Continuous brittle crushing occurs in the movement of an ice sheet against an offshore structure. Matlock’s ice-structure interaction model is used to simulate the behavior of the ice crushing by modeling ice teeth indentation contacting a spring-mass-dashpot structure. The dynamic behavior of this ice-structure interaction system is studied using Fourier analysis to predict the response of specific periodicity. The system’s equations of motion are established based on the assumption of continuous ice indentation. This assumption allows immediate contact of the structure with the next tooth at the time of fracturing of a previous tooth. The time histories of tooth deflections are expressed through the non-linear dynamic equations. The kinematic initial conditions of the system can be predicted at targeted periodicity via the Fourier analysis. Furthermore, given a representative system, the amplitudes of the structural dynamic vibration predicted by the Fourier analysis compare well to more precise periodic solutions found by the mathematical closed-form simulation.
Introduction

This paper considers the nonlinearity of a simplified ice-structure interaction model (Matlock et al., 1971) and predicts the kinematic response for the vibrations at specific periodicity. Previous studies indicate that periodic behavior of the ice-structure interaction is highly non-linear and difficult to predict due to geometrical variability in the intermittent ice breakage and contact to the structure (Karr et al., 1993). The average and maximum magnitude of the structural contact forces are determined for the structural motion responses (Jonkman, 2009 and Yu, 2014). The periodic cycles, the average contact forces and the magnitude of the oscillating force are key factors for estimating structural fatigue life.

It can be observed that the steady-state responses previously obtained were found by selection of initial conditions in gridded space. However, due to the limited experimental volume, it’s not feasible to examine all possible combinations of inputs. Our aim is to predict the behavior of the dynamic response at any specific periodicity by expanding the dynamic equations of motion using Fourier analysis.

Based on Matlock’s model, Karr et al. (1993) discussed the actual force time histories which show oscillations and are highly dependent upon the initial velocities and physical properties of the ice-structure dynamic system. Forces intermittently rise and drop with respect to the deflection and breakage of the ice-teeth; the cyclical forces thus form an intermittent repeating process. This dynamic system is complicated by the ice deformation response, variation in ice-properties, geometry of the contact interface, as well as the dynamics of repeated impacts in each cycle. The imperfect system may have random variation in the ice-pitch, ice-stiffness and ice-strength to reflect the complexity of a real problem. However, the perfect system discussed here is argued to be representative of the more complicated imperfect system by showing similar characteristics.

Many mathematical approaches have been applied to solve non-linear dynamic system response with similar features of intermittent contact forces. Wang (1994) used the Trigonometric Collocation method to eliminate the need to evaluate the integrals of systems of mild nonlinearity. Wong et al. (1991) applied the Incremental Harmonic Balance (IHB) method to obtain all possible harmonic responses of unsymmetrical piecewise-linear systems. However, these methods are computationally expensive and cannot predict the specific periodicity and the oscillating amplitude. Karr et al. (1995) and Yu (2014) discussed periodic solutions for the Matlock’s ice-structure interaction model from the closed-form solution. Similarly, the orbits of the steady-state periodic responses are not predictable a priori due to the numerical integrations over time steps and the non-linear nature of the dynamic relations. The periodic solutions are found only by simulation from arbitrary initial conditions.

While it has been noted in previous research that an overshoot effect will occur at the jump discontinuity using finite Fourier series, the Gibbs constant can be applied to reduce the overshoot effect (Foster and Richards, 1991). David and Shu (1997) discuss the sufficiency of achieving the same order of accuracy as in the case of smooth functions by applying expansion coefficients. We apply the traditional Gibbs constant \( g = 0.1790 \) to adjust the overshoot effect in calculation of the initial position of the structure in the dynamic system.
Mathematical Model

Based on Matlock’s (1971) ice-structure interaction model, a first-order approximation for the dynamic ice-structure interaction modeling is a mass-spring-dashpot system with a single degree of freedom (Figure 1).

![Figure 1a](image1.png)  
**Figure 1a.** Ice brittle crushing against an offshore structure  
**Figure 1b.** Simplified Matlock’s dynamic model for ice-structure interaction

The model parameters shown in Figure 1 are: \( M \) — oscillator mass; \( C \) — oscillator damping coefficient; \( K_1 \) — stiffness of oscillator spring; \( K_2 \) — ice teeth stiffness; \( y(t) \) — displacement of the mass oscillator; \( z(t) \) — displacement of the ice sheet; \( \Delta(t) \) — deflection of ice-tooth; \( \bar{P} \) — distance between teeth interval (ice pitch); \( u \) — constant velocity of the ice-sheet in the \( y \) direction.

Following the normalization procedure of Karr et al. (1993), we define the non-dimensional system parameters with respect to the structure’s stiffness \( K_1 \) and the maximum ice forcing \( F_{\text{max}} \) on the oscillator due to the ice teeth deflection at its maximum:

\[
\begin{align*}
  k_{\text{ice}} &= \frac{K_2}{K_1} \\
  \delta &= \frac{\Delta}{\Delta_{\text{max}}} \\
  \bar{y} &= \frac{F_{\text{max}}}{K_1} \\
  x &= \frac{y}{y} \\
  U &= \frac{u}{y} \\
  p &= \frac{p}{y} \\
  z_0 &= \frac{z(t=0)}{y}
\end{align*}
\]

[1]

Time is normalized with respect to the natural angular velocity of the structure \( \omega_n \), \( N \) is the number of ice-breakage during each cycle of movement, hence:

\[
\tau = \omega_n t \quad \omega_n^2 = \frac{K_1}{M} \quad T = \frac{Np}{u}
\]

[2]

where \( T \) is the normalized period for a single cycle. Substituting the parameters in Eq. [1] and Eq. [2] into the equations of motion, we obtain the governing differential equations with non-dimensionalized parameters as follows:

\[
\ddot{x}(\tau) + 2\zeta \dot{x}(\tau) + x(\tau) = \begin{cases} 0, & \delta \leq 0 \text{ or } \delta = 1 \\ k_{\text{ice}}[z_0 + Ut - x(\tau) - p(i - 1)], & 0 < \delta < 1 \end{cases}
\]

[3]
In Eq. [3] the normalized damping coefficient $\zeta$ is $\frac{c}{2M\omega_n}$, and $z_0$ is the initial position of the ice sheet at $\tau = 0$, and $i$ is the tooth number active when in contact with the mass ($i = 1, 2, 3...$). At $\tau = 0$, the first tooth is in immediate contact with the mass with $\delta(0) = 0$ ($i = 1$).

Defining $d = p(i - 1)$, the tooth deflection at the initial point $\delta(0)$ is 0. The kinematic expression for tooth deflection $\delta(\tau)$ is:

$$\delta(\tau) = [z_0 - x(\tau) + Ut - d]k_{ice} \quad [4]$$

In an effort to expand the deflection $\delta(\tau)$ in a Fourier series, it’s assumed that no teeth separate from the mass during each cycle of movement. This assumption implies immediate contact with the following tooth at the fracture of a previous tooth and it is justified in the perfect dynamic system where the evenly distributed teeth pitch $\bar{p}$ equals the maximum tooth deflection $\Delta_{max}$. Rearranging Eq. [3] by applying the constraint of $0 \leq \delta \leq 1.0$ yields:

$$\dot{x}(\tau) + 2\zeta\ddot{x}(\tau) + (1 + k_{ice})x(\tau) = k_{ice}[z_0 + Ut - d] \quad [5]$$

For a periodicity of N-teeth breakage per cycle (P-N response), we have:

$$UT = Np \quad [6]$$

The breakage occurs at time $\tau = \alpha_i T$, where $i = 1..(N - 1), N; \alpha_i$ is the time ratio within one cycle of period $T$ when the $i^{th}$ tooth breakage occurs, and $\alpha_N = 1$. The last two terms in Eq. [5] can then be expressed by the Heaviside step function as follows:

$$Ut - d = Ut - pH[\tau - \alpha_1 T] - pH[\tau - \alpha_2 T]... - pH[\tau - \alpha_N T] \quad [7]$$

Defining $g(\tau) = k_{ice}(z_0 + Ut - d)$, $g(\tau)$ is expanded in a Fourier series:

$$g(\tau) = k_{ice}(z_0 + Ut - d) = z_0k_{ice} + Utk_{ice} - k_{ice}p \sum_{i=1}^{i=N} H[\tau - \alpha_i T]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega \tau) + b_n \sin(n\omega \tau)] \quad [8]$$

The Fourier coefficients are calculated as:

$$a_0 = 2k_{ice}z_0 - Npk_{ice} + 2pk_{ice} \sum_{i=1}^{N} \alpha_i$$

$$a_n = \frac{pk_{ice}}{\pi n} \sum_{i=1}^{N} \sin(2\pi n \alpha_i)$$

$$b_n = -\frac{pk_{ice}}{\pi n} \sum_{i=1}^{N} \cos(2\pi n \alpha_i) \quad [9]$$

Substituting Eq. [9] into Eq. [5], the steady state displacement trajectory $x(\tau)$ is:

$$x(\tau) = \frac{a_0}{2(1 + k_{ice})} + \sum_{n=1}^{\infty} \frac{[a_n \cos(n\omega \tau - \phi_n) + b_n \sin(n\omega \tau - \phi_n)]}{\beta_n} \quad [10]$$
where
\[
\tau = \frac{2\pi}{T}
\]
\[
\beta_n = [(1 + k_{ice}) - (nr)^2]^2 + (2\zeta nr)^2
\]
\[
\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1 + k_{ice} - (nr)^2}\right)
\]

The changes in displacement between two points of breakage are expressed by the following relation, where \(q = 1, 2, \ldots (N - 1)\):
\[
x(t)|_{t=a_q T} - x(t)|_{t=0} = p(q - N\alpha_q)
\]

Substituting Eq. [14] into Eq. [10], yields (N-1) equations for a specific Period-N response:
\[
F(a_q) = \\
\sum_{n=1}^{N} \frac{1}{\sqrt{\beta_n}} [\sum_{i=1}^{n} \sin(2\pi a_i) \{\cos\phi_n - \cos(2\pi a_q - \phi_n)\} + \sum_{i=1}^{N} \cos(2\pi a_i) \{\sin\phi_n + \sin(2\pi a_q - \phi_n)\}] - \frac{\pi}{k_{ice}} (q - N\alpha_q) = 0
\]

Furthermore, recalling the kinematic relationship for tooth deflection in Eq. [4] and the maximum deflection limit \(\delta_{max} = 1.0\), we obtain the initial location \(z_0\) for the ice sheet at time \(\tau = 0\) as follows:
\[
z_0 = -\frac{N p k_{ice}}{2} + p k_{ice} \sum_{i=1}^{N} \alpha_i + (1 + k_{ice}) \sum_{n=1}^{\infty} \frac{(a_n \cos\phi_n - b_n \sin\phi_n)}{\sqrt{\beta_n}}
\]

The initial velocity of the oscillator at \(\tau = 0\) is calculated as:
\[
\dot{x}(t)_{\tau=0} = \sum_{n=1}^{\infty} \frac{nr(a_n \sin\phi_n + b_n \cos\phi_n)}{\sqrt{\beta_n}}
\]

**Periodic Motion Response Predictions**

To seek the motion response for a specific periodicity, we first assume that the number of tooth breakages is \(N\) for each cycle. The \(N^{th}\) element \(a_N\) of the vector \(\alpha\) equals 1.0, and the remaining elements \(\alpha_{1\ldots N-1}\) are unknowns. The corresponding breaking time ratios \(\alpha_i\) can be determined numerically from the \((N-1)\) non-linear equations \(F(\alpha)\), as expressed in Eq. [15]. The corresponding time history of teeth deflection \(\delta(\tau)\) is thus determined through the set \(\alpha\), but the \(\alpha\) must be examined to verify that the responses are within the constraints of \(0.0 \leq \delta(\tau) \leq 1.0\). In the following sample calculations for a given system, periodic solutions of \(N=1\) (P-1) to \(N=5\) (P-5) have been examined and the calculated displacements are compared with the results from the closed-form solutions.

The system parameters used in the sample periodic motion predictions for both the Fourier analysis and the closed-form simulation are:
\[
U = \frac{10}{54\pi}, \quad p = \frac{2}{9}, \quad k_{ice} = 4.5, \quad \zeta = 0.06
\]
Effort is given to verify the accuracy of the predicted amplitude of motion and the occurrence of tooth-breakage for specific periodicity. The fixed point of breakage for the closed-form P-1 solution is $x(0) = 0.56, \dot{x}(0) = -0.015$. It is observed that the predicted displacement of P-1 response by Fourier analysis is in close agreement with the displacement simulated from the closed form solution (Figure 2a). However, at the time of tooth fracturing, it is observed that the displacement-time derivative from the Fourier simulation is less than the velocity obtained by the closed-form solution. The normalized velocity at breakage is -0.015 from the closed-form solution, and it is -0.37 from the Fourier simulation. The difference in velocity is caused by the Gibbs effect of overshooting at the point of discontinuity due to tooth-breakage. The ice-tooth deformation forcing obtained by Fourier analysis is gradual at the breakage of $\delta(x = a_i T)$ rather than shifting directly to zero. The overshooting effect in time history of the tooth-deflection is estimated to be 0.09, which agrees with the product of Gibbs constant $g$ times one-half of the jump size at the point of breakage (Figure 2b). There is thus a source of error in estimating the velocity of the mass at breakage due to the Gibbs effect. In fact, inputting the kinematic initial condition at breakage from the calculated P-1 response into the closed-form simulator, a periodic solution of P-5 is obtained.

![Figure 2a](image1.png)  ![Figure 2b](image2.png)

**Figure 2a.** The P-1 response $x(\tau)$ obtained by Fourier series and closed-form solution

**Figure 2b.** Tooth deflection $\delta(\tau)$ for a P-1 response by Fourier analysis

The $\alpha$ components for P-3 response are calculated as $\alpha_1 = 0.095, \alpha_2 = 0.41$, which compares well to the vibrations in Karr et al.’s (1993) steady-state P-3 response (Figure 3). Less than 6.6% of difference in the amplitude of motion is found, and the tooth breakage occurrences are in close agreement. Also, similar observations are found for P-2 response.

In addition to this periodic response, another possible P-3 response is calculated from the Fourier analysis (Figure 4a). Moreover, we find possible P-4 solutions which are missing from the previous closed-form solutions (Karr et al., 1993). One typical simulation is shown in Figure 4b. It is observed that both the P-3 and P-4 responses resemble a portion of the oscillating motion from the closed-form P-25 steady-state response (Figure 5a). This P-25 response is obtained by using the Fourier calculated breakage initial conditions from a P-3 response: $x(0) = 0.49, \dot{x}(0) = -0.48$. Another closed-form solution with static initial condition $x(0) = 0, \dot{x}(0) = 0$ is shown in Figure 5b. It is noticed that this response consists of transient indentations during which the mass sweeps through 5, 4, 2 and 3 tooth-breakages respectively. The amplitudes of the
transient response from Figure 5b resemble the motion amplitudes from the P-3 to P-4 responses calculated by Fourier analysis (Figure 4) with the same number of tooth-breakage in one single sweeping cycle.

**Figure 3.** Comparison of a P-3 response by closed-form simulator \((x(0) = 0.66, \dot{x}(0) = 0.021)\) and by Fourier series analysis \((\alpha = [0.095, 0.41, 1.00])\)

**Figure 4a.** A P-3 response calculated by Fourier analysis \((\alpha = [0.83, 0.91, 1.00])\)

**Figure 4b.** A P-4 response calculated by Fourier analysis \((\alpha = [0.094, 0.83, 0.92, 1.00])\)

**Figure 5a.** Time history \(x(t)\) by Closed-form solution with input \(x(0) = 0.49, \dot{x}(0) = -0.48\)

**Figure 5b.** Time history \(x(t)\) by Closed-form solution with input \(x(0) = 0, \dot{x}(0) = 0\)
Finally, the time history of displacements for a P-5 response predicted by the Fourier analysis compares well to the steady-state closed-form solutions in terms of the motion of response and the tooth breakage occurrence (Figure 6). Furthermore, the predicted motion of amplitude for P-5 by Fourier analysis is in agreement with the transient response shown in Figure 5b for P-1 response with 5 teeth breaking in the first sweep. Closed-form solutions for steady-state P-1, P-2, P-3, P-5 and P-25 responses have been recorded by random initial inputs. The P-5 response features the maximum oscillating magnitude from the Fourier periodic solutions from P-1 through P-5 responses. The transient motion resembling the P-5 response shown in Figure 5b is thus not negligible. Therefore the Fourier analysis can be used to estimate the extreme motions of the dynamic system for both transient and steady-state response.

![Graph](image)

**Figure 6a.** A P-5 response by Closed-form simulator (initial condition \( x_0 = 0.93, x_0 = -0.012 \)).

**Figure 6b.** A P-5 response predicted by Fourier analysis (\( \alpha = [0.027, 0.058, 0.087, 0.11, 1.00] \))

**Conclusions**

In this paper, we expand the equations of motion in Fourier series, and set up the relationships among the system parameters to evaluate the responses for specific steady-state periodicity. Our approach establishes the non-linear dynamic equations through Fourier analysis with respect to the number of tooth-breakages \( N \) per cycle. This method allows rapid estimation for the range of motion and the evaluation of structural contact forces. The amplitudes predicted by our Fourier analysis solution correspond well to the simulation results obtained from closed-form solutions with random initial condition selections. Furthermore, the time ratios of breakage are accurately predicted thus the cyclic behavior can be analyzed accordingly. Also, with the calculated structural periodic responses, the mean value and the magnitude of the oscillating contact forces can be obtained. These output parameters are key factors for strength and fatigue life assessment. The previously un-detected periodic response of a P-4 is found through our Fourier solution. Further effort should be given to validate the basin of attractions given a representative system and more specific evaluation of the error range in the velocity predictions due to the Gibbs effect.
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References


