gamma+2} R^{-2}\right],\end{aligned}$$

where c_1 is the constant from Lemma 2.2.3, which depends on δ and r, and C_2 is the constant from Lemma 2.2.1, which depends on K_0. Since $|v_0| \ge r/2$, we have $|v_0|^2 \ge \frac{r}{r+1}\langle v_0\rangle^2$. Therefore, we can choose κ sufficiently large, depending only on c_1, C_2, and r, such that $c_1 \kappa t_0^{-1} |v_0|^2 \ge c_1 \kappa |v_0|^2 \ge 2C_2 \langle v_0\rangle^2$. Using this, as well as $|v_0| \le 2R$, we obtain

$$\operatorname{tr}(\bar{a}^f D_v^2 \psi) \ge \delta(\zeta(x_0) - A_1 t_0) e^{-\kappa t_0^{-1}|v_0|^2} \left[c_1 \kappa^2 t_0^{-2} R^{\gamma+2}(\xi_R(v_0) - A_2 t_0) - C_2 R^{\gamma}\right]. \tag{5.0.12}$$

For the left side of (5.0.11), we have

$$\begin{aligned}\partial_t \psi + v_0 \cdot \nabla_x \psi = \delta e^{-\kappa t_0^{-1}|v_0|^2} & \left[(-A_1 + v_0 \cdot \nabla_x \zeta(x_0))(\xi_R(v_0) - A_2 t)\right. \\ & \left. + (-A_2 + \kappa t_0^{-2} |v_0|^2 (\xi_R(v_0) - A_2 t_0))(\zeta(x_0) - A_1 t_0)\right].\end{aligned}$$

With $A_1 \ge 4R/r$, we have

$$-A_1 + v_0 \cdot \nabla_x \zeta(x_0) \le -A_1 - 2\frac{v_0 \cdot x_0}{r^2} \le -A_1 + \frac{4R}{r} \le 0,$$

so that

$$\partial_t \psi + v_0 \cdot \nabla_x \psi \le \delta(\zeta(x_0) - A_1 t_0)\left[4\kappa t_0^{-2} R^2 (\xi_R(v_0) - A_2 t_0) - A_2\right] e^{-\kappa t_0^{-1}|v_0|^2}.$$

Combining this with (5.0.11) and (5.0.12), we obtain

$$\begin{aligned}(\zeta(x_0) - A_1 t_0) & \left[c_1 \kappa^2 t_0^{-2} R^{\gamma+2}(\xi_R(v_0) - A_2 t_0) - C_2 R^{\gamma}\right] \\ & \le (\zeta(x_0) - A_1 t_0)\left[4\kappa t_0^{-2} R^2 (\xi_R(v_0) - A_2 t_0) - A_2\right]\end{aligned}$$

or

$$[c_1\kappa R^\gamma - 4]\,\kappa t_0^{-2}R^2(\xi_R(v_0) - A_2t_0) + [A_2 - C_2R^\gamma] \le 0,$$

which is a contradiction if $\kappa > 4/(c_1R^\gamma)$ and $A_2 > C_2R^\gamma$. We have established (5.0.10), and after sending $\varepsilon \to 0$, we have shown $f(t,x,v) \ge \psi(t,x,v)$ in Ω. In more detail,

$$f(t,x,v) \ge \delta(\zeta(x) - A_1t)(\xi_R(v) - A_2t)e^{-\kappa t^{-1}|v|^2},$$

with A_1, A_2, and κ as above. If $z = (t,x,v)$ is such that

$$|x| \le r/4, \quad t \le \min\left\{\frac{1}{4A_1}, \frac{1}{2A_2}\right\}, \quad \text{and } \frac{r}{2} \le |v| \le R,$$

then we clearly have $f(t,x,v) \ge \dfrac{\delta}{4}e^{-\kappa t^{-1}|v|^2}$. Transforming from $\widetilde{f}$ back to the original solution f, we obtain the statement of the lemma. □

Finally, we prove our main lower bounds result:

Theorem 5.0.1. *Assume that $f : [0,T] \times \mathbb{R}^6 \to [0,\infty)$ is periodic in x and satisfies*

$$f(0,x,v) \ge \delta, \quad |x| < r, |v| < r,$$

for some $\delta, r > 0$, and

$$\sup_{t,x} \int_{\mathbb{R}^3} (1 + |v|^\gamma) f(t,x,v)\,\mathrm{d}v \le K_0,$$

for some $K_0 > 0$. Then there exists $T' \in (0,T]$ and $R(t) > 0$ such that for every $t \in (0,T']$ and $x \in \mathbb{R}^3$, there exists a $v_x \in B_{R(t)}$ such that

$$f(t,x,v) \ge \eta(t) > 0, \quad |v - v_x| < r',$$

where the function η is uniformly positive on any compact subset of $(0,T']$. The time T' and the functions $\eta(t), R(t)$ depend on δ, r, and K_0.

Proof. Using Lemma 5.0.1 with $\sigma = 1$ and $x_0 = v_0 = 0$, we obtain lower bounds of the form

$$f(t,x,v) \geq \frac{\delta}{2}, \quad t \in [0,\tau], x \in B_{r/3}(0), v \in B_{r/3}(0),$$

where $\tau \leq Cr^2$ for a constant $C > 0$ depending on $\|f\|_{L^\infty_q}$.

Now, let $(t_1, x_1) \in (0,\tau] \times (\mathbb{R}^3 \setminus B_r)$ be fixed, and let $v_1 = x_1/t_1$. Lemma 5.0.2 with $R = \max\{1, 4|v_1|\}$ gives, with $\tau' = \min\{\tau, Cr/R\}$,

$$f(t,x,v) \geq \frac{\delta}{8} e^{-\kappa |v|^2/t}, \quad 0 < t \leq \tau', |x| \leq r/12, |v| \leq R, \tag{5.0.13}$$

where κ is the constant from Lemma 5.0.2, and depends on δ, r, and $\|f\|_{L^\infty_q}$. Let $t_* = \min\{\tau', t_1/2\}$. Letting $v_* = x_1/(t_1 - t_*)$, we then have $|v_*| \leq 2|x_1|/t_1 \leq R/2$.

For the next step, spreading lower bounds to the neighborhood of (t_1, x_1), we further restrict the time domain by defining

$$\sigma = \max\left\{1, \frac{r}{12}\sqrt{\frac{2C}{t_1 K_0 R^{\gamma+2}}}\right\}, \tag{5.0.14}$$

where C is the constant from Lemma 5.0.1, and requiring that t_1 satisfy the inequality

$$t_1 \leq \min\left\{\left(\frac{2C(r/12)^2}{2K_0 R^{\gamma+2}}\right)^{1/3}, \frac{36C}{K_0 R^\gamma}\right\}. \tag{5.0.15}$$

Since $R = \max\{1, 4|x_1|/t_1\}$, this inequality means that t_1 has to be smaller than a constant times $|x_1|^{-(\gamma+2)/(1-\gamma)}$, which we can always guarantee because our x domain is bounded, resulting in a condition $t_1 \leq \tau''$, where τ'' depends on K_0, r, C, and the size of the spatial domain[1]. This step is the only reason we require $\gamma < 1$.

We would like to apply Lemma 5.0.1 to $f(t_* + t, x, v)$, with $x_0 = 0$, $v_0 = v_*$, and $r/12$ replacing r, and we would like the resulting lower bound to hold up to time $t_1/2 \geq t_1 - t_*$. For this, we need to check three conditions:

[1] In the case of an unbounded spatial domain where f_0 satisfies a "well-distributed" hypothesis as in [30], τ'' would depend on the maximal distance from any x location to a location where f_0 satisfies suitable positive lower bounds.

- $t_1/2 \le \sigma$, which holds as a result of (5.0.15).
- $r/(12\sigma) \le R/2$, so that $B_{r/(12\sigma)}(v_*) \subset B_R(0)$. With our choice of σ, this inequality is equivalent to

$$t_1 \le \frac{36C}{K_0 R^\gamma},$$

which is true because of (5.0.15).

- $t_1/2 \le \dfrac{CK_0^{-1}(r/(12\sigma))^2}{\langle |v_*| + r/(12\sigma)\rangle^{\gamma+2}}$, where C is the constant from Lemma 5.0.1. This holds because of (5.0.15) and the previous bullet point.

We are now able to apply Lemma 5.0.1 with (5.0.13) and obtain

$$f(t_1, x, v) \ge \frac{\delta}{16} e^{-\kappa R^2/t_*}, \quad \text{if } \frac{|v - v_*|^2}{r^2/\sigma^2} + \frac{|x - t_1 v|^2}{r^2} < \frac{1}{4}.$$

In particular, recalling that $x_1 = t_1 v_1$ and $\sigma \ge 1$, we have that if $|x - x_1| < r/8$ and $|v - v_*| < r/(8\sigma)$, then $|x - t_1 v| < r/4$. Therefore,

$$f(t_1, x, v) \ge \frac{\delta}{16} e^{-\kappa R^2/t_*}, \quad |v - v_*| < \frac{r}{8\sigma}, |x - x_1| < \frac{r}{8}. \tag{5.0.16}$$

Recall that $R = \max\{1, 4|x_1|/t_1\}$ and $t_* = \min\{\tau', t_1/2\}$, so that, after tracing the dependence on all constants, we obtain $\eta(t_1)$ and r' as in the statement of the theorem. Since $(t_1, x_1) \in \mathbb{T}^3 \times [0, \tau'']$ was arbitrary, the proof is complete. □

Chapter 6

Proof of existence

6.1 Approximating the initial data.

Let $\psi(x,v) \geq 0$ be a standard smooth mollifier on $\mathbb{R}^6$ satisfying $\int_{B_1(0)} \psi dxdv = 1$ and $\psi = 0$ outside $B_1(0)$. For any $\varepsilon > 0$, let

$$\psi_\varepsilon(x,v) = \varepsilon^{-6}\psi(x/\varepsilon, v/\varepsilon).$$

Also, for any $r > 0$, let $\zeta_r(v)$ be a smooth, radially decreasing cutoff function with $\zeta = 0$ outside B_r and $\zeta = 1$ in $B_{r/2}$. Now define the approximate initial data

$$f_0^\varepsilon(x,v) = \zeta_{1/\varepsilon}(v)[\psi_\varepsilon * f_0](x,v),$$

where $*$ denotes convolution in (x,v) variables. By standard arguments, $f_0^\varepsilon \to f_0$ pointwise as $\varepsilon \to 0$.

6.2 Applying existence theory for regular initial data

We need to apply the main result of [10], quoted above as Theorem 2.1.1.

Since we defined our f_0^ε as the product of $\zeta_{1/\varepsilon}(v)$, which is further defined as radially decreasing cutoff function with $\zeta = 0$ outside B_r and $\zeta = 1$ in $B_{r/2}$ for $r > 0$. This causes the exponential function $e^{d_0\langle v\rangle}$ to be bounded by $e^{d_0\langle 1/\varepsilon\rangle}$. Moreover, the L^2 norm of $\partial_x^\alpha \partial_v^\beta f_0^\varepsilon$ is bounded by its L^∞ norm, which is finite because the convolution kernel ψ_ε is C^∞.

Therefore, our f_0^ε satisfies the hypothesis of the theorem, thus giving us some solution on a time interval $[0, T_\varepsilon]$. This solution is called f^ε. Note that the time of existence is ε-dependent.

6.3 Applying a priori estimates to f_ε

Apply both the pointwise lower bounds (Theorem 5.0.1) and the sub-exponential upper bounds (Lemma 4.0.1) to f_ε. To do this, it is necessary to check that the hypotheses are all satisfied.

The following conditions need to be checked for the lower bounds theorem:

- $f^\varepsilon \in L_q^\infty([0, T_\varepsilon] \times \mathbb{R}^6)$.
- $f_0^\varepsilon(x, v) \geq \delta 1_{|x|<r, |v|<r}$ for some $r > 0$.

Firstly, to prove $f^\varepsilon \in L_q^\infty([0, T_\varepsilon] \times \mathbb{R}^6)$: $f^\varepsilon \in L_q^\infty([0, T_\varepsilon] \times \mathbb{R}^6)$ if and only if $|\langle v\rangle^q \cdot f^\varepsilon(t, x, .v)| \leq K$.

We know that due to Sobolov embedding, we have $H_{x,v}^{10}(R^6) \subset H_{x,v}^4(R^6) \subset L^\infty([0, T_\varepsilon] \times \mathbb{R}^6)$.

Since $e^{(d_0 - \kappa t)\langle v\rangle} f \in H_{x,v}^{10}(\mathbb{R}^6) \subset L^\infty(\mathbb{R}^6)$, then for $t < d_0/(2\kappa)$, we would have $f \leq K e^{-(d_0 - \kappa t)\langle v\rangle} \leq K e^{-(d_0/2)\langle v\rangle}$, and therefore

$$\langle v\rangle^q \cdot f \leq \varepsilon^{(d_0/2)\langle v\rangle} \cdot f \leq K.$$

Therefore, $f^\varepsilon \in L_q^\infty$.

Secondly, we know from the lower bound lemma that $f_0(x, v) \geq \delta 1_{|x|<r, |v|<r}$ for some $r > 0$, therefore, since the convolution $f_0^\varepsilon(x, v)$ coverges to $f_0(x, v)$ uniformly, we conclude that $f_0^\varepsilon(x, v) \geq \delta 1_{|x|<r/2, |v|<r/2}$, for ε small enough.

The hypotheses of Lemma 4.0.1 are satisfied by assumption.

6.4 Extending f_ε to a uniform time interval

We want to show that the approximate solutions f^ε exist on a uniform time interval $[0, T]$. For this, we use the smoothing estimate of [41, Corollary 1.2], which says: if a solution f satisfies

$$\begin{aligned} &0 < m_0 \le \int_{\mathbb{R}^3} f(t,x,v)dv \le M_0, \quad \int_{\mathbb{R}^3} |v|^2 f(t,x,v)dv \le E_0, \\ &\int_{\mathbb{R}^3} f(t,x,v)\log f(t,x,v)dv \le H_0, \end{aligned} \tag{6.4.1}$$

uniformly in t and x, then f is C^∞ with regularity estimates depending only on m_0, M_0, E_0, and H_0. To apply this result, we need to show f^ε satisfies all four of these inequalities on $[T_\varepsilon/2, T_\varepsilon] \times \mathbb{R}^3$, using the upper and lower bounds above.

First, we want to prove that $\int_{\mathbb{R}^3} f_0^\varepsilon(t,x,v)dv \le M_0$.

From above, $f^\varepsilon \le Ke^{-\rho\langle v\rangle^\beta}$ for some $\rho > 0$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^3} f^\varepsilon(t,x,v)dv &\le \int_{\mathbb{R}^3} Ke^{-\rho\langle v\rangle^\beta}dv \\ &= K \cdot \int_0^\infty e^{-\rho\langle r\rangle^\beta} \cdot r^2 4\pi dr \\ &= 4\pi \cdot K \int_0^\infty e^{-\rho\langle r\rangle^\beta} \cdot r^2 dr \le C_\rho K, \end{aligned}$$

for some C_ρ depending only on ρ.

Secondly, we want to prove that $0 < m_0 \le \int_{\mathbb{R}^3} f(t,x,v)dv$. From the previous section, we know that we can apply Theorem 5.0.1, implying that for all $x_0 \in \mathbb{T}^3$ and $t \in [T_\varepsilon/2, T_\varepsilon]$, the lower bound $f(t,x_0,v) \ge \delta 1_{|x-x_0|<r,|v-v_0|<r}$ holds for some $v_0 \in \mathbb{R}^3$ and $\delta, r > 0$. So, since $f^\varepsilon \ge 0$, $\int_{\mathbb{R}^3} f^\varepsilon(t,x,v)dv \ge \int_{B_r(v_0)} f^\varepsilon(t,x,v)dv \ge \int_{B_r(v_0)} \frac{\delta}{2}dv = \frac{\delta}{2} \cdot \frac{4}{3}\pi r^3 =: m_0$.

Thirdly, we want to prove $\int_{\mathbb{R}^3} |v|^2 f(t,x,v)dv \le E_0$.

As above, we have $e^{\rho\langle v\rangle^\beta} f^\varepsilon(t) \leq K$ for some $\rho > 0$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^3} |v|^2 f^\varepsilon(t,x,v)dv &\leq \int_{\mathbb{R}^3} |v|^2 K e^{-\rho_\varepsilon\langle v\rangle^\beta} dv \\
&= K \cdot \int_0^\infty e^{-\rho\langle r\rangle^\beta} \cdot r^4 4\pi dr \\
&= 4\pi \cdot K \int_0^\infty e^{-\rho\langle r\rangle^\beta} \cdot r^4 dr \leq C_\rho K,
\end{aligned}$$

for some $C_\rho > 0$ depending on ρ.

The fourth goal is to show $\int_{\mathbb{R}^3} f(t,x,v) \log f(t,x,v)dv \leq H_0$. Note: We only integrate over the space when $f > 1$, because integrating over the space$f < 1$ results in a negative function.

$$\begin{aligned}
\int_{\mathbb{R}^3} f \cdot \log f dv &\leq \int_{f>1} f \cdot \log f dv \\
&\leq \int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot \log(K e^{-\rho\langle v\rangle^\beta}) dv \\
&\leq \int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot (\log K + \log e^{-\rho\langle v\rangle^\beta}) dv \\
&\leq \int_{f>1} K e^{-\rho|v|^2} \cdot (\log K - \rho\langle v\rangle^\beta) dv \\
&= \int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot \log K - \int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot \rho|v|^2) dv.
\end{aligned}$$

Note: $-\int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot \rho\langle v\rangle^\beta dv$ yields a negative value, hence it will be disregarded. So,

$$\int_{f>1} K e^{-\rho\langle v\rangle^\beta} \cdot \log K \leq K \log K \cdot \int_{f>1} e^{-\rho\langle v\rangle^\beta} dv \leq K \log K \cdot \int_{\mathbb{R}^3} e^{-\rho\langle v\rangle^\beta} dv.$$

Next, we re-apply the existence theorem of Chaturvedi quoted above, with initial data $f(T_\varepsilon, x, v)$, to show f can be extended to a time interval $[0, T_\varepsilon + T_1]$, with T_1 depending on the H^{10} norm of $e^{\rho|v|} f(T_\varepsilon, x, v)$. This quantity is independent of ε, by [41] and our above estimates of the mass, energy, and entropy densities. Therefore, we have shown the approximate solutions f^ε exist on some uniform time interval $[0, T]$.

6.5 Limit as $\varepsilon \to 0$

Since f^ε solves the Landau equation, we can apply the regularity estimate of Proposition 3.2.1. The hypotheses of this proposition are satisfied, by our upper and lower bounds for f^ε and Lemma 2.2.3.

Next, we use the following compactness theorem: If a sequence of functions f^ε is uniformly bounded in the norm

$$\|f^\varepsilon\|_{C_k^{2,\beta}(\Omega)} = [D_v^2 f^\varepsilon]_{C_k^\beta(\Omega)} + [(\partial_t + v \cdot \nabla_x) f^\varepsilon]_{C_k^\beta(\Omega)} + \|f^\varepsilon\|_{L^\infty(\Omega)},$$

for some bounded subset $\Omega \subset \mathbb{R}^7$, then they converge in $C_k^{2,\beta'}(\Omega)$ for any $\beta < \beta'$ to a limit f. Since Ω is arbitrary, f is defined on all of $(0, T] \times \mathbb{R}^6$. In particular, this implies pointwise convergence as $\varepsilon \to 0$. In fact, by the smoothness result of [41], f^ε is C^∞ for positive times, and this is preserved in the limit as $\varepsilon \to 0$.

Next, we need to show the limit f solves the Landau equation. This follows from the pointwise convergence of $D_v^2 f^\varepsilon \to D_v^2 f$, $(\partial_t + v \cdot \nabla_x) f^\varepsilon \to (\partial_t + v \cdot \nabla_x) f$, and the convergence of the coefficients $\bar{a}^{f^\varepsilon}$ and $\bar{c}^{f^\varepsilon}$ (which follows from the Dominated Convergence Theorem).

Finally, we establish that f matches the initial data in the sense of integration against test functions, as in the statement of Theorem 1.1.1. This is established by multiplying the equation for f^ε by the test function ϕ, integrating by parts, and using the above convergence facts to take the limit as $\varepsilon \to 0$.

Chapter 7

Propagation of Hölder regularity

In the next two chapters, with the goal of proving uniqueness, we place stronger assumptions on the initial data f_0 of our solution: for some $\delta, r, R > 0$, assume that for all $x \in \mathbb{R}^3$, there is a $v_x \in B_R$ so that

$$f_0(x, v) \geq \delta, \quad v \in B_r(v_x). \tag{7.0.1}$$

Furthermore, we assume

$$e^{\rho_0 \langle v \rangle^\beta} f_0 \in L^\infty(\mathbb{R}^6). \tag{7.0.2}$$

for some $\rho_0 > 0$ and $\beta \in [\gamma, 1]$, and

$$e^{\rho \langle v \rangle^\beta} f_0 \in C_k^{3\alpha}(\mathbb{R}^6), \tag{7.0.3}$$

for some $\alpha \in (0, 1/3)$ and $\rho > 0$.

The purpose of the current section is to show that the Hölder continuity that we assume at time zero is propagated forward to positive times. For technical reasons, we work with the specific solution that we constructed above in Theorem 1.1.1 (which has an approximating sequence $f^\varepsilon \to f$, with each f^ε smooth and rapidly decaying) rather than a general classical solution. The precise statement of the result is as follows:

Theorem 7.0.1. *Let $f_0(x,v) \geq 0$ satisfy the lower bound condition* (7.0.1) *for some $\delta, r, R > 0$, as well as the pointwise upper bound* (7.0.2) *for some $\rho_0 > 0$, and the Hölder continuity assumption* (7.0.3) *for some $\alpha \in (0, 1/3)$, $\beta \in [\gamma, 1]$, and*

$$\rho = \frac{4\alpha}{24 - 9\alpha}\rho_0.$$

Let $f : [0,T] \times \mathbb{R}^6 \to [0, \infty)$ be the classical solution to the Landau equation (1.0.1) *guaranteed by Theorem 1.1.1.*

Then there exists $T_H > 0$ such that

$$\|e^{(\rho/4)\langle v\rangle^\beta} f\|_{C_k^\alpha([0,\min(T,T_H)]\times\mathbb{R}^6)} \leq C,$$

The constants T_H and C depend on ρ_0, α, β, δ, r, R, $\|e^{\rho\langle v\rangle^\beta} f_0\|_{C^{3\alpha}_{k,x,v}(\mathbb{R}^6)}$, and $\|e^{\rho_0\langle v\rangle^\beta} f_0\|_{L^\infty(\mathbb{R}^6)}$.

The strategy to prove Theorem 7.0.1 is based on bounding a weighted finite difference of f via a barrier argument. For $(t, x, v) \in \mathbb{R}^7$ and $h, k \in B_1(0) \subset \mathbb{R}^3$, and some $\sigma > 0$ to be determined, define

$$\begin{aligned} \tau f(t,x,v,h,k) &= f(t, x+h, v+k) \\ \delta f(t,x,v,h,k) &= \tau f(t,x,v,h,k) - f(t,x,v), \\ g(t,x,v,h,k) &= e^{(\rho-\sigma t)\langle v\rangle^\gamma} \frac{|\delta f(t,x,v)|^2}{(|h|^2+|k|^2)^\alpha}, \end{aligned} \tag{7.0.4}$$

where ρ and α are the exponents appearing in (7.0.3).

The function g is chosen in order to control a weighted Hölder seminorm of f in (x, v) variables. This seminorm controlled by g is with respect to the Euclidean distance $d_E((x_1, v_1),(x_2, v_2)) = \sqrt{|x_1 - x_2|^2 + |v_1 - v_2|^2}$, rather than the kinetic distance d_k. For any $\Omega \subset \mathbb{R}^6$, let us introduce the notation

$$[h]_{C^\alpha_{E,x,v}(\Omega)} = \sup_{(x_1,v_1),(x_2,v_2)\in\Omega} \frac{|h(x_1,v_1) - h(x_2,v_2)|}{d_E((x_1,v_1),(x_2,v_2))^\alpha},$$

for the Hölder seminorm with respect to the standard Euclidean metric, as well as the norm $\|h\|_{C^\alpha_{E,x,v}(\Omega)} = \|h\|_{L^\infty(\Omega)} + [h]_{C^\alpha_{E,x,v}(\Omega)}$. Euclidean Hölder norms are used only in the current section. They are needed because of the specific form of the denominator of g, which is imposed on us by the proof of Lemma 7.0.2. A quick calculation shows that

$$c_1[h]_{C^\alpha_{k,x,v}(\mathbb{R}^6)} \le [h]_{C^\alpha_{E,x,v}(\mathbb{R}^6)} \le C_2[h]_{C^{3\alpha}_{k,x,v}(\mathbb{R}^6)}. \tag{7.0.5}$$

for constants c_1, C_2 depending on α. The second inequality here is the reason for the loss of Hölder exponent from 3α to α in Theorem 7.0.1, since the proof is based on propagating the C^α_E seminorm of f.

The key property of the function g, that it controls a weighted Hölder seminorm of f, is made precise by the following elementary lemma:

Lemma 7.0.1. *For any $\rho, \sigma > 0$, $T \in [0, \frac{\rho}{2\sigma}]$, $\alpha \in (0,1]$, and $\beta \in [0,1]$, the function g defined by (7.0.4) satisfies*

$$c_1 \sup_{0\le t\le T} [e^{(\rho/2)\langle v\rangle^\beta} f(t)]_{C^\alpha_{E,x,v}(\mathbb{R}^6)} \le \|g\|_{L^\infty([0,T]\times\mathbb{R}^6\times B_1(0)^2)} \le C_2 \sup_{0\le t\le T} [e^{\rho\langle v\rangle^\beta} f(t)]_{C^\alpha_{E,x,v}(\mathbb{R}^6)},$$

where the constants c_1 and C_2 depend on ρ, β, and α.

A direct calculation shows that g satisfies the equation

$$\begin{aligned}(\partial_t + v\cdot\nabla_x + k\cdot\nabla_h)g + \sigma\langle v\rangle^\gamma g + \frac{2\alpha h\cdot k}{|h|^2+|k|^2}g \\ = 2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f(t,x,v)}{(|h|^2+|k|^2)^\alpha}\left[\mathrm{tr}(\bar a^{\delta f} D_v^2 \tau f + \bar a^f D_v^2 \delta f) + \bar c^{\delta f}\tau f + \bar c^f \delta f\right]\end{aligned} \tag{7.0.6}$$

Using this equation, we will show that $g(t,x,v,h,k)$ is bounded above by a certain barrier function $G(t)$ up to a certain time value. The exact form of G will be dictated by the estimates that are available for g at a first crossing point. Therefore, we derive these estimates first:

Lemma 7.0.2. *Let f_0 be as in Theorem 7.0.1, and let f be a solution of the Landau equation* (1.0.1) *on $[0,T]\times\mathbb{R}^6$ for some $T>0$, with initial data f_0. Let g be defined by* (7.0.4)*, with*

$$\rho = \frac{4\alpha}{24-9\alpha}\rho_0,$$

where ρ_0 is the constant from (7.0.2)*.*

There exists $\sigma_0 > 0$ sufficiently large, such that if $\sigma \geq \sigma_0$, and $\zeta_0 = (t_0, x_0, v_0, h_0, k_0) \in [0,T]\times\mathbb{R}^6\times B_1(0)^2$ is a point with $t_0 \leq \min\left\{1, \dfrac{\rho}{2\sigma}\right\}$ and such that $g(t_0,\cdot)$ achieves its maximum over $\mathbb{R}^6 \times \overline{B_1(0)}^2$ at (x_0, v_0, h_0, k_0), then

$$\partial_t g \leq C_0\left(g + t_0^{-1+\mu(\alpha)} g^{1+\nu(\alpha)}\right) \qquad \text{at } \zeta_0,$$

where $\mu(\alpha) \in (0,1)$ and $\nu(\alpha) > 0$ depend only on α, and $C_0 > 0$ and σ_0 depend on ρ_0, β, δ, r, R, and $\|e^{\rho_0\langle v\rangle^\beta} f_0\|_{L^\infty(\mathbb{R}^6)}$.

Proof. First, the assumption (7.0.2) for f_0 implies, via Lemma 4.0.1, the decay estimate

$$\|e^{\rho_0\langle v\rangle^\beta} f\|_{L^\infty([0,T]\times\mathbb{R}^6)} \leq K_0,$$

for some K_0 depending only on T and the initial data. Here, we may assume $T \leq 1$, since the proof only needs this bound on f up to time $t_0 \leq 1$. Throughout the proof, we absorb dependence on this K_0 into constants.

Furthermore, because of our assumption that $t \leq t_0 \leq \dfrac{\rho}{2\sigma}$, we have $e^{(\rho-\sigma t_0)\langle v\rangle^\gamma} \geq e^{(\rho/2)\langle v\rangle^\gamma}$ for any $v \in \mathbb{R}^3$. This will be used repeatedly.

For the remainder of this proof, all evaluations of g, τf, and δf are assumed to be at ζ_0, and all evaluations of f are assumed to be at $z_0 = (t_0, x_0, v_0)$ unless otherwise noted.

Since ζ_0 is a local maximum point of g, we have

$$\nabla_x g = \nabla_v g = \nabla_h g = \nabla_k h = 0, \quad D_v^2 g \leq 0, D_k^2 g \leq 0, \quad \text{at } \zeta_0. \tag{7.0.7}$$

Therefore, evaluating (7.0.6) at ζ_0 results in

$$\partial_t g + \sigma\langle v\rangle^\gamma g \leq -\frac{2\alpha h\cdot k}{|h|^2+|k|^2} g + 2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f(t,x,v)}{(|h|^2+|k|^2)^\alpha}\left[\mathrm{tr}(\bar{a}^{\delta f} D_v^2 \tau f + \bar{a}^f D_v^2 \delta f) + \bar{c}^{\delta f}\tau f + \bar{c}^f \delta f\right]. \tag{7.0.8}$$

We want to bound the expression on the right by a power of $g(\zeta_0)$. First, we have the simple estimate

$$-\frac{2\alpha h\cdot k}{|h|^2+|k|^2}g\leq \alpha g.$$

Next, recalling that

$$0=\nabla_v g=\left[\gamma(\rho-\sigma t)\langle v\rangle^{\gamma-2}v|\delta f|^2+2\delta f\nabla_v(\delta f)\right]\frac{e^{(\rho-\sigma t)\langle v\rangle^{\gamma}}}{(|h|^2+|k|^2)^{\alpha}},$$

we see that

$$\nabla_v(\delta f)=-\frac{1}{2}\gamma(\rho-\sigma t)\langle v\rangle^{\gamma-2}v(\delta f). \tag{7.0.9}$$

We also calculate

$$\begin{aligned}D_v^2 g=&\left[\gamma\langle v\rangle^{\gamma-2}(\rho-\sigma t)(I+(\gamma-2)v\otimes v\langle v\rangle^{-2})(\delta f)^2+\gamma^2(\rho-\sigma t)^2\langle v\rangle^{2\gamma-4}v\otimes v(\delta f)^2\right.\\&\left.+4\gamma(\rho-\sigma t)\langle v\rangle^{\gamma-2}v\otimes\nabla_v(\delta f)\delta f+2\nabla_v(\delta f)\otimes\nabla_v(\delta f)+2\delta f D_v^2(\delta f)\right]\frac{e^{(\rho-\sigma t)\langle v\rangle^{\gamma}}}{(|h|^2+|k|^2)^{\alpha}},\end{aligned}$$

which implies

$$\begin{aligned}2\delta f\,\mathrm{tr}(\bar{a}^f D_v^2\delta f)=&\frac{(|h|^2+|k|^2)^{\alpha}}{e^{(\rho-\sigma t)\langle v\rangle^{\gamma}}}\mathrm{tr}(\bar{a}^f D_v^2 g)-\gamma\langle v\rangle^{\gamma-2}(\rho-\sigma t)[\mathrm{tr}(\bar{a}^f)+(\gamma-2)v\cdot(\bar{a}^f v)\langle v\rangle^{-2}](\delta f)^2\\&-\gamma^2(\rho-\sigma t)^2\langle v\rangle^{2\gamma-4}(\delta f)^2 v\cdot(\bar{a}^f v)-4\gamma(\rho-\sigma t)\langle v\rangle^{\gamma-2}\delta f v\cdot(\bar{a}^f\nabla_v(\delta f))\\&-2\nabla_v(\delta f)\cdot(\bar{a}^f\nabla_v(\delta f)).\end{aligned}$$

Since $\bar{a}^f\geq 0$ and $D_v^2 g\leq 0$ (recall (7.0.7)), several of the terms in the last expression have a good sign, and we are left with

$$2\delta f\,\mathrm{tr}(\bar{a}^f D_v^2\delta f)\leq\gamma(2-\gamma)(\rho-\sigma t)\langle v\rangle^{\gamma-4}v\cdot(\bar{a}^f v)(\delta f)^2-4\gamma(\rho-\sigma t)\langle v\rangle^{\gamma-2}\delta f v\cdot(\bar{a}^f\nabla_v(\delta f)).$$

For the second term in this right-hand side, we use (7.0.9), yielding

$$\begin{aligned}2\delta f\,\mathrm{tr}(\bar{a}^f D_v^2\delta f)&\leq\gamma(2-\gamma)(\rho-\sigma t)\langle v\rangle^{\gamma-4}v\cdot(\bar{a}^f v)(\delta f)^2+2\gamma^2(\rho-\sigma t)^2\langle v\rangle^{2\gamma-4}(\delta f)^2 v\cdot(\bar{a}^f v)\\&=\left[\gamma(2-\gamma)(\rho-\sigma t)\langle v\rangle^{\gamma-4}+2\gamma^2(\rho-\sigma t)^2\langle v\rangle^{2\gamma-4}\right](\delta f)^2 v\cdot(\bar{a}^f v).\end{aligned}$$

With Lemma 2.2.2, this gives

$$2\delta f \mathrm{tr}(\bar{a}^f D_v^2 \delta f) \le C\|f\|_{L^\infty_q([0,t_0]\times\mathbb{R}^6)} \langle v\rangle^{2\gamma-2}(1+\langle v\rangle^\gamma)(\delta f)^2,$$

for some arbitrarily chosen $q > 5+\gamma$. Absorbing $\|f\|_{L^\infty_q([0,T]\times\mathbb{R}^6)} \le CK_0$ into the constant, we then have

$$2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f(t,x,v)}{(|h|^2+|k|^2)^\alpha}\mathrm{tr}(\bar{a}^f D_v^2\delta f) \le C\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}}{(|h|^2+|k|^2)^\alpha}\langle v\rangle^{3\gamma-2}(\delta f)^2 \le C\langle v\rangle^\gamma g,$$

since $\gamma \le 1$.

Next, we address the term in (7.0.8) with $\mathrm{tr}(\bar{a}^{\delta f} D_v^2 \tau f)$. Noting that $\delta f = g^{1/2}(|h|^2 + |k|^2)^{\alpha/2} e^{-(\rho-\sigma t)\langle v\rangle^\gamma/2}$, we have

$$\begin{aligned}
|\bar{a}^{\delta f}(\zeta_0)| &\le C\int_{\mathbb{R}^3} |w|^{\gamma+2}|\delta f(t_0,x_0,v_0-w,h_0,k_0)|\,\mathrm{d}w\\
&= C\int_{\mathbb{R}^3} |w|^{\gamma+2} g^{1/2}(t_0,x_0,v_0-w,h_0,k_0)\frac{(|h_0|^2+|k_0|^2)^{\alpha/2}}{e^{(\rho-\sigma t)\langle v_0-w\rangle^\gamma/2}}\,\mathrm{d}w\\
&\le C\|g(t_0,\cdot)\|^{1/2}_{L^\infty(\mathbb{R}^6\times B_1^2)}(|h_0|^2+|k_0|^2)^{\alpha/2}\int_{\mathbb{R}^3}\frac{|w|^{\gamma+2}}{e^{(\rho-\sigma t)\langle v_0-w\rangle^\gamma/2}}\,\mathrm{d}w,\\
&\le Cg(\zeta_0)^{1/2}(|h_0|^2+|k_0|^2)^{\alpha/2}\langle v_0\rangle^{\gamma+2},
\end{aligned} \tag{7.0.10}$$

for a constant depending on ρ. We have used the fact that $g(\zeta_0)$ is the maximum value of $g(t_0,\cdot)$ over $\mathbb{R}^6\times B_1(0)^2$. This implies

$$\begin{aligned}
2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h|^2+|k|^2)^\alpha}\mathrm{tr}(\bar{a}^{\delta f}D_v^2\tau f) &\le 2Cg(\zeta_0)^{1/2}\langle v_0\rangle^{\gamma+2}|D_v^2\tau f(\zeta_0)|\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h_0|^2+|k_0|^2)^{\alpha/2}}\\
&\le Cg(\zeta_0)\|\langle v\rangle^{\gamma+2}e^{(\rho-\sigma t_0)\langle v\rangle^\gamma/2}D_v^2 f(t_0,\cdot)\|_{L^\infty(\mathbb{R}^6)}.
\end{aligned} \tag{7.0.11}$$

To bound this second-order norm of f, we apply Proposition 3.2.2 with $\alpha/2$ replacing α. We can apply Proposition 3.2.2 because of the lower bounds satisfied by f_0, which imply suitable lower ellipticity estimates for $\bar{a}^f$ via Theorem 5.0.1 and Lemma 2.2.3. We therefore have

$$\begin{aligned}
&\|\langle v\rangle^{\gamma+2} e^{(\rho-\sigma t_0)\langle v\rangle^\gamma/2} D_v^2 f(t_0,\cdot)\|_{L^\infty(\mathbb{R}^6)} \\
&\qquad\qquad \le \|e^{(\rho/2)\langle v\rangle^\beta} D_v^2 f(t_0,\cdot)\|_{L^\infty(\mathbb{R}^6)} \\
&\qquad\qquad \le C t_0^{-1-\alpha^2/(24-2\alpha)} \left(1 + \|e^{(3/\alpha-1)\rho\langle v\rangle^\beta} f\|_{C_k^{\alpha/2}([0,t_0]\times\mathbb{R}^6)}^{P(\alpha/2)}\right).
\end{aligned}$$

Since the weight $e^{(3/\alpha-1)\rho\langle v\rangle^\beta}$ grows too fast for us to close our estimates, we interpolate via Lemma 2.4.1, then apply Proposition 3.3.1 to translate to a Hölder norm in (x,v) variables:

$$\begin{aligned}
&\|\langle v\rangle^{\gamma+2} e^{(\rho-\sigma t_0)\langle v\rangle^\gamma/2} D_v^2 f(t_0,\cdot)\|_{L^\infty(\mathbb{R}^6)} \\
&\quad \le C t_0^{-1-\alpha^2/(24-2\alpha)} \left(1 + \|e^{(\rho/4)\langle v\rangle^\beta} f\|_{C_k^{\alpha}([0,t_0]\times\mathbb{R}^6)}^{P(\alpha/2)/2} \|e^{(6/\alpha-9/4)\rho\langle v\rangle^\beta} f\|_{L^\infty([0,t_0]\times\mathbb{R}^6)}^{P(\alpha/2)/2}\right) \\
&\quad \le C t_0^{-1-\alpha^2/(24-2\alpha)} \left(1 + \|e^{(\rho/2)\langle v\rangle^\beta} f\|_{L^\infty([0,t_0],C_{k,x,v}^{\alpha}(\mathbb{R}^6))}^{P(\alpha/2)/2}\right),
\end{aligned}$$

where we absorbed $\|e^{(6/\alpha-9/4)\rho\langle v\rangle^\beta} f\|_{L^\infty} \le K_0$ into the constant, since $(6/\alpha - 9/4)\rho = \rho_0$. From (7.0.5) and Lemma 7.0.1, we have

$$\begin{aligned}
\|e^{(\rho/2)\langle v\rangle^\beta} f\|_{L^\infty([0,t_0],C_{k,x,v}^{\alpha}(\mathbb{R}^6))} &\le \|e^{(\rho/2)\langle v\rangle^\beta} f\|_{L^\infty([0,t_0],C_{E,x,v}^{\alpha}(\mathbb{R}^6))} \\
&\le g(\zeta_0)^{1/2},
\end{aligned}$$

since ζ_0 is the location of the maximum value of g over $[0,t_0]\times\mathbb{R}^6\times(B_1(0))^2$. Returning to (7.0.11), we have shown

$$2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h|^2+|k|^2)^\alpha}\operatorname{tr}(\bar{a}^{\delta f} D_v^2 \tau f) \le C t_0^{-1-\alpha^2/(6-\alpha)}(g(\zeta_0) + g(\zeta_0)^{1+P(\alpha)/4}).$$

We now address the zeroth-order terms in (7.0.8). First, with Lemma 2.2.2,

$$2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h|^2+|k|^2)^\alpha}\bar{c}^f \delta f = 2\bar{c}^f g \le C\|f\|_{L_q^\infty([0,T]\times\mathbb{R}^6)}\langle v\rangle^\gamma g \le C\langle v_0\rangle^\gamma g,$$

where $q > 5+\gamma$ is arbitrary, and $\|f\|_{L_q^\infty} \le CK_0$ is absorbed into the constant as above. Next, proceeding in a similar way to (7.0.10), we have

$$\begin{aligned}|\bar{c}^{\delta f}(\zeta_0)| &\leq C\int_{\mathbb{R}^3}|w|^\gamma|\delta f(t_0,x_0,v_0-w,h_0,k_0)|\,\mathrm{d}w\\ &= C\int_{\mathbb{R}^3}|w|^\gamma\frac{(|h_0|^2+|k_0|^2)^{\alpha/2}}{e^{(\rho-\sigma t)\langle v_0-w\rangle^\gamma/2}}\,\mathrm{d}w\\ &\leq Cg(\zeta_0)^{1/2}(|h_0|^2+|k_0|^2)^{\alpha/2}\langle v_0\rangle^\gamma,\end{aligned}$$

and

$$\begin{aligned}2\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h|^2+|k|^2)^\alpha}\bar{c}^{\delta f}\tau f &\leq 2Cg(\zeta_0)^{1/2}\langle v_0\rangle^\gamma\tau f(\zeta_0)\frac{e^{(\rho-\sigma t)\langle v\rangle^\gamma}\delta f}{(|h_0|^2+|k_0|^2)^{\alpha/2}}\\ &\leq Cg(\zeta_0)\|\langle v\rangle^\gamma e^{(\rho-\sigma t)\langle v\rangle^\gamma/2}\tau f\|_{L^\infty([0,T]\times\mathbb{R}^6)}\\ &\leq Cg(\zeta_0)\|e^{\rho\langle v\rangle^\gamma/2}f\|_{L^\infty}\leq Cg(\zeta_0),\end{aligned}$$

since $\rho/2 \leq \rho_0$. Overall, we have shown that, at the point ζ_0,

$$\partial_t g + \sigma\langle v\rangle^\gamma g \leq C\langle v\rangle^\gamma g + Ct_0^{-1-\alpha^2/(24-2\alpha)}(g + g^{1+P(\alpha/2)/2}),$$

and the proof is complete after choosing $\sigma = C$. □

We are now ready to prove the main result of this section:

Proof of Theorem 7.0.1. First, we may assume f is smooth and decaying exponentially as $|v| \to \infty$, by passing to the approximating sequence f^ε from the proof of Theorem 1.1.1. These qualitative properties are used only to get a first crossing point in our barrier argument, so they do not quantitatively affect the estimate we are proving. Therefore, the estimate is preserved in the limit as $\varepsilon \to 0$, since $f^\varepsilon \to f$ pointwise. For simplicity, we write f instead of f^ε in this proof.

With g defined as in (7.0.4) with ρ as in the statement of the theorem, and σ as in Lemma 7.0.2, we note that

$$\|g(0,\cdot)\|_{L^\infty(\mathbb{R}^6\times B_1(0)^2)} < \infty,$$

as a result of (7.0.5) and our assumption (7.0.3) on f_0.

We want to show that g is bounded on some positive time interval. To show this, define the barrier $G(t)$ as the solution to the initial value problem

$$\begin{cases} \partial_t G = 2C_0(G + t^{-1+\mu(\alpha)} G^{1+\nu(\alpha)}), \\ G(0) = 1 + \|g(0,\cdot)\|_{L^\infty(\mathbb{R}^6 \times B_1(0))} + 4\|e^{(\rho/2)\langle v\rangle^\beta} f\|^2_{L^\infty([0,T]\times\mathbb{R}^6)}, \end{cases} \tag{7.0.12}$$

where $C_0 > 0$, $\mu(\alpha) \in (0,1)$, and $\nu(\alpha) > 0$ are the constants from Lemma 7.0.2. The norm $\|e^{(\rho/2)\langle v\rangle^\beta} f\|_{L^\infty}$ is finite from (7.0.2) and Lemma 4.0.1

The solution $G(t)$ to (7.0.12) exists on a time interval $[0, T_G]$, with T_G depending on α, C_0, and $G(0)$. Let $T^* = \min\{T_G, T, \rho/(2\sigma)\}$. We want to show that $g(t,x,v,h,k) < G(t)$ whenever $t \in [0, T^*]$. If this is false, then there must be a point $\zeta_0 = (t_0, x_0, v_0, h_0, k_0) \in [0,T^*] \times \mathbb{R}^6 \times \overline{B_1(0)}^2$, with $t_0 > 0$, where g and G cross for the first time. The existence of this point follows in a standard way from the compactness of the domain in the (t,x,h,k) variables (recall that f and g are periodic in x), as well as the decay of g for large $|v|$ (which follows by the qualitative rapid decay and smoothness of f).

Next, we point out that the crossing point ζ_0 occurs with both h_0 and k_0 in the interior of $B_1(0)$, since if $|h_0|^2 + |k_0|^2 \geq 1$, the definition (7.0.4) of g would imply, since $t_0 \leq T^* \leq \rho/(2\sigma)$,

$$G(t_0) = g(\zeta_0) \leq e^{(\rho - \sigma t_0)\langle v_0\rangle^\beta} (\delta f)^2(\zeta_0) \leq 4\|e^{(\rho/2)\langle v_0\rangle^\beta} f\|^2_{L^\infty([0,T]\times\mathbb{R}^6)} \leq G(0),$$

which is impossible since $G(t)$ is strictly increasing.

Since ζ_0 is the first crossing point between g and G, we have

$$\partial_t G(t_0) \leq \partial_t g(\zeta_0),$$

and because G is independent of (x,v,h,k), the point (x_0, v_0, h_0, k_0) is a maximum point for $g(t_0,\cdot)$, and we can apply Lemma 7.0.2:

$$\partial_t G(t_0) \leq C_0 \left(g(\zeta_0) + t_0^{-1+\mu(\alpha)} g(\zeta_0)^{1+\nu(\alpha)} \right) = C_0 \left(G(t_0) + t_0^{-1+\mu(\alpha)} G(t_0)^{1+\nu(\alpha)} \right) < \partial_t G(t_0),$$

by (7.0.12). This contradiction implies $g(t,x,v,h,k) < G(t)$ whenever $t \in [0,T^*]$. There is a time $T_2 \in (0,T_G)$ depending on α, C_0, and $G(0)$, such that $G(t_2) = 2G(0)$. Define

$$T_H = \min\left\{T_2, \frac{\rho}{2\sigma}\right\},$$

and $T^{**} = \min\{T, T_H\}$. With Lemma 7.0.1 and Lemma 4.0.1, we have

$$\begin{aligned}\sup_{0\le t\le T^{**}} [e^{(\rho/2)\langle v\rangle^\beta} f(t)]^2_{C^\alpha_{E,x,v}(\mathbb{R}^6)} &\le 2G(0)\\ &\le C\left(\|g(0,\cdot)\|_{L^\infty(\mathbb{R}^6\times B_1(0))} + \|e^{(\rho/2)\langle v\rangle^\beta} f\|^2_{L^\infty([0,T^{**}]\times\mathbb{R}^6)}\right)\\ &\le C\left(\|e^{\rho\langle v\rangle^\beta} f_0\|^2_{C^\alpha_{E,x,v}(\mathbb{R}^6)} + \|e^{(\rho/2)\langle v\rangle^\beta} f_0\|^2_{L^\infty(\mathbb{R}^6)}\right).\end{aligned}$$

Next, we translate this inequality to kinetic Hölder norms with (7.0.5):

$$\sup_{0\le t\le T^{**}} [e^{(\rho/2)\langle v\rangle^\beta} f(t)]^2_{C^\alpha_{k,x,v}(\mathbb{R}^6)} \le C\|e^{\rho\langle v\rangle^\beta} f_0\|_{C^{3\alpha}_{k,x,v}(\mathbb{R}^6)}.$$

Finally, we apply Proposition 3.3.1 to obtain

$$\|e^{(\rho/4)\langle v\rangle^\beta} f(t)\|^2_{C^\alpha_{k,x,v}([0,T^{**}]\times\mathbb{R}^6)} \le C\|e^{\rho\langle v\rangle^\beta} f_0\|_{C^{3\alpha}_{k,x,v}(\mathbb{R}^6)}.$$

as desired. The weight on the left side has been changed to $e^{(\rho/4)\langle v\rangle^\beta}$ to absorb the polynomial moments lost when applying Proposition 3.3.1.

Finally, we recall that T_H depends on ρ, σ, α, C_0, and $G(0)$. Since $G(0)$ is bounded in terms of $[e^{\rho\langle v\rangle^\beta} f_0]_{C^{3\alpha}_k(\mathbb{R}^6)}$ and $\|e^{\rho_0\langle v\rangle^\beta} f_0\|_{L^\infty(\mathbb{R}^6)}$, the proof is complete. □

Chapter 8

Uniqueness

This chapter proves the uniqueness of classical solutions. First, let us prove an auxiliary lemma:

Lemma 8.0.1. *Let $\phi(v) = e^{\rho \langle v \rangle^\gamma}$. For any $\mu > 0$, there holds*

$$\left\| \frac{h * |\cdot|^\mu}{\sqrt{\phi}} \right\|_{L^2_v(\mathbb{R}^3)} \leq C \|\sqrt{\phi} h\|_{L^2_v(\mathbb{R}^3)},$$

whenever h is such that the right-hand side is finite. Here, C is a constant depending only on ρ, γ, and μ.

Proof. From Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{1}{\phi} (h * |\cdot|^\mu)^2(v) \,\mathrm{d}v &= \int_{\mathbb{R}^3} \frac{1}{\phi} \left(\int_{\mathbb{R}^3} |v-w|^\mu h(w) \,\mathrm{d}w \right)^2 \mathrm{d}v \\
&= \int_{\mathbb{R}^3} \frac{1}{\phi} \left(\int_{\mathbb{R}^3} \frac{|v-w|^\mu}{\sqrt{\phi}(w)} \sqrt{\phi}(w) h(w) \,\mathrm{d}w \right)^2 \mathrm{d}v \\
&\leq C \int_{\mathbb{R}^3} \frac{1}{\phi(v)} \left\| \frac{|v-w|^\mu}{\sqrt{\phi}(w)} \right\|^2_{L^2_w(\mathbb{R}^3)} \|\sqrt{\phi} h\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}v \\
&\leq C \|\sqrt{\phi} h\|^2_{L^2(\mathbb{R}^3)} \int_{\mathbb{R}^3} \frac{1}{\phi(v)} \langle v \rangle^{2\mu} \,\mathrm{d}v \leq C \|\sqrt{\phi} h\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

with C as in the statement of the lemma. □

Now we are ready to prove our main uniqueness result:

Proof of Theorem 1.1.2. The difference $h = f - g$ satisfies the equation

$$\partial_t h + v \cdot \nabla_x h = Q(f,f) - Q(g,g) = Q(h,f) + Q(g,h). \tag{8.0.1}$$

Let $\rho \in (0, \rho_0)$ and $\sigma > 0$ be constants to be determined later, and define the weight $\phi(t,v) = e^{(\rho - \sigma t)\langle v\rangle^\gamma}$. We assume throughout the proof that $0 \le t \le T_U = \min\{\rho/(2\sigma), T_H\}$, where T_H is the constant from Theorem 7.0.1. In particular, this implies that $\phi \le e^{(\rho/2)\langle v\rangle^\gamma}$.

Multiplying (8.0.1) by $h\phi$ and integrating over $\mathbb{T}^3 \times \mathbb{R}^3$, the term $\iint \phi h v \cdot \nabla_x h \,\mathrm{d}v\,\mathrm{d}x$ vanishes since ϕ is independent of x, yielding

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\phi}h\|_{L^2}^2 + \frac{\sigma}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \langle v\rangle^\gamma \phi h^2 \,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h[Q(h,f) + Q(g,h)]\,\mathrm{d}v\,\mathrm{d}x \tag{8.0.2}$$

Looking at the terms on the right, we start with $\iint \phi h Q(h,f)\,\mathrm{d}v\,\mathrm{d}x$. For this term, we use the non-divergence form of the collision operator:

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h Q(h,f)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h \operatorname{tr}(\bar{a}^h D_v^2 f)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h \bar{c}^h f\,\mathrm{d}v\,\mathrm{d}x = I_1 + I_2.$$

For I_1, we first bound $D_v^2 f$ by combining Proposition 3.2.2 and Theorem 7.0.1 with $\alpha/3$ replacing α:

$$\begin{aligned}
\|(\phi D_v^2 f)(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^6)} &\le C\|e^{(\rho/2)\langle v\rangle^\gamma} D_v^2 f(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^6)} \\
&\le C(1 + t^{\kappa(\alpha)})\|e^{\rho'\langle v\rangle^\gamma} f\|_{C_k^{\alpha/3}([0,t]\times\mathbb{R}^6)} \\
&\le C(1 + t^{\kappa(\alpha)}),
\end{aligned}$$

where $\kappa(\alpha) = -1 + (\alpha/3)^2/(6 - \alpha/3)$ and $\rho' = (9/\alpha - 1)\rho$. From Theorem 7.0.1, we see that the constant C depends on $\|e^{\rho''\langle v\rangle^\gamma} f_0\|_{L^\infty(\mathbb{R}^6)}$, where $\rho'' = (24 - 9\alpha)/(\alpha)\rho'$. We choose the parameter ρ in the definition of our weight ϕ by setting $\rho'' = \rho_0$ and solving for ρ.

Returning to the term I_1, we bound $\bar{a}^h$ using Lemma 8.0.1 with $\mu = \gamma + 2$, giving

$$\begin{aligned}
\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h \mathrm{tr}(\bar{a}^h D_v^2 f)\,\mathrm{d}v\,\mathrm{d}x &\le \|(\phi D_v^2 f)(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^6)} \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \sqrt{\phi} h \frac{|\bar{a}^h|}{\sqrt{\phi}}\,\mathrm{d}v\,\mathrm{d}x \\
&\le C(1+t^{\kappa(\alpha)}) \int_{\mathbb{T}^3} \|\sqrt{\phi}h\|_{L^2_v(\mathbb{R}^3)} \left\| \frac{|\bar{a}^h|}{\sqrt{\phi}} \right\|_{L^2_v(\mathbb{R}^3)} \mathrm{d}x \\
&\le C(1+t^{\kappa(\alpha)}) \int_{\mathbb{T}^3} \|\sqrt{\phi}h\|^2_{L^2_v(\mathbb{R}^3)}\,\mathrm{d}x \\
&\le C(1+t^{\kappa(\alpha)}) \|\sqrt{\phi}h\|^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}.
\end{aligned}$$

For I_2, we have

$$\begin{aligned}
\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h \bar{c}^h f\,\mathrm{d}v\,\mathrm{d}x &\le \|(\phi f)(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \sqrt{\phi} h \frac{\bar{c}^h}{\sqrt{\phi}}\,\mathrm{d}v\,\mathrm{d}x \\
&\le \|(\phi f)(t)\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3)} \int_{\mathbb{T}^3} \|\sqrt{\phi}h\|_{L^2_v(\mathbb{R}^3)} \left\| \frac{\bar{c}^h}{\sqrt{\phi}} \right\|_{L^2_v(\mathbb{R}^3)} \mathrm{d}x \\
&\le C\|\phi f\|_{L^\infty} \int_{\mathbb{T}^3} \|\sqrt{\phi}h\|^2_{L^2_v(\mathbb{R}^3)}\,\mathrm{d}x = C\|\phi f\|_{L^\infty} \|\sqrt{\phi}h\|^2_{L^2},
\end{aligned}$$

by Lemma 8.0.1 with $\mu = \gamma$.

Next, we address the term $\iint \phi h Q(g,h)\,\mathrm{d}v\,\mathrm{d}x$ in (8.0.2). Here, we use the divergence form of the collision operator:

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h Q(g,h)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \phi h[\nabla_v\cdot(\bar{a}^g\nabla_v h) + \bar{b}^g h + \bar{c}^g h]\,\mathrm{d}v\,\mathrm{d}x = J_1 + J_2 + J_3.$$

For J_1, we integrate by parts in v:

$$J_1 = -\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} [\phi\nabla_v h + h\nabla_v\phi]\cdot(\bar{a}^g\nabla_v h)\,\mathrm{d}v\,\mathrm{d}x.$$

Since $\bar{a}^g$ is a postive-definite matrix, we have $(\nabla_v h + \varepsilon\nabla_v\phi)\cdot[\bar{a}^g(\nabla h + \varepsilon\nabla_v\phi)] \ge 0$ for any $\varepsilon > 0$, which implies the following inequality:

$$-\nabla_v\phi\cdot(\bar{a}^g\nabla_v h) \le \frac{\varepsilon}{2}\nabla_v\phi\cdot(\bar{a}^g\nabla_v\phi) + \frac{1}{2\varepsilon}\nabla_v h\cdot(\bar{a}^g\nabla_v h).$$

Choosing $\varepsilon = h/(2\phi)$, this gives

$$
\begin{aligned}
-\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} & h\nabla_v\phi\cdot(\bar{a}^g\nabla_v h)\,\mathrm{d}v\,\mathrm{d}x \\
&\le \int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\phi\nabla_v h\cdot(\bar{a}^g\nabla_v h)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\frac{\nabla_v\phi}{\phi}\cdot(\bar{a}^g\nabla_v\phi)\,\mathrm{d}v\,\mathrm{d}x,
\end{aligned}
\tag{8.0.3}
$$

and

$$
J_1 \le \frac{1}{4}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\frac{\nabla_v\phi}{\phi}\cdot(\bar{a}^g\nabla_v\phi)\,\mathrm{d}v\,\mathrm{d}x.
$$

From the upper for $\bar{a}^g$ in Lemma 2.2.1, we have

$$
\begin{aligned}
J_1 &\le \frac{\gamma^2}{4}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\phi\langle v\rangle^{2\gamma-4}v\cdot(\bar{a}^g v)\,\mathrm{d}v\,\mathrm{d}x \\
&\le C_g\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\phi\langle v\rangle^{3\gamma-2}\,\mathrm{d}v\,\mathrm{d}x \le C_g\|\sqrt{\phi}h\|^2_{L^2_{\gamma/2}(\mathbb{T}^3\times\mathbb{R}^3)}.
\end{aligned}
$$

For J_2, the growth of $\bar{b}^g \approx \langle v\rangle^{\gamma+1}$ presents a difficulty, since the good term on the left side of (8.0.2) only allows us to absorb a weight like $\langle v\rangle^\gamma\phi$. To get around this, we integrate by parts repeatedly and use the facts that $\bar{b}^g_i = -\sum_{j=1}^3 \partial_{v_j}\bar{a}^g_{ij}$ and $\nabla_v\cdot\bar{b}^g = \bar{c}^g$:

$$
\begin{aligned}
J_2 &= \frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\phi\bar{b}^g\cdot\nabla_v(h^2)\,\mathrm{d}v\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\left(\sum_{j=1}^3\partial_{v_j}\bar{a}^g_{ij}\right)\cdot\nabla_v\phi\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\phi\bar{c}^f h^2\,\mathrm{d}v\,\mathrm{d}x \\
&= -\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h\nabla_v\phi\cdot(\bar{a}^g\nabla_v h)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\mathrm{tr}(\bar{a}^g D_v^2\phi)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\phi\bar{c}^g h^2\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

In this right-hand side, the first term is equal to the term handled above in (8.0.3), and we estimate it in the same way. The third term is equal to $-\frac{1}{2}J_3$. For the middle term, we use the expression (2.4.3) (with $\rho - \sigma t$ replacing ρ) for $\mathrm{tr}(\bar{a}^g D_v^2 \phi)$ and discard negative terms to obtain

$$\begin{aligned}
-\frac{1}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} & h^2 \mathrm{tr}(\bar{a}^g D_v^2\phi)\,\mathrm{d}v\,\mathrm{d}x \\
&= -\frac{(\rho-\sigma t)\gamma}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\phi\langle v\rangle^{\gamma-4}\left[\left((\gamma-2)+(\rho-\sigma t)\beta\langle v\rangle^{\beta}\right)v\cdot(\bar{a}^g v)+\langle v\rangle^2\mathrm{tr}(\bar{a}^g)\right]\mathrm{d}v\,\mathrm{d}x \\
&\le \frac{(\rho-\sigma t)\gamma(\gamma-2)}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} h^2\phi\langle v\rangle^{\gamma-4} v\cdot(\bar{a}^g v)\,\mathrm{d}v\,\mathrm{d}x \\
&\le C_g\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\langle v\rangle^{2\gamma-2}h^2\phi\,\mathrm{d}v\,\mathrm{d}x = C_g\|\sqrt{\phi}h\|^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)},
\end{aligned}$$

after using Lemma 2.2.2. For J_3, Lemma 2.2.2 implies

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\phi h^2\bar{c}^g\,\mathrm{d}v\,\mathrm{d}x \le C\|g\|_{L^\infty_q}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\langle v\rangle^\gamma\phi h^2\,\mathrm{d}v\,\mathrm{d}x \le C\|g\|_{L^\infty_q}\|\sqrt{\phi}h\|^2_{L^2_{\gamma/2}(\mathbb{T}^3\times\mathbb{R}^3)}.$$

Putting everything together, we have

$$\frac{1}{2}\|\sqrt{\phi}h\|^2_{L^2}+\frac{\sigma}{2}\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\langle v\rangle^\gamma\phi h^2\,\mathrm{d}v\,\mathrm{d}x \le C\left(1+t_0^{\kappa(\alpha)}\right)\|\sqrt{\phi}h\|^2_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}+C\|\sqrt{\phi}h\|^2_{L^2_{\gamma/2}(\mathbb{T}^3\times\mathbb{R}^3)}.$$

The constant in this inequality depends only on α, ρ_0, and $\|e^{\rho_0\langle v\rangle^\gamma}f_0\|_{C^\alpha_{k,x,v}(\mathbb{R}^6)}$. Choosing $\sigma = C$ and applying Gronwall's inequality, we obtain

$$\|\sqrt{\phi}(t)h(t)\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)} \le \|\sqrt{\phi}(0)h(0)\|_{L^2(\mathbb{T}^3\times\mathbb{R}^3)}\exp(C(t_0+t_0^{1+\kappa(\alpha)}) = 0,$$

so that $h(t,x,v)=0$ for all t and almost every (x,v). By continuity, $h(t,x,v)=0$ everywhere, and we conclude $f=g$ pointwise. Since T_U depends on ρ, σ, and T_H, we conclude the proof. □

Chapter 9

Conclusion

In this dissertation, the Landau equation from plasma physics in the case of hard potentials was analyzed. In particular, a solution was constructed for the initial-boundary value problem with periodic boundary conditions in the x variable.

The key contributions were: an existence result on a time interval $[0, T]$ with initial data that is much more general than was allowed by previous work in the literature, (ii) a uniqueness result in the case that the initial state is vacuum-free and Hölder continuous, (iii) results about the spreading of positivity, which imply both that vacuum regions in the initial data are instantly filled, and that vacuum cannot spontaneously form, (iv) a result showing that Hölder continuity at time zero is propagated forward in time, which was used as a lemma in the proof of uniqueness but is also interesting in its own right.

The broader question of global-in-time existence (i.e. replacing the time interval $[0, T]$ with $[0, \infty)$ in our results) is a difficult unsolved problem, but the results in this dissertation may shed some light on this problem, either by providing tools that could be used on the way to proving global existence, or by ruling out some possible types of singularities. The strategies developed here could also potentially be adapted to the non-cutoff Boltzmann equation with hard potentials.

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